



# Multiple steady-states in phytoplankton population induced by photoinhibition <sup>☆</sup>

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## Abstract

We study the effect of photoinhibition in a nonlocal reaction–diffusion–advection equation, which models the dynamics of a single phytoplankton species in a water column where the growth of the species depends solely on light. Our results show that, in contrast to the case of no photoinhibition, where at most one positive steady state can exist, the model with photoinhibition possesses at least two positive steady states in certain parameter ranges. Our approach involves bifurcation theory and perturbation–reduction arguments.

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## 1. Introduction

3 Phytoplankton are microscopic plants that float in oceans and lakes and form the base of the  
4 aquatic food chain. Since they transport significant amounts of atmospheric carbon dioxide into  
5 the deep oceans, they may also play a crucial role in the climate dynamics. Phytoplankton species  
6 typically compete for nutrients and light [4,5,16,19,28,29]. But in oligotrophic ecosystems with  
7 ample supply of light, they tend to compete only for nutrients [20,22], and in eutrophic envi-  
8 ronments with ample nutrients supply, they compete only for light [8,15]. In a water column, a  
9 phytoplankton population diffuses due to turbulent mixing caused by wind and wave actions. In  
10 many cases, phytoplankton also sinks due to its own weight.

11 In this paper, we consider a single sinking phytoplankton species in an eutrophic water col-  
12 umn. Our analysis is based on a nonlocal reaction–diffusion–advection model given by Huisman  
13 and colleagues in [8,14], but the growth function  $g(I)$  of phytoplankton species in the model is  
14 modified to include photoinhibition into consideration.

15 Photoinhibition is characterized by a decreasing rate of photosynthesis with increasing light,  
16 which occurs in many phytoplankton species that are sensitive to strong light. This phenomenon  
17 is caused by damage to the photosynthetic machinery of cells and by protective mechanisms to  
18 avoid this damage [23,27].

19 Without photoinhibition, the growth function  $g(I)$  is generally assumed to be strictly increas-  
20 ing in  $I$ , representing the fact that increase of the light level  $I$  leads to better growth of the  
21 phytoplankton. In such a case this model was investigated recently through rigorous mathemat-  
22 ical analysis in [6,7,13,21] (see also earlier work in [17,18,26] and references therein), which  
23 show that the phytoplankton population either stabilizes at a unique positive steady-state or con-  
24 verges to 0 as time goes to infinity, depending on whether the loss rate is below or above a critical  
25 level.

26 With photoinhibition, observations in many laboratory studies [10,11,24] suggest that the  
27 function  $g(I)$  should be increasing before  $I$  reaches a certain critical level  $I_* > 0$  where  $g(I)$   
28 has a maximum, after which  $g(I)$  decreases and converges to 0 as  $I \rightarrow \infty$ . In a completely  
29 mixed water column, the reaction–diffusion–advection model reduces to an ODE model, and the  
30 effect of photoinhibition was studied in the recent papers [9,12]. With a growth function  $g(I)$   
31 as above, [9] demonstrates that the phytoplankton population may have two stable steady-states  
32 (one positive, the other 0), plus another unstable positive steady state, causing a bistable dynami-  
33 cal behavior with the phytoplankton population stabilizing at one or the other stable steady state,  
34 depending on its initial value. (The multi-species case was also considered in [9] and [12].)

35 In this paper, we examine the effect of photoinhibition in an incompletely mixed water col-  
36 umn, through a single species reaction–diffusion–advection model, where photoinhibition is  
37 incorporated into the growth function  $g(I)$  as described above. We show that the phenomenon  
38 of multiple positive steady-states observed in completely mixed water column persists, and their  
39 stability suggests a bistable dynamical behavior.

40 Multiplicity results for similar reaction–diffusion equations are usually obtained by making  
41 use of the upper and lower solution technique, combined with tools from global analysis (such as  
42 the topological degree theory or global bifurcation theory). However, such upper and lower solu-  
43 tion techniques are difficult to apply here due to the nonlocal nature of the problem. To overcome  
44 this difficulty, apart from employing local and global bifurcation analysis, we also use a perturba-  
45 tion and reduction approach, which is new to this kind of nonlocal reaction–diffusion equations.  
46 It is our hope that the mathematical techniques developed here may find more applications in  
47 phytoplankton models.

We now describe the model in more detail. Consider a vertical water column with a cross section of one unit area and depth  $h$ . Let  $p(x, t)$  be the population density of the phytoplankton at depth  $x \in [0, h]$  and time  $t$ . Then the change of density is governed by the following reaction–diffusion–advection problem

$$\begin{cases} p_t = Dp_{xx} - \sigma p_x + [g(I_0 e^{-k_0 x - k \int_0^x p(s,t) ds}) - d]p, & 0 < x < h, t > 0, \\ Dp_x(x, t) - \sigma p(x, t) = 0, & x = 0 \text{ or } h, t > 0, \\ p(x, 0) = p_0(x) \geq 0, & 0 \leq x \leq h, \end{cases} \quad (1.1)$$

where  $d > 0$  is the loss rate of the species, the positive constants  $D, \sigma$  represent the diffusion rate and the sinking rate, respectively.

The term

$$I(x, t) = I_0 e^{-k_0 x - k \int_0^x p(s,t) ds} \quad (1.2)$$

is known as the light intensity, with  $k_0 \geq 0$  the background turbidity,  $k > 0$  the light attenuation coefficient of the phytoplankton species, and  $I_0 > 0$  the incident light intensity.

The function  $g(I)$  governs the growth rate according to the change of light level  $I$ . We always assume that  $g$  is  $C^1$ . Taking into account of photoinhibition we assume, as in [9], that  $g(I)$  has the following properties:

$$\begin{cases} \text{(i)} & g(0) = 0, \\ \text{(ii)} & \text{there exists } I_* > 0 \text{ such that } (I_* - I)g'(I) > 0 \text{ for } I \neq I_*, \\ \text{(iii)} & \lim_{I \rightarrow \infty} g(I) = 0. \end{cases} \quad (1.3)$$

The boundary conditions at  $x = 0$  and  $x = h$  imply that there is no population flux at the surface or bottom of the water column.

