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Spectral Characterization of Controllability and Observability for Frequency Regulation Dynamics

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***Abstract*— We give a full characterization using spectral graph theory of the controllability and observability of the swing and power flow dynamics in frequency regulation. In particular, we show that the controllability/observability of the system depends on two orthogonal conditions: 1) intrinsic structure of the system graph 2) algebraic coverage of buses with controllable loads/sensors. Condition 1) encodes information on graph symmetry and is shown to hold for almost all practical systems. Condition 2) captures how buses interact with each other through the network and can be verified using the eigenvectors of the graph Laplacian matrix. Based on this framework, the optimal placement of controllable loads and sensors in the network can be formulated as a set cover problem. We demonstrate how our results identify the critical buses in real systems by performing simulation in the IEEE 39-bus New England interconnection test system. We show that for this testbed, a single well chosen bus is capable of providing full controllability/observability.**

I. Introduction

Our energy system is at the cusp of historical transformation into a more sustainable and intelligent form. On one hand, the penetration of renewable energies brings increasing level of flow fluctuations to the grid, which calls for technologies improving controllability and observability of the grid. On the other hand, integration of smart devices that are able to sense, communicate and control enables more flexible and decentralized control mechanisms to be implemented. Frequency regulation, for instance, has been revisited recently in [1]–[4] with focus on load side participation, which exhibits faster response and more localized deviation compared to conventional generator side frequency regulation.

This line of work motivated a series of efforts dedicated to the theoretical understanding of load-side participation in frequency regulation. For example, a primal-dual based framework for studying the load control problem is proposed in [5]–[7], based on which a decentralized frequency-based load control scheme is developed. Later on, a virtual flow reformulation is proposed in [8] to account for congestion control jointly with frequency regulation. Passivity approaches are investigated in [9], [10] for primary and in [11] for secondary frequency regulation with load-side participation, which often apply more generally to higher order models compared to Lyapunov based methods in [5]–[8]. There is also a large body of literature on distributed

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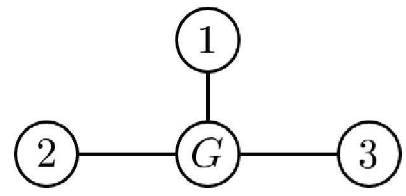


Fig. 1. An example network with symmetry.

frequency control employing other approaches, see [12], [13], for example.

It is almost always assumed that all buses have controllable generation/loads and sensors in existing work [5]–[13], which is not practical for structure preserving models in power systems. Indeed, feasible placements of popular controllable loads such as electric vehicle charging stations [14] and aggregated households [3] are typically limited to certain geographical areas and the penetration of such controllable loads takes investment and time. Moreover, the enormous amount of sensing devices needed for full load-side participation in a large scale network can be very costly [14], [15]. Hence we are interested in power network models where not all nodes have controllable loads or sensing devices.

When only a subset of buses are controllable, it is less clear how much controllability we have over the system. As a motivating example, let us consider the highly symmetric network shown in Fig. 1, where the nodes 1, 2 and 3 are assumed to have the same load, damping, inertia, initial phase etc., and assume we can only control the node with label G. Then because of symmetry, no matter how we alter the mechanical power injection at node G, the power flows on all transmission lines would be the same and therefore the system cannot be controllable. This, of course, is a highly contrived example - nevertheless, it is a manifestation of an intrinsic network property that leads to a loss of controllability, as we will show in Section III.

Similarly, when sensors are only installed at a subset of buses, we lose a certain degree of observability of the system. Consider again the network shown in Fig. 1 under the same symmetric setting and assume we can only measure the frequency deviation at the bus G. For any initial state that conforms to the system input and output observed, we can always permute the states among the buses 1, 2 and 3 (which effectively relabels those buses), and still get a feasible solution. Intrinsically, there is no way for the observer to distinguish those initial states because of the symmetry.

In this work, we develop a theory quantifying the impact of such limited controllable loads/sensors coverage over the system controllability/observability in frequency regulation. In particular, we show that the controllability/observability of the swing and network dynamics is precisely characterized by two orthogonal conditions: 1) intrinsic structure of the system graph; 2) algebraic coverage, which we define in Section III, of buses with controllable loads/sensors. Condition 1) encodes information on the graph symmetry and is shown to hold for almost all practical systems. Condition 2) captures how buses interact with each other through the network and can be verified using the eigenvectors of the graph Laplacian matrix. We would like to remark that our results do not explicitly hint on how optimal decentralized control scheme should be designed. Indeed, the standard control associated with our results is typically centralized and open-loop. The focus of this work is more towards a fundamental understanding on structural properties of our power system.

The rest of the paper is organized as follows. We first review the system model and relevant spectral graph theory concepts in Section II. In Section III, we present the exact conditions for the system to be controllable. The practical interpretations of these conditions are discussed in Section IV. The parallel results in the system observability are given in Section V. We present two applications of our characterizations in Section VI. The first application as presented in Section VI-A is more analytical, which reduces the problem of optimal placement for controllable loads and sensors to a set cover problem. The second application as presented in Section VI-B is an evaluation in the IEEE 39-bus New England interconnection test system, showing how a single well chosen critical bus based on our theory is capable of regulating the frequency of the whole grid. We conclude in Section VII.

II. Model and Problem Setup

In this section, we present the system model as adopted in [5]–[8] and review relevant concepts from spectral graph theory. We also refine the existing models to include the limited coverage of controllable loads and sensors.

Let $\mathbb{R}$ denote the set of real numbers. For a set $\mathcal{N}$, its cardinality is denoted as $|\mathcal{N}|$. We reserve caligraphic symbols like $\mathcal{F},\mathcal{U},\mathcal{O}$ for sets related to the physical system (for instance buses with controllable loads). Uppercase symbols like A, B, C usually refer to matrices, but can also refer to a vector space or a set in the proofs. For two matrices A, B with proper dimensions, $[A\ B]$ means the concatenation of A, B in a row, and $[A; B]$ means the concatenation of A, B in a column. A variable without subscript usually denotes a vector with appropriate components, e.g., $\omega = (\omega_j, j \in \mathcal{N}) \in \mathbb{R}^{|\mathcal{N}|}$. For a matrix A, we denote A^T, A^{-1}, $A^{-1/2}$ and $A^\dagger$ as its transpose, inverse, inverse of square root and Moore-Penrose pseudo-inverse respectively, provided they are properly defined. For a time-dependent signal $\omega(t)$, we use $\dot{\omega}$ to denote its time derivative $\frac{d\omega}{dt}$. For any vector x, we use $\mathrm{diag}(x)$ to denote the diagonal matrix with entries from x as the main diagonal.

