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State observers in the design of eigenstructures for enhanced sensitivity

D. Bernal

Civil and Environmental Engineering Department, Center for Digital Signal Processing, Northeastern University, Boston, MA, United States

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ABSTRACT

The problem of closed-loop enhanced sensitivity design is as follows: Given a linear time invariant system, find a (realizable) feedback gain such that: (1) the closed-loop is stable in the reference and the potentially damaged states, and (2) the eigenstructure includes a subset of poles, with desirable derivatives, that lie in a part of the plane where identification is feasible. This paper shows that pole derivatives with respect to system parameters for a controller/observer system, contrary to the assumption often made, depend on both the controller and the observer gains, i.e. the separation principle holds for placing the poles but does not extend to the pole derivatives. Closed-form expressions for the derivatives with due consideration to both gains are presented. Examination shows that the sum of these derivatives is independent of both gains, is constant along the nonlinear paths traced by the poles as damage increases and, provided the damage affects only the stiffness, is nearly zero.

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1. Introduction

Eigenstructure assignment is a well-known scheme in which a static linear gain that satisfies design objectives is obtained by directly specifying closed loop poles and eigenvectors. Eigenstructures that, in addition to satisfying some performance objectives are least sensitive to uncertainties in some parameters are also of interest, and work related to this goal can be found in [1–5], among others. The flip side of sensitivity minimization, sensitivity maximization, has been proposed in Structural Health Monitoring to improve the resolution of damage characterization from identified eigenvalue shifts [6–10]. This paper examines how the presence of an observer in the loop, necessary when the operating mode is estimated state feedback, affects the sensitivities. Eigenvalue sensitivities depend on the right and left side eigenvectors of the controller/observer system and in this regard design for sensitivity enhancement is an eigenvector placement problem. Pole locations remaining relevant, however, since apart from the constraints imposed on them by identifiability and stability, their positions determine the subspaces wherein the closed-loop eigenvectors must lay [11].

The design of closed-loop eigenstructures for monitoring requires decisions on the cost function to be maximized (minimized), decisions on the extent and distribution of damage for which closed-loop stability must be satisfied, and specification of the limits the hardware imposes on the controller. While specific choices on these items are made in the numerical example, the objective of this paper is not to propose design criteria but to present consistent expressions for the evaluation of the derivatives of the poles of a controller/observer system. In as far as these derivatives go the common practice has been to assume that the separation principle, which holds true for pole positions, extends to the derivatives and, therefore, that

E-mail address: bernal@neu.edu

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results for the controller poles do not depend on the observer gain. It is shown here that this is not so, and while the reason is best appreciated in the context of the derivations, the essence can be stated from the outset, namely: pole derivatives depend on both gains because parameter shifts resulting from damage are not known to the observer and, as a consequence, pole positions in the perturbed state depend on both gains. It's opportune to note that the problem of robust pole placement in the presence of an observer [3,4] differs from the one considered here in that the issue in the former is how to select gains such that the deviations between the target and the realized controller/observer poles from inevitable model error are minimized; the focus here being, instead, on how small changes in parameters, taking place after the controller/observer is formulated, translate into movement of the poles.

In this paper we parameterize the controller/observer transition matrix to reflect the fact that the observer is unaware of changes due to damage and derive the consistent closed form expressions for the pole derivatives. Comparison between the expressions obtained and the ones that hold for full state measurements show that the effect of the observer is captured by a matrix that is the stabilizing solution of a Sylvester equation [12]. Numerical results show that the effect of the observer on the Jacobian of the controller poles, and on the extent of damage for which the closed-loop is stable, can be large. A number of lemmas extracted from the derivation are presented and proved; the most significant, given its implications in the design for stability, shows that the sum of the discrete time (DT) derivatives is independent of the controller and of the observer gains. A numerical section exemplifying the analytical examinations is included.

