Lebesgue-Approximation-Based Model Predictive Control for Nonlinear Sampled-Data Systems with Measurement Noises

Lixing Yang, Xiaofeng Wang

***Abstract*— In computer-controlled systems model predictive control (MPC) algorithms are usually implemented in a discrete-time manner even when the plant is continuous-time. It means that (i) the time instants to sample the states and compute the optimal solution to a finite horizon optimal control problem (FHOCP) must be triggered intermittently and (ii) the computation of the control inputs, including solving the FHOCP, should be completely discrete-time. Inappropriate discretization may place a heavy computational burden on the processor and possibly lead to violation of system constraints, instability, and infeasibility of MPC. This paper presents a discrete-time MPC algorithm, based on Lebesgue approximation, for nonlinear sampled-data systems. In this algorithm, the sampling instants are triggered by a self-triggered scheme. The predictive model in the FHOCP is iterated in an aperiodic manner subject to the Lebesgue approximation model. Sufficient conditions are derived on feasibility and stability of the closed-loop systems with the guarantee of exclusion of Zeno behavior.**

I. Introduction

Model predictive control (MPC) is an efficient tool used in a wide range of applications, such as process control, power grids, transportation systems, and manufacturing, to name a few. It can, to some extent, optimize the performance of control systems subject to constraints on states and inputs. A standard implementation of MPC predicts the (near) optimal control inputs over a finite horizon based on a predictive model that represents the behavior of the actual dynamical system of interest.

In computer-controlled systems, MPC algorithms are often implemented in discrete-time. With this observation, sampled-data MPC has been investigated where algorithms were developed to identify the sampling time instants. Periodic sampling was studied in [1], [2]. However, this approach could be conservative in some applications since it may trigger the computation of the solution to a finite horizon optimal control problem (FHOCP) more frequent than necessary and therefore lead to significant over-provisioning to the processor.

As a result, event-triggered MPC algorithms were developed to address the issue where the sampling and computation instants are determined by the occurrence of some pre-defined events [3], [5]–[9]. Different from event-triggering approaches, self-triggered MPC predicts the next the sampling instant based on the past information [10]–[14]. All these works focus on determining the sampling/computation instants, while the controller itself remains to be continuous-time. In other words, at each sampling/computation time instant, the controller still needs to solve a continuous-time FHOCP. Discretization of the FHOCP in MPC with fixed period was considered in [15], [16] for continuous-time linear systems. To extend to nonlinear case, our recent work proposed a Lebesgue-approximation-based MPC approach (LAMPC) [17] where the discretization of the FHOCP is based on the Lebesgue-approximation model (LAM) of the continuous-time plant [18]. Different from periodic approaches, the LAM iterates both the predictive states and the future sampling time instants. This aperiodic approach may potentially enlarge the predicted horizon.

This paper extends the LAMPC approach in [17] to nonlinear sampled-data systems with measurement noises. We study how the noises affect accuracy of the LAM and the performance of the MPC. In the proposed algorithm the sampling time instants is triggered by a self-triggering scheme and the LAM is used in the FHOCP as an approximation of the continuous-time plant. Sufficient conditions on feasibility and stability of the resulting closed-loop systems are derived. It is shown that the system is uniformly ultimately bounded with appropriate thresholds in the LAM.

The paper is structured as follows. Problem formulation is presented in Section II. The LAMPC algorithm is introduced in Section III. Section IV analyzes feasibility of our approach and Section V discusses system stability. Simulation results are provided in Section VI. Finally, conclusions are drawn in Section VII.

II. Problem Formulation

Definition 2.1: A continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{K}_\infty$ if it is strictly increasing and $\alpha(0) = 0$, $\lim_{s\to\infty} \alpha(s) = \infty$.

Definition 2.2: The state $x(t)$ of a system $\dot{x} = f(x)$ is called uniformly ultimately bounded (UUB) with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0 , such that $\|x(t_0)\| \leq a$ implies $\|x(t)\| \leq b$ for any $t \geq t_0 + T$.

Consider a nonlinear continuous system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \quad (1)$$

Lixing Yang and Xiaofeng Wang are with the Department of Electrical Engineering at the University of South Carolina, Columbia, SC, 29208, email: lixing@email.sc.edu, wangxi@cec.sc.edu. The authors gratefully acknowledge the partial financial support of the National Science Foundation (IIS-1525900 and ECCS1739886).

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the system state, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ is the control input, and $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$. The sets $\mathcal{X}$ and $\mathcal{U}$ are the constraint sets on the state and input, respectively. We assume $f(x, u)$ is locally Lipschitz with respect to x, i.e., there exists a positive constant L_f so that for $\forall x, y \in \mathcal{X}$ and $u \in \mathcal{U}$, the following inequality holds:

$$\|f(x, u) - f(y, u)\| \leq L_f \|x - y\|.$$

The main idea of MPC is described as follows: At time t_k, the system samples the state and obtains the sampled state $\bar{x}(t_k)$. Using $\bar{x}(t_k)$ as the initial point, the controller solves the FHOCP over the prediction horizon $[t_k, t_k + T_k]$, where T_k is the horizon length, and identify the next sampling time instant t_{k+1}. Then the calculated optimal control inputs will be applied to the plant over the time interval $[t_k, t_{k+1})$. The next sampling and prediction will start at t_{k+1}.

