Mechanical Systems and Signal Processing 105 (2018) 481–487

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

Eigenvalue sensitivity of sampled time systems operating in closed loop

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ARTICLE INFO

Article history:
Received 5 March 2017
Received in revised form 7 November 2017
Accepted 9 November 2017
Available online 22 December 2017

Keywords:
Eigenvalue sensitivity
Matrix exponential
Closed loop instability

ABSTRACT

The use of feedback to create closed-loop eigenstructures with high sensitivity has received some attention in the Structural Health Monitoring field. Although practical implementation is necessarily digital, and thus in sampled time, work thus far has center on the continuous time framework, both in design and in checking performance. It is shown in this paper that the performance in discrete time, at typical sampling rates, can differ notably from that anticipated in the continuous time formulation and that discrepancies can be particularly large on the real part of the eigenvalue sensitivities; a consequence being important error on the (linear estimate) of the level of damage at which closed-loop stability is lost. As one anticipates, explicit consideration of the sampling rate poses no special difficulties in the closed-loop eigenstructure design and the relevant expressions are developed in the paper, including a formula for the efficient evaluation of the derivative of the matrix exponential based on the theory of complex perturbations. The paper presents an easily reproduced numerical example showing the level of error that can result when the discrete time implementation of the controller is not considered.

1. Introduction

A classic problem in control theory is the selection of static gains leading to closed-loop eigenstructures with desired characteristics. In structures subjected to narrow band disturbances, for example, one seeks to keep all the close-loop poles away from the peaks of the excitation spectrum. Hardware deployed for vibration control can also be used to perform closed-loop Structural Health Monitoring (SHM) in periods when its primary function is unnecessary. In this instance the objective is not to exert control on the response but to create an eigenstructure whose sensitivity facilitates interrogation regarding the existence of damage. As one gathers, this is an application where eigenvalue sensitivity plays a central role.

The idea of doubling up the hardware to perform SHM in closed-loop was introduced by Ray and Tian [1] who recommended, based on single-degree-of-freedom considerations, that the gain be selected to shift the poles towards lower frequencies. In a subsequent examination, Jiang, Tang and Wang [2] used the fact that the expression for the closed-loop sensitivity is a function of the right and left eigenvectors and exploited the freedom in eigenvector placement offered by multiple actuators to increase sensitivity while penalizing the magnitude of the control gain. Other work on the optimization of the gain for sensitivity and some exploratory experimental work can be found in [3–8].

Studies carried out thus far on closed-loop sensitivity enhancement have been based on the continuous time (CT) framework. Practical implementations are, however, invariably digital and the question opens up as to how the digital to analog

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https://doi.org/10.1016/j.ymssp.2017.11.014
0888-3270/

(D/A) conversion and the sampling rate affect the results. This paper shows that the eigenvalue sensitivities realized in the discrete time (DT) implementation can differ significantly from the CT results, even at sampling rates that can be considered fast relative to the shortest resonant period in the system. Particularly important in this regard being the fact that the discrepancy is often large on the real part of the sensitivity, and thus on the level of parameter changes (damage in the Structural Health Monitoring application) for which the closed-loop is predicted to remain stable in a linear estimate. In what follows we will refer to the eigenvalues and sensitivities that would be obtained by an ideal identification algorithm operating on the sampled data as the "realized" closed-loop eigenvalues and sensitivities. As noted, the realized properties depend on the sampling rate and on the specifics of the D/A conversion, and on this last aspect we adopt the simplest of all the options that satisfy the causality constraint, the widely used zero order hold (ZOH).

A feature in the computation of DT sensitivities that does not exist in the CT model is the need for the derivative of the matrix exponential. Included in the paper are two exact expressions for this derivative, the first in terms of an integral [9] and the second as an infinite series in terms of lie brackets [10]. Because it is somewhat tangential to the paper's objective we do not discuss the matter in detail but note that their numerical evaluation is not efficient compared to an estimation based on the theory of complex perturbations [11–13], which is thus recommended for applications. The paper derives the expressions that give the realized closed-loop sensitivity, presents a brief discussion on the linear estimation of parameter changes leading to closed-loop instability and includes an easily reproducible numerical example exemplifying the main points.