2. Sensitivity of controller/observer system

Let S be a non-defective but otherwise arbitrary square matrix that is a function of some parameter, θ. We shall refer to the derivatives of the eigenvalues of S with respect to θ as sensitivities; not to be confused with the matrix in the Laplace domain that carries the same name [13,14]. The sensitivity of the jth eigenvalue with respect to θ writes

$$\lambda_j' = \varphi_j^T S' \psi_j \tag{1}$$

where ψ_j and φ_j are the right and left side eigenvectors and the prime indicates differentiation with respect to θ. We now particularize Eq. (1) to the case where S is the transition matrix of a finite dimensional linear time invariant system under estimated state feedback. We begin with the expression that governs the evolution of the state in DT, namely

$$x_{k+1} = A_d x_k + B_d u_k + B_\omega \omega_k + B_f f_k \tag{2}$$

where $A_d \in R^{N\times N}$ is the transition matrix, $B_d, \in R^{N\times r}$ and $B_\omega \in R^{N\times z}$ are the control input and disturbance influence matrices, $u_k \in R^{r\times 1}$ are the control inputs and $\omega_k \in R^{z\times 1}$ the disturbances, which are typically assumed to be zero mean, Gaussian, and white, with covariance Q. Finally, if there are any deterministic exogenous excitations, $f_k \in R^{h\times 1}$, then $B_f \in R^{N\times h}$ is the associated influence matrix. Given the stochastic disturbances (plus the measurement noise) the poles identified from finite length signals are random variables and, consequently, so are the identified pole movements due to plant parameter changes (damage in this application). The expressions derived next are deterministic and correspond to the expectation level from an unbiased identification. With $K \in R^{N\times m}$ as the time invariant gain of the observer, the evolution of the estimated state is governed by

$$\hat{x}_{k+1} = A_d \hat{x}_k + B_d u_k + B_f f_k + K(y_k - \hat{y}_k) \tag{3}$$

where the true and the estimated outputs are denoted y_k and $\hat{y}_k \in R^{m\times 1}$. For full estimated state feedback we have

$$u_k = -G \cdot \hat{x}_k \tag{4}$$

where $G \in R^{r\times N}$ is the control gain, $\hat{x}$ is the estimated state and the minus sign is, of course, conventional. We restrict the output y_k to be a linear combination of the state plus some measurement noise, v_k, typically assumed zero mean, Gaussian, and white, with covariance R. Excluding cases with direct transmission, or assuming the direct transmission contribution is subtracted from the output, one has

$$y_k = C_d x_k + v_k \tag{5}$$

From previous results it follows that the state and the estimated state form a 2N linear system having the state space recurrence

$$\begin{Bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{Bmatrix} = \begin{bmatrix} A_d & -B_d G \\ KC_d & A_{do} - KC_d - B_d G \end{bmatrix} \begin{Bmatrix} x_k \\ \hat{x}_k \end{Bmatrix} + \begin{bmatrix} B_f \\ B_f \end{bmatrix} \{f_k\} + \begin{bmatrix} B_\omega & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \omega_k \\ v_k \end{Bmatrix} \tag{6}$$

where the reader will note that we've introduced notation to distinguish between the transition matrix that reflects changes due to damage, A_d, and the invariant transition matrix of the state estimator, A_{do}. Needless to say, in the reference state $A_{do} = A_d$. Observations on the poles and eigenvectors of the system in Eq. (6) are most easily made after introducing the basis transformation

$$\begin{Bmatrix} x_k \\ e_k \end{Bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{Bmatrix} x_k \\ \hat{x}_k \end{Bmatrix} \tag{7}$$

where $e_k = x_k - \hat{x}_k$ is the state error. Substituting Eq. (7) into Eq. (6) and recognizing that the matrix in Eq. (7) is involutory one finds

$$\begin{Bmatrix} x_{k+1} \\ e_{k+1} \end{Bmatrix} = \begin{bmatrix} A_d - B_d G & B_d G \\ 0 & A_{do} - KC_d \end{bmatrix} \begin{Bmatrix} x_k \\ e_k \end{Bmatrix} + \begin{bmatrix} B_f \\ 0 \end{bmatrix} \{f_k\} + \begin{bmatrix} B_\omega & 0 \\ B_\omega & -K \end{bmatrix} \begin{Bmatrix} \omega_k \\ v_k \end{Bmatrix} \tag{8}$$