We are interested in the multiplicity of positive steady states of (1.1). To simplify notations we assume that  $D = h = I_0 = 1$ . We stress that this is for simplicity of the notations only; our method can deal with the general case without extra difficulties. Thus we will study the positive solutions of the nonlocal elliptic boundary value problem

$$\begin{cases} p_{xx} - \sigma p_x + p[g(I) - d] = 0, & 0 < x < 1, \\ qp_x(0) - \sigma p(0) = 0, & p_x(1) - \sigma p(1) = 0, \end{cases} \quad (1.4)$$

with

$$I = I(x) = e^{-k_0 x - k \int_0^x p(s) ds}. \quad (1.5)$$

Our first existence and multiplicity result is obtained by a standard argument involving local and global bifurcation theory of Crandall and Rabinowitz [1,2,25]. The multiplicity result is local in nature.

**Theorem 1.1.** Suppose that (1.3) holds and  $I_* < e^{-k_0}$ . Then there exist some positive constants  $d_* < d^* < g(I_*)$  such that (1.4) has at least one positive solution for  $d \in (0, d_*]$ , two positive solutions for  $d \in (d_*, d^*)$ , and no positive solution for  $d > g(I^*)$ .

1 Here  $d_*$  is uniquely determined by an eigenvalue problem, and  $d^* > d_*$  is sufficiently close  
 2 to  $d_*$ , whose existence arises from a bifurcation analysis along the line of trivial solutions  
 3  $\{(d, 0) : d \in \mathbb{R}^1\}$ , which shows that a branch of positive solutions  $\{(d, p)\}$  bifurcates from the  
 4 line of trivial solutions at  $(d_*, 0)$ , and it goes rightward initially but has to become unbounded  
 5 through  $d$  converging to 0. It can be shown that for  $d \in (d_*, d^*)$ , one of the positive steady state  
 6 is unstable (see [Lemma 2.3](#)), but we have no information on the stability of the other positive  
 7 steady state, though we believe it is stable.

8 By making use of a perturbation-reduction approach, we can obtain a multiplicity result which  
 9 is global in nature, together with information on the asymptotic profile and stability of the solu-  
 10 tions, but only for large  $\sigma$ .

<sup>12</sup> **Theorem 1.2.** Suppose that (1.3) holds and  $I_* < e^{-k_0}$ .

(i) *Define*

$$G(\mu) = \frac{1}{\mu} \int_0^\mu g(e^{-k_0 - s}) ds.$$

Then there exists a unique  $\mu_* > \ln(e^{-k_0}/I_*)$  such that

$$G'(\mu_*) = 0, \quad \quad G'(\mu)(\mu_* - \mu) > 0 \quad \text{for } \mu \in (0, +\infty) \setminus \{\mu_*\}.$$

Moreover, for each  $d \in (g(e^{-k_0}), G(\mu_*))$ , the equation  $d = G(\mu)$  has exactly two positive solutions  $\mu_1, \mu_2$ , and  $0 < \mu_1 < \mu_* < \mu_2$ .

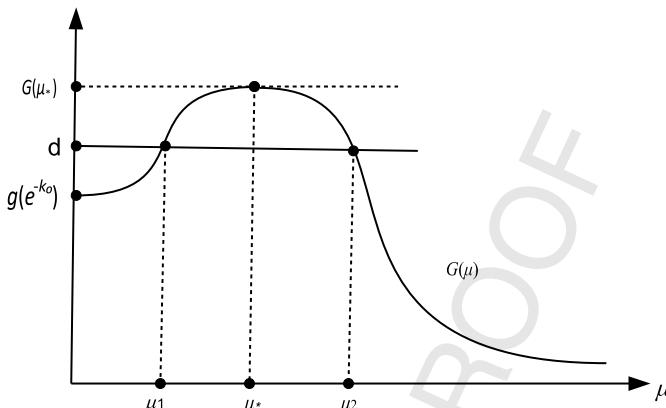
27 (ii) For each  $d \in (g(e^{-k_0}), G(\mu_*))$ , there exists  $\epsilon^* > 0$  such that for every  $\sigma > 1/\epsilon^*$ , (1.4) has  
 28 two positive solutions of the form

$$p_1(x) = \frac{\mu_1}{k} \sigma e^{\sigma(x-1)} + z_{1,\sigma}(x), \quad p_2(x) = \frac{\mu_2}{k} \sigma e^{\sigma(x-1)} + z_{2,\sigma}(x).$$

with  $z_{i,\sigma}$  satisfying  $\lim_{\sigma \rightarrow \infty} \int_0^1 |z_{i,\sigma}(x)| dx = 0$ ,  $i = 1, 2$ .

**Remark 1.3.** If  $I_* \geq 1$ , we can show that (1.4) has at most one positive solution for any  $\sigma$ . Indeed, we can modify  $g(I)$  for  $I > I_*$  to obtain a new  $C^1$  function  $\tilde{g}(I)$  which is strictly increasing in  $I$  for all  $I > 0$ . When  $I_* \geq 1$ , it is easily seen that if  $\rho$  is a positive solution of (1.4), then it is also a positive solution of (1.4) with  $g(I)$  replaced by  $\tilde{g}(I)$ . Hence we are back in the no-photoinhibition case and can apply the result in [13,7] to conclude that there is at most one positive solution, and the dynamics of (1.1) is simple. Biologically this fact is rather natural, as 1 is the highest possible level of light intensity felt by the species in the water column, so only the values of  $g(I)$  for  $I \in (0, 1]$  contribute to the growth of the species.

