A Convex Reformulation of the Controller Synthesis Problem for MIMO Single-Delay Systems with Implementation in SOS

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***Abstract*— In this paper, we propose a new dual class of stability condition for MIMO single-delay systems which is based on the implicit existence of a Lyapunov-Krasovskii functional but does not explicitly construct such a functional. This new type of stability condition allows the controller synthesis problem to be formulated as a convex optimization problem with little or no conservatism using a variable transformation. Furthermore, we show how to invert this variable transformation in order to obtain the stabilizing controller. The stability and controller synthesis conditions are then enforced using the SOS framework exploiting recent advances in this field. Numerical testing verifies there is little to no conservatism in either the "dual" stability test or the controller synthesis condition.**

I. Introduction

Systems with delay have been studied for some time [1], [2], [3]. Recently, there have been many results on the use of optimization and semidefinite programming for stability of linear and nonlinear time-delay systems. Although the computational question of stability of a linear state-delayed system is believed to be NP-hard, several techniques have been developed which use LMI methods [4] to construct sequences of polynomial-time algorithms which provide sufficient stability conditions and appear to converge to necessity as the complexity of the algorithms increase. Examples of such sequential algorithms include the piecewise-linear approach [2], the delay-partitioning approach [5], the Wirtinger-based method [6] and the SOS approach [7]. In addition, there are frequency-domain approaches such as [8], [9]. These algorithms are sufficiently reliable so that for the purposes of this paper, we consider the problem of stability analysis of linear discrete-delay systems to be solved.

The purpose of this paper is to explore methods by which the success in stability analysis of time-delay systems may be used to attack what may be considered the relatively underdeveloped field of robust and optimal controller synthesis. Although there have been a number of results on controller synthesis for time-delay systems [10], none of these results has been able to resolve the fundamental bilinearity of the synthesis problem. That is, controller synthesis requires us to find both a Lyapunov operator $\mathcal{P}$ and a feedback operator $\mathcal{K}$. Unfortunately, however, the bilinear, non-convex term $\mathcal{P}\mathcal{B}\mathcal{K}$ appears in the synthesis conditions. Without convexity in the decision variables, it is difficult to construct provably stabilizing controllers without significant conservatism, much less address the problems of robust and quadratic stabilization. Some papers use iterative methods to alternately optimize the Lyapunov operator and controller as in [11] or [12] (via a "tuning parameter"). However, this iterative approach is not guaranteed to converge. Meanwhile, approaches based on frequency-domain methods, discrete approximation, or Smith predictors result in controllers which are not provably stable or are sensitive to variations in system parameters or in delay. Finally, we mention that delays often occur in both state and input and to date most methods do not provide a unifying formulation of the controller synthesis problem with both state and input delay.

This paper covers several significant results in a relatively short conference format. As a result, discussion is often compressed and the proofs are shortened, omitted, or referenced to prior work. Full proofs and discussion will be treated in an expanded future journal format. There are six main results we must cover. First, we give a general dual stability condition for a broad class of infinite-dimensional systems. Roughly speaking, this result says $\dot{x} = \mathcal{A}x$ is stable if there exists a $\mathcal{P} > 0$ such that $\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* < 0$ where if X is the set of solutions, $\mathcal{P}(X) = X$. We then apply this dual criterion to single delay systems to get a dual version of Lyapunov-Krasovskii theory. We then use LMIs to parameterize the set of positive operators with polynomial multipliers and kernels for which $\mathcal{P} > 0$ and $\mathcal{P}(X) = X$. We solve the resulting LMIs numerically for several examples and demonstrate that the results are not significantly conservative. We then perform a variable substitution to get synthesis conditions of the form $\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* + \mathcal{B}\mathcal{Z} + \mathcal{Z}^*\mathcal{B}^* < 0$ where $\mathcal{Z} = \mathcal{K}\mathcal{P}$. We then parameterize $\mathcal{Z}$ using polynomial multipliers and kernels and solve the LMI as a test for stabilizability. We propose a method for inverting $\mathcal{P}$ and use this to find the controller $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$. We then test the stabilizing controller on a numerical example.

A. Notation

Notation includes the Hilbert spaces L_2 of square integrable functions and $W_2 := \{x : x, \dot{x} \in L_2\}$ with domains clear from context. $\mathcal{C}[X]$ denotes the continuous functions on X. S^n denotes the symmetric matrices of dimension $n \times n$. $I_n \in \mathbb{S}^n$ denotes the identity matrix.

II. Lyapunov Krasovskii Functionals

In this paper, we consider stability and control of linear discrete-delay systems of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t) \qquad \text{for all } t \geq 0,$$
$$x(t) = \phi(t) \qquad \text{for all } t \in [-\tau, 0] \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\phi \in \mathcal{C}[-\tau, 0]$. We associate with any solution x and any time $t \geq 0$, the 'state'

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of System (1), $x_t \in \mathcal{C}[-\tau, 0]$, where $x_t(s) = x(t+s)$. When $u = 0$, systems of the form (1) have a unique solution for any $\phi \in \mathcal{C}[-\tau, 0]$ and global, local, asymptotic and exponential stability are all equivalent.

Stability of Equations (1) may be certified through the use of Lyapunov-Krasovskii functionals - an extension of Lyapunov theory to systems with infinite-dimensional state-space. In particular, it is known that stability of linear time-delay systems is *equivalent* to the existence of a quadratic Lyapunov-Krasovskii functional of the form

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T M(s) \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta)\, ds\, d\theta, \qquad (2)$$

where the Lie (upper-Dini) derivative of the functional is negative along any solution x of (1) and the unknown functions M and N are continuous in their respective arguments. One may also assume $M = M^T$ and $N(s,\theta) = N(\theta, s)^T$.

Primal Lyapunov-Krasovskii Form: For reference, the primal stability condition is

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s)\\ \tau M_{21}(s) & \tau M_{22}(s)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(s)\end{bmatrix} ds + \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T N(s,\theta)\phi(\theta) d\theta ds$$

such that $V(\phi) \geq \|\phi(0)\|^2$ and

$$V_D(\phi) = \int_{-\tau}^{0} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix}^T \begin{bmatrix} D_{11}+D_{11}^T & D_{12} & \tau D_{13}(s)\\ D_{12}^T & -M_{22}(-\tau) & \tau D_{23}(s)\\ \tau D_{13}(s)^T & \tau D_{23}(s)^T & -\tau \dot{M}_{22}(s)\end{bmatrix} \begin{bmatrix}\phi(0)\\ \phi(-\tau)\\ \phi(s)\end{bmatrix} ds$$
$$- \tau \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds}N(s,\theta) + \frac{d}{d\theta}N(s,\theta)\right)\phi(\theta) d\theta ds \leq -\epsilon \|\phi\|^2$$

$$D_{11} = M_{11}A_0 + M_{12}(0) + \frac{1}{2}M_{22}(0),$$
$$D_{12} = M_{11}A_1 - M_{12}(-\tau), \quad D_{23} = A_1^T M_{12}(s) - N(-\tau, s)$$
$$D_{13} = A_0^T M_{12}(s) - \dot{M}_{12}(s) + N(0, s).$$