2. Effective closed-loop sensitivity

The state space recurrence in discrete time for a Linear Time Invariant system operating in closed-loop writes

$$x_{k+1} = A_d x_k + B_{d,u} u_k + B_{d,f} f_k \tag{1}$$

where $A_d \in R^{NxN}$, $B_{d,u} \in R^{Nxr}$ and $B_{d,f} \in R^{Nxq}$ are the transition, control to state, and external forces to state matrices, respectively, and x, u and f $\in R^{Nx1}, \in R^{rx1}, \in R^{qx1}$ are the state, the control forces, and the exogenous excitation, with N = system order, r = number of actuators and q = number of external actions. For a control action based on static constant gain one has

$$u_k = -Ky_k = -KCx_k \tag{2}$$

where $C \in R^{mxN}$ is the state to output matrix, with m = number of measurements, and the minus sign is, of course, conventional. In writing Eq. (2) we've assumed that the measurements do not include collocated accelerations. Substituting Eq. (2) into Eq. (1) the transition matrix in closed-loop writes

$$\bar{A}_d = A_d - B_d KC \tag{3}$$

Let θ be a parameter of the description of the system in CT, differentiating Eq. (3) with respect to θ writes

$$\bar{A}'_d = A'_d - B'_d KC \tag{4}$$

where independence of C from the parameter implies that acceleration measurements have been excluded. Given a non-defective matrix $\bar{A}_d(\theta)$ with eigenvalues $\lambda_d(\theta)$ and left and right side eigenvectors $\varphi(\theta)$ and $\psi(\theta)$ the derivative of the j^{th} eigenvalue with respect to θ writes

$$\lambda'_j = \varphi_j^T \bar{A}'_d \psi_j \tag{5}$$

where we've left out explicit reference to the parameter to simplify the notation. The relation between B_d in Eq. (4) and its continuous time counterpart is a function of how the control action is delivered. It is common to operate on the premise that this action is applied though a D/A zero order hold circuit, which, neglecting delays, leads to the relation [14,15]

$$B_d = A_c^{-1}(A_d - I)B_c \tag{6}$$

with B_c = the control input to state matrix in continuous time. Differentiating Eq. (6) writes

$$B'_d = A_c^{-1}(A'_d B_c + A_d B'_c - B'_c - A'_c B_d) \tag{7}$$

and substituting Eq. (7) into Eq. (4) gives

$$\bar{A}'_d = A'_d - A_c^{-1}\{A'_d B_c + (A_d - I)B'_c - A'_c B_d\}KC \tag{8}$$

The open-loop transition matrix in discrete time is given by

$$A_d = e^{A_c \Delta t} \tag{9}$$

where Δt is the sampling time step. The derivative of Eq. (9) is needed to evaluate Eq. (8) and in doing so it is necessary to keep in mind that the derivative of the exponential matrix does not follow the elementary calculus rules but writes [9]

$$\frac{d}{d\theta}(e^{Q(\theta)}) = \int_0^1 e^{\alpha Q(\theta)} \frac{dQ(\theta)}{d\theta} e^{(1-\alpha)Q(\theta)} d\alpha \tag{10}$$

where (in this instance) we have $Q(\theta) = A_c(\theta)\Delta t$. The result in Eq. (10) can also be expressed as an infinite series that writes

$$\frac{d}{d\theta}(e^{Q}) = \left\{ Q' + \frac{1}{2!}[Q,Q'] + \frac{1}{3!}[Q,[Q,Q']] + \cdots \right\} \cdot e^{Q} \tag{11}$$

where we've omitted explicit reference to the dependence of Q on the parameter θ to simplify the notation and where the operation [X, Y], known as the commutator of the matrices X and Y, also known as the lie bracket of X, Y [10] is defined as

$$[X, Y] \doteq XY - YX \tag{12}$$

Much more computationally efficient than the numerical evaluation of the integral in Eq. (10) or the summation of enough terms in Eq. (11) is to estimate the matrix exponential derivative using the complex perturbation scheme [11–13]. This approach involves only one evaluation of the matrix exponential (albeit with complex entries) and in this case writes

$$A'_d = \lim_{\varepsilon \to 0} \Im\left(\frac{e^{(A_c + A'_c \varepsilon i)\Delta t}}{\varepsilon} \right) \tag{13}$$