44 The results in [Theorem 1.2](#) suggest that for large  $\sigma$ , the two solutions  $p_1$  and  $p_2$  form a “ $\supset$ ”-  
 45 shaped curve in the  $(d, p)$ -space as  $d$  is varied in the range  $(g(e^{-k_0}), G(\mu_*))$ , which resembles  
 46 the solution curve of the equation  $d = G(\mu)$  in the  $(d, \mu)$ -plane. [Fig. 1](#) shows the graph of the  
 47 curve  $G(\mu)$  and the two solutions  $\mu_1$  and  $\mu_2$  of  $d = G(\mu)$  for the case  $I_* < e^{-k_0}$ .

Fig. 1. Illustrative graph of  $G(\mu)$  and the solutions of  $d = G(\mu)$ .

It is interesting to note that, by Theorem 3.2 of [13], as  $\sigma \rightarrow +\infty$ , the bifurcation value  $d_*$  in Theorem 1.1 converges to  $g(e^{-k_0})$ , suggesting that the global bifurcation curve in Theorem 1.1 for large  $\sigma$  looks like the solution curve of the equation  $d = G(\mu)$  in the  $(d, \mu)$ -plane, and the two positive solutions in Theorem 1.2 are from the “lower” and “upper” branches of the global bifurcation curve.

Let us also note that for large  $\sigma$ , the solutions  $p_1(x)$  and  $p_2(x)$  are well approximated by  $\frac{\mu_1}{k} \sigma e^{\sigma(x-1)}$  and  $\frac{\mu_2}{k} \sigma e^{\sigma(x-1)}$ , respectively, which have values close to 0 away from  $x = 1$ , while their values at  $x = 1$  go to  $\infty$  as  $\sigma \rightarrow \infty$ . The fact that the populations concentrate at the bottom of the water column is due to the assumption of large sinking rate  $\sigma$ . The information on the asymptotic profiles of  $p_1(x)$  and  $p_2(x)$  enables us to investigate their stability, see Theorem 4.1 in Section 4, which suggests that  $p_1$  is unstable,  $p_2$  is stable, as expected for the bistable phenomenon. (Note that 0 is a stable steady state for  $d$  in the range of Theorem 1.2.)

The rest of the paper is organized as follows. In Section 2 we use a bifurcation approach to prove Theorem 1.1, with  $d$  as the bifurcation parameter. To overcome some of the limitations encountered in Section 2 in the bifurcation approach, in Section 3 we use a perturbation and reduction approach to study the steady-state solutions of (1.1) with large sinking rate and prove Theorem 1.2. In Section 4 we consider the linearized stability of the two positive steady states found in Section 3.

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## 2. Multiple steady-states through a bifurcation approach

In this section we use a bifurcation approach to prove Theorem 1.1. We focus on the existence of positive solutions by studying the bifurcation of a branch of positive solutions of (1.4) from the trivial solution  $p = 0$ , with coefficient  $d$  as the bifurcation parameter.

We first consider the stability of  $p = 0$ , which is determined by the sign of the largest eigenvalue, denoted by  $\lambda = d_*$ , of the linear eigenvalue problem

$$\begin{cases} \varphi_{xx} - \sigma \varphi_x + g(e^{-k_0 x})\varphi = \lambda \varphi, & 0 < x < 1, \\ \varphi_x(0) - \sigma \varphi(0) = 0, & \varphi_x(1) - \sigma \varphi(1) = 0. \end{cases} \quad (2.1)$$

1 By the well-known Krein–Rutman theorem it is known that  $d_*$  exists and is positive. Its corre-  
 2 sponding eigenfunction can be chosen to be positive in  $[0, 1]$ , denoted by  $\varphi_*$ , which is uniquely  
 3 determined by the normalization  $\max_{[0,1]} \varphi_* = 1$ . To investigate the structure of the set of solu-  
 4 tions of (1.4) near  $(d, p) = (d_*, 0)$ , we first introduce a few notations. Set  $X = \{u \in C^2([0, 1]) : u_x - \sigma u = 0$  at  $x = 0, 1\}$ ,  $Y = C([0, 1])$ . Define mapping  $F(d, p) : (0, \infty) \times X \rightarrow Y$  by  
 5

$$F(d, p) = p_{xx} - \sigma p_x + p[g(I) - d].$$

9 Clearly,  $F(d, 0) = 0$  for  $d \in (0, \infty)$ . Since

$$F_p(d, 0)\varphi = \varphi_{xx} - \sigma \varphi_x + [g(e^{-k_0 x}) - d]\varphi, \quad (2.2)$$

13 we see that (i) the kernel of  $F_p(d_*, 0)$  is spanned by  $\varphi_*$ , and is thus one dimensional; (ii) the  
 14 range of  $F_p(d_*, 0)$ , denoted by  $R(F_p(d_*, 0))$ , is given by  
 15

$$\left\{ \zeta \in Y : \int_0^1 e^{-\sigma x} \varphi_* \zeta dx = 0 \right\},$$

20 and is thus of co-dimension one. Furthermore,  $F_{pd}(d_*, 0)\varphi_* = -\varphi_* \notin R(F_p(d_*, 0))$ . By The-  
 21 rem 1.7 of [1], we obtain the result:  
 22

23 **Lemma 2.1.** *Let  $Z$  be any complement of  $\text{span}\{\varphi_*\}$  in  $X$ . Then there exists some  $\delta > 0$  and  
 24 continuously differentiable functions  $d : (-\delta, \delta) \rightarrow \mathbb{R}$  and  $\psi : (-\delta, \delta) \rightarrow X$  such that  $d(0) = d_*$ ,  
 25  $\psi(0) = 0$ , and  $F(d(s), p(s)) = 0$ , where  $p(s) = s\varphi_* + s\psi(s)$ . Moreover,  $F^{-1}(\{0\})$  near  $(d_*, 0)$   
 26 consists precisely of the curves  $p = 0$  and  $(d(s), p(s))$ ,  $s \in (-\delta, \delta)$ . Furthermore,*  
 27

$$d'(0) \int_0^1 e^{-\sigma x} \varphi_*^2 dx = -k \int_0^1 e^{-\sigma x} e^{-k_0 x} g'(e^{-k_0 x}) \varphi_*^2 \left( \int_0^x \varphi_* \right) dx. \quad (2.3)$$