The use of Lyapunov-Krasovskii functionals can be simplified by considering stability in the semigroup framework - a generalization of the concept of differential equations. A 'strongly continuous semigroup' is an operator, $S(t) : Z \to Z$, defined by the Hilbert space Z, which for any solution of Eqn. (1), x, satisfies $x_{t+s} = S(s)x_t$. Note that for a given Z, the semigroup may not exist even if the solution exists for any initial conditions in Z. Associated with a semigroup on Z is an operator $\mathcal{A}$, called the 'infinitesimal generator' which satisfies

$$\frac{d}{dt}S(t)\phi = \mathcal{A}S(t)\phi$$

for any $\phi \in X$. The space $X \subset Z$ is often referred to as the domain of the generator $\mathcal{A}$, and is the space on which the generator is defined and need not be a closed subspace of Z. In this paper we will refer to X as the 'state-space'. For System (1), following the approach in [13], we define $Z := \{\mathbb{R}^n \times L_2\}$ and

$$\mathcal{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}(s) := \begin{bmatrix} A_0 x_1 + A_1 x_2(-\tau)\\ \dot{x}_2(s)\end{bmatrix}.$$

The state-space is $X := \left\{\begin{bmatrix}x_1^T & x_2^T\end{bmatrix}^T \in L_2^{2n} : x_2 \in W_2 \text{ and } x_2(0) = x_1\right\}$ which is not closed in Z. Using these definitions of $\mathcal{A}$, Z and X, the "complete-quadratic" Lyapunov functional (2) can be compactly represented as

$$V(\phi) = \left\langle \begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}, \mathcal{P}_{\{M,N\}}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}\right\rangle_{L_2}$$

where we define the notation

$$\mathcal{P}_{\{M,N\}}\begin{bmatrix}x_1\\ x_2\end{bmatrix} := M(s)\begin{bmatrix}x_1\\ x_2(s)\end{bmatrix} + \int_{-\tau}^{0} N(s,\theta)\begin{bmatrix}x_1\\ x_2(\theta)\end{bmatrix} d\theta.$$

That is, the Lyapunov functional is defined by a multiplier and integral operator whose multiplier and kernel are unknown. Likewise, the derivative of the functional can be represented as

$$\dot{V}(\phi) = \left\langle \begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}, \mathcal{P}\mathcal{A}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}\right\rangle_{L_2} + \left\langle \mathcal{A}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}, \mathcal{P}\begin{bmatrix}\phi(0)\\ \phi\end{bmatrix}\right\rangle_{L_2}.$$

In fact, it is known [13] that a strongly continuous semigroup defined by a linear operator $\dot{x} = \mathcal{A}x$ on Hilbert space X is exponentially stable if and only if there exists a positive self-adjoint operator $\mathcal{P}$ such that

$$\langle \mathcal{A}x, \mathcal{P}x\rangle_Z + \langle x, \mathcal{P}\mathcal{A}x\rangle_Z \leq -\epsilon \|x\|$$

for all $x \in X$ and some $\epsilon > 0$.

III. A Dual Stability Condition

In this section, we propose a general form of dual stability condition which relies on the implicit existence of a Lyapunov-Krasovskii functional.

Theorem 1: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on L_2 with domain X. Further suppose there exists a linear bounded coercive operator P with image $P(X) = X$ which is self-adjoint with respect to the L_2 inner product and

$$\langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle \leq -\epsilon \langle x, x\rangle$$

for all $x \in X$ and some $\epsilon > 0$. Then the dynamical system $\dot{x}(t) = \mathcal{A}x$ generates an exponentially stable semigroup.

Proof: Because P is coercive, self-adjoint and $P(X) = X$, the inverse exists, is coercive, bounded, self-adjoint and $P^{-1} : X \to X$. Define the Lyapunov function

$$V(y) = \langle y, P^{-1}y\rangle$$

where $y \in X$ and with derivative

$$\dot{V}(y) = \langle \dot{y}, P^{-1}y\rangle + \langle y, P^{-1}\dot{y}\rangle = \langle \mathcal{A}y, P^{-1}y\rangle + \langle P^{-1}y, \mathcal{A}y\rangle.$$

Now define $x = P^{-1}y \in X$. Then $y = Px$ and

$$\begin{aligned}\dot{V}(y) &= \langle \mathcal{A}y, P^{-1}y\rangle + \langle P^{-1}y, \mathcal{A}y\rangle\\ &= \langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle\\ &\leq -\epsilon\langle x, x\rangle = -\epsilon\langle y, P^{-1}P^{-1}y\rangle \leq -\alpha\langle y, y\rangle\end{aligned}$$

where the last inequality holds for some $\alpha > 0$ by boundedness of P. Negativity of the derivative of the Lyapunov function implies exponential stability in the square norm of the state by, e.g. [13] or by the invariance principle. ■

IV. Operators with $P = P^*$ and $P(X) = X$

In order to satisfy the dual stability condition, we must restrict ourselves to a class of operators which are self-adjoint with respect to the given inner-product and which preserve the structure of the state-space (map X to X). Note that the constraint $P(X) = X$ may add conservatism since X is not a closed subspace of Z. However, X is compactly embedded in Z and it may be possible to use a density argument to show this constraint is not conservative.

First recall that the state-space is $X := \{\begin{bmatrix}x_1^T & x_2^T\end{bmatrix}^T \in L_2^{2n} \;:\; x_2 \in W_2 \text{ and } x_2(0) = x_1\}$. To preserve this structure, we consider operators of the form

$$(Px)(s) := \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix} \tag{3}$$
$$= \begin{bmatrix} \tau(R(0,0)+S(0))x_1 + \int_{-\tau}^0 R(0,s)x_2(s)ds \\ \tau R(s,0)x_2(0) + \tau S(s)x_2(s) + \int_{-\tau}^0 R(s,\theta)x_2(\theta)d\theta, \end{bmatrix}$$

where R and S are continuous. It is easy to show that P is a bounded linear operator and $P : X \to X$. Furthermore, P is self-adjoint, as indicated in the following lemma.

Lemma 2: Suppose that $R(s,\theta) = R(\theta,s)^T$ and $S(s) \in \mathbb{S}^n$. Then the operator P, as defined in Equation (3), is self-adjoint with respect to the L_2 inner product.