where ε can be taken very small without incurring finite precision difficulties. The DT sensitivity realized in a conventional identification is the result from Eq. (5), with the matrix from Eq. (8). The sensitivities that a user observes in an identification campaign are the DT results mapped to CT according to Eq. (9), namely

$$\lambda_c = \frac{\log(\lambda_d)}{\Delta t} \tag{14}$$

and

$$\lambda'_c = \frac{\lambda'_d}{\lambda_d \Delta t} \tag{15}$$

In summary, the sensitivities that can be used to judge the effectiveness of any postulated closed-loop gain are those from Eq. (15) and not the sensitivity of the closed-loop transition matrix in CT used thus far in the literature. Note that the values in Eqs. (14) and (15) are only "operationally equivalent" CT properties since the mapping of Eq. (5) does not hold for the input to state matrix that appears in the closed-loop transition matrix expression.

3. Closed-loop stability

Among the items that need to be addressed to make closed-loop monitoring viable is ensuring that the operation is stable for the range of parameter variations that may occur as a result of damage. The problem can be generically described as one of ensuring that

$$S \cup P = S \tag{16}$$

where P is the region in parameter space for which the design requires that the system be stable and S is the region for which the closed-loop is in fact stable. One expects stability to be an important constraint in using feedback to enhance sensitivity because sensitivity and stability are conflicting goals. Since the theme of this paper is not the design of closed-loop eigenstructures we do not pursue the issues that arise in the characterization of P and S but limit our discussion to showing that if any estimation of S is to be carried out using sensitivities then these sensitivities must account for the manner in which the control forces are delivered.

3.1. Linear estimate of parameter changes at incipient instability

To make the computations in the numerical section transparent we outline the approach used there to obtain instability bound information. The approach is a first order Taylor expansion of the eigenvalue locus which does not require explicit discussion. A couple of observations that can enhance clarity, however, may be worth noting and we make them within a narrative that outlines the approach.

The linear approximation of the eigenvalue locus for any change in the parameter vector $\Delta\theta \in R^{qx1}$ writes

$$\lambda \cong \lambda_0 + J \cdot \Delta\theta \tag{17}$$

where the subscript 0 is used to indicate the reference state and $J \in \mathbb{C}^{nxq}$ is the Jacobian that lists the eigenvalue derivatives with respect to each parameter as its columns. Designating Υ as the vector space that contains all the $\Delta\theta$ vectors for which incipient instability is predicted (by linear analysis) one has

$$\Upsilon = \{\Delta\theta | \Re(\lambda_j) = 0 \vee \Im(\lambda_j) = 0 \; j = 1, 2, \ldots, q\} \tag{18}$$

where all entries on $\Delta\theta$ must be $\leqslant 0$. It's worth noting that we've designated the event of reaching the real line (not only the imaginary) as incipient instability. This is justified by the fact that when an eigenvalue pair migrates to the real line (becoming a repeated pole) additional parameter changes split the repeated pole into two real ones that move in opposite directions and very small additional perturbations make the one that moves to the right cross the imaginary. For all practical purposes, therefore, hitting the real line is a stability limit.

There are infinite vectors $\Delta\theta$ that satisfy Eq. (18) and their tips define, in hyperspace, the surface of the linear estimate of S. The simplest and probably most informative vectors in this space are those in the set of minimum cardinality, namely the set in which only one entry in $\Delta\theta$ is non-zero. The non-zero entry in these vectors being the minimum scalar for which the associated column of the Jacobian moves some eigenvalue to an instability boundary.

4. Numerical examination

We consider a simple system for which the computations are easily reproduced, if desired, specifically, a 2-DOF system driven by one actuator, as shown in Fig. 1. Due to the single actuator the closed-loop eigenvectors are fully specified by the location of the closed-loop poles and to make the scenario even simpler we assume full state feedback, which makes the matrix C in Eq. (8) equal to the identity. We take the position that a user computes the gain and the resulting sensitivities in a CT framework and then illustrate how these values compare to the real situation where the control action is delivered through a ZOH circuit, as a function of the sampling rate. Referring to the upper quadrant we place one closed-loop pole at the location of the open-loop that has the lowest frequency and the other such that the closed-loop eigenvalue gap is collinear with the open-loop one and equals 25% of its magnitude. The resulting gain is listed in Fig. 1 together with the open loop and the "equivalent" closed-loop poles.