Since the transition matrices in Eqs. (6) and (8) are related by a similarity the poles are preserved and are, as can be seen from the fact that the transition matrix in Eq. (8) is block triangular, the union of the closed-loop poles of the controller and the observer; a result known as the separation principle [15]. Also clear from the block triangular nature is the fact that the eigenvector matrix of the transition in Eq. (8) has the form

$$\Phi = \begin{bmatrix} \Psi_a & Z \\ 0 & \Psi_b \end{bmatrix} \tag{9}$$

where Ψ_a and Ψ_b are the eigenvectors of the controller and of the observer, namely

$$(A_d - B_d G)\Psi_a = \Psi_a \Lambda_a \tag{10}$$

and

$$(A_{do} - KC_d)\Psi_b = \Psi_b \Lambda_b \tag{11}$$

Inspection of Eq. (8) shows that the matrix Z in Eq. (9) satisfies the Sylvester equation

$$L_1 Z - Z L_2 = E \tag{12}$$

where, in this case,

$$L_1 = A_d - B_d G, \quad L_2 = \Lambda_b, \quad E = -B_d G \Psi_b \tag{13a-c}$$

It is known that Sylvester's equation has a unique solution provided there are no common eigenvalues in L_1 and L_2; a constraint that in this case translates to the requirement that the poles of the controller and of the observer, Λ_a and Λ_b have no common entries. The constraint is here easily satisfied because the poles of the observer are typically much more damped than those of the controller. We move forward, therefore, on the premise that there are no coincident poles. It is useful to note that the solution of Sylvester's equation is particularly simple in this case because the matrix L_2 is diagonal and this allows Z to be formed one column at a time. Specifically, from Eqs. (12) and (13) one gathers that the jth column is given by

$$z_j = (I \cdot \lambda_{b,j} - A_d + B_d G)^{-1} B_d G \cdot \psi_{b,j} \tag{14}$$

We now move to the task of evaluating the derivatives of the poles of the system in Eq. (6). From Eqs. (7) and (9) one finds that the right side eigenvector matrix is

$$\Psi = \begin{bmatrix} \Psi_a & Z \\ \Psi_a & Z - \Psi_b \end{bmatrix} \tag{15}$$

The inverse of the matrix in Eq. (15), which is the transpose of the left side eigenvectors, can be expressed in block partitioned form as

$$\Psi^{-1} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \tag{16}$$

Differentiating the transition matrix in Eq. (6) and substituting Eqs. (15) and (16) in the expression for the pole derivative writes

$$\begin{Bmatrix} diag(\Lambda_a') \\ diag(\Lambda_b') \end{Bmatrix} = diag\left(\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} A_d' & -B_d' G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi_a & Z \\ \Psi_a & Z - \Psi_b \end{bmatrix} \right) \tag{17}$$

The partitions in Eq. (16) are easily found to be

$$X_1 = \Psi_a^{-1}(I - Z\Psi_b^{-1}), \quad X_2 = \Psi_a^{-1} Z \Psi_b^{-1}, \quad X_3 = -X_4 = \Psi_b^{-1} \tag{18a-c}$$