Proof: The operator $P : X \to X$ is self-adjoint with respect to the inner product $\langle \cdot,\cdot\rangle_{L_2}$ if $\langle y, Px\rangle_{L_2} = \langle Py, x\rangle_{L_2}$ for any $x, y \in X$. By exploiting the structure of P and X, we have the following.

$$\langle y, Px\rangle_{L_2} = \int_{-\tau}^0 \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix}^T \cdot$$
$$\begin{bmatrix} \tau(R(0,0)+S(0))x_1 + \int_{-\tau}^0 R(0,\theta)x_2(\theta)d\theta \\ \tau R(s,0)x_2(0) + \tau S(s)x_2(s) + \int_{-\tau}^0 R(s,\theta)x_2(\theta)d\theta \end{bmatrix} ds$$
$$= \int_{-\tau}^0 \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix}^T \begin{bmatrix} \tau(R(0,0)+S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s)\end{bmatrix}\begin{bmatrix} x_1 \\ x_2(s)\end{bmatrix} ds$$
$$+ \int_{-\tau}^0\int_{-\tau}^0 \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & R(s,\theta)\end{bmatrix}\begin{bmatrix} x_1 \\ x_2(\theta)\end{bmatrix} ds d\theta$$
$$= \int_{-\tau}^0 \left(\begin{bmatrix} \tau(R(0,0)+S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s)\end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix}\right)^T \cdot$$
$$\begin{bmatrix} x_1 \\ x_2(s)\end{bmatrix} ds + \int_{-\tau}^0\int_{-\tau}^0 \begin{bmatrix} 0 \\ R(s,\theta)^T y_2(s)\end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2(\theta)\end{bmatrix} ds\, d\theta$$
$$= \int_{-\tau}^0 \begin{bmatrix} (\tau R(0,0)+S(0))y_1 + \tau R(0,s)y_2(s) \\ \tau R(s,0)y_1 + \tau S(s)y_2(s) + \int_{-\tau}^0 R(\theta,s)^T y_2(\theta)d\theta\end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2(s)\end{bmatrix} ds$$
$$= \int_{-\tau}^0 \begin{bmatrix} \tau(R(0,0)+S(0))y_1 + \int_{-\tau}^0 R(0,\theta)y_2(\theta)d\theta \\ \tau R(s,0)y_2(0) + \tau S(s)y_2(s) + \int_{-\tau}^0 R(s,\theta)y_2(\theta)d\theta\end{bmatrix}^T$$
$$\cdot \begin{bmatrix} x_1 \\ x_2(s)\end{bmatrix} ds = \langle Py, x\rangle_{L_2}$$

■

V. Dual Stability Conditions: Single Delay

In this section, we apply the structured operator in Section IV to the dual stability condition in Thm. 1 to establish conditions for stability in the single-delay case. Note that we do not yet discuss how to enforce these conditions.

Theorem 3: Suppose there exist $\epsilon > 0$ and functions $S \in W_2^{n\times n}[-\tau,0]$ and $R \in W_2^{n\times n}[[-\tau,0]\times[-\tau,0]]$ where $R(s,\theta) = R(\theta,s)^T$ and $S(s) \in \mathbb{S}^n$ such that $\langle x, \mathcal{P}x\rangle_{L_2^{2n}} \geq \epsilon\|x\|^2$ for all $x \in X$ where $\mathcal{P}$ is defined as in Eqn. (3) and

$$\left\langle \begin{bmatrix} x \\ \phi(-\tau) \\ \phi\end{bmatrix}, \mathcal{D}\begin{bmatrix} x \\ \phi(-\tau) \\ \phi\end{bmatrix}\right\rangle_{L_2^{3n}} \leq -\epsilon\left\|\begin{bmatrix} x \\ \phi\end{bmatrix}\right\|^2_{L_2^{2n}}$$

for all $\begin{bmatrix}x^T & \phi^T\end{bmatrix}^T \in X$ and where

$$\left[\mathcal{D}\begin{bmatrix} x \\ y \\ \phi\end{bmatrix}\right](s) = \begin{bmatrix} D_0\begin{bmatrix} x \\ y\end{bmatrix} + \int_{-\tau}^0 V(s)\phi(s)ds \\ \tau V(s)^T\begin{bmatrix} x \\ y\end{bmatrix} + \tau\dot{S}(s)\phi(s) + \int_{-\tau}^0 G(s,\theta)\phi(\theta)d\theta\end{bmatrix}$$

where

$$D_0 := \begin{bmatrix} C_{11}+C_{11}^T & C_{12} \\ C_{12}^T & C_{22}\end{bmatrix}, \quad V(s) = \begin{bmatrix} C_{13}(s) \\ 0\end{bmatrix},$$
$$C_{11} := \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) + \frac{1}{2}S(0),$$
$$C_{12} := \tau A_1 S(-\tau), \qquad C_{22} := -S(-\tau),$$
$$C_{13}(s) := A_0 R(0,s) + A_1 R(-\tau,s) + \partial_s R(s,0)^T,$$
$$G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta).$$

Then the system defined by Eqn. (1) is exponentially stable.

Proof: Define the operators $\mathcal{A}$ and $\mathcal{P}$ as above. By Lemma 2, $\mathcal{P}$ is self-adjoint and $\mathcal{P} : X \to X$. This, combined with $\mathcal{P}$ coercive can be used to show $\mathcal{P}(X) = X$. Then by Theorem 1 the system is exponentially stable if

$$\left\langle \mathcal{AP}\begin{bmatrix} x \\ \phi\end{bmatrix}, \begin{bmatrix} x \\ \phi\end{bmatrix}\right\rangle_{L_2^{2n}} + \left\langle \begin{bmatrix} x \\ \phi\end{bmatrix}, \mathcal{AP}\begin{bmatrix} x \\ \phi\end{bmatrix}\right\rangle_{L_2^{2n}} \leq -\left\|\begin{bmatrix} x \\ \phi\end{bmatrix}\right\|_{L_2^{2n}}$$

for all $\begin{bmatrix} x \\ \phi\end{bmatrix} \in X$. We begin by constructing $\mathcal{AP}x$.

$$\left(\mathcal{AP}\begin{bmatrix} x \\ \phi\end{bmatrix}\right)(s) := \begin{bmatrix} y_1 \\ y_2(s)\end{bmatrix}, \qquad \text{where}$$
$$y_1 = \tau A_0(R(0,0)+S(0))x + \int_{-\tau}^0 A_0 R(0,s)\phi(s)ds$$
$$+ A_1\Big(\tau R(-\tau,0)\phi(0) + \tau S(-\tau)\phi(-\tau) + \int_{-\tau}^0 R(-\tau,\theta)\phi(\theta)d\theta\Big)$$
$$y_2(s) = \tau\frac{d}{ds}R(s,0)\phi(0) + \tau\dot{S}(s)\phi(s) + \tau S(s)\dot{\phi}(s)$$
$$+ \int_{-\tau}^0 \frac{d}{ds}R(s,\theta)\phi(\theta)d\theta.$$