We use the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ to describe the power transmission network, where $\mathcal{N} = \{1, \ldots, n\}$ is the set of buses and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ denotes the set of transmission lines. The terms bus/node and line/edge are used interchangeably in this paper. We assume without loss of generality that $\mathcal{G}$ is connected and simple. An edge in $\mathcal{E}$ is denoted either as e or (i, j). We further assign an arbitrary orientation over $\mathcal{E}$ so that if $(i, j) \in \mathcal{E}$ then $(j, i) \notin \mathcal{E}$. For any subset of buses $S \in \mathcal{N}$, we denote its characteristic function using the corresponding symbol 1_S. Let $n = |\mathcal{N}|, m = |\mathcal{E}|$ be the number of buses and transmission lines, respectively. The incidence matrix of $\mathcal{G}$ is a $n \times m$ matrix C defined as

$$C_{ie} = \begin{cases} 1 & \text{if node } i \text{ is the source of } e \\ -1 & \text{if node } i \text{ is the target of } e \\ 0 & \text{otherwise} \end{cases}$$

For each bus $j \in \mathcal{N}$, we denote the frequency deviation as ω_j and denote the inertia constant as $M_j > 0$. The symbol P_j^m is overloaded to denote the mechanical power injection if j is a generator bus and denote the aggregate change in uncontrollable load if j is a load bus. For each transmission line $(i, j) \in \mathcal{E}$, we denote as P_{ij} the branch flow deviation and denote as B_{ij} the link susceptance.

At each bus, there are three types of additional components which affect the system dynamics.

1) **Controllable Load**. Such component incurs extra load denoted by d_j and the level of d_j is controllable. The set of buses with controllable loads is denoted as $\mathcal{U}$.
2) **Frequency Sensitive Load**. Such component is sensitive to local frequency deviations and incurs additional load of $\hat{d}_j = D_j \omega_j$. We do not allow direct control to such loads and denote the set of buses with frequency sensitive loads as $\mathcal{F}$.
3) **Sensor**. Such component measures the local frequency deviation ω_j. The set of buses equipped with sensors is denoted as $\mathcal{S}$.

Summarizing the above different components, the swing and network dynamics is given by

$$-M_j \dot{\omega}_j = 1_{\mathcal{F}}(j)\hat{d}_j + 1_{\mathcal{U}}(j) d_j - P_j^m + \sum_{e \in \mathcal{E}} C_{je} P_e, \; j \in \mathcal{N}$$

$$\dot{P}_{ij} = B_{ij}(\omega_i - \omega_j), \quad (i, j) \in \mathcal{E}$$

and the system state is observed through

$$y_j = 1_{\mathcal{S}}(j)\omega_j, \qquad j \in \mathcal{N}$$

Readers are referred to [6] for more detailed justification and derivation of this model.

Now using x to denote the system state $x = [\omega; P]$, and putting F, U, S, M, D and B to be the diagonal matrices with $1_{\mathcal{F}}(j)$, $1_{\mathcal{U}}(j)$, $1_{\mathcal{S}}(j)$, M_j, D_j and B_{ij} as diagonal entries respectively, we can rewrite the system dynamics in the state-space form

$$\dot{x} = Ax - \begin{bmatrix} M^{-1}U \\ 0 \end{bmatrix} d + \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} P^m \tag{1a}$$

$$y = \begin{bmatrix} S & 0 \end{bmatrix} x \tag{1b}$$

where

$$A = \begin{bmatrix} -M^{-1}FD & -M^{-1}C \\ BC^T & 0 \end{bmatrix}$$

and is referred to as the system matrix of (1) in the sequel.

In our characterizations, the scaled graph Laplacian matrix defined as $L = M^{-1/2}CBC^TM^{-1/2}$ plays a key role. It is more explicitly given by

$$L_{ij} = \begin{cases} -\frac{B_{ij}}{\sqrt{M_iM_j}} & i \neq j, (i,j) \in \mathcal{E} \text{ or } (j,i) \in \mathcal{E} \\ \frac{1}{M_i}\sum_{j:j\in N(i)} B_{ij} & i = j \\ 0 & \text{otherwise} \end{cases}$$

where $N(i)$ is the set of neighbors of i. For any vector $x \in \mathbb{R}^n$, we have

$$x^TLx = \sum_{(i,j)\in\mathcal{E}} B_{ij}\left(\frac{x_i}{\sqrt{M_i}} - \frac{x_j}{\sqrt{M_j}}\right)^2 \geq 0$$

This implies that L is a positive semidefinite matrix and thus diagonalizable. It is well-known that $\text{rank}(L) = n-1$ for a connected graph [16] and therefore 0 is a simple eigenvalue of L. We denote $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ as its eigenvalues and put $\beta := \{\beta_1, \beta_2, \cdots, \beta_n\}$ to be an orthonormal set of its eigenvectors with β_s affording λ_s. The notation $\mathcal{N} = \{1, 2, \ldots, n\}$ is abused to also denote the index set of β. Whether $\mathcal{N}$ denotes the set of buses or denotes an index set for β will be clear from the context. The following property of the spectrum of L turns out to be particularly useful in this work.

Definition 2.1: The matrix L is said to have *simple spectrum* if all the eigenvalues of L are distinct.

We recommend the readers to take $M = I_n$ and $B = I_m$ in first reading, under which our results are significantly cleaner yet all key points (except Proposition 4.1) are captured. Throughout the analysis, we make the following assumption:

Sensitive Load Frequency sensitive components only exist at buses with controllable loads. That is, we assume $\mathcal{F} \subset \mathcal{U}$.