Substituting Eqs. (18) into Eq. (17) writes

$$diag(\Lambda_a') = diag(\Psi_a^{-1} L (A_d' - B_d' G)\Psi_a) \tag{19}$$

and

$$diag(\Lambda_b') = diag(\Psi_b^{-1}(A_d'(I - L) + B_d' G L)\Psi_b) \tag{20}$$

where we've taken

$$L = I - Z\Psi_b^{-1} \tag{21}$$

Although damage induces (in general) changes in all the poles, one generally focuses on the controller poles because the observer ones are often highly damped and thus difficult to identify. At this juncture we note that the matrix L in Eq. (21) captures all the effect that the observer has on the derivatives, namely, for L = I Eq. (19) gives the derivatives of the controller that hold true when the full state is measured. Evaluation of Eqs. (19) and (20) requires the derivatives of B_d and A_d. The first depends on how the control action is delivered and for the commonly used zero order hold circuit writes

$$B_d' = A_c^{-1}(A_d' B_c - A_c' B_d + (A_d - I)B_c') \tag{22}$$

The second involves the matrix exponential and is most conveniently evaluated using the complex perturbation approach, which in our case writes [16–18]

$$A_d' = \lim_{\varepsilon \to 0} \Im\left(\frac{e^{(A_c + A_c'\varepsilon i)\Delta t}}{\varepsilon}\right) \tag{23}$$

where the subscript c has been used to indicate the continuous time version of the respective matrices.

2.1. Direct solution for $Z\Psi_b^{-1}$

Examination of Eq. (19) shows that the derivatives of the controller poles do not depend on Z and Ψ_b^{-1} separately but only on their product. If one is not interested in the derivatives of the observer poles (as is generally the case) this product can be obtained directly without solving the eigenvalue problem of the observer. Specifically, one notes that Eq. (12) can be written as

$$(A_d - B_d G)Z\Psi_b^{-1} - Z\Psi_b^{-1}\Psi_b\Lambda_b\Psi_b^{-1} + B_d G = 0 \tag{24}$$

or, using Eq. (11), as

$$(A_d - B_d G)Z\Psi_b^{-1} - Z\Psi_b^{-1}(A_{d0} - KC) + B_d G = 0 \tag{25}$$

which is a Sylvester equation for $Z\Psi_b^{-1}$. In Eq. (25) $A_d = A_{d0}$.

2.2. Pseudo continuous time

The sensitivities obtained from measurements using an unbiased identification algorithm have an expectation equal to the DT results from Eq. (19) and (20) with a variance that approaches zero asymptotically as the signals duration approaches infinity. In practice it is customary to report results on the s-plane because physical significance is more readily judged. The standard mapping relating the poles in discrete and continuous time is

$$\lambda_c = \frac{log(\lambda_d)}{\Delta t} \tag{26}$$

from where it follows that

$$\lambda_c' = \frac{\lambda_d'}{\lambda_d \Delta t} \tag{27}$$

Note that while Eqs. (26) and (27) give true system properties when the operation is in open loop, in the closed-loop case there is no real "continuous time" as the transition matrix includes the matrix B, which depends on how the control action is delivered.

2.3. Is Z = 0 Possible?

We close by noting that the possibility of an observer that has no effect on the controller sensitivities, namely, one for which Z = 0 does not exist. Indeed, to realize Z = 0 it is necessary, by inspection of Eqs. (12) and (13c), to have $B_d G \Psi_b = 0$, and this is precluded by the fact that $B_d G$ is not zero and Ψ_b is full column rank.

3. Constraints on the sum of the sensitivities

Some constraints that can aid qualitative reasoning on how the poles evolve as damage progresses are derived next.

Lemma#1. *The sum of the sensitivities of the controller/observer system is independent of the gain of the controller and of the gain of the observer.*

Proof. Recognizing that Eq. (17) is a similarity transformation and the fact that the trace of a matrix is equal to the sum of its eigenvalues one has

$$\sum \{\, diag(\Lambda'_a) \quad diag(\Lambda'_b) \,\} = trace(A'_d) \tag{28}$$

and independence follows since A_d is the transition matrix of the open loop. □

Lemma#2. *For stiffness related damage the result from Eq. (28) is (almost) independent of the spatial distribution of the damage and is close to zero - exact zero reached asymptotically as* $\Delta t \to 0$.