Thus $\left\langle \begin{bmatrix} x \\ \phi\end{bmatrix}, \mathcal{AP}\begin{bmatrix} x \\ \phi\end{bmatrix}\right\rangle := \tau x^T y_1 + \int_{-\tau}^0 \phi(s)^T y_2(s)ds.$ Examining these terms separately and using $x = \phi(0)$, we have

$$\tau x^T y_1 = \tau^2 x^T A_0(R(0,0)+S(0))x$$
$$+ \tau\int_{-\tau}^0 x^T A_0 R(0,s)\phi(s)ds + \tau x^T A_1 \tau R(-\tau,0)\phi(0)$$
$$+ \tau^2 x^T A_1 S(-\tau)\phi(-\tau) + \tau\int_{-\tau}^0 x^T A_1 R(-\tau,\theta)\phi(\theta)d\theta$$
$$= \int_{-\tau}^0\Big(x^T\tau A_0(R(0,0)+S(0))x + \tau x^T A_0 R(0,s)\phi(s)\Big)ds$$
$$+ \int_{-\tau}^0 (x^T\tau A_1 R(-\tau,0)x + x^T\tau A_1 S(-\tau)\phi(-\tau))ds$$
$$+ \int_{-\tau}^0 \tau x^T A_1 R(-\tau,s)\phi(s)ds = \int_{-\tau}^0 \begin{bmatrix} x \\ x(-\tau) \\ x(s)\end{bmatrix}^T \cdot$$
$$\begin{bmatrix} \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) & *^T & *^T \\ \frac{\tau}{2}S(-\tau)^T A_1^T & 0 & *^T \\ \frac{\tau}{2}R(0,s)^T A_0^T + \frac{\tau}{2}R(-\tau,s)^T A_1^T & 0 & 0\end{bmatrix}\begin{bmatrix} x \\ x(-\tau) \\ x(s)\end{bmatrix} ds.$$

Examining the second term, we get

$$\int_{-\tau}\phi(s)^T y_2(s)ds = \int_{-\tau}\phi(s)^T\tau\left(\partial_s R(s,0)\phi(0)+\dot{S}(s)\phi(s)\right)ds$$
$$+\int_{-\tau}^{0}\phi(s)^T\tau S(s)\dot{\phi}(s)ds+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T\frac{d}{ds}R(s,\theta)\phi(\theta)d\theta ds$$
$$=\int_{-\tau}\tau\phi(s)^T\left(\partial_s R(s,0)\phi(0)+\dot{S}(s)\phi(s)\right)ds$$
$$+\frac{\tau}{2}x^T S(0)x-\frac{\tau}{2}\phi(-\tau)^T S(-\tau)\phi(-\tau)$$
$$-\frac{1}{2}\int_{-\tau}^{0}\tau\phi(s)^T\dot{S}(s)\phi(s)ds$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T\frac{d}{ds}R(s,\theta)\phi(\theta)d\theta ds$$
$$=\int_{-\tau}^{0}\begin{bmatrix}x\\x(-\tau)\\x(s)\end{bmatrix}^T\begin{bmatrix}\frac{1}{2}S(0) & *^T & *^T\\ 0 & -\frac{1}{2}S(-\tau) & *^T\\ \frac{\tau}{2}\partial_s R(s,0) & 0 & \frac{\tau}{2}\dot{S}(s)\end{bmatrix}\begin{bmatrix}x\\x(-\tau)\\x(s)\end{bmatrix}ds$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T\frac{d}{ds}R(s,\theta)\phi(\theta)d\theta ds.$$

Combining both terms, and using symmetry of the inner product, we get

$$\left\langle\begin{bmatrix}x\\\phi\end{bmatrix},\mathcal{AP}\begin{bmatrix}x\\\phi\end{bmatrix}\right\rangle+\left\langle\mathcal{AP}\begin{bmatrix}x\\\phi\end{bmatrix},\begin{bmatrix}x\\\phi\end{bmatrix}\right\rangle=$$
$$\int_{-\tau}^{0}\begin{bmatrix}x\\x(-\tau)\\x(s)\end{bmatrix}^T\begin{bmatrix}C_{11}+C_{11}^T & C_{12} & \tau C_{13}(s)\\ C_{12}^T & C_{22} & 0_n\\ \tau C_{13}(s)^T & 0_n & \tau\dot{S}(s)\end{bmatrix}\begin{bmatrix}x\\x(-\tau)\\x(s)\end{bmatrix}ds$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T\left(\frac{d}{ds}R(s,\theta)+\frac{d}{d\theta}R(s,\theta)\right)\phi(\theta)d\theta ds$$
$$=\left\langle\begin{bmatrix}x\\\phi(-\tau)\\\phi\end{bmatrix},\mathcal{D}\begin{bmatrix}x\\\phi(-\tau)\\\phi\end{bmatrix}\right\rangle_{L_2^{3n}}\le-\epsilon\left\|\begin{bmatrix}x\\\phi\end{bmatrix}\right\|_{L_2^{2n}}^2.$$

Therefore, we conclude that Thm. 1 is satisfied and hence System (1) is exponentially stable. ■

Dual Lyapunov-Krasovskii Form: To summarize the results of Theorem 3 in a more traditional Lyapunov-Krasovskii format, the system is stable if there exists a

$$V(\phi)$$
$$=\int_{-\tau}^{0}\begin{bmatrix}\phi(0)\\\phi(s)\end{bmatrix}^T\begin{bmatrix}\tau(R(0,0)+S(0)) & \tau R(0,s)\\ \tau R(s,0) & \tau S(s)\end{bmatrix}\begin{bmatrix}\phi(0)\\\phi(s)\end{bmatrix}ds$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T R(s,\theta)\phi(\theta)d\theta ds$$

such that $V(\phi)\ge\left\|\begin{bmatrix}\phi(0)\\\phi\end{bmatrix}\right\|^2$ and

$$V_D(\phi)$$
$$=\int_{-\tau}^{0}\begin{bmatrix}\phi(0)\\\phi(-\tau)\\\phi(s)\end{bmatrix}^T\begin{bmatrix}C_{11}+C_{11}^T & C_{12} & \tau C_{13}(s)\\ C_{12}^T & C_{22} & 0_n\\ \tau C_{13}(s)^T & 0_n & \tau\dot{S}(s)\end{bmatrix}\begin{bmatrix}\phi(0)\\\phi(-\tau)\\\phi(s)\end{bmatrix}ds$$
$$+\int_{-\tau}^{0}\int_{-\tau}^{0}\phi(s)^T\left(\frac{d}{ds}R(s,\theta)+\frac{d}{d\theta}R(s,\theta)\right)\phi(\theta)d\theta ds$$
$$\le-\epsilon\left\|\begin{bmatrix}\phi(0)\\\phi\end{bmatrix}\right\|.$$

Note that unlike the standard Lyapunov-Krasovskii functions, the derivative of the dual functional has tri-diagonal structure.