We assume that the sampled state contains measurement noise, i.e.,

$$\bar{x}(t_k) = x(t_k) + w(t_k)$$

where $w(t_k)$ is the noise satisfying [1]

$$\|w(t_k)\| \leq \sigma. \tag{2}$$

In our discrete-time MPC framework, the sampling time instants and the calculation of the FHOCP must be discrete. In general, the next sampling time t_{k+1} can be expressed as

$$t_{k+1} = t_k + \phi_k, \tag{3}$$

where $\phi_k \in \mathbb{R}^+$ is the inter-sampling time interval to be defined. The cost function of the FHOCP at the kth computation is

$$J\left[\hat{u}|\bar{x}(t_k)\right] = \int_{t_k}^{t_k+T_k} L(\hat{x}, \hat{u}) d\tau + V_f(\hat{x}(t_k + T_k)), \tag{4}$$

where $\hat{x} : \mathbb{R}^+ \to \mathbb{R}^n$ and $\hat{u} : \mathbb{R}^+ \to \mathbb{R}^m$ are the predictive state and input, respectively, $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ is the running cost function, and $V_f : \mathbb{R}^n \to \mathbb{R}^+$ is the terminal cost function.

The objective is to design a completely discrete-time and cost-efficient MPC algorithm to stabilize system (1), in the presence of measurement noises and state/input constraints.

III. LAMPC Algorithm

This section presents the LAMPC algorithm. To begin with, we first define the LAM for the FHOCP: Starting at time t_k,

$$\begin{aligned} \hat{x}_k^{i+1} &= \hat{x}_k^i + d\hat{x}_k^i \frac{\hat{D}_k^i}{\|d\hat{x}_k^i\|}, \quad \hat{x}_k^0 = \bar{x}(t_k) \\ t_k^{i+1} &= t_k^i + \frac{\hat{D}_k^i}{\|d\hat{x}_k^i\|}, \quad t_k^0 = t_k \\ d\hat{x}_k^i &= f(\hat{x}_k^i, \hat{u}_k^i), \end{aligned} \tag{5}$$

where $\hat{x}_k^i$, $\hat{u}_k^i$, and $\hat{D}_k^i = D(\hat{x}_k^i, \hat{u}_k^i)$ are the predictive state, the predictive input, and the state-dependent discretization level of LAM at the ith iteration, respectively. The threshold function $D : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ will be discussed later.

Let $N \in \mathbb{N}$ be the number of iterations in the LAM. The horizon length can be described as $T_k = t_k^N - t_k^0$. Using zero-order-hold approximation $\hat{x}(\tau) = \hat{x}_k^i$ and $\hat{u}(\tau) = \hat{u}_k^i$ for any $\tau \in [t_k^i, t_k^{i+1})$, the cost function (4) can be rewritten as

$$\begin{aligned} J[\hat{u}_k|\bar{x}(t_k)] &= \sum_{i=0}^{N-1} \int_{t_k^i}^{t_k^{i+1}} L(\hat{x}(\tau), \hat{u}(\tau)) d\tau + V_f(\hat{x}(t_k^N)) \\ &= \sum_{i=0}^{N-1} L(\hat{x}_k^i, \hat{u}_k^i)(t_k^{i+1} - t_k^i) + V_f(\hat{x}_k^N) \\ &= \sum_{i=0}^{N-1} L(\hat{x}_k^i, \hat{u}_k^i) \frac{\hat{D}_k^i}{\|d\hat{x}_k^i\|} + V_f(\hat{x}_k^N). \end{aligned} \tag{6}$$

Then we can state the FHOCP at time t_k as a discrete-time optimal control problem:

$$\begin{aligned} V(\bar{x}(t_k)) = \min_{\hat{u}_k^i \in \mathcal{U},\ i=0,\dots,N-1} & J[\hat{u}_k|\bar{x}(t_k)] \\ \text{subject to} \quad & \hat{x}_k^{i+1} = \hat{x}_k^i + d\hat{x}_k^i \frac{\hat{D}_k^i}{\|d\hat{x}_k^i\|} \\ & \hat{x}_k^0 = \bar{x}(t_k) \\ & \hat{x}_k^i \in \mathcal{X}_i, i = 1, \dots, N-1 \\ & \hat{x}_k^N \in \mathcal{X}_N = \mathcal{X}_f \end{aligned} \tag{7}$$

where $\mathcal{X}_i$ are the constraint sets on the predictive state $\hat{x}_k^i$ and $\mathcal{X}_N$ is the terminal set.

Let $\hat{u}_k^{i,*}$ $(i = 0, \dots, N-1)$ be the optimal solutions and $\hat{x}_k^{i,*} \in \mathcal{X}_i$ be the corresponding optimal states. Accordingly, $d\hat{x}_k^i$ and $\hat{D}_k^i$ in (5) become

$$\hat{D}_k^{i,*} = D(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}) \quad \text{and} \quad d\hat{x}_k^{i,*} = f(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}).$$

With the optimal solution, we can define the next sampling time instant t_{k+1}, following the time iteration in (5):

$$t_{k+1} = t_k^{1,*} = t_k + \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}. \tag{8}$$

The LAMPC algorithm is summarized as follows.