III. Controllability

In this section, we analyze the state-space dynamics given in (1) and characterize its controllability using the spectrum of the scaled Laplacian matrix L.

Before presenting our characterization, we first clarify what we mean by the controllability of (1). The classical definition of controllability requires the whole state space $\mathbb{R}^{n+m}$ being reachable from any initial point. This turns out to be too strong and is not suitable for our application. Indeed, from the branch flow dynamics

$$\dot{P} = BC^T\omega$$

we see that

$$B^{-1}(P(t) - P(0)) = \int_0^t C^T\omega(s)ds \in \text{range}(C^T)$$

If we assume the system is in the nominal state at time $t = 0$, that is $x(0) = [\omega(0); P(0)] = 0$, then we know $B^{-1}P(t) \in \text{range}(C^T)$ for any t. In other words, the scaled branch flow vector is confined in the range of C^T because of the system physics. This motivates the following definition.

Definition 3.1: The dynamics (1) is said to be *P-controllable* or *controllable in power system sense*, if for any $t > 0$, initial state $x(0) = [\omega(0); P(0)]$ and target state $x(t) = [\omega(t); P(t)]$ satisfying

$$B^{-1}(P(t) - P(0)) \in \text{range}(C^T)$$

there exists a control u such that

$$x(t) = \phi(x(0), u, t)$$

where $\phi(x(0), u, t)$ is the system state at time t given initial state $x(0)$ and control input u.

Our first result generalizes the classical Kalman criteria to the context of P-controllability. It shows that to determine the system P-controllability, it suffices to form the controllability matrix with the scaled Laplacian matrix L (instead of the full system matrix A) and we can ignore the drift term P^m (even when it is time-variant) in (1a).

Proposition 3.2: The dynamics (1) is P-controllable if and only if the matrix

$$W = \begin{bmatrix} M^{-1/2}U & -LM^{-1/2}U & \cdots & (-L)^{n-1}M^{-1/2}U \end{bmatrix}$$

has full row rank.

The proof of this proposition is presented in Appendix I. This result tells us that to decide the P-controllability of (1), it amounts to compute the rank of W. Recall $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of L and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is an orthonormal set of corresponding eigenvectors. Let Q be the matrix with β_j's as columns and Λ be the diagonal matrix with λ_j's as diagonal entries, i.e. $L = Q\Lambda Q^T$. We introduce the concept of algebraic coverage.

Definition 3.3: With all previous notations, the *algebraic coverage* of a bus $j \in \mathcal{N}$, denoted as $\text{cov}(j)$, is defined to be the set

$$\text{cov}(j) := \{s \in \mathcal{N} : \beta_{s,j} \neq 0\}$$

where $\beta_{s,j}$ is the j-th entry of β_s.

Now we are ready to give the spectral characterization for the P-controllability of (1).

Theorem 3.4: With all the previous notations, the dynamics (1) is P-controllable if and only if

1) The scaled Laplacian matrix L has simple spectrum
2) The algebraic coverage from controllable loads is full

$$\mathcal{N} = \bigcup_{j\in\mathcal{U}} \text{cov}(j) \tag{2}$$

Proof: Recall U is the diagonal matrix encoding the placement of controllable loads. Let V be the Vandermonde matrix

$$V = \begin{bmatrix} 1 & -\lambda_1 & \lambda_1^2 & \cdots & (-\lambda_1)^{n-1} \\ 1 & -\lambda_2 & \lambda_2^2 & \cdots & (-\lambda_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\lambda_n & \lambda_n^2 & \cdots & (-\lambda_n)^{n-1} \end{bmatrix}$$

and

$$u_j = Q^TM^{-1/2}U_j, \quad \forall j \in \mathcal{N}$$

where U_j is the j-th column of U.

Since Q is orthogonal, we know $(-L)^k = Q(-\Lambda)^k Q^T$ and as a result

$$W = Q\left[Q^T M^{-1/2} U \quad \cdots \quad (-\Lambda)^{n-1} Q^T M^{-1/2} U\right] \quad (3)$$

For any integer p, q, we denote as $r(p,q)$ the unique number $r \in \{1, 2, \cdots, q\}$ such that $p = qk + r$ for some integer k. Define a permutation matrix $\Pi \in \mathbb{R}^{n^2 \times n^2}$ given by

$$\Pi_{ij} = \begin{cases} 1 & j = (r(i,n) - 1)n + \lfloor (i-1)/n \rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, multiplying Π on the right hand side of W collects columns corresponding to each u_j together. With such notations, we have

$$\begin{aligned} &\left[Q^T M^{-1/2} U \quad \cdots \quad (-\Lambda)^{n-1} Q^T M^{-1/2} U\right] \Pi \\ &= \left[\operatorname{diag}(u_1) \quad \cdots \quad \operatorname{diag}(u_n)\right] (I_n \otimes V) \end{aligned} \quad (4)$$

where $\otimes$ means the tensor multiplication and (4) can be checked by directly comparing each component. Now note both Q and Π are invertible, from (3) we know the rank of W is the same as the rank of (4). Therefore by Proposition 3.2, we see the dynamics (1) is P-controllable if and only if (4) has full rank. It is well known that the determinant of the Vandermonde matrix V is given by

$$\det(V) = \prod_{i<j} [(-\lambda_i) - (-\lambda_j)]$$

and therefore V is invertible if and only if the tuple $(\lambda_s : s \in \mathcal{N})$ has distinct entries. Also it is easy to see that $I_n \otimes V$ has full rank if and only if V has full rank, thus $I_n \otimes V$ is invertible if and only if L has simple spectrum.

Next, it can be checked that the nonzero rows of

$$\left[\operatorname{diag}(u_1) \quad \operatorname{diag}(u_2) \quad \cdots \quad \operatorname{diag}(u_n)\right] \quad (5)$$

are independent because their nonzero entries appear in "orthogonal" positions. Therefore (5) has full row rank if and only if all the rows have nonzero entries, or in other words

$$\mathcal{N} = \bigcup_{j \in \mathcal{N}} \operatorname{supp}(u_j)$$

where $\operatorname{supp}(u_j)$ is the support of u_j.