VI. LMI conditions for Positivity of Multiplier and Integral Operators

In this Section, we define LMI-based conditions for positivity of operators of the form

$$\left(\mathcal{P}_{M,N}x\right)(s):=M(s)x(s)+\int_{-\tau}^{0}N(s,\theta)x(\theta)d\theta. \quad (4)$$

where $x\in L_2^n[-\tau,0]$ and M and N are continuous. Note that we initially consider positivity of the operator on $L_2^{m+n}[-\tau_K,0]$ and not the subspace $\mathbb{R}^m\times L_2^n[-\tau_K,0]$.

To enforce positivity we use the result in [14] which is based on the observation that a positive operator will always have a square root.

Theorem 4: For any functions $Y_1:[-\tau,0]\to\mathbb{R}^{m_1\times n}$ and $Y_2:[-\tau,0]\times[-\tau,0]\to\mathbb{R}^{m_2\times n}$, square integrable on $[-\tau,0]$ with $g(s)\ge0$ for $s\in[-\tau,0]$, suppose that

$$M(s)=g(s)Y_1(s)^T Q_{11}Y_1(s)$$
$$N(s,\theta)=g(s)Y_1(s)Q_{12}Y_2(s,\theta)+g(\theta)Y_2(\theta,s)^T Q_{12}^T Y_1(\theta)$$
$$+\int_{-\tau_K}^{0}g(\omega)Y_2(\omega,s)^T Q_{22}Y_2(\omega,\theta)\,d\omega$$

where $Q_{ij}\in\mathbb{R}^{m_i\times m_j}$ and

$$Q=\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{bmatrix}\ge0.$$

Then for $\mathcal{P}_{M,N}$ as defined in Equation (4), $\langle x,\mathcal{P}_{M,N}x\rangle_{L_2^n}\ge0$ for all $x\in L_2^n[-\tau,0]$.

The proof of Theorem 4 can be found in [14].

Thm. 4 gives a linear parametrization of a cone of positive operators using positive semidefinite matrices. For this paper, we choose $Y_1(s):=Z_d(s)\otimes I_n$ and $Y_2(s,\theta):=Z_d(s,\theta)\otimes I_n$ where Z_d is the vector of monomials of degree d or less. For the interval $s\in[-\tau,0]$, we can choose $g_1(s)=1$ or $g_2=-s(s+\tau)$. Inclusion of $g\ne1$ is a variation of the classical Positivstellensatz approach to local positivity, as can be found in, e.g. [15], [16], [17]. To improve accuracy, we typically use a combination of both although we may set $Q_{12},Q_{21},Q_{22}=0$ for the latter to reduce the number of variables. To simplify notation, throughout the paper, we will use the notation $\{M,N\}\in\Xi_{d,n}$ to denote the LMI constraints on the coefficients of the polynomials M,N implied by the conditions of Thm. 4 using a combination of $g(s)=1$ and $g=-s(s+\tau)$ as

$$\Xi_{d,n}:=\left\{\{M,N\}:\begin{array}{c}M=M_1+M_2,\ N=N_1+N_2,\ \text{where }\{M_1,N_1\}\\ \text{and }\{M_2,N_2\}\text{ satisfy Thm. 4 with }g_1=1\text{ and}\\ g_2=-s(s+\tau)\text{, respectively}\\ \text{and }Y_1=Z_d\otimes I_n,\ Y_2=Z_d\otimes I_n.\end{array}\right\}$$

A. A Class of Spacing Functions

Thm. 4 enforces positivity over L_2^n. However, we only need positivity on the subspaces $\mathbb{R}^n\times L_2^n$ and $\mathbb{R}^n\times\mathbb{R}^n\times L_2^n$. To this end, we introduce a set of free-variables we call spacing functions which can be added to the multipliers and kernels without changing the integral and which act as a projection onto the lower-dimensional subspace.

Theorem 5: Suppose that F and H are defined as

$$F(s) = \begin{bmatrix} K(s) + \int_{-\tau}^{0}\int_{-\tau}^{0} \frac{L_{11}(\omega,t)}{\tau} d\omega dt & \int_{-\tau}^{0} L_{12}(\omega, s) d\omega \\ \int_{-\tau}^{0} L_{21}(s,\omega) d\omega & 0 \end{bmatrix}$$

$$H(s,\theta) = -\begin{bmatrix} L_{11}(s,\theta) & L_{12}(s,\theta) \\ L_{21}(s,\theta) & 0 \end{bmatrix}$$

for some square-integrable functions K and L_{ij} where $K(s) \in \mathbb{R}^{m\times m}$, $L_{11}(s,\theta) \in \mathbb{R}^{m\times m}$, and $L_{12}(s,\theta) \in \mathbb{R}^{m\times n}$ such that $\int_{-\tau}^{0} K(s) ds = 0$. If

$$\mathcal{T}z(s) := F(s)z(s) + \int_{-\tau}^{0} H(s,\theta)z(\theta)\, d\theta$$

then for any $z \in \mathbb{R}^m \times L_2^n$,

$$\langle z, \mathcal{T}z\rangle_{L_2^{m+n}} = 0.$$

Proof: The proof is straightforward. ∎

For simplicity, we use $\{F, H\} \in \Theta_{m,n}$ to denote the conditions of Thm. 5 which is a set of linear equality constraints on the coefficients of the polynomials which define F and H.

$$\Theta_{m,n} := \{\{F, H\} \,:\, F, H \text{ satisfy the conditions of Thm. 5.}\}$$

VII. Dual Stability using SOS/LMIs

We now state an LMI representation of the dual stability condition for a single delay system.

Theorem 6: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, functions $S \in W_2^{n\times n}[-\tau, 0]$, $R \in W_2^{n\times n}[[-\tau, 0] \times [-\tau, 0]]$, $\{F_1, H_1\} \in \Theta_{n,n}$, and $\{F_2, H_2\} \in \Theta_{2n,n}$ where $R(s,\theta) = R(\theta, s)^T$ and $S(s) \in \mathbb{S}^n$ such that

$$\{M, N\} \in \Xi_{d,2n} \quad \text{and} \quad \{-D, -E\} \in \Xi_{d,3n}$$

where

$$M(s) = \begin{bmatrix} \tau R(0,0) + \tau S(0) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} + F_1(s) - \epsilon I_{2n},$$

$$N(s,\theta) = \begin{bmatrix} 0_n & 0_n \\ 0_n & R(s,\theta) \end{bmatrix} + H_1(s,\theta),$$

$$D(s) := \begin{bmatrix} D_0 & \tau V(s) \\ \tau V(s)^T & \tau \dot{S}(s) + \epsilon I_n \end{bmatrix} + F_2(s),$$

$$D_0 := \begin{bmatrix} C_{11} + C_{11}^T + \epsilon I_n & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}, \quad V(s) = \begin{bmatrix} C_{13}(s) \\ 0 \end{bmatrix},$$

$$C_{11} := \tau A_0(R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2}S(0),$$

$$C_{12} := \tau A_1 S(-\tau), \quad C_{22} := -S(-\tau),$$

$$C_{13}(s) := A_0 R(0,s) + A_1 R(-\tau, s) + \partial_s R(s,0)^T,$$

$$E(s,\theta) := \begin{bmatrix} 0_{2n} & 0_{2n,n} \\ 0_{n,2n} & G(s,\theta) \end{bmatrix} + H_2(s,\theta)$$

$$G(s,\theta) := \frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta).$$

Then the system defined by Eqn (1) is exponentially stable.