TABLE I

LAMPC Algorithm Routine

At time $t = t_k$,	
1	Sample the state and obtain $\bar{x}(t_k)$;
2	Solve the FHOCP (7) for $\hat{x}_k^{i,*}$ and $\hat{u}_k^{i,*}$;
3	Apply the optimal solution $\hat{u}_k^{0,*}$ to the plant, i.e, set $u(t) = \hat{u}_k^{0,*}$ over $[t_k, t_{k+1})$;
4	Start the next sampling and computation cycle at time t_{k+1} defined by (8);

IV. Feasibility

In order to guarantee feasibility of LAMPC, the constraint sets $\mathcal{X}_i$ in the FHOCP must be correctly defined. Notice that

[1] Notice that if the system is output-feedback, given an observer, $\bar{x}(t_k)$ can be viewed as the estimate of $x(t_k)$ and $w(t_k)$ can be the estimation error. As long as the estimation error is bounded, the results in this paper are still applicable.

we do not have direct access to $x(t)$, but only access to $\hat{x}_k^i$. Therefore, to ensure $x(t) \in \mathcal{X}$, the idea is to reduce the constraint set $\mathcal{X}$ by certain amount to obtain $\mathcal{X}_i$ and hopefully $\hat{x}_k^{1,*} \in \mathcal{X}_1$ can imply $x(t) \in \mathcal{X}$. To begin with, we first study the error dynamics between the LAM (5) and the continuous-time dynamics (1). Let $z(t)$ be defined as

$$z(t) = \bar{x}(t_k) + d\hat{x}_k^{0,*}(t - t_k), \ \forall t \in [t_k, t_{k+1}]. \tag{9}$$

Note that $z(t_k) = \hat{x}_k^{0,*} = \bar{x}(t_k)$ and $z(t_{k+1}) = \hat{x}_k^{1,*}$.

Lemma 4.1: Consider system (1) and the signal $z(t)$, then for any $t \in [t_k, t_{k+1}]$, the following inequality holds

$$\|x(t) - z(t)\| \leq \epsilon_k \triangleq \hat{D}_k^{0,*}\left(e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}} - 1\right) + \sigma e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}}, \tag{10}$$

where σ is defined in (2).

Proof: By the definition of $z(t)$, we know that $\dot{z} = d\hat{x}_k^{0,*} = f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})$ for $t \in [t_k, t_{k+1}]$. Let us set the error to be $e(t) = x(t) - z(t)$, then we can get its dynamics as:

$$\dot{e}(t) = f(x(t), \hat{u}_k^{0,*}) - f(\bar{x}(t_k), \hat{u}_k^{0,*}),$$

which leads to the inequality

$$\begin{aligned}
&\frac{d}{dt}\|e(t)\| \leq \|\dot{e}(t)\| = \|f(x(t), \hat{u}_k^{0,*}) - f(\bar{x}(t_k), \hat{u}_k^{0,*})\| \\
&\leq L_f\|x(t) - \bar{x}(t_k)\| = L_f\|x(t) - z(t) + z(t) - \bar{x}(t_k)\| \\
&\leq L_f\|e(t)\| + L_f\|z(t) - z(t_k)\| \\
&\leq L_f\|e(t)\| + L_f\|d\hat{x}_k^{0,*}(t - t_k)\| \\
&\leq L_f\|e(t)\| + L_f\|d\hat{x}_k^{0,*}(t_{k+1} - t_k)\| \\
&= L_f\|e(t)\| + L_f\|d\hat{x}_k^{0,*}\|\frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} = L_f\|e(t)\| + L_f\hat{D}_k^{0,*}.
\end{aligned}$$

Solving the inequality with $\|e(t_k)\| \leq \sigma$, we obtain

$$\begin{aligned}
\|e(t)\| &\leq \hat{D}_k^{0,*}(e^{L_f(t-t_k)} - 1) + \sigma e^{L_f(t-t_k)} \\
&\leq \hat{D}_k^{0,*}(e^{L_f(t_{k+1}-t_k)} - 1) + \sigma e^{L_f(t_{k+1}-t_k)} \\
&= \hat{D}_k^{0,*}\left(e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}} - 1\right) + \sigma e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}}
\end{aligned}$$

for any $t \in [t_k, t_{k+1}]$. ■

Let

$$\epsilon = \max_{x \in \mathcal{X},\, u \in \mathcal{U}} D(x,u)\left(e^{\frac{L_f D(x,u)}{\|f(x,u)\|}} - 1\right) + \sigma e^{\frac{L_f D(x,u)}{\|f(x,u)\|}} \tag{11}$$

and

$$\mathcal{X}_1 \triangleq \mathcal{X} - 2\epsilon \tag{12}$$

where $\mathcal{X} - 2\epsilon$ is the Pontryagin difference between $\mathcal{X}$ and a ball $\mathcal{B}(2\epsilon)$ centered at the origin with the radius 2ϵ. With the result of Lemma 4.1, if $\bar{x}(t_0) \in \mathcal{X} - \epsilon$, $x(t) \in \mathcal{X}$ can be guaranteed for any $t \in [t_k, t_{k+1})$.

To ensure feasibility, we assume that for any $x, y \in \mathcal{X}$, and $u \in \mathcal{U}$, there exists a positive constant L_s such that

$$\left\| D(x,u)\frac{f(x,u)}{\|f(x,u)\|} - D(y,u)\frac{f(y,u)}{\|f(y,u)\|} \right\| \leq L_s\|x - y\| \tag{13}$$

and there exists a function $h: \mathbb{R}^n \to \mathbb{R}^m,\ h(0) = 0$ so that

$$0 \in \text{int}(\mathcal{X}_f) \tag{14}$$

$$\begin{aligned}
&\mathcal{X}_f + (\sigma + \epsilon)(L_s + 1)^{N-1} \subset \mathcal{X}_U \\
&\qquad \triangleq \{x \in \mathcal{X}_{N-1} | h(x) \in \mathcal{U}\}
\end{aligned} \tag{15}$$

$$\begin{aligned}
&x \in \mathcal{X}_f + (\sigma + \epsilon)(L_s + 1)^{N-1} \Rightarrow \\
&\qquad x + \frac{D(x, h(x))f(x, h(x))}{\|f(x, h(x))\|} \in \mathcal{X}_f.
\end{aligned} \tag{16}$$

Then we construct an admissible control input for the LAM at the $(k+1)$st computation with the initial condition $\hat{x}_{k+1}^0 = \bar{x}(t_{k+1})$:

$$\hat{u}_{k+1}^i = \begin{cases} \hat{u}_k^{i,*},\ i = 0, 1, ..., N-2 \\ h(\hat{x}_{k+1}^i).\ i = N-1 \end{cases} \tag{17}$$

We compute the optimal input from 0 to $(N-1)_{th}$ in each computation cycle. As a result, the ith control input calculated in the current cycle can be mapped to the $(i-1)$st input in next cycle.