From $u_j = Q^T M^{-1/2} U_j$ we can compute

$$u_{j,s} = \begin{cases} \frac{\beta_{s,j}}{\sqrt{M_j}} & j \in \mathcal{U} \\ 0 & j \notin \mathcal{U} \end{cases}$$

from which we see

$$\operatorname{supp}(u_j) = \begin{cases} \{s \in \mathcal{N} : \beta_{s,j} \neq 0\} & j \in \mathcal{U} \\ \emptyset & j \notin \mathcal{U} \end{cases}$$

But this implies

$$\bigcup_{j \in \mathcal{N}} \operatorname{supp}(u_j) = \bigcup_{j \in \mathcal{U}} \{s \in \mathcal{N} : \beta_{s,j} \neq 0\} = \bigcup_{j \in \mathcal{U}} \operatorname{cov}(j)$$

As a result, (4) has full rank if and only if L has simple spectrum and (2). This completes the proof. ∎

IV. Interpretations

The characterization given in Theorem 3.4 is purely algebraic. In this section, we explain the practical meanings of our controllability criteria.

A. *L should have simple spectrum*

Recall we mentioned in the introductory section that for the graph in Fig. 1, the system (1) cannot be controllable because of the network symmetry. It turns out that L having simple spectrum roughly means the associated graph has few symmetries and this condition can be interpreted as a general criteria on whether the network topology has too much symmetry so it loses certain controllability. Indeed, it is proven in [17] that if L has simple spectrum, then any nontrivial automorphism of $\mathcal{G}$ has order two[1]. This specifically rules out star graphs with more than three nodes and symmetric weights (that is M_j's are the same for all $i \in \mathcal{N}$ and B_e's are the same for all $e \in \mathcal{E}$), including the graph in Fig. 1, from having simple spectrum. As another example, it can be shown that the Laplacian matrices associated with line graphs (under arbitrary B and M) always have simple spectrum [18]. For more results relating properties of the graph automorphism group to the spectrum of L, we refer the readers to [16]–[18].

This condition is much less restrictive than one would expect. In fact, one can check that the associated L matrix for all test cases coming with the Matpower 6.0 package [19] (including almost all IEEE and RTE testbeds) has simple spectrum. We now establish a density result to explain such abundance of practical systems with simple spectrum.

Consider a fixed transmission network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with line susceptance matrix B and inertia matrix M. Let

$$\Omega = \prod_{e \in \mathcal{E}} (-B_e, \infty)$$

be the space of feasible perturbations to B (so that we have positive line susceptances). We add a random perturbation $\omega \in \Omega$ drawn according to certain probability measure μ to the line susceptances, which can come from either measurement noise or manufacturing error, and consider the resulting scaled Laplacian matrix

$$L(\omega) = M^{-1/2} C (B + \operatorname{diag}(\omega)) C^T M^{-1/2}$$

The following result shows that for a large family of perturbation distributions, $L(\omega)$ has simple spectrum almost surely.

Proposition 4.1: Let ν be the Lebesgue measure over $\mathbb{R}^m$. Assume the probability measure $\mu : \Omega \to [0,1]$ is absolutely continuous with respect to ν and is Borel[2]. Let

$$E = \{\omega \in \Omega : L(\omega) \text{ has simple spectrum}\}$$

Then E is Borel and $\mu(E) = 1$.

[1]The result in [17] requires the assumption $M = I_m$. One can, however, prove similar results for general M by assigning M_j as node weights and requiring an automorphism to preserve both line and node weights.

[2]For two measures μ and ν on the same measurable space Ω, μ is said to be absolutely continuous with respect to ν if and only if for any measurable set $E \subset \Omega$, $\nu(E) = 0$ implies $\mu(E) = 0$. A measure μ is said to be a Borel measure if all Borel sets are measurable with respect to μ. See [20] for more details.

The proof of this result is presented in Appendix II. Since a probability Borel measure is absolutely continuous with respect to the Lebesgue measure if and only if it affords a probability density function[3], for almost all practical probability models of such perturbation (e.g. truncated Gaussian noise with arbitrary covariance, bounded uniform distribution, truncated Laplace distribution), $L(\omega)$ has simple spectrum almost surely. Similar perturbation results on M also hold.

Therefore, under mild assumptions on perturbations to the system parameters, the L matrix associated with a practical system almost always has simple spectrum.

B. The algebraic coverage of controllable loads should be full

Intuitively, the algebraic coverage of a bus j reflects the set of eigenvectors of L (which are usually interpreted as the spectra of the network graph $\mathcal{G}$ in spectral graph theory) that the bus j can "interact" with. When the algebraic coverage of controllable loads is full, the control signals can interact with the entire spectra of the network and thus are able to drive the system to any state. As an illustration to this intuitive meaning of the algebraic coverage, we present an alternative interpretation for entries in the pseudo-inverse of L (which is denoted as $X := L^{\dagger}$) and demonstrate that such interpretation is natural in certain scenarios. Fix two buses $i, j \in \mathcal{N}$. For each $s \in \mathcal{N}$, we put $\kappa_{ij}^s = \beta_{s,j}\beta_{s,k}$, which can be interpreted as the "mutual influence" between i and j through the spectrum s. We have $\kappa_{ij}^s \neq 0$ if and only if $s \in \text{cov}(i) \cap \text{cov}(j)$, and when this holds, s lies in the common coverage of i and j and thus is a "bridging" spectrum. Recall $L = Q\Lambda Q^T$ and therefore $X = Q\Lambda^{\dagger}Q^T$, which then implies

$$X_{ij} = \sum_{s \in \text{cov}(i) \cap \text{cov}(j), s \neq 1} \frac{\kappa_{ij}^s}{\lambda_s}$$

In other words, X_{ij} can be interpreted as the weighted average of "mutual influence" between i and j over all "bridging" spectra.