32 **Proof.** It suffices to check (2.3). Dividing  $F(d(s), p(s)) = 0$  by  $s$  and differentiating the result  
 33 with respect to  $s$  at  $s = 0$ , using  $p = s\varphi_* + s\psi(s)$  we have  
 34

$$(\psi'(0))_{xx} - \sigma(\psi'(0))_x + \psi'(0)[g(e^{-k_0 x}) - d_*] + \varphi_* \left[ -kg'(e^{-k_0 x}) e^{-k_0 x} \int_0^x \varphi_* - d'(0) \right] = 0.$$

40 Multiplying the above equation by  $e^{-\sigma x} \varphi_*$  and integrating by parts we obtain (2.3).  $\square$   
 41

42 In the terminology of [2], 0 is an  $F_{pd}(d_*, 0)$ -simple eigenvalue of the operator  $F_p(d_*, 0)$ . By  
 43 Corollary 1.13 and Theorem 1.16 of [2] we have  
 44

45 **Proposition 2.2.** *There exist some positive constants  $\delta_1$  and  $\delta_2$  and continuously differen-  
 46 tiable functions  $\gamma : (d_* - \delta_1, d_* + \delta_1) \rightarrow \mathbb{R}$ ,  $\mu : (-\delta_2, \delta_2) \rightarrow \mathbb{R}$ ,  $v : (d_* - \delta_1, d_* + \delta_1) \rightarrow X$ ,  
 47  $w : (-\delta_2, \delta_2) \rightarrow X$  such that*

$$\begin{aligned} F_p(d, 0)v(d) &= -\gamma(d)v(d), \\ F_p(d(s), p(s))w(s) &= -\mu(s)w(s), \\ \lim_{s \rightarrow 0, \mu(s) \neq 0} \frac{-sd'(s)\gamma'(d_*)}{\mu(s)} &= 1, \end{aligned} \tag{2.4}$$

where  $\gamma(d_*) = \mu(0) = 0$ ,  $v(d_*) = w(0) = \varphi_*$ .

The next result suggests that for  $s > 0$  small, the nontrivial (positive) solution  $p(s) = s\varphi_* + s\psi(s)$  is unstable under suitable conditions.

**Lemma 2.3.** Suppose that (1.3) holds and  $I_* < e^{-k_0}$ . Then for any sufficiently small  $s > 0$ ,  $\mu(s) \leq 0$ .

**Proof.** By (2.2) and the definition of  $d_*$ , we see that  $\gamma(d) = d - d_*$ . If (1.3) holds,  $g'(e^{-k_0 x}) < 0$  for  $x \in (0, 1)$ , which together with (2.3) implies that  $d'(0) > 0$ . By (2.4), we see that  $\mu(s) < 0$  for  $s > 0$  small.  $\square$

**Lemma 2.4.** Suppose that (1.3) holds. If  $d \notin (0, g(I_*)]$ , then (1.4) has no positive solution.

**Proof.** Let  $p$  denote a positive solution of (1.4). Integrating (1.4) in  $(0, h)$  and applying the boundary condition in (1.4), we have

$$\int_0^1 p[g(I) - d] dx = 0.$$

Since  $g(I) \in (0, g(I_*))$  and  $I \not\equiv I^*$ , we see that  $d \in (0, g(I_*))$ . Therefore, (1.4) has no positive solution when  $d \notin (0, g(I_*))$ .  $\square$

**Lemma 2.5.** *Given any  $\eta > 0$ , there exists some positive constant  $C(\eta)$  such that every positive solution  $p$  of (1.4) with  $d \geq \eta$  satisfies  $\|p\|_{L^\infty(0,1)} \leq C(\eta)$ .*

The proof of Lemma 2.5 is identical to that of Lemma 4.2 of [13] and is omitted.

**Proof of Theorem 1.1.** By Lemma 2.1 and the global bifurcation result of Rabinowitz [25], (1.4) has an unbounded connected branch of positive solutions, denoted by  $\Gamma = \{(d, p)\} \subset \mathbb{R} \times C^1([0, 1])$ , which bifurcates from the trivial solution branch  $\{(d, 0)\}$  at  $(d_*, 0)$ . Since (1.4) has no positive solutions when  $d \notin (0, g(I_*))$  (Lemma 2.4) and all positive solutions of (1.4) are uniformly bounded when  $d$  is positive and bounded away from zero (Lemma 2.5), we see that  $\Gamma$  can only become unbounded as  $d \rightarrow 0^+$ . As  $\Gamma$  is connected, (1.4) has at least one positive solution for every  $d \in (0, d_*)$ . Denote  $\{(d, p) \in \Gamma : 0 < d < d_*\}$  by  $\Gamma_1$ . By Lemma 2.1,  $\Gamma$  contains a branch of positive solutions, denoted by  $\Gamma_2$ , which is given by  $(d(s), p(s))$  for  $s > 0$  small. By (1.3) and  $I_* < e^{-k_0}$ , we have  $d'(0) > 0$ . In particular, there exists some  $d^* > d_*$  such that the projection of  $\Gamma_2$  onto the  $d$ -axis is given by  $(d_*, d^*)$ . Hence  $\Gamma_1$  and  $\Gamma_2$  must be disjoint. As  $\Gamma$  is connected and  $\Gamma_1, \Gamma_2 \subset \Gamma$ , we see that the projection of  $\Gamma \setminus \Gamma_2$  onto the  $d$ -axis must contain

1  $(0, d^*)$ , i.e. (1.4) has at least two positive solutions for  $d \in (d_*, d^*)$ . By Lemma 2.4, (1.4) has no  
 2 positive solution for  $d > g(I_*)$ . Thus we complete the proof of Theorem 1.1.  $\square$

3  
 4 The proof of Theorem 1.1 indicates that there exists  $d^{**} \in [d^*, g(I_*)]$  such that (1.4) has no  
 5 positive solution lying on  $\Gamma$  for  $d > d^{**}$ , and it has at least one positive solution for  $d \in (0, d^{**}]$ .  
 6 In such a case, it is natural to expect that there exist two positive solutions for  $d \in (d_*, d^{**})$ ,  
 7 and the global bifurcation curve is “ $\supset$ ”-shaped, with a turning point at  $d = d^{**}$ . Moreover, one  
 8 expects the following typical conclusions:

9 For  $d \in (d_*, d^{**})$ , there are two positive solutions, and the solution on the “upper” branch of  
 10 the global bifurcation curve is stable, while that located on the “lower” branch is unstable. Note  
 11 also that the zero solution  $p = 0$  is always stable for  $d > d_*$ .