With $\hat{u}_{k+1}^i$ and $\hat{x}_{k+1}^0 = \bar{x}(t_{k+1})$, the LAM can generate the states at the $(k+1)$st computation cycle as $\hat{x}_{k+1}^i$. Then we will study the difference between $\hat{x}_{k+1}^{i-1}$ and $\hat{x}_k^{i,*}$.

Lemma 4.2: Given inequality (13), the following inequality holds

$$\|\hat{x}_{k+1}^{i-1} - \hat{x}_k^{i,*}\| \leq (\sigma + \epsilon_k)(L_s + 1)^{i-1},\ i = 1, 2, ..., N. \tag{18}$$

Proof: The proof is similar to the proof of Lemma 2 in [17] and therefore omitted due to space limit. ■

With the bound in Lemma 4.2, we can define the reduced constraint sets in the FHOCP at each iteration:

$$\mathcal{X}_i \triangleq \mathcal{X} - (\sigma + \epsilon)\left(\sum_{p=1}^{i}(L_s + 1)^{p-1} + 1\right),\ i = 1, ..., N-1. \tag{19}$$

Theorem 4.1: With $\mathcal{X}_i$ and $\mathcal{X}_f$ defined in (19) and (14)-(16), respectively, the FHOCP in (7) is always feasible.

Proof: We will prove that if $\hat{x}_k^{i,*} \in \mathcal{X}_i$ for $i = 1, 2, \cdots, N$, there is a feasible solution of the optimization problem in $k+1$, which is $\hat{u}_{k+1}^i$ defined in (17), based on the optimal solution in k, $\hat{u}_k^{i,*}$.

First, we show that $\hat{u}_{k+1}^i \in \mathcal{U}$ for $i = 0, 1, \cdots, N-1$. Based on equation (17), $\hat{u}_{k+1}^i = \hat{u}_k^{i+1,*} \in \mathcal{U}$ for $i = 0, 1, \cdots, N-2$ because of the feasibility of $\hat{u}_k^{i,*}$. By Lemma 4.2 and $\epsilon_k \leq \epsilon$, we know

$$\|\hat{x}_{k+1}^{N-1} - \hat{x}_k^{N,*}\| \leq (\sigma + \epsilon)(L_s + 1)^{N-1}.$$

Since $\hat{x}_k^{N,*} \in \mathcal{X}_f$, $\hat{x}_{k+1}^{N-1} \in \mathcal{X}_f + \epsilon(L_s + 1)^{N-1}$ holds. By equation (15),

$$\hat{x}_{k+1}^{N-1} \in \mathcal{X}_f + (\sigma + \epsilon)(L_s + 1)^{N-1} \subseteq \mathcal{X}_U \tag{20}$$

and therefore $\hat{u}_{k+1}^{N-1} \in \mathcal{U}$.

Next, we show that $\hat{x}_{k+1}^i \in \mathcal{X}_i$ for $i = 1, \cdots, N-1$ and $\hat{x}_{k+1}^N \in \mathcal{X}_f$. Because equation (20), we have $\hat{x}_{k+1}^N \in X_f$ by equation (16). Also, by Lemma 4.2, we know

$$\|\hat{x}_{k+1}^i - \hat{x}_k^{i+1,*}\| \leq (\sigma + \epsilon)(L_s + 1)^i$$

for $i = 1, \cdots, N-1$. Also, notice that

$$\hat{x}_k^{i+1,*} \in \mathcal{X}_{i+1} = \mathcal{X} - (\sigma + \epsilon) \left(\sum_{p=1}^{i+1} (L_s + 1)^{p-1} + 1 \right)$$

for $i = 1, \cdots, N-2$ and $\hat{x}_k^{N,*} \in \mathcal{X}_f \subseteq \mathcal{X}_{N-1}$. So

$$\begin{aligned} \hat{x}_{k+1}^i &\in \mathcal{X}_{i+1} + (\sigma + \epsilon)(L_s + 1)^i \\ &= \mathcal{X} - (\sigma + \epsilon) \Big(\textstyle\sum_{p=1}^{i} (L_s + 1)^{p-1} + 1 \Big) = \mathcal{X}_i \end{aligned}$$

for $i = 1, \cdots, N-1$, which completes the proof. ■

V. Stability

This section discusses stability of the closed-loop system. Before presenting the main results, we will first define the threshold function $D(x, u)$ because this function will directly affect system stability. If the threshold $\hat{D}_k^i$ is too large, the LAM may significantly deviate from the actual system, which may lead to instability.