Proof: The proof follows immediately from Thms. 4 and 5 applied to Thm. 3 ∎

VIII. Numerical Testing of Dual Stability

In this section, we apply the dual stability condition to two numerical examples in order to verify that the proposed dual stability conditions are not significantly conservative. In each case, we list the maximum provable stable value as a function of degree d. The computation time is listed in CPU seconds on an Intel i7-5960X 3.0GHz processor. This time corresponds to the interior-point (IPM) iteration in SeDuMi and does not account for preprocessing, postprocessing, or for the time spent on polynomial manipulations formulating the SDP using SOSTOOLS. Such polynomial manipulations can significantly exceed SDP computation time.

a) Example A: First, we consider a scalar example which is known to be stable for $\tau \le \frac{\pi}{2}$.

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	4	analytic
$\tau_{\max}$	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

b) Example B: Next, we consider a well-studied 2-state, single delay system.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.6581	1.716	1.7178	1.7178	1.7178
$\tau_{\min}$	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

To illustrate computational scaling, tests were performed on 10-state and 20-state single delay systems using polynomial degree 2. Computation times were 22s and 951s, respectively.

IX. Full-State Feedback

Given a dual stability condition, it is easy to construct a synthesis condition for full-state feedback.

Corollary 7: Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on L_2 with domain X and $B : U \to X$. Further suppose there exists a bounded coercive operator $P : X \to X$ which is self-adjoint with respect to the L_2 inner product and an operator $Z : X \to U$ such that

$$\langle (\mathcal{A}P + BZ)\, x, x\rangle + \langle x, (\mathcal{A}P + BZ)\, x\rangle \le -\langle x, x\rangle$$

for all $x \in X$. Let $K = ZP^{-1}$. Then the dynamical system $\dot{x}(t) = (\mathcal{A} + BK)\, x$ generates an exponentially stable semigroup.

Proof: The proof follows immediately from Theorem 1 with $Z = KP$. ∎

X. Existence of a Stabilizing Controller

For time-delay systems, there are several different formulations of the controller synthesis problem. Some of these are not full-state feedback. For example, if we seek a K such that $u(t) = Kx(t)$ is stabilizing, this is, in fact, output feedback, as it only uses part of the state x_t. Others, such as input delay are full-state feedback with a delay in the input operator B. In this paper, however, we will consider the simplest form of state-feedback where

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t).$$

and we seek a map $u(t) = Kx_t$ where $K : X \to \mathbb{R}^m$. In this case, $B : \mathbb{R}^m \to X$ has the simple form

$$(Bu)(s) := \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}.$$

In the following theorem, we suppose that variable operator $Z : X \to \mathbb{R}^m$ has the form

$$(Zx)(s) = Z_0 x_1 + Z_1 x_2(-\tau) + \int_{-\tau}^{0} Z_2(s) x_2(s) ds.$$

Theorem 8: Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrices $Z_0 \in \mathbb{R}^{m\times n}$, $Z_1 \in \mathbb{R}^{m\times n}$ and polynomials $Z_2 \in W_2^{m\times n}[-\tau, 0]$, $S \in W_2^{n\times n}[-\tau, 0]$, $R \in W_2^{n\times n}[[-\tau, 0] \times [-\tau, 0]]$, $\{F_1, H_1\} \in \Theta_{n,n}$, and $\{F_2, H_2\} \in \Theta_{2n,n}$ where $R(s, \theta) = R(\theta, s)^T$ and $S(s) \in \mathbb{S}^n$ such that

$$\{M, N\} \in \Xi_{d,2n} \quad \text{and} \quad \{-D - L, -E\} \in \Xi_{d,3n} \tag{5}$$

where M, N, D, E are as defined in Thm. 6 and

$$L(s) := \begin{bmatrix} L_{11} + L_{11}^T & *^T & *^T \\ L_{12}^T & 0 & *^T \\ L_{13}(s)^T & 0 & 0 \end{bmatrix}$$
$$L_{11} = B_0 Z_0, \quad L_{12} = B_0 Z_1, \quad L_{13}(s) = \tau B_0 Z_2(s). \tag{6}$$

Then the delayed System (1) is full-state feedback stabilizable. Furthermore, let

$$(P_1^{-1}x)(s) = Y_0(s)x_1 + Y_1(s)x_2(s) + \int_{-\tau}^{0} Y_2(s, \theta)x(\theta)d\theta$$

be the inverse of

$$(P_1 x)(s) = \tau R(s,0)x_2(0) + S(s)x_2(s) + \int_{-\tau}^{0} R(s, \theta)x_2(\theta)d\theta$$

(Which can be found via Thm. 9). Then a stabilizing controller is

$$u(t) = K_0 x(t) + K_1 x(t - \tau) + \int_{-\tau}^{0} K_2(s)x(t + s)ds$$

where

$$K_0 = Z_0 Y_0(0) + Z_1 Y_0(-\tau) + \int_{-\tau}^{0} Z_2(s)Y_0(s)ds + Z_0 Y_1(0)$$
$$K_1 = Z_1 Y_1(-\tau)$$
$$K_2(s) = Z_0 Y_2(0, s) + Z_1 Y_2(-\tau, s) + Z_2(s)Y_1(s) + \int_{-\tau}^{0} Z_2(\theta)Y_2(\theta, s)d\theta.$$

Note on Inverse: The proof is stated using the decomposition of $(Px)(x) = \begin{bmatrix} (P_1 x)(0) \\ (P_1 x)(s) \end{bmatrix}$ where P_1 is as defined in the theorem. This is because we have an analytic expression for the inverse of P_1 (in Thm. 9). Recently, new expressions have been proposed for the inverse of P in [18], and the reader may want to consider use of these expressions. Of course, in this case, we still have $(P^{-1}x)(x) = \begin{bmatrix} (P_1^{-1}x)(0) \\ (P_1^{-1}x)(s) \end{bmatrix}$, and one may therefore use the expression for P^{-1} to extract the expression for P_1^{-1}.