12 Such a global bifurcation picture can be partially proved by techniques of [3] for increasing  
 13 operators, if the problem at hand has the usual order-preserving property. Unfortunately, due  
 14 to the nonlocal nature of our problem, this nice property is lost and we cannot use such tools  
 15 involving the order-preserving property as in [3] or the usual upper and lower solution technique.  
 16 We remark that even the modified order-preserving property used in [6] is lost here due to the  
 17 fact that  $g(I)$  is no longer increasing in  $I$  for all  $I > 0$ .

18 To overcome these difficulties, in the next two sections, we employ a perturbation and reduc-  
 19 tion approach, which strongly suggests the validity of the global bifurcation picture described  
 20 above, at least for large  $\sigma$ .

### 22 3. Multiple steady-states via a perturbation-reduction approach

23  
 24 In this section, we use a perturbation and reduction approach to study the positive solutions  
 25 of (1.4). Their stability will be considered in Section 4 later. We will examine the problem with  
 26 a large  $\sigma$ , and write it in the form

$$28 \quad 29 \quad \sigma = \epsilon^{-1},$$

30 with  $\epsilon > 0$  small.

31 Let  $p(x)$  be a positive solution of (1.4), and define

$$34 \quad u(x) = p(\epsilon x + 1)e^{-x/2}.$$

36 A simple calculation shows that

$$38 \quad 39 \quad \begin{cases} u_{xx} - \frac{1}{4}u + \epsilon^2[g(\tilde{I}) - d]u = 0, & x \in (-\epsilon^{-1}, 0), \\ 40 \quad u_x - \frac{1}{2}u = 0, & x \in \{-\epsilon^{-1}, 0\}, \end{cases} \quad (3.1)$$

42 where

$$45 \quad 46 \quad 47 \quad \tilde{I} = \exp\left(-k_0 - \epsilon k_0 x - k\epsilon \int_{-\epsilon^{-1}}^x u(s)e^{s/2}ds\right).$$

1 We will look for a solution to (3.1) of the form

$$2 u(x) = \epsilon^{-1} \eta e^{x/2} + z(x)$$

3 with  $\eta > 0$  and

$$4 z \in H := \left\{ z : \int_{-\epsilon^{-1}}^0 z(x) e^{x/2} dx = 0, \int_{-\epsilon^{-1}}^0 |z(x)| e^{x/2} dx < \infty \right\}.$$

5 Since the function  $e^{x/2}$  satisfies

$$6 u_{xx} - \frac{1}{4}u = 0 \quad \text{in } (-\epsilon^{-1}, 0), \quad u_x - \frac{1}{2}u = 0 \quad \text{at } x = -\epsilon^{-1}, 0,$$

7 we necessarily have

$$8 \begin{cases} -z_{xx} + \frac{1}{4}z = \epsilon^2 [g(\tilde{I}) - d]u, & x \in (-\epsilon^{-1}, 0), \\ z_x - \frac{1}{2}z = 0, & x \in \{-\epsilon^{-1}, 0\}, \end{cases}$$

9 with  $\tilde{I}$  as above, and  $u(x) = \epsilon^{-1} \eta e^{x/2} + z(x)$ .

10 We now define

$$11 f(x) = f(x; \epsilon, \eta, z) := \epsilon^2 \{g(\tilde{I})[\epsilon^{-1} \eta e^{x/2} + z(x)]^+ - d[\epsilon^{-1} \eta e^{x/2} + z(x)]\}$$

12 (here  $u^+ = \max\{u, 0\}$ ) and consider the auxiliary problem

$$13 \begin{cases} -z_{xx} + \frac{1}{4}z = f(x) - \lambda e^{x/2}, & x \in (-\epsilon^{-1}, 0), \\ z_x - \frac{1}{2}z = 0, & x \in \{-\epsilon^{-1}, 0\}, \end{cases} \quad (3.2)$$

14 with  $\lambda$  determined by

$$15 \lambda \int_{-\epsilon^{-1}}^0 e^x dx = \int_{-\epsilon^{-1}}^0 f(x) e^{x/2} dx. \quad (3.3)$$

16 Then from (3.2) we find that  $z \in H$  has the expression

$$17 z(x) = \int_x^0 f(y) 2 \sinh\left(\frac{x-y}{2}\right) dy - \lambda \int_x^0 e^{y/2} 2 \sinh\left(\frac{x-y}{2}\right) dy + \alpha e^{x/2},$$

18 where  $\sinh(x) = (e^x - e^{-x})/2$  and  $\alpha \in \mathbb{R}^1$  is chosen such that  $\int_{-1/\epsilon}^0 z(x) e^{x/2} dx = 0$ . A simple  
19 calculation gives

$$z(x) = \int_x^0 f(y) 2 \sinh\left(\frac{x-y}{2}\right) dy + \lambda e^{x/2} (x - 1 + e^{-x}) + \alpha e^{x/2}.$$