Let

$$D(x, u) = \frac{\|f(x, u)\|}{L_f} \log \left(\max \left\{ \frac{\rho L(x, u)}{\|f(x, u)\|}, \delta \right\} + 1 \right) \quad (21)$$

where δ is an arbitrarily small positive constant and ρ is a positive constant to be determined. Such a choice of the threshold can avoid Zeno behavior since $\frac{D(x,u)}{\|f(x,u)\|} \geq \frac{\log(\delta+1)}{L_f}$ always holds and therefore $t_k^{i+1} - t_k^i \geq \frac{\log(\delta+1)}{L_f}$ according to equation (5).

Let $L(x, u)$ be selected such that there exist class $\mathcal{K}_\infty$ functions $\beta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\beta(\|x\|) \leq \frac{L(x, u)}{\|f(x, u)\|}, \quad (22)$$

$$\gamma(\|x\|) \leq D(x, u), \text{ and} \quad (23)$$

$$(24)$$

$$\frac{L(x, u)}{L_f} \log \left(\max \left\{ \frac{\rho L(x, u)}{\|f(x, u)\|}, \delta \right\} + 1 \right) \quad (25)$$

is locally Lipschitz with respect to x with the Lipschitz constant L_c for any $x \in \mathcal{X}$ and $u \in \mathcal{U}$.

Assume that besides the conditions in (15) and (16) the function $h(x)$ also satisfies

$$V_f \left(x + \tfrac{D(x,h(x)) f(x,h(x))}{\|f(x,h(x))\|} \right) - V_f(x) \leq -\tfrac{L(x,h(x)) D(x,h(x))}{\|f(x,h(x))\|}. \quad (26)$$

Theorem 5.1: Suppose that the hypotheses in Theorem 4.1 hold. If for any $x, y \in \mathcal{X}/\{0\}$ and $u \in \mathcal{U}/\{0\}$, there exist positive constants L_{V_f}, ρ and a class $\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$|V_f(x) - V_f(y)| \leq L_{V_f} \|x - y\| \quad (27)$$

$$V_f(x) \leq \alpha(\|x\|) \quad (28)$$

$$\rho < \frac{1}{\sum_{i=1}^{N-1} L_c (L_s + 1)^{i-1} + L_{V_f} (L_s + 1)^{N-1}} \quad (29)$$

hold, the system in (1) under the LAMPC algorithm (7) is UUB.

Proof: Let $J[\hat{u}_{k+1} | \bar{x}(t_{k+1})]$ be the cost of the FHOCP generated by $\hat{u}_{k+1}^i$ in (17) with the initial condition $\bar{x}(t_{k+1})$. Consider

$$\begin{aligned} & J[\hat{u}_{k+1} | \bar{x}(t_{k+1})] - V(\bar{x}(t_k)) \\ &= \sum_{i=0}^{N-1} L\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) \tfrac{\hat{D}_{k+1}^i}{\|d\hat{x}_{k+1}^i\|} + V_f\left(\hat{x}_{k+1}^N\right) - V(\bar{x}(t_k)) \\ &= \sum_{i=0}^{N-2} L\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) \tfrac{\hat{D}_{k+1}^i}{\|d\hat{x}_{k+1}^i\|} + L\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) \tfrac{\hat{D}_{k+1}^{N-1}}{\|d\hat{x}_{k+1}^{N-1}\|} \\ &\quad + V_f\left(\hat{x}_{k+1}^N\right) - V(\bar{x}(t_k)) + V_f\left(\hat{x}_{k+1}^{N-1}\right) - V_f\left(\hat{x}_{k+1}^{N-1}\right) \\ &\quad + L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \tfrac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \tfrac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} \\ &= \left. \begin{array}{c} \sum_{i=0}^{N-2} L\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) \tfrac{\hat{D}_{k+1}^i}{\|d\hat{x}_{k+1}^i\|} + V_f\left(\hat{x}_{k+1}^{N-1}\right) \\ + L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \tfrac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} - V(x(t_k)) \end{array} \right\} \Phi \\ &\quad + \underbrace{L\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) \tfrac{\hat{D}_{k+1}^{N-1}}{\|d\hat{x}_{k+1}^{N-1}\|} + V_f(\hat{x}_{k+1}^N) - V_f(\hat{x}_{k+1}^{N-1})}_{\Psi} \\ &\quad - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \tfrac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}. \end{aligned} \quad (30)$$

By inequality (26), we have $\Psi \leq 0$, because $\hat{u}_{k+1}^{N-1} = h(\hat{x}_{k+1}^{N-1})$ and $\hat{x}_{k+1}^N = \hat{x}_{k+1}^{N-1} + \frac{D\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) f\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right)}{\left\|f\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right)\right\|}$.

Therefore, the preceding inequality can be simplified as

$$J[\hat{u}_{k+1} | \bar{x}(t_{k+1})] - V(\bar{x}(t_k)) \leq \Phi - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \tfrac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}. \quad (31)$$

Consider Φ. Note that the first term in Φ can be written as

$$\sum_{i=0}^{N-2} L\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) \tfrac{\hat{D}_{k+1}^i}{\|d\hat{x}_{k+1}^i\|} = \sum_{i=1}^{N-1} L\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right) \tfrac{\hat{D}_{k+1}^{i-1}}{\|d\hat{x}_{k+1}^{i-1}\|}.$$

By (7), $V(\bar{x}(t_k)) = \sum_{i=0}^{N-1} \frac{L\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right) \hat{D}_k^{i,*}}{\|d\hat{x}_k^{i,*}\|} + V_f\left(\hat{x}_k^{N,*}\right)$. Therefore,

$$\begin{aligned} \Phi &= \sum_{i=1}^{N-1} L\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right) \tfrac{\hat{D}_{k+1}^{i-1}}{\|d\hat{x}_{k+1}^{i-1}\|} + V_f\left(\hat{x}_{k+1}^{N-1}\right) \\ &\quad - \sum_{i=1}^{N-1} L\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right) \tfrac{\hat{D}_k^{i,*}}{\|d\hat{x}_k^{i,*}\|} - V_f\left(\hat{x}_k^{N,*}\right). \end{aligned} \quad (32)$$