The real and the imaginary parts of the sensitivity with respect to reductions in the stiffness of the first and second levels are depicted in Figs. 2 and 3, respectively, for both poles, as a function of the sampling time step and are compared to the sensitivity predicted by the CT formulation. To judge the range selected for the time steps we note that the shortest period in the CT closed-loop is 0.86 s; sampling rates in practice for a system like this would likely be in the order of 0.03–0.06 s (1/30–1/15 of the shortest relevant period).

A cursory examination of Figs. 2 and 3 shows that the CT result deviates significantly from what is actually realized, even at relatively high sampling rates, and, in particular, that the discrepancies are large in the real part, which often dominates instability estimates.

4.1. Stability

For specificity we obtain the linear estimate of the instability bound for a time step of 0.025 s, which is around 34 times faster than the shortest period and can thus be considered fast sampling. The Jacobian is

$$J = \begin{bmatrix} 2.87 + 8.61i & -2.87 - 6.27i \\ -2.86 + 1.91i & 2.87 + 14.11i \end{bmatrix} 10^{-2} \tag{19}$$

and the reference state realized in "equivalent" CT poles are

$$\lambda_0 = \begin{Bmatrix} -0.094 + 4.68i \\ -1.257 + 7.23i \end{Bmatrix} \tag{20}$$

From these results the estimate of the vector of maximum parameter changes based on a linear analysis is

$$\Delta\theta = \begin{Bmatrix} 44 \\ 3.3 \end{Bmatrix} \tag{21}$$

where the first and second entries are governed by the second and the first pole reaching the imaginary, respectively.

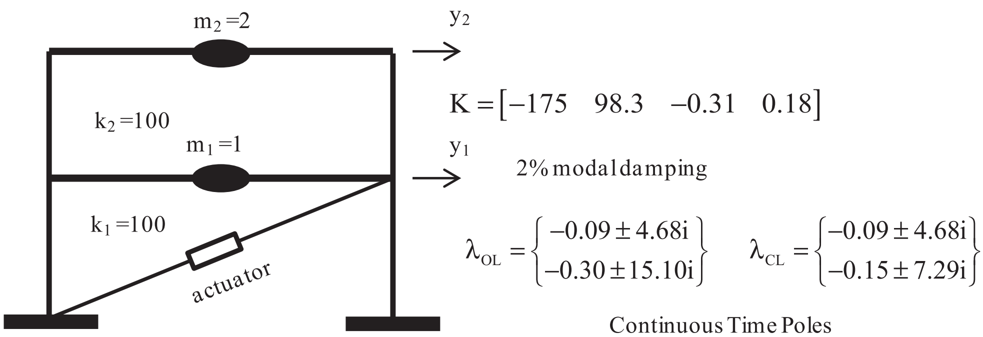

Fig. 1. Structure of numerical example.