Proof. The transition matrix in continuous time for a system with mass, damping and stiffness $\{M, C_{dam}, K_s\}$ writes

$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K_s & -M^{-1}C_{dam} \end{bmatrix} \tag{29}$$

so for stiffness related damage one has

$$A'_c = \begin{bmatrix} 0 & 0 \\ -M^{-1}K'_s & 0 \end{bmatrix} \tag{30}$$

A two term Taylor approximation of the DT transition is

$$A_d \cong I + A_c \Delta t \tag{31}$$

and thus

$$A'_d \cong A'_c \Delta t \tag{32}$$

so

$$trace(A'_d) \cong trace(A'_c)\Delta t \tag{33}$$

where the right-hand side is identically zero as shown by Eq. (30). □

Lemma#3. *On the premise that stiffness is linear on the parameters one has that for stiffness related damage the sum of the updated sensitivities, as damage evolves, is nearly constant, with exact invariance realized as* $\Delta t \to 0$.

Proof. The statement follows as an immediate consequence of lemma#2 and the premise that the stiffness is linear on the parameters so that the derivative is independent of the stiffness. □

We close by noting that while the previous results apply strictly to the sum of the derivatives of the controller and the observer, the derivatives of the observer poles are typically small compared to those of the controller and one is justified, at least for qualitative reasoning, to treat these results as applying to the derivatives of the controller separately.

4. Numerical illustration

The structure in Fig. 1 is selected to give some quantitative context to the analytical work. We restrict attention to the two closed-loop poles having the lowest frequencies and to two damage distributions, loss of stiffness in levels #1 or level #3, making the inspected eigenvalue Jacobian 2×2. The essential point is to illustrate that an observer/controller, deployed to track parameter changes, cannot be judged from the behavior of the associated state feedback controller. Design of the full state feedback controller is carried out as follows:

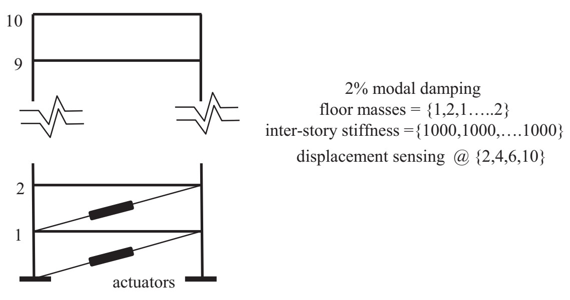

Fig. 1. Structure used for numerical illustrations.

(1) Free parameters: $\Theta = \{\theta_1 \quad \cdots \quad \theta_6\}$ with $\lambda_j^{CL} = \theta_j \lambda_j^{OL}$ $j = 1,2$ $\Theta \in \Re^{1\times 6}$ $\varphi_1^{CL} = V_1\{\theta_3 \quad \theta_4\}^T$ $\varphi_2^{CL} = V_2\{\theta_5 \quad \theta_6\}^T$ where V_j are the subspaces where the closed loop modes can be placed, namely $V_j = Null[A_d - I\cdot\lambda_j^{CL} \quad -B_d]$
(2) Constraints: $\Re(\lambda^{CL}) \leqslant 0$ for 25% damage in levels #1 or #3, and $\|G\| \leqslant 2000$, where G is the controller gain.
(3) Objective: $\arg\max\limits_{\Theta} z := \{z|\min(\sigma)\}$ where $\sigma =$ singular values of the Jacobian.
(4) All poles not part of the optimization are placed at the location of the open loop ones and the constants that collapse the allowable eigenvector bases are taken to maximize the projection of the closed-loop eigenvectors on the corresponding open loop ones.

Note that restriction of θ_{3-6} to the real domain and the fact that the poles in the optimization are placed on specified radial lines, are not constraints imposed by the problem but are adopted (at some loss in optimality) to reduce the number of free parameters. Finally, although a single constant suffices here to collapse the eigenvector basis into a realization, two constants were chosen to reduce the range over which the search is carried out. Appendix A includes, for convenience, a summary of the standard SVD approach used to map the eigenstructure parametrization to the gain [11].