When $M = I_m$, the "mutual influence" interpretation of entries of X turns out to be natural for some useful quantities in contingency analysis. For instance, under the DC power flow model, when we shift power generation of Δp_{ij} from bus $i \in \mathcal{N}$ to bus $j \in \mathcal{N}$, the power flow on a link $(k, l) \in \mathcal{E}$ would change accordingly, say by ΔP_{kl}. The ratio between ΔP_{kl} and Δp_{ij} is defined to be the generation-shift sensitivity factor [21] and is given as [4]

$$\frac{\Delta P_{kl}}{\Delta p_{ij}} = B_{kl} \left(X_{ik} + X_{jl} - X_{il} - X_{jk} \right) \tag{6}$$

One can note that the "mutual influence" terms among the involved buses i, j, k, l appear naturally in (6). Such "mutual influence" terms in fact also encode very detailed information on the spanning tree distributions of $\mathcal{G}$. We refer the readers to [22] for more detailed discussion.

[3]This is a result of the Radon-Nikodym Theorem. See [20] for example.

[4]In [21], the bus j is taken to be the slack bus and X is defined in a slight different way. One can however follow the same derivation in [21] to derive (6).

V. OBSERVABILITY

In this section, we present our characterization in the observability of (1). The development in this section is in parallel to Section III and thus we omit all proofs.

As in the case of controllability, the classical definition of observability is too strong. A more suitable notion of observability in our applications is given as follows.

Definition 5.1: The dynamics (1) is said to be *P-observable* or *observable in power system sense* if for any $t > 0$, an initial state $x(0) = [\omega(0); P(0)]$ such that

$$B^{-1} \left(P(t) - P(0) \right) \in \text{range}(C^T)$$

can be uniquely determined from the system input $d(s)$ and output $y(s)$ over $0 < s \leq t$.

We can then give the spectral characterization for the P-observability as follows.

Theorem 5.2: The dynamics (1) is P-observable if and only if

1) The scaled Laplacian matrix L has simple spectrum
2) The algebraic coverage from sensors is full

$$\mathcal{N} = \bigcup_{j \in \mathcal{S}} \text{cov}(j)$$

The second item in this criteria for observability again confirms our intuition that algebraic coverage encodes information on how buses interact with each other through the network.

VI. APPLICATIONS

In this section, we present two applications of our results. The first application is on the optimal placement of controllable loads/sensors so that controllability/observability of (1) is achieved. We show this problem can be reduced to a set cover problem. The second application is over the IEEE 39-bus New England interconnection test system, where we demonstrate a single critical bus chosen based on our theory is capable of regulating the frequency of the whole grid.

A. Optimal placement of controllable loads and sensors

Given a power transmission network $\mathcal{G}$, if the associated L matrix does not have simple spectrum, then by Theorem 3.4 and Theorem 5.2, such intrinsic deficiency of $\mathcal{G}$ forbids the dynamics (1) from being controllable/observable, no matter how many controllable loads or sensors we install. Fortunately, as Proposition 4.1 suggests, such deficiency usually does not occur for practical systems.

Now assume $\mathcal{G}$ has simple spectrum. By Theorem 3.4, the dynamics (1) is P-controllable if and only if the union of algebraic coverage from controllable loads is full. Therefore the problem of choosing the minimum set of buses to place controllable loads such that (1) is P-controllable can be formulated as

$$\min_{J} \quad |J| \tag{7a}$$

$$\text{s.t.} \quad \bigcup_{j \in J} \text{cov}(j) = \mathcal{N} \tag{7b}$$

This is an instance of the well-studied set cover problem, one of Karp's 21 NP-complete problems [23]. Although

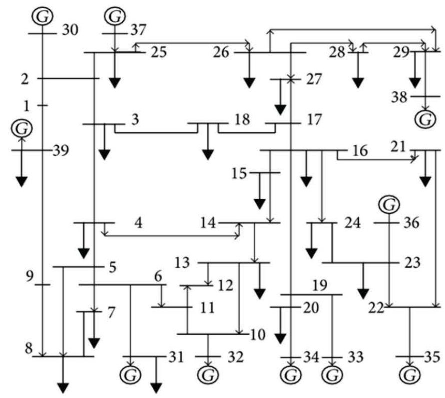
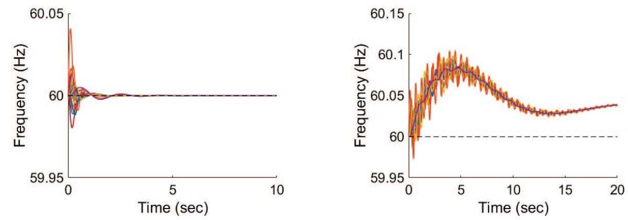


Fig. 2. Line diagram of the IEEE 39-bus New England interconnection test system.



(a) With control.

(b) Without control.

Fig. 3. Comparison of the system evolution with and without control at bus 35 after adding a step increase of 1 pu to the generation at bus 30.

Theorem 3.4 does not completely resolve (7), it shows that approximation algorithms devised for set cover problems can be readily applied to our setting to obtain placements with good quality.

A similar argument applied to the P-observability of (1) leads to the same optimization problem as (7). Therefore we are led to the following corollary, which is intuitive but non-trivial without Theorem 3.4 and Theorem 5.2.

Corollary 6.1: For the dynamics (1), the set of optimal placements of controllable loads and the set of optimal placement of sensors are the same.

This result tells us that, in practice, we should always install sensors at the buses with controllable loads, and vice versa.

B. Secondary frequency regulation with a single bus

We now demonstrate how our results can identify critical buses for controllability by evaluating over the IEEE 39-bus New England interconnection test system, as shown in Fig. 2. There are 10 generators and 29 load nodes in the system, and in contrast to our linearized model for theoretical study, the simulation adopts more realistic nonlinear dynamics.

One can check that the L matrix associated with this network has simple spectrum (which is as expected according to Proposition 4.1) and that the bus 35 has full algebraic coverage, i.e. all the eigenvectors β_s of L have nonzero entry at position 35. Therefore Theorem 3.4 implies that even if we can only inject control at bus 35, the system is still P-controllable. Thus we should be able to drive the whole system back to the nominal state after arbitrary disturbance. In order to verify this, we add a step increase of 1 pu to the generation at bus 30, and compare the system evolution with or without control at bus 35. In contrast to the standard control associated with the controllability Gramian, the control we adopt here utilizes only local frequency deviation. Details about the control scheme design can be found in [24]. The simulation results are shown in Fig. 3.