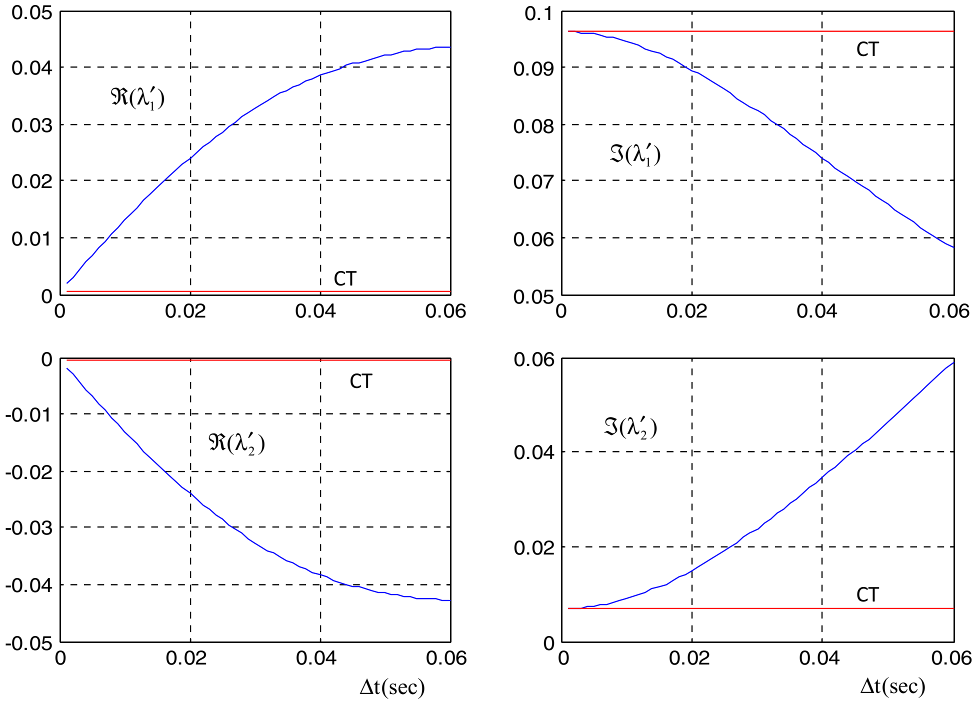

Fig. 2. Real and imaginary parts of the sensitivity of the two closed-loop eigenvalues for the system of Fig. 1 for changes in k_1.

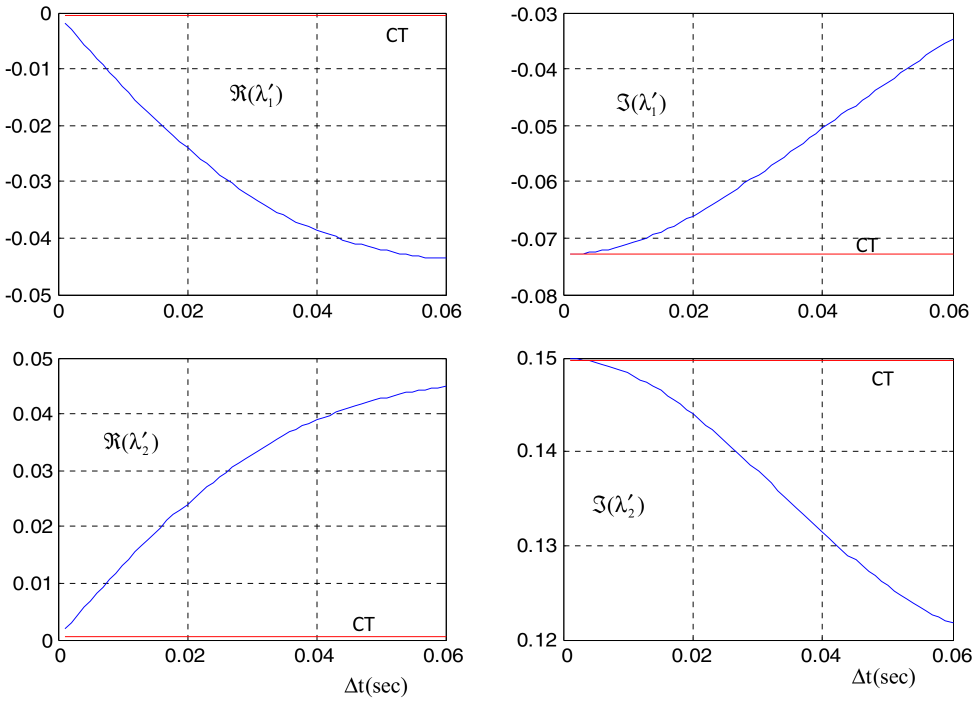

Fig. 3. Real and imaginary parts of the sensitivity of the two closed-loop eigenvalues for the system of Fig. 1, for changes in k_2.

If the effect of sampling is neglected, i.e. if the user assumes that the results of the CT model hold with sufficient accuracy the results are:

$$J_{CL} = \begin{bmatrix} 0.06 + 9.62i & -0.06 - 7.28i \\ -0.06 + 0.68i & 0.06 + 14.97i \end{bmatrix} 10^{-2} \tag{22}$$