Consider the function $\frac{L(x,u) D(x,u)}{\|f(x,u)\|}$. Applying the definition of $D(x, u)$ in (21) yields

$$\tfrac{L(x,u) D(x,u)}{\|f(x,u)\|} = \frac{L(x, u)}{L_f} \log \left(\max \left\{ \frac{\rho L(x, u)}{\|f(x, u)\|}, \delta \right\} + 1 \right).$$

By (25), this function is locally Lipschitz. Therefore,

$$\begin{aligned} \Phi &\leq \sum_{i=1}^{N-1} \left| \tfrac{L\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right) \hat{D}_{k+1}^{i-1}}{\|d\hat{x}_{k+1}^{i-1}\|} - \tfrac{L\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right) \hat{D}_k^{i,*}}{\|d\hat{x}_k^{i,*}\|} \right| \\ &\qquad + \left| V_f\left(\hat{x}_{k+1}^{N-1}\right) - V_f\left(\hat{x}_k^{N,*}\right) \right| \\ &\leq \sum_{i=1}^{N-1} L_c \left\| \hat{x}_{k+1}^{i-1} - \hat{x}_k^{i,*} \right\| + L_{V_f} \left\| \hat{x}_{k+1}^{N-1} - \hat{x}_k^{N,*} \right\|. \end{aligned}$$

By Lemma 4.2, $\left\| \hat{x}_{k+1}^{i-1} - \hat{x}_k^{i,*} \right\| \leq (\sigma + \epsilon_k)(L_s + 1)^{i-1}$.

Therefore,

$$\Phi \leq (\sigma + \epsilon_k) \underbrace{\left(\sum_{i=1}^{N-1} L_c (L_s + 1)^{i-1} + L_{V_f} (L_s + 1)^{N-1} \right)}_{\theta}.$$

With this inequality, (31) can be further simplified as

$$\begin{aligned} &J[\hat{u}_{k+1} | \bar{x}(t_{k+1})] - V(\bar{x}(t_k)) \\ &\leq - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} + (\sigma + \epsilon_k)\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} V(\bar{x}(t_{k+1})) - V(\bar{x}(t_k)) &= \min_{\hat{u}_{k+1}} J[\hat{u}_{k+1} | \bar{x}(t_{k+1})] - V(\bar{x}(t_k)) \\ &\leq -L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} + (\sigma + \epsilon_k)\theta. \end{aligned}$$

Applying the definition of $D(x, u)$ and ϵ_k into this inequality,

$$\begin{aligned} &V(\bar{x}(t_{k+1})) - V(\bar{x}(t_k)) \\ &\leq - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} \\ &\quad + \left(\hat{D}_k^{0,*} \left(e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}} - 1 \right) + \sigma e^{\frac{L_f \hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|}} \right) \theta + \sigma\theta \\ &= - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} + \hat{D}_k^{0,*} \max\left\{ \frac{\rho L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|}, \delta \right\} \theta \\ &\quad + \sigma \max\left\{ \frac{\rho L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|}, \delta \right\} \theta + 2\sigma\theta \end{aligned}$$

Note that $\hat{x}_k^{0,*} = \bar{x}(t_k)$. Consider the case when $\|\bar{x}(t_k)\| \geq \beta^{-1}(\frac{\delta}{\rho})$, i.e., $\rho\beta(\|\bar{x}(t_k)\|) \geq \delta$. By inequality (22), the preceding inequality implies

$$\begin{aligned} &V(\bar{x}(t_{k+1})) - V(\bar{x}(t_k)) \\ &\leq - L\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \frac{\hat{D}_k^{0,*}}{\|d\hat{x}_k^{0,*}\|} + \hat{D}_k^{0,*} \frac{\rho L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|} \theta \\ &\quad + \sigma \frac{\rho L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|} \theta + 2\sigma\theta \\ &= - \frac{L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|} \left(\hat{D}_k^{0,*} - \hat{D}_k^{0,*}\rho\theta - \sigma\rho\theta \right) + 2\sigma\theta. \end{aligned}$$

Consider the case when $\|\bar{x}(t_k)\| > \gamma^{-1}(\frac{2\sigma\rho\theta}{(1-\rho\theta)})$. By inequality (23), we know

$$(1 - \rho\theta)\hat{D}_k^{0,*} > 2\sigma\rho\theta.$$

Therefore, the preceding inequality implies

$$\begin{aligned} V(\bar{x}(t_{k+1})) - V(\bar{x}(t_k)) &\leq - \frac{L(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})}{\|f(\hat{x}_k^{0,*}, \hat{u}_k^{0,*})\|} \sigma\rho\theta + 2\sigma\theta \\ &\leq - \beta(\|\bar{x}(t_k)\|)\sigma\rho\theta + 2\sigma\theta. \end{aligned}$$