As one can see from the figure, despite the geographical distance between the disturbance and the controllable node, the control scheme successfully drives the grid back to nominal state within 5 seconds. In contrast, when no control is posed, the bus frequencies still stabilize because of governor dynamics, but not to the nominal state. Moreover, the stabilization process takes considerably longer time. Such difference demonstrates that with a single bus 35 chosen based on our theory, frequency regulation over the grid can actually be achieved.

VII. Conclusion

In this work, we develop full characterizations on the impact of limited controllable loads/sensors coverage over the controllability/observability for the swing and power flow dynamics in frequency regulation. We present two applications of our theoretical results: 1) an analytical application which reduces the problem of optimal placement of controllable loads and sensors to a set cover problem; 2) an evaluation over the IEEE 39-bus New England interconnection test system where secondary frequency control over the whole network can be achieved by a single critical bus chosen based on our theory.

Our results can be extended in several directions. First, the linearized model (1) is usually accurate near the nominal operation point, but may incur noticeable error when the system is far away from the equilibrium. Such scenarios may arise after a system failure. It is thus interesting to see how our results can be generalized to nonlinear dynamics. Second, the control suggested by our result can be very costly. It is of interest to understand what the cheapest control should be if we already know the system is controllable, and how the placement of controllable loads affects this optimal cost. Third, in applications, we usually only focus on the controllability over a subset of transmission lines that are subject to congestion. We should understand whether we can refine the theory so that the results can be tailored for such partial controllability.

References

[1] B. J. Kirby, "Spinning reserve from responsive loads," *Report of Oak Ridge National Laboratory*, 2003.

[2] PNNL, "Grid friendly controller helps balance energy supply and demand." [Online]. Available: http://readthis.pnl.gov/MarketSource/ReadThis/B3099_not_print_quality.pdf

[3] D. Trudnowski, M. Donnelly, and E. Lightner, "Power-system frequency and stability control using decentralized intelligent loads," in *2005/2006 IEEE/PES Transmission and Distribution Conference and Exhibition*, May 2006, pp. 1453–1459.

[4] N. Lu and D. J. Hammerstrom, "Design considerations for frequency responsive grid friendlytm appliances," in *2005/2006 IEEE/PES Transmission and Distribution Conference and Exhibition*, May 2006, pp. 647–652.

[5] C. Zhao, U. Topcu, and S. Low, "Swing dynamics as primal-dual algorithm for optimal load control," in *Smart Grid Communications (SmartGridComm), 2012 IEEE Third International Conference on*. IEEE, 2012, pp. 570–575.
[6] C. Zhao, U. Topcu, N. Li, and S. Low, "Design and stability of load-side primary frequency control in power systems," *Automatic Control, IEEE Transactions on*, vol. 59, no. 5, pp. 1177–1189, 2014.
[7] Z. Wang, F. Liu, L. Chen, and S. Mei, "Distributed economic automatic generation control: A game theoretic perspective," in *2015 IEEE Power Energy Society General Meeting*, July 2015, pp. 1–5.
[8] E. Mallada, C. Zhao, and S. Low, "Optimal load-side control for frequency regulation in smart grids," in *Communication, Control, and Computing (Allerton), 2014 52nd Annual Allerton Conference on*. IEEE, 2014, pp. 731–738.
[9] A. Kasis, E. Devane, C. Spanias, and I. Lestas, "Primary frequency regulation with load-side participation part i: stability and optimality," *IEEE Transactions on Power Systems*, vol. PP, no. 99, pp. 1–1, 2016.
[10] E. Devane, A. Kasis, M. Antoniou, and I. Lestas, "Primary frequency regulation with load-side participation part ii: beyond passivity approaches," *IEEE Transactions on Power Systems*, vol. PP, no. 99, pp. 1–1, 2016.
[11] A. Kasis, E. Devane, and I. Lestas, "Stability and optimality of distributed schemes for secondary frequency regulation in power networks," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 3294–3299.
[12] T. L. Vincent, K. Poolla, S. Mohagheghi, and E. Bitar, "Stability guarantees for primary frequency control with randomized flexible loads," in *2016 American Control Conference (ACC)*, July 2016, pp. 2328–2333.
[13] P. Šulc, S. Backhaus, and M. Chertkov, "Optimal distributed control of reactive power via the alternating direction method of multipliers," *IEEE Transactions on Energy Conversion*, vol. 29, no. 4, pp. 968–977, Dec 2014.
[14] H. Yang, C. Y. Chung, and J. Zhao, "Application of plug-in electric vehicles to frequency regulation based on distributed signal acquisition via limited communication," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 1017–1026, May 2013.
[15] Y. Ota, H. Taniguchi, T. Nakajima, K. M. Liyanage, and A. Yokoyama, "An autonomous distributed vehicle-to-grid control of grid-connected electric vehicle," in *2009 International Conference on Industrial and Information Systems (ICIIS)*, Dec 2009, pp. 414–418.
[16] F. R. K. Chung, *Spectral Graph Theory*. American Mathematical Society, 1997.
[17] C. O. Aguilar and B. Gharesifard, "Graph controllability classes for the laplacian leader-follower dynamics," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1611–1623, June 2015.
[18] N. Biggs, *Algebraic Graph Theory*, 2nd ed. Cambridge University Press, 1993.
[19] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Transactions on power systems*, vol. 26, no. 1, pp. 12–19, 2011.
[20] H. Royden and P. Fitzpatrick, *Real Analysis*. Prentice Hall, 2010.
[21] A. Wood and B. Wollenberg, *Power Generation, Operation, and Control*. Wiley-Interscience, 1996.
[22] L. Guo, C. Liang, and S. Low, "Monotonicity properties and spectral characterization of power redistribution in cascading failures," in *55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE, 2017.
[23] T. H. Cormen, C. Stein, R. L. Rivest, and C. E. Leiserson, *Introduction to Algorithms*, 2nd ed. McGraw-Hill Higher Education, 2001.
[24] J. Pang, L. Guo, and S. Low, "Optimal load control for frequency regulation under limited control coverage," in *10th Bulk Power Systems Dynamics and Control Symposium*, 2017.
[25] H. Federer, *Geometric measure theory*, ser. Grundlehren der mathematischen Wissenschaften. Springer, 1969.