4.1. Observers

Two observers are formulated, a Kalman filter [19] for process and measurement noise $Q = B_\omega(100\cdot I_{10})B_\omega^T$ and $R = 10^{-5}\cdot I_4$, which corresponds to around 5% SNR in the open loop response, and a Luenberger observer [20], with poles having the same magnitude as the open-loop ones and damping increased by a factor of 10 (from 2 to 20%). The 2-norm of the Kalman and the Luenberger observers are 20.3 and 93.8 respectively.

4.2. Effect of the observer on the CT Jacobian

The Jacobians, identified by their superscripts are:

$$J^{SF} = \begin{bmatrix} 0.05+4.00i & 0.05+5.40i \\ -0.04-4.92i & -0.05-4.62i \end{bmatrix}\cdot 10^{-2} \quad J^{KAL} = \begin{bmatrix} -0.06+3.15i & -0.09+3.02i \\ 0.10-3.89i & 0.080-2.56i \end{bmatrix}\cdot 10^{-2}$$

$$J^{LUEN} = \begin{bmatrix} 1.31-10.17i & -9.35+3.37i \\ 2.28+14.60i & 9.61-6.58i \end{bmatrix}\cdot 10^{-2} \quad J^{OL} = \begin{bmatrix} 0.035i & 0.095i \\ 0.096i & 0.087i \end{bmatrix}\cdot 10^{-2}$$

A cursory examination shows that the Jacobian for state feedback does not, as expected, reflect the sensitivities that hold when either the Kalman or the Luenberger observer is in the loop. Other {Q,R} pairs and other Luenberger poles would lead to different results and the point here is not to suggest one observer over the other but to stress that the results obtained on the premise of full state measurements are, in general, far from those realized in the controller/observer implementation.

4.3. Closed-Loop stability

Stability in the reference condition is guaranteed since the separation principle holds and both the observer and the controller poles are stable. The situation once damage ensues differs. In this particular example what is found is that the controller + Kalman system satisfies the stability constraints but the controller + Luenberger becomes unstable at small stiffness losses. The behavior is easily rationalized by noting that the largest negative real part in the Jacobian for the controller + Kalman (indicating that a loss of stiffness moves the pole towards the imaginary) is $-9e^{-4}$ while for the Luenberger is more than one hundred times larger at $-935e^{-4}$. We take the opportunity to note that the path of a closed-loop pole in the complex plane, as damage increases, can be highly nonlinear and thus predictions of instability thresholds based on Jacobians can be poor. A good example is the second closed-loop pole when damage takes place in level#3 as shown in Fig. 2b.

Two comments regarding the stability check are opportune. First, because of potentially high nonlinearity checking stability at the largest damage considered does not exclude the possibility of unstable behavior at smaller damages. Second, checking maximum damage one parameter at a time does not ensure stability for distributions that involve multiple damages, even if the multiple damage patterns have smaller parameter change vector norms than the single damage cases. At the time of writing it is not clear whether these are important or marginal concerns.

5. Conclusions

The paper has shown that the separation principle does not hold in the computation of derivatives of a controller/observer because the observer properties are independent of the damaged. Design of closed-loop eigenstructures for sensitivity based on estimated state feedback must, therefore, consider the gain of the observer. Expressions to compute the pole derivatives accounting for both gains are presented in the paper. A result of some interest is the fact that the sum of the DT sensitivities is independent of both the controller and the observer gains and that it is nearly zero. The noteworthy observation in this result is that if in the closed-loop one or more poles move at a high rate away from the imaginary, as damage