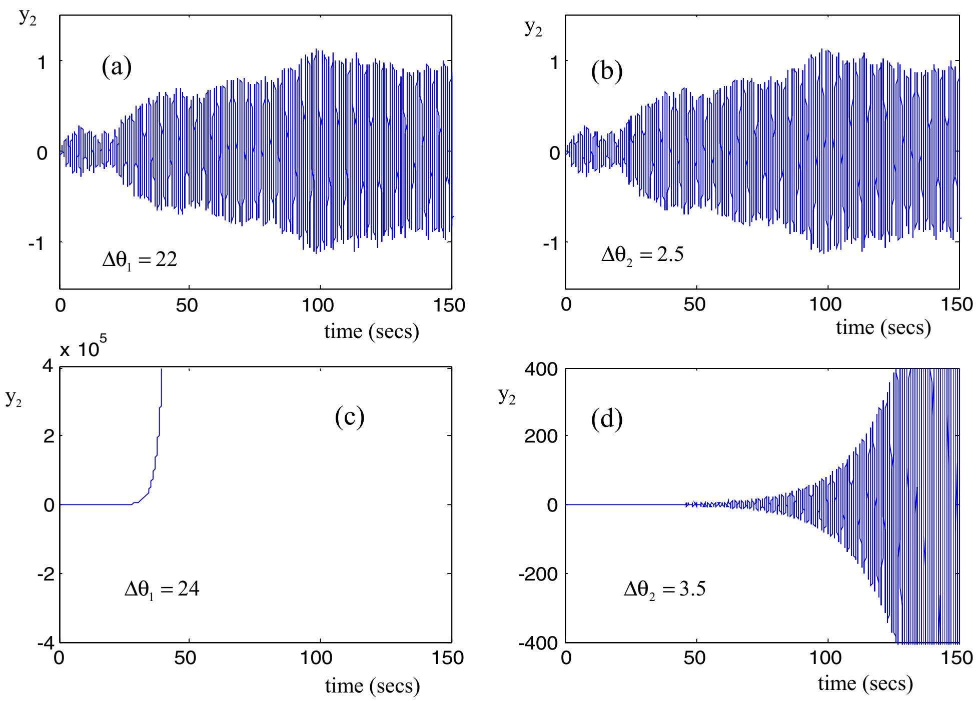

Fig. 4. (a) and (c) responses at the second coordinate of the system in Fig. 1 for two close values of stiffness reduction in level#1, (b) and (d) responses at the second coordinate for two close values of stiffness reduction in level#2.

$$\lambda_{0,CL} = \begin{Bmatrix} -0.094 + 4.68i \\ -0.146 + 7.29i \end{Bmatrix} \tag{23}$$

and the estimated parameter changes at incipient instability are then

$$\Delta\theta_{CL} = \begin{Bmatrix} 48.7 \\ 48.7 \end{Bmatrix} \tag{24}$$

The first and second entries determined by the first pole reaching the real line and the second the imaginary, respectively.

Comparison of the results of Eq. (21) with those in Eq. (24) shows that the CT predictions do not capture the fact that the system is marginally stable when it comes to losses in stiffness in the second level. Note that since the reference value of both parameters is 100, parameter changes are also % reductions needed to reach instability, and in the second level the prediction is only 3.3%. Also worth explicit note is the fact that the reasonable agreement in the change needed to reach instability in the first parameter is incidental since the boundary that is reached in the CT case (the real line) is not the actual one.

To validate the previous predictions, while minimizing circularity, we do not evaluate instability by computing eigenvalues but rather simulate the response in closed-loop and detect instability by increasing the parameter changes (one at a time) until instability is reached. Responses at either side of the detected instability boundary are depicted in Fig. 4. As can be seen, instability takes place at a reduction of around 23% in the first parameter and between 2.5 and 3.5% in the second. The prediction of the linear analysis is accurate in the (practically important) case where only a small change is needed and, not surprisingly, is poor in predicting the limit of the first parameter, which is relatively large. The important observation being, of course, that the instability estimate for the second parameter based on the CT sensitivities is in gross error.

5. Conclusions

The primary objective of this paper is to bring attention to the fact that the predictions based on a CT formulation of closed-loop eigenstructure sensitivity can be poor indicatives of what is realized in the actual DT implementation. It is important, therefore, that the DT nature of the implementation be explicitly considered. As one anticipates, the design and assessment of closed-loop eigenstructure with due consideration for the DT nature of the controller presents no particular difficulty and the relevant expressions are given in the paper, including an efficient approach to evaluate the matrix exponential derivative, which is needed. The paper stresses the fact that the error in the CT estimate of the DT realized sensitivity can be particularly large in the real part, and therefore, on the linear estimate of the damage level for which the closed-loop is anticipated stable.

Acknowledgement

This research was funded by NSF grant 1634277 ''Monitoring the Health of Structural Systems from the Geometry of Sensor Traces", this support is gratefully acknowledged.

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