Proof: Define the operators $\mathcal{A}, P, B$ and Z as above. By Condition 5 and by Thms. 4 and 5, and since M, N are as defined in Thm. 6 for the given ϵ, the operator P is coercive. This implies that by Cor. 7 for the input $u(t) = ZP^{-1}x_t$, the closed system is exponentially stable if

$$\langle \mathcal{A}Px, x\rangle + \langle x, \mathcal{A}Px\rangle + \langle BZx, x\rangle + \langle x, BZx\rangle \le -\langle x, x\rangle .$$

The $\mathcal{A}Px$ terms have already been detailed in the proof of Thm. 3. We now expand the remaining $\langle x, BZx\rangle$ terms:

$$BZx = \begin{bmatrix} B_0 Z_0 x_1 + B_0 Z_1 x_2(-\tau) + \int_{-\tau}^{0} B_0 Z_2(s)x_2(s)ds \\ 0 \end{bmatrix}$$

This yields

$$\langle x, BZx\rangle$$
$$= \int_{-\tau}^{0} \left(x_1^T B_0 Z_0 x_1 + x_1^T B_0 Z_1 x_2(-\tau) + \tau x_1^T B_0 Z_2(s)x_2(s)\right) ds$$
$$= \int_{-\tau}^{0} \begin{bmatrix} x_1 \\ x(-\tau) \\ x(s) \end{bmatrix}^T \begin{bmatrix} L_{11} & *^T & *^T \\ \frac{1}{2}L_{21}^T & 0 & *^T \\ \frac{1}{2}L_{31}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x(-\tau) \\ x(s) \end{bmatrix} ds.$$

Combining $\langle \mathcal{A}Px, x\rangle$ and $\langle x, BZx\rangle$, by Condition 5, by application of Thms. 4 and 5 and by Thm. 3, we have stability of the closed-loop system. We now construct K directly as

$$u = Kx = ZP^{-1}x = Z \begin{bmatrix} (P_1^{-1}x)(0) \\ (P_1^{-1}x)(s) \end{bmatrix} = Z \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= Z_0 y_1 + Z_1 y_2(-\tau) + \int_{-\tau}^{0} Z_2(s)y_2(s)ds$$
$$= Z_0 \left(Y_0(0)x_1 + Y_1(0)x_1 + \int_{-\tau}^{0} Y_2(0, s)x_2(s)ds \right)$$
$$+ Z_1 \left(Y_0(-\tau)x_1 + Y_1(-\tau)x_2(-\tau) + \int_{-\tau}^{0} Y_2(-\tau, s)x_2(s)ds \right)$$
$$+ \int_{-\tau}^{0} Z_2(s)Y_0(s)x_1 ds + \int_{-\tau}^{0} Z_2(s)Y_1(s)x_2(s)ds$$
$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} (Z_2(s)Y_2(s, \theta))\, x_2(\theta)\, ds\, d\theta$$
$$= \left(Z_0 Y_0(0) + Z_1 Y_0(-\tau) + \int_{-\tau}^{0} Z_2(s)Y_0(s)ds + Z_0 Y_1(0) \right) x_1$$
$$+ Z_1 Y_1(-\tau)x_2(-\tau)$$
$$+ \int_{-\tau}^{0} \left(Z_0 Y_2(0, s) + Z_1 Y_2(-\tau, s) + Z_2(s)Y_1(s) + \int_{-\tau}^{0} Z_2(\theta)Y_2(\theta, s)d\theta \right) x_2(s)ds$$

■

XI. Inverting the Positive Operator

In order to construct the controller defined in Thm. 8, one must obtain the inverse of the operator

$$(P_1 x)(s) = \tau R(s,0)x_2(0) + \tau S(s)x_2(s) + \int_{-\tau}^{0} R(s, \theta)x_2(\theta)d\theta$$

Since $(Px)(s) = \begin{bmatrix} (P_1 x_2)(0) \\ (P_1 x_2)(s) \end{bmatrix}$, $(P^{-1}y)(s) = \begin{bmatrix} (\bar{P}^{-1}y_2)(0) \\ (\bar{P}^{-1}y_2)(s) \end{bmatrix}$ and hence invertibility of P is equivalent to invertibility of

P_1 and since P is invertible, P_1 is likewise. Moreover, it is substantially simpler to invert P_1 since this inversion is on the space $W_2[-\tau, 0]$ and not the mixed state-space. Moreover, in [19], we gave an analytic formulation of the inverse of operators of the form

$$M(s)x_2(s) + \int_{-\tau}^{0} N(s,\theta)x_2(\theta)d\theta$$

when M and N satisfy certain stronger positivity conditions. For the weaker conditions defined by Thm. 4, the inverse was constructed in, e.g. [20] using a power series expansion. We quote the result in [19].

Theorem 9: Consider the linear operator P defined by

$$Px(s) = M(s)x(s) + \int_I N(s,\theta)x(\theta)d\theta,$$

where $M(s) > 0$ for all $s \in I$ and N has a representation $N(s,\theta) = Z(s)^T RZ(\theta)$. Define the linear operator $\hat{P}$ by

$$\hat{P}x(s) = M(s)^{-1}x(s) + \int_I \hat{N}(s,\theta)x(\theta)d\theta$$

Where

$$\begin{aligned} \hat{N}(s,\theta) &= M(s)^{-1}Z(s)^T QZ(\theta)M(\theta)^{-1} \\ Q &= -R(S^{-1}+R)^{-1}S^{-1} \\ S &= \int_I Z(s)M(s)^{-1}Z(s)^T ds. \end{aligned}$$

Then $\hat{P}Px = P\hat{P}x = x$ for any integrable function x.

In this paper, we expand this inversion formula to cover a broader class of operator which includes the term $\tau R(s,0)x_2(0)$ and show that this inverse has the required form. Specifically, we have the following.

Theorem 10: Define $L = L_1 + L_2$, where

$$\begin{aligned} (L_1x)(s) &:= K(s)x(0) \\ (L_2x)(s) &:= M(s)x(s) + \int_I N(s,\theta)x(\theta)\,d\theta. \end{aligned}$$

Suppose that L_2 is invertible with

$$(L_2^{-1}x)(s) = Q(s)x(s) + \int_I R(s,\theta)x(\theta)\,d\theta$$

and that $\rho(J) < 1$, where

$$J := Q(0)K(0) + \int_I R(0,s)K(s)\,ds.$$

Then

$$\begin{aligned} &((L_1 + L_2)^{-1}x)(s) := \\ &\qquad Y_0(s)x(0) + Y_1(s)x(s) + \int_I Y_2(s,\theta)x(\theta)\,d\theta \end{aligned}$$

where

$$\begin{aligned} Y_0(s) &= -H(s)(I+J)^{-1}Q(0) \qquad Y_1(s) = Q(s) \\ Y_2(s,\theta) &= R(s,\theta) - H(s)(I+J)^{-1}R(0,\theta) \\ H(s) &= Q(s)K(s) + \int_I R(s,\theta)K(\theta)\,d\theta. \end{aligned}$$