For  $z \in H$ , we now define the nonlinear operator

$$F_{\epsilon, \eta}(z)(x) := F_{\epsilon, \eta}^1(z)(x) + \alpha(\epsilon, \eta, z) e^{x/2},$$

with

$$F_{\epsilon, \eta}^1(z)(x) := \int_x^0 f(y; \epsilon, \eta, z) 2 \sinh\left(\frac{x-y}{2}\right) dy + \lambda(\epsilon, \eta, z) e^{x/2} (x - 1 + e^{-x}),$$

where  $\lambda(\epsilon, \eta, z)$  is determined by (3.3) with  $f(x) = f(x; \epsilon, \eta, z)$ , and  $\alpha(\epsilon, \eta, z)$  is determined by

$$\int_{-1/\epsilon}^0 e^{x/2} F_{\epsilon, \eta}^1(z)(x) dx + \alpha \int_{-1/\epsilon}^0 e^x dx = 0,$$

that is,

$$\alpha(\epsilon, \eta, z) := -(1 - e^{-1/\epsilon})^{-1} \int_{-1/\epsilon}^0 e^{x/2} F_{\epsilon, \eta}^1(z)(x) dx. \quad (3.4)$$

We will show that for all small  $\epsilon > 0$  and all  $\eta > 0$  in a certain interval  $J$ , the operator  $F_{\epsilon, \eta}$  is a contraction mapping in  $H$ , and hence it has a unique fixed point  $z = z_{\epsilon, \eta}$ . We will then choose  $\eta = \eta_\epsilon$  so that  $\lambda(\epsilon, \eta, z) = 0$  for  $z = z_{\epsilon, \eta}$  and  $\eta = \eta_\epsilon$ . We will show that this gives a positive solution  $u = \epsilon^{-1} \eta_\epsilon e^{x/2} + z_{\epsilon, \eta_\epsilon}(x)$  to (3.1) for all small  $\epsilon > 0$ . A key point in this approach is that with our assumptions on  $g(I)$ , for each small  $\epsilon > 0$ , for  $d > 0$  in a certain range, we always have two solutions for  $\eta_\epsilon$ , and hence this approach yields two positive steady-states for (3.1).

We will show that for  $\epsilon > 0$  small, the equation  $\lambda(\epsilon, \eta, z_{\epsilon, \eta}) = 0$  is a perturbation of the equation

$$d = \int_0^{k\eta} \frac{1}{k\eta} g(\xi_0 e^{-s}) ds, \quad \xi_0 = e^{-k_0} \in (0, 1].$$

Denote

$$\mu = k\eta, \quad G(\mu) = \frac{1}{\mu} \int_0^\mu g(\xi_0 e^{-s}) ds \quad \text{and} \quad G_0(\mu) = \int_0^\mu g(\xi_0 e^{-s}) ds,$$

1 then the above equation is equivalent to  
 2

3 
$$d = G(\mu). \quad (3.5)$$
  
 4

5 In order to determine the range  $J$  for  $\eta$  so that  $F_{\epsilon, \eta}$  is a contraction mapping, and multiple  
 6 solutions to (3.1) exist, we need to obtain several simple properties of the functions  $G(\mu)$  and  
 7  $G_0(\mu)$  first.

8  
 9 *3.1. Properties of  $G_0(\mu)$  and  $G(\mu)$*   
 10

11Q2 Our assumptions on  $g$  imply that  $g(I) \leq c_0 I$  for some  $c_0 > 0$  and all  $I > 0$ . It follows that  
 12Q3

13  
 14 
$$G_0(+\infty) = \int_0^{+\infty} g(\xi_0 e^{-s}) ds \in (0, \infty).$$
  
 15  
 16

17 Clearly  
 18

19 
$$G(0) = g(\xi_0) > 0, \quad G(\mu) > 0 \quad \text{for } \mu > 0, \quad G(+\infty) = 0$$
  
 20

21 and  
 22

23  
 24 
$$G'(\mu) = \mu^{-2} [\mu G'_0(\mu) - G_0(\mu)] =: \mu^{-2} \tilde{G}(\mu).$$
  
 25

26 If  $I_* \geq \xi_0$ , then  
 27

28 
$$G'_0(\mu) = g(\xi_0 e^{-\mu}) > 0, \quad G''_0(\mu) = -g'(\xi_0 e^{-\mu}) \xi_0 e^{-\mu} < 0 \quad \forall \mu \geq 0.$$
  
 29

30 Hence  
 31

32 
$$\tilde{G}'(\mu) = \mu G''_0(\mu) < 0 \quad \text{for } \mu > 0.$$
  
 33

34 It follows that  
 35

36 
$$\tilde{G}(\mu) < \tilde{G}(0) = 0 \quad \text{for } \mu > 0.$$
  
 37

38 Thus  $G'(\mu) = \mu^{-2} \tilde{G}(\mu) < 0$  for  $\mu > 0$ . This implies that for each  $d \in (0, d_*):= (0, g(\xi_0))$ ,  
 39 (3.5) has a unique solution  $\mu > 0$ , and for each  $d \geq d_*$ , (3.5) has no solution  $\mu > 0$ .  
 40

41 If  $I_* < \xi_0$ , then  $G'_0(\mu) > 0$  for  $\mu \geq 0$ , and  
 42

43 
$$G''_0(\mu) > 0 \quad \text{for } \mu \in (0, \ln(\xi_0/I_*)), \quad G''_0(\mu) < 0 \quad \text{for } \mu > \ln(\xi_0/I_*).$$
  
 44

45 Thus  
 46

47 
$$\tilde{G}'(\mu) > 0 \quad \text{for } \mu \in (0, \ln(\xi_0/I_*)), \quad \tilde{G}'(\mu) < 0 \quad \text{for } \mu > \ln(\xi_0/I_*).$$
  
 48

1 Using  $G_0''(\mu) < 0$  for all large  $\mu > 0$  and  $G_0(+\infty) > 0$ , it is easily seen that  $\tilde{G}(\mu) < 0$  for all  
 2 large  $\mu$ . Thus there exists a unique  $\mu_* > \ln(\xi_0/I_*)$  such that for  $\mu \in (0, \mu_*)$ ,  $\tilde{G}(\mu) > 0$ , and for  
 3  $\mu > \mu_*$ ,  $\tilde{G}(\mu) < 0$ . It follows that  
 4

5  $G'(\mu) > 0 \quad \text{for } \mu \in (0, \mu_*), \quad G'(\mu) < 0 \quad \text{for } \mu > \mu_*.$   
 6