To summarize, when

$$\|\bar{x}(t_k)\| > \max\left\{ \beta^{-1}(\tfrac{2}{\rho}), \gamma^{-1}(\tfrac{2\sigma\rho\theta}{(1-\rho\theta)}), \beta^{-1}(\tfrac{\delta}{\rho}) \right\},$$

we have means

$$V(\bar{x}(t_{k+1})) - V(\bar{x}(t_k)) < 0. \tag{33}$$

By (26), we know

$$V_f\left(\hat{x}_k^{i+1}\right) - V_f(\hat{x}_k^i) \leq -L(\hat{x}_k^i, h(\hat{x}_k^i)) \frac{D(\hat{x}_k^i, h(\hat{x}_k^i))}{\|f(\hat{x}_k^i, h(\hat{x}_k^i))\|}.$$

Summing up the inequality above for $i = 0, 1, \cdots, N-1$,

$$V_f\left(\hat{x}_k^N\right) - V_f(\hat{x}_k^0) \leq - \sum_{i=0}^{N-1} L(\hat{x}_k^i, h(\hat{x}_k^i)) \frac{D(\hat{x}_k^i, h(\hat{x}_k^i))}{\|f(\hat{x}_k^i, h(\hat{x}_k^i))\|}.$$

Thus, by the definition of $V(\bar{x}(t_k))$,

$$\begin{aligned} V(\bar{x}(t_k)) &\leq \sum_{i=0}^{N-1} L(\hat{x}_k^i, h(\hat{x}_k^i)) \frac{D(\hat{x}_k^i, h(\hat{x}_k^i))}{\|f(\hat{x}_k^i, h(\hat{x}_k^i))\|} + V_f\left(\hat{x}_k^N\right) \\ &\leq V_f(\hat{x}_k^0) = V_f(\bar{x}(t_k)) \leq \alpha(\|\bar{x}(t_k)\|) \end{aligned} \tag{34}$$

holds, which, together with inequality (33), implies that $\bar{x}(t_k)$ will be uniformly ultimately bounded. By Lemma 4.1, we know that $x(t)$ will also be UUB. ■

VI. Simulation

This section shows how the LAMPC works on a nonlinear system with measurement noises. We consider the crane model in [19] with the excitation angle ϕ and the horizontal trolley position p:

$$\begin{aligned} \dot{p}(t) &= v(t) \\ \dot{v}(t) &= u(t) \\ \dot{\phi}(t) &= \omega(t) \\ \dot{\omega}(t) &= -gsin(\phi(t)) - u(t)cos(\phi(t)) - b\omega(t), \end{aligned}$$

where $x = (p, v, \phi, \omega)^\top$ is the state and u is the control input. The control input must satisfy $-0.5 \leq u(t) \leq 0.5$. Besides, we use the parameters $m = 1kg, L = 1m, b = 0.2J$ and $g = 9.81m/s^2$. The measurement noise $w(t_k)$ satisfies $\|w(t_k)\| \leq 0.05$.

The running cost function and the terminal cost function are defined as

$$L(x, u) = |f(x, u)| \cdot (|x| + |u| + 1), \; V_f(x) = 5|x|.$$

The threshold function $D(x, u)$ is defined by equation (21). The BARON solver [20] is used to solve the nonlinear FHOCP.

Figure 1 plot the state and input trajectories of the system. It is obvious that the system converges to a small neighborhood of the origin and the constraints are not violated. Figure 2 plots the trajectory of $V(\bar{x}(t_k))$. We can see that $V(barx(t_k))$ keeps decreasing until being close to zero. This is consistent to the theoretical results. Figure 3 shows the history of the inter-sampling time intervals generated by the self-triggered scheme (top) and the length of prediction horizons at each sampling instants (bottom). It is clear that those intervals are time-varying until the state stays around the steady state. Also notice that they are strictly greater than zero.

VII. Conclusions

This paper presents the LAMPC algorithm for nonlinear continuous-time systems with measurement noises. A self-