Proof: The proof is a straightforward application of the power series expansion of $(L_1 + L_2)^{-1}$.

$$\begin{aligned} (L_1 + L_2)^{-1} &= L_2^{-1}\sum_{i=0}^{\infty}(-L_1L_2^{-1})^i = \left[\sum_{i=0}^{\infty}(-L_2^{-1}L_1)^i\right] L_2^{-1} \\ &= L_2^{-1} - L_2^{-1}L_1\left[\sum_{i=0}^{\infty}(-L_2^{-1}L_1)^i\right] L_2^{-1} \end{aligned}$$

which converges if $\rho(L_2^{-1}L_1) < 1$. However, we have that

$$\begin{aligned} &(L_2^{-1}L_1x)(s) = Q(s)(L_1x)(s) + \int_{-\tau}^{0} R(s,\theta)(L_1x)(\theta)\,d\theta \\ &= Q(s)K(s)x(0) + \int_{-\tau}^{0} R(s,\theta)K(\theta)x(0)\,d\theta \\ &= \left(Q(s)K(s) + \int_{-\tau}^{0} R(s,\theta)K(\theta)\,d\theta\right)x(0). \end{aligned}$$

Repeating the operator yields

$$\begin{aligned} &(L_2^{-1}L_1(L_2^{-1}L_1x))(s) \\ &= \left(Q(s)K(s) + \int_{-\tau}^{0} R(s,\theta)K(\theta)\,d\theta\right)(L_2^{-1}L_1x)(0). \end{aligned}$$

However,

$$\begin{aligned} (L_2^{-1}L_1x)(0) &= \left(Q(0)K(0) + \int_{-\tau}^{0} R(0,\theta)K(\theta)\,d\theta\right)x(0) \\ &= Jx(0). \end{aligned}$$

Therefore,

$$\begin{aligned} &(L_2^{-1}L_1(L_2^{-1}L_1x))(s) \\ &\qquad = \left(Q(s)K(s) + \int_{-\tau}^{0} R(s,\theta)K(\theta)\,d\theta\right)Jx(0). \end{aligned}$$

By induction, we conclude

$$\begin{aligned} &(L_2^{-1}L_1(L_2^{-1}L_1)^k x)(s) \\ &\qquad = \left(Q(s)K(s) + \int_I R(s,\theta)K(\theta)\,d\theta\right)J^k x(0). \end{aligned}$$

Including the summation, we get

$$\begin{aligned} &L_2^{-1}L_1\left[\sum_{i=0}^{\infty}(-L_2^{-1}L_1)^i\right]x \\ &\quad = \left(Q(s)K(s) + \int_I R(s,\theta)K(\theta)\,d\theta\right)\left[\sum_{i=0}^{\infty}(-J)^k\right]x(0) \\ &\quad = H(s)(I+J)^{-1}x(0), \end{aligned}$$

where $H(s) = Q(s)K(s) + \int_I R(s,\theta)K(\theta)\,d\theta$. Therefore, simple substitution and algebraic manipulation yields

$$\begin{aligned} &\left((L_1 + L_2)^{-1}x\right)(s) = Q(s)x(s) + \int_I R(s,\theta)x(\theta)\,d\theta \\ &- H(s)(I+J)^{-1}\left[Q(0)x(0) + \int_I R(0,\theta)x(\theta)\,d\theta\right] \\ &= -H(s)(I+J)^{-1}Q(0)x(0) + Q(s)x(s) \\ &+ \int_I \left(R(s,\theta) - H(s)(I+J)^{-1}R(0,\theta)\right)x(\theta)\,d\theta \\ &= Y_0(s)x(0) + Y_1(s)x(s) + \int_I Y_2(s,\theta)x(\theta)\,d\theta. \end{aligned}$$

■

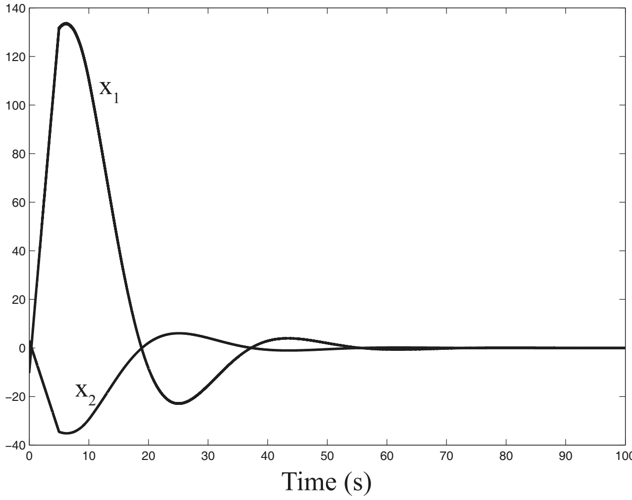

Fig. 1. A Matlab DDE23 simulation of System (7) with Controller (8) and delay $\tau = 5s$.

XII. Numerical Results

c) Synthesis Condition: After a non-exhaustive search, we have yet to find a result which cannot be replicated using the method described here. However, this comparison is somewhat unfair, as most "state-feedback" results in the literature typically only use $x(t)$ or $x(t-\tau)$ and hence are working with more limited information. Often such results are appropriately justified by a presumed lack of knowledge of the delay. However, in such a case, the approach is not truly state feedback and should rather be considered output feedback, a topic we leave for future work. To illustrate our approach, we consider the commonly referenced dynamical system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \quad (7)$$

This system was stabilized using non-convex iterative/"tuning parameter" methods in e.g. [11] and [12] for $\tau < 1$ (using only $x(t)$). We applied the methods of this paper for $\tau = 5$ using simple degree 2 polynomials and obtained the following exponentially stabilizing controller.

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau) \quad (8)$$
$$+ \int_{-5}^{0} \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds$$

These results were obtained using a combination of Matlab, MuPad and SOSTOOLS to perform the optimization and controller reconstruction. The polynomial inversion was performed in MuPad and approximated using polynomial functions to simplify presentation. Simulations for fixed initial conditions were performed and can be seen in Figure 1.

XIII. Conclusion

In conclusion, we have proposed a new form of duality which allows us to convexify the controller synthesis problem for infinite-dimensional systems. This dual principle requires a Lyapunov operator which is positive, invertible, self-adjoint and preserves the structure of the state-space. We have used Sum-of-Squares to parameterize a class of such operators. We applied these results to generate full-state feedback controllers for single-delay systems. Numerical tests indicate the algorithm compares favorably with results in the literature, although this comparison is somewhat specious as we were unable to find any literature which uses true full-state feedback for control. The contribution of the present paper is not in the accuracy of the results, however. Rather the contribution is in the convexification of the synthesis problem which opens the door for dynamic output-feedback H_∞ synthesis for infinite-dimensional systems.

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