7 Define  $\hat{d}_* := G(\mu_*)$ ; then for each  $d \in (g(\xi_0), \hat{d}_*)$ , (3.5) has exactly two solutions  $\mu_1 \in (0, \mu_*)$   
 8 and  $\mu_2 \in (\mu_*, +\infty)$ , and (3.5) has a unique solution  $\mu > 0$  for  $d \in \{\hat{d}_*\} \cup (0, g(\xi_0))$ . Moreover,  
 9

10  $G'(\mu_1) > 0 > G'(\mu_2).$  (3.6)  
 11

12 From now on, we always assume that  
 13

14  $I_* < \xi_0 \quad \text{and} \quad d \in (g(\xi_0), \hat{d}_*).$  (3.7)  
 15

16 Thus (3.5) has exactly two solutions  $0 < \mu_1 < \mu_2$ , and apart from (3.6), we have  
 17

18  $\mu > G(\mu) \quad \text{for } \mu \in (0, \mu_1) \cup (\mu_2, \infty), \quad \mu < G(\mu) \quad \text{for } \mu \in (\mu_1, \mu_2).$  (3.8)  
 19

20 Clearly part (i) of Theorem 1.2 follows from the above discussions. The rest of this section is  
 21 devoted to the proof of part (ii) of Theorem 1.2. We set  
 22

23 
$$J = \left[ \frac{\mu_1}{2k}, \frac{2\mu_2}{k} \right].$$
  
 24  
 25

26 3.2.  $F_{\epsilon, \eta}$  maps a subset of  $H$  into itself  
 27

28 It is easily seen that  $H = H_\epsilon$  endowed with the norm  
 29

30 
$$\|z\| = \int_{-\epsilon^{-1}}^0 |z(x)| e^{x/2} dx$$
  
 31  
 32  
 33

34 is a Banach space.  
 35

36 **Lemma 3.1.** *There exist  $\epsilon_0 > 0$  and  $M > 0$  such that for each  $\epsilon \in (0, \epsilon_0]$  and  $\eta \in J$ ,  $F_{\epsilon, \eta}$  maps  
 37  $\Omega := \{z \in H : \|z\| \leq M\}$  into itself.*  
 38

39 **Proof.**  
 40

41 
$$e^{x/2} F_{\epsilon, \eta}^1(z)(x) = e^{x/2} \int_x^0 f(y; \epsilon, \eta, z) 2 \sinh\left(\frac{x-y}{2}\right) dy + \lambda(\epsilon, \eta, z) [(x-1)e^x + 1]$$
  
 42  
 43  
 44  
 45

46  $= S_1(x) + S_2(x).$

47 We have, for  $\epsilon > 0$  small,

$$\begin{aligned}
1 & |\lambda(\epsilon, \eta, z)| = (1 - e^{-\epsilon^{-1}})^{-1} \left| \int_{-\epsilon^{-1}}^0 \epsilon^2 \{g(\cdot)[\epsilon^{-1} \eta e^{x/2} + z(x)]^+ - d[\epsilon^{-1} \eta e^{x/2} + z(x)]\} e^{x/2} dx \right| \\
2 & \leq 2 \int_{-\epsilon^{-1}}^0 \epsilon^2 [c_0 e^{-\epsilon k \int_{-\epsilon^{-1}}^x (\epsilon^{-1} \eta e^{y/2} - |z(y)|) e^{y/2} dy} + d] (\epsilon^{-1} \eta e^x + |z(x)| e^{x/2}) dx \\
3 & \leq 2 \int_{-\epsilon^{-1}}^0 [c_0 e^{\epsilon k \|z\|} + d] (\epsilon \eta e^x + \epsilon^2 |z(x)| e^{x/2}) dx \\
4 & \leq C(e^{\epsilon k \|z\|} + 1)(\epsilon + \epsilon^2 \|z\|),
\end{aligned}$$

for some  $C > 0$  independent of  $\eta \in J$  and small  $\epsilon > 0$ . In the following, we will use  $C$  to denote a generic positive constant that is independent of  $\eta \in J$  and small  $\epsilon > 0$ , whose value may change from line to line.

From the above estimate we obtain

$$\begin{aligned}
1 & \int_{-\epsilon^{-1}}^0 |S_2(x)| dx \leq |\lambda(\epsilon, \eta, z)| \int_{-\epsilon^{-1}}^0 |(x-1)e^x + 1| dx \\
2 & \leq C(e^{\epsilon k \|z\|} + 1)(1 + \epsilon \|z\|).
\end{aligned}$$

For the term  $S_1(x)$  we have

$$\begin{aligned}
1 & |S_1(x)| = \left| e^{x/2} \int_x^0 f(y; \epsilon, \eta, z) 2 \sinh\left(\frac{x-y}{2}\right) dy \right| \\
2 & \leq e^{x/2} \int_x^0 \epsilon^2 |g(\cdot) - d| (\epsilon^{-1} \eta e^{y/2} + |z(y)|) 2 \left| \sinh\left(\frac{x-y}{2}\right) \right| dy \\
3 & \leq \epsilon^2 \int_x^0 C(e^{\epsilon k \|z\|} + 1) (\epsilon^{-1} e^y + |z(y)| e^{y/2}) |e^{x-y} - 1| dy \\
4 & \leq C(e^{\epsilon k \|z\|} + 1)(1 + \epsilon \|z\|) \epsilon.
\end{aligned}$$

Thus

$$\int_{-\epsilon^{-1}}^0 |S_1(x)| dx \leq C(e^{\epsilon k \|z\|} + 1)(1 + \epsilon \|z\|).$$

1 We therefore have  
 2  
 3  
 4

$$\|F_{\epsilon,\eta}^1(z)\| \leq C(e^{\epsilon k\|z\|} + 1)(1 + \epsilon\|z\|).$$

5 By (3.4), we have  
 6  
 7

$$\|\alpha(\epsilon, \eta, z)e^{x/2}\| \leq \|F_{\epsilon,\eta}^1(z)\| \leq C(e^{\epsilon k\|z\|} + 1)(1 + \epsilon\|z\|).$$