APPENDIX I
PROOF OF PROPOSITION 3.2

Consider the vector space $Z := \mathbb{R}^n \times \text{range}(BC^T) \subset \mathbb{R}^{n+m}$. Then from the very definition we see that (1) is P-controllable if and only if the affine space

$$Z + [0; P(0)] = Z + x(0)$$

is reachable from $x(0)$ (the equality is because $[\omega(0); 0] \in Z$). Denote the set of Lebesgue integrable functions from $[0,t]$ to $\mathbb{R}^n$ as $I_{[0,t]}$. Now for any control input $d \in I_{[0,t]}$, the solution to (1a) is given by

$$\begin{aligned} x(t) = &-\int_0^t e^{(t-s)A}\begin{bmatrix} M^{-1}U \\ 0 \end{bmatrix} d(s)ds \\ &+ \int_0^t e^{(t-s)A}\begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} P^m(s)ds + e^{tA}x(0) \end{aligned} \quad (8)$$

We will show Proposition 3.2 by inspecting each term in (8).

Lemma 1.1: Let

$$R_1 := \left\{ \int_0^t e^{(t-s)A}\begin{bmatrix} M^{-1}U \\ 0 \end{bmatrix} d(s)ds : d \in I_{[0,t]} \right\}$$

be the set of possible values of the first term in (8). Then R_1 is a subspace of Z and $R_1 = Z$ if and only if

$$W = [M^{-1/2}U, -LM^{-1/2}U, \cdots, (-L)^{n-1}M^{-1/2}U]$$

has full row rank.

Proof: Since the set of buses with frequency sensitive components is contained in the set of buses with controllable loads, we can absorb the $D_j\omega_j$ term into d_j for all $j \in \mathcal{F}$ without affecting the system controllability. Therefore we can assume $F = 0$ in (1). Now by induction, we can compute $A^{(2k)}$ to be

$$\begin{bmatrix} M^{-1/2}(-L)^k M^{1/2} & 0 \\ 0 & (-BC^T M^{-1}C)^k \end{bmatrix}$$

and compute $A^{(2k+1)}$ to be

$$\begin{bmatrix} 0 & -M^{-1}C(-BC^T M^{-1}C)^k \\ BC^T M^{-1/2}(-L)^k M^{1/2} & 0 \end{bmatrix}$$

Put $\tilde{B} := [M^{-1}U; 0]$. It is a classical result that R_1 is the same as the range of the controllability matrix of (1) given as

$$\tilde{W} = [\tilde{B} \quad A\tilde{B} \quad \cdots A^{n-1}\tilde{B}]$$

Multiplying $\tilde{B}$ to the powers of A and discarding the zero columns, we see the range of $\tilde{W}$ is equal to the range of

$$\begin{bmatrix} \tilde{M}\tilde{U} & 0 & -\tilde{M}L\tilde{U} & \cdots & 0 \\ 0 & BC^T\tilde{M}\tilde{U} & 0 & \cdots & BC^T\tilde{M}(-L)^{n-1}\tilde{U} \end{bmatrix} \quad (9)$$

where $\tilde{M} := M^{-1/2}$ and $\tilde{U} := M^{-1/2}U$. Since BC^T is a common factor for the last m rows, we see the range of (9) and thus R_1 is a subspace of Z.

One can check that the dimension of Z is $2n-1$. Since R_1 is a subspace of Z, we know $R_1 = Z$ if and only if the rank of (9) is $2n-1$. Now define

$$W = [\tilde{U}, -L\tilde{U}, L^2\tilde{U}, \cdots, (-L)^{n-1}\tilde{U}]$$

Since both $\tilde{M}$ and B are invertible, it is easy to see that the rank of (9) is given by $\text{rank}(W) + \text{rank}(C^T W)$. Moreover, from $\text{rank}(W) \le n$ and $\text{rank}(C^T W) \le \text{rank}(C^T) \le n-1$, we know the matrix in (9) has rank $2n-1$ if and only if

$$\text{rank}(W) = n, \qquad \text{rank}(C^T W) = n-1$$

Finally note $\text{rank}(C^T W) = n-1$ is equivalent to having $\text{rank}(W) = n$. We thus see $\text{rank}(W) = 2n-1$ if and only if $\text{rank}(W) = n$, or in other words, $R_1 = Z$ if and only if W has full row rank. ■

Lemma 1.2: For arbitrary $P^m \in I_{[0,t]}$, we have

$$\int_0^t e^{(t-s)A} \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} P^m(s) ds \in Z$$

Proof: The proof is similar to the part where we show $R_1 \subset Z$ in Lemma 1.1 and we omit the details here. ■

Lemma 1.3: For arbitrary $x(0) \in \mathbb{R}^{n+m}$, we have

$$e^{tA}x(0) - x(0) \in Z$$

Proof: It is easy to check that the convergence radius of $g(x) := \sum_{i=1}^{\infty} \frac{x^{i-1}}{i!}$ is infinite and thus the matrix series $g(A) := \sum_{i=1}^{\infty} \frac{A^{i-1}}{i!}$ converges and is well-defined. Now note

$$e^{tA} - I = \sum_{i=1}^{\infty} \frac{A^i}{i!} = Ag(A)$$

thus $e^{tA}x(0) - x(0) = Ag(A)x(0) \in \text{range}(A)$. It is easy to check $\text{range}(A) \subset Z$, which then implies $e^{tA}x(0) - x(0) \in Z$. ■