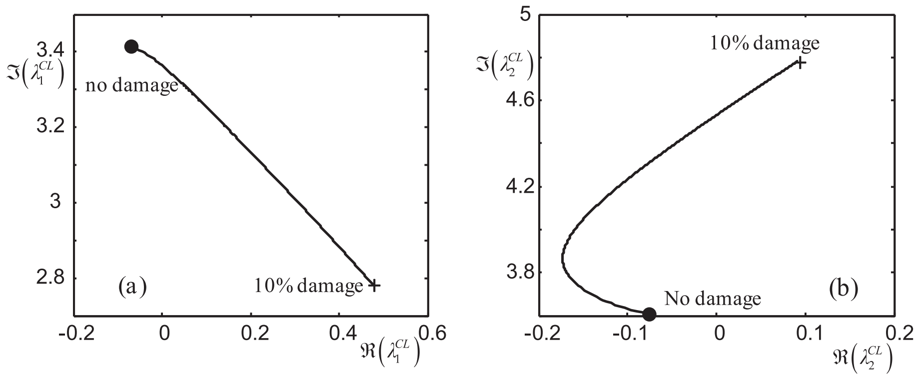

Fig. 2. Path of the two closed loop poles monitored as damage in level#3 increases from 0 to 10%.

takes place, then some other poles have to make up for it by moving towards the imaginary with the obvious stability implications.

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Appendix A. Mapping of closed-loop eigenstructure to controller gain

Taking the entries of the controller gain as free parameters is generally prohibitive from a dimensionality perspective but an option along the same line is to parameterize it as

$$G = \sum_{j=1}^{\ell} \alpha_j G_j \tag{A.1}$$

where ℓ depends on what the nonlinear optimization algorithm can handle and α_j are the free parameters. Apart from the obvious question of how to best select the G_j matrices, an inconvenient feature of this alternative is that realizations can prove unstable. A parametrization where stability can be guaranteed in every trial uses pole locations and the constants that collapse the eigenvector basis as the free parameters and we summarize it next for convenience in reviewing the numerical example. Let, $\Lambda^{CL} = \{ \lambda_1^{CL} \quad \cdots \quad \lambda_N^{CL} \}$ be a trial closed-loop eigenstructure; to ensure that the associated controller gain, G, is real Λ^{CL} must be closed under conjugation and is evident that stability is realized by taking $\Re(\lambda_j^{CL}) \leqslant 0$. Since the controller is implemented in DT it is appropriate to extract the gain by operating in DT so that the effect of the inter-sample behavior of the control can be appropriately considered [21]. The eigenvalue problem writes

$$(A_d - B_d G)\psi_j^{CL} = \psi_j^{CL} \lambda_j^{CL} \tag{A.2}$$

from where

$$\begin{bmatrix} A_d - I \cdot \lambda_j^{CL} & -B_d \end{bmatrix} \begin{Bmatrix} \psi_j^{CL} \\ G\psi_j^{CL} \end{Bmatrix} = 0 \tag{A.3}$$

with

$$V_j = Null\left(\begin{bmatrix} A_d - I \cdot \lambda_j^{CL} & -B_d \end{bmatrix} \right) = \begin{bmatrix} S_j \\ Q_j \end{bmatrix} \tag{A.4}$$

where $S_j \in \mathbb{C}^{Nxr}$ *and* $Q_j \in \mathbb{C}^{rxr}$ it follows that

$$\begin{bmatrix} S_j \\ Q_j \end{bmatrix} \{\theta_j\} = \begin{Bmatrix} \psi_j^{CL} \\ b_j \end{Bmatrix} \tag{A.5}$$

where it's evident that

$$b_j = G \cdot \psi_j^{CL} \tag{A.6}$$

Taking

$$\Psi^{CL} = \begin{bmatrix} \psi_1^{CL} & \cdots & \psi_N^{CL} \end{bmatrix} \quad and \quad \Gamma = \begin{bmatrix} b_1 & \cdots & b_N \end{bmatrix} \tag{A.7-A.8}$$

one has

$$G \cdot \Psi^{CL} = \Gamma \tag{A.9}$$

and thus, on the premise that the eigenvectors have been selected so that Ψ^{CL} is full rank

$$G = \Gamma \cdot (\Psi^{CL})^{-1} \tag{A.10}$$

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