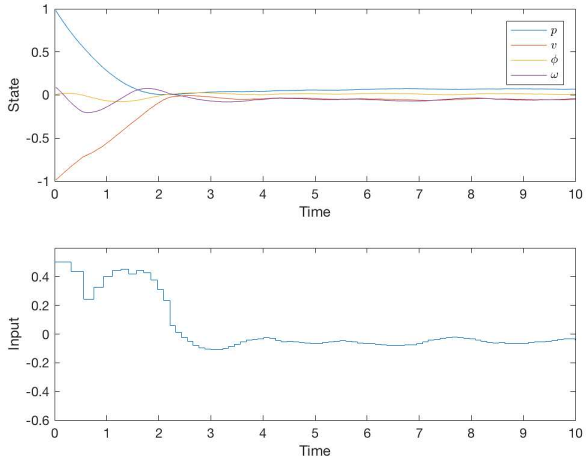

Fig. 1. The state and input trajectory generated by LAMPC

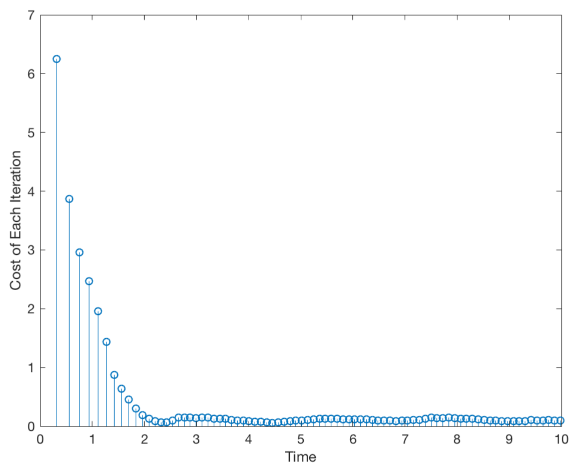

Fig. 2. The history of $V(\bar{x}(t_k))$

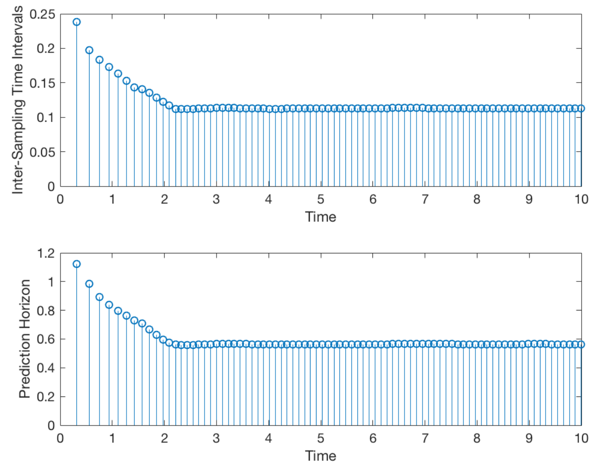

Fig. 3. The inter-sampling time intervals and the prediction horizons

triggered method is used to trigger the sampling and computation and the LAM is designed to discretize the FHOCP. We show that with appropriate design of the threshold function in the LAM, even when the sampled state contains measurement noises, the LAMPC can still guarantee the system to be UUB. As mentioned in context, this work can be applied to output-feedback systems as long as the observer is well designed such that the error between the estimated state and the actual state is uniformly bounded as stated in (2).

REFERENCES

[1] F. A. Fontes, "A general framework to design stabilizing nonlinear model predictive controllers," *Systems & Control Letters*, vol. 42, no. 2, pp. 127–143, 2001.

[2] M. Rubagotti, D. M. Raimondo, A. Ferrara, and L. Magni, "Robust model predictive control with integral sliding mode in continuous-time sampled-data nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 556–570, 2011.

[3] F. D. Brunner, W. Heemels, and F. Allgöwer, "Robust event-triggered MPC for constrained linear discrete-time systems with guaranteed average sampling rate," *IFAC-PapersOnLine*, vol. 48, no. 23, pp. 117–122, 2015.

[4] A. Eqtami, D. V. Dimarogonas, and K. J. Kyriakopoulos, "Novel event-triggered strategies for model predictive controllers," in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*. IEEE, 2011, pp. 3392–3397.

[5] A. Ferrara, G. P. Incremona, and L. Magni, "Model-based event-triggered robust MPC/ISM," in *Proceedings of European Control Conference*. IEEE, 2014, pp. 2931–2936.

[6] D. Lehmann, E. Henriksson, and K. H. Johansson, "Event-triggered model predictive control of discrete-time linear systems subject to disturbances," in *Proceedings of European Control Conference (ECC)*. IEEE, 2013, pp. 1156–1161.

[7] H. Li and Y. Shi, "Event-triggered robust model predictive control of continuous-time nonlinear systems," *Automatica*, vol. 50, no. 5, pp. 1507–1513, 2014.

[8] J. Sijs, M. Lazar, and W. Heemels, "On integration of event-based estimation and robust MPC in a feedback loop," in *Proceedings of the 13th ACM international conference on Hybrid systems: computation and control*. ACM, 2010, pp. 31–40.

[9] J. B. Berglind, T. Gommans, and W. Heemels, "Self-triggered MPC for constrained linear systems and quadratic costs," *IFAC Proceedings Volumes*, vol. 45, no. 17, pp. 342–348, 2012.

[10] F. D. Brunner, W. Heemels, and F. Allgöwer, "Robust self-triggered MPC for constrained linear systems," in *Proceedings of European Control Conference*. IEEE, 2014, pp. 472–477.

[11] A. Eqtami, S. Heshmati-alamdari, D. V. Dimarogonas, and K. J. Kyriakopoulos, "Self-triggered model predictive control for nonholonomic systems," in *Proceedings of 2013 European Control Conference*. IEEE, 2013, pp. 638–643.

[12] T. Gommans and W. Heemels, "Resource-aware MPC for constrained nonlinear systems: A self-triggered control approach," *Systems & Control Letters*, vol. 79, pp. 59–67, 2015.

[13] E. Henriksson, D. E. Quevedo, H. Sandberg, and K. H. Johansson, "Self-triggered model predictive control for network scheduling and control1," *Proceedings of the 8th IFAC Symposium on Advanced Control of Chemical Processes*, vol. 45, no. 15, pp. 432–438, 2012.

[14] P. Sopasakis, P. Patrinos, and H. Sarimveis, "MPC for sampled-data linear systems: Guaranteeing constraint satisfaction in continuous-time," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 1088–1093, 2014.

[15] M. Farina and R. Scattolini, "Tube-based robust sampled-data MPC for linear continuous-time systems," *Automatica*, vol. 48, no. 7, pp. 1473–1476, 2012.

[16] X. Wang and L. Yang, "Sporadic model predictive control using Lebesgue approximation," in *Proceedings of American Control Conference*. IEEE, 2017, pp. 5768–5773.

[17] X. Wang and B. Zhang, "Lebesgue approximation model of continuous-time nonlinear dynamic systems," *Automatica*, vol. 64, pp. 234–239, 2016.

[18] B. Houska, H. J. Ferreau, and M. Diehl, "An auto-generated real-time iteration algorithm for nonlinear MPC in the microsecond range," *Automatica*, vol. 47, no. 10, pp. 2279–2285, 2011.

[19] M. Tawarmalani and N. V. Sahinidis, "A polyhedral branch-and-cut approach to global optimization," *Mathematical Programming*, vol. 103, no. 2, pp. 225–249, 2005.