Now put

$$R := \left\{\phi(x(0), d, t) : d \in I_{[0,t]}\right\}$$

to be the set of reachable states from $x(0)$ according to (8). From the above lemmas, we see that for any control d, we have

$$\phi(x(0), d, t) - x(0) \in Z$$

and therefore $R - x(0) \subset Z$. Moreover, since

$$\int_0^t e^{(t-s)A} \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} P^m(s) ds + e^{tA}x(0) - x(0) \in Z$$

$R - x(0) = Z$ if and only if $R_1 = Z$, which in turn is equivalent to W having full row rank. As a result, we see (1) is P-controllable if and only if W has full row rank. ■

Appendix II
Proof of Proposition 4.1

Fix a network $\mathcal{G}$ and the associated B, M matrices. For any $\omega \in \Omega$, let

$$\chi(t;\omega) := \det\left(M^{-1/2}C(B + \text{diag}(\omega))C^T M^{-1/2} - tI\right)$$

be the characteristic polynomial of the perturbed Laplacian matrix $L(\omega)$ and let $\text{disc}(\omega) := \text{disc}\,(\chi(t;\omega))$ be the discriminant of χ. Recall $\text{disc}(\omega)$ is a polynomial in the coefficients of χ and therefore is a polynomial of the entries in ω. Moreover, $\text{disc}(\omega) = 0$ if and only if χ has multiple roots, or equivalently $L(\omega)$ does not have simple spectrum. Put

$$E = \{\omega \in \Omega : L(\omega) \text{ has simple spectrum}\}$$

We then have $E = \Omega \setminus \text{disc}^{-1}(0)$. Since disc is a polynomial, we see E is Borel.

Lemma 2.1: The polynomial $\text{disc}(\omega)$ is not identically zero.

Proof: To show $\text{disc}(\omega)$ is not identically zero, it suffices to show that we can find $\omega \in \Omega$ such that $\text{disc}(\omega) \neq 0$, which is equivalent to the existence of a link susceptance matrix B_0 such that $L_0 := M^{-1/2}CB_0C^T M^{-1/2}$ has simple spectrum. We use mathematical induction to prove the existence of such B_0. To facilitate the discussion, for any subgraph $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$ of $\mathcal{G}$ with line susceptance matrix B', we refer to the matrix

$$L' = M'^{-1/2}C'B'(C')^T M'^{-1/2}$$

as the Laplacian matrix of $\mathcal{G}'$, where M' is the square submatrix of M corresponding to the nodes in $\mathcal{N}'$, and C' is the incidence matrix of $\mathcal{G}'$.

First we pick a random link e_1 in $\mathcal{E}$ and assign arbitrary susceptance to e_1. The subgraph $\mathcal{G}_1$ generated by e_1 has only one connected component (which is formed by e_1 itself) and therefore the Laplacian matrix L_1 of $\mathcal{G}_1$ has one zero eigenvalue and one nonzero eigenvalue, which are distinct.

Next, consider any subgraph $\mathcal{G}_k = (\mathcal{N}_k, \mathcal{E}_k)$ of $\mathcal{G}$ with $k < m$ many links and let $e_k = (i_k, j_k) \in \mathcal{E}\setminus\mathcal{E}_k$ be a link of $\mathcal{G}$ not in $\mathcal{G}_k$. Assume $\mathcal{G}_k$ has simple spectrum. We claim that by choosing the susceptance for e_k properly, the graph $\mathcal{G}_{k+1}$ obtained by adjoining e_k (and possibly one of the vertices i_k, j_k) to $\mathcal{G}_k$ still has simple spectrum. Indeed, for the case where both i_k and j_k are in $\mathcal{N}_k$, we know the Laplacian matrix L_{k+1} for $\mathcal{G}_{k+1}$ is given as

$$\begin{aligned} L_{k+1} = L_k + \Big(&- \frac{B_{i_k j_k}}{\sqrt{M_{i_k} M_{j_k}}}(E_{i_k j_k} + E_{j_k i_k}) \\ &+ \frac{B_{i_k j_k}}{M_{i_k}} E_{i_k i_k} + \frac{B_{i_k j_k}}{M_{j_k}} E_{j_k j_k}\Big) := L_k + \Delta L_k \end{aligned} \quad (10)$$

where for any i, j, E_{ij} is the matrix with 1 at the intersection of i-th row and j-th column and 0 otherwise. Let δ_k be the minimum gap between the eigenvalues of L_k. Choose $B_{i_k j_k}$ small enough so that the spectral norm of ΔL_k is less than $\delta_k/2$. Then by Weyl's inequality, we know that each eigenvalue of L_k is perturbed by at most $\delta_k/2$ from adding ΔL_k. As a result, the eigenvalues of L_{k+1} are still distinct.

If i_k is not in $\mathcal{N}_k$, then $L_{k+1} = \overline{L}_k + \Delta L_k$, where $\overline{L}_k$ is the matrix obtained from L_k by appending a row and a column of zeros and ΔL_k is the same as in (10). It is easy to see that $\overline{L}_k$ and L_k share the same nonzero eigenvalues and $\overline{L}_k$ has two zero eigenvalues. Similar to the previous case, by choosing $B_{i_k j_k}$ small enough, we can ensure the distinct nonzero eigenvalues of $\overline{L}_k$ after perturbation of ΔL_k are still distinct and nonzero. Note L_{k+1} has only one zero eigenvalue, thus by choosing $B_{i_k j_k}$ even smaller if necessary, the new nonzero eigenvalue coming from the perturbation of ΔL_k can be made arbitrarily small and thus distinct from other eigenvalues. We have thus justified our claim.

Now by induction, we see that we can always pick the line susceptances properly so that the resulting L has simple spectrum. This completes the proof. ■

It is well-known from algebraic geometry that the root set of a polynomial which is not identically zero has Lebesgue measure zero [25]. In particular, for the polynomial $\text{disc}(\omega)$ which is not identically zero, we have $\nu\left(\text{disc}^{-1}(0)\right) = 0$. Since μ is absolutely continuous with respect to ν, we see $\mu\left(\text{disc}^{-1}(0)\right) = 0$ or equivalently, L has simple spectrum with probability 1 as $E = \Omega \setminus \text{disc}^{-1}(0)$. ■