8 Therefore, taking  $M = 3C$  then from the above inequalities we can easily find  $\epsilon_0 > 0$  sufficiently  
 9 small (depending on  $C$ ) such that for all  $\epsilon \in (0, \epsilon_0]$  and  $\eta \in J$ ,  
 10

$$\|F_{\epsilon,\eta}(z)\| \leq M \quad \text{if } \|z\| \leq M. \quad \square$$

### 14 3.3. $F_{\epsilon,\eta}$ is a contraction mapping

15 In this subsection we show that there exists  $\epsilon^0 \in (0, \epsilon_0]$  such that for each  $\epsilon \in (0, \epsilon^0]$  and each  
 16  $\eta \in J$ ,  $F_{\epsilon,\eta}$  is a contraction mapping on  $\Omega$ .  
 17

18 Suppose that  $z_1, z_2 \in \Omega$ , and denote for  $i = 1, 2$ ,  
 19

$$I_i(x) = \exp\left(-k_0 - \epsilon k_0 x - k \int_{-\epsilon^{-1}}^x \eta e^y dy\right) e^{-\epsilon k \int_{-\epsilon^{-1}}^x z_i(y) e^{y/2} dy}.$$

20 Clearly  
 21  
 22

$$\begin{aligned} |g(I_1) - g(I_2)| &\leq C|I_1 - I_2| \\ &\leq C|e^{-\epsilon k \int_{-\epsilon^{-1}}^x z_1(y) e^{y/2} dy} - e^{-\epsilon k \int_{-\epsilon^{-1}}^x z_2(y) e^{y/2} dy}| \\ &\leq C e^{\epsilon k(\|z_1\| + \|z_2\|)} \epsilon k \int_{-\epsilon^{-1}}^x |z_1(y) - z_2(y)| e^{y/2} dy \\ &\leq C e^{\epsilon k 2M} \epsilon \|z_1 - z_2\| \\ &\leq \epsilon C \|z_1 - z_2\|. \end{aligned}$$

36 We write  
 37

$$\begin{aligned} &[F_{\epsilon,\eta}^1(z_1)(x) - F_{\epsilon,\eta}^1(z_2)(x)] e^{x/2} \\ &= \epsilon^2 \int_x^0 \{g(I_1)(\epsilon^{-1} \eta e^{y/2} + z_1)^+ - g(I_2)(\epsilon^{-1} \eta e^{y/2} + z_2)^+ - d(z_1 - z_2)\} e^{y/2} (e^{x-y} - 1) dy \\ &\quad + [\lambda(\epsilon, \eta, z_1) - \lambda(\epsilon, \eta, z_2)] e^x (x - 1 + e^{-x}) \\ &= \epsilon^2 \int_x^0 [g(I_1) - g(I_2)] (\epsilon^{-1} \eta e^{y/2} + z_1)^+ e^{y/2} (e^{x-y} - 1) dy \end{aligned}$$

$$\begin{aligned}
& + \epsilon^2 \int_x^0 \{g(I_2)[(\epsilon^{-1} \eta e^{y/2} + z_1)^+ - (\epsilon^{-1} \eta e^{y/2} + z_2)^+] - d(z_1 - z_2)\} e^{y/2} (e^{x-y} - 1) dy \\
& + [\lambda(\epsilon, \eta, z_1) - \lambda(\epsilon, \eta, z_2)] e^x (x - 1 + e^{-x}) \\
& = T_1(x) + T_2(x) + T_3(x).
\end{aligned}$$

Then

$$\begin{aligned}
|T_1(x)| & \leq \epsilon^2 C \epsilon \|z_1 - z_2\| \int_x^0 (\epsilon^{-1} \eta e^y + |z_1| e^{y/2}) dy \\
& \leq C \epsilon^2 \|z_1 - z_2\|.
\end{aligned}$$

Since

$$|g(I_2)| + d \leq C I_2 + d \leq C e^{\epsilon k \|z_2\|} + d \leq C,$$

we have

$$\begin{aligned}
|T_2(x)| & \leq C \epsilon^2 \int_x^0 \{ |(\epsilon^{-1} \eta e^{y/2} + z_1)^+ - (\epsilon^{-1} \eta e^{y/2} + z_2)^+| + |z_1 - z_2| \} e^{y/2} |e^{x-y} - 1| dy \\
& \leq C \epsilon^2 \int_x^0 |z_1(y) - z_2(y)| e^{y/2} dy \\
& \leq C \epsilon^2 \|z_1 - z_2\|.
\end{aligned}$$

To estimate  $T_3(x)$ , we notice that

$$\begin{aligned}
& [\lambda(\epsilon, \eta, z_1) - \lambda(\epsilon, \eta, z_2)] \int_{-\epsilon^{-1}}^0 e^x dx \\
& = \epsilon^2 \int_{-\epsilon^{-1}}^0 \{g(I_1)(\epsilon^{-1} \eta e^{y/2} + z_1)^+ - g(I_2)(\epsilon^{-1} \eta e^{y/2} + z_2)^+ - d(z_1 - z_2)\} e^{y/2} dy \\
& = \epsilon^2 \int_{-\epsilon^{-1}}^0 [g(I_1) - g(I_2)] (\epsilon^{-1} \eta e^{y/2} + z_1)^+ e^{y/2} dy \\
& + \epsilon^2 \int_{-\epsilon^{-1}}^0 \{g(I_2)[(\epsilon^{-1} \eta e^{y/2} + z_1)^+ - (\epsilon^{-1} \eta e^{y/2} + z_2)^+] - d(z_1 - z_2)\} e^{y/2} dy.
\end{aligned}$$

1 Therefore similar to the estimates for  $T_1(x)$  and  $T_2(x)$  above, we obtain  
 2

$$3 |\lambda(\epsilon, \eta, z_1) - \lambda(\epsilon, \eta, z_2)| \leq C\epsilon^2 \|z_1 - z_2\|. \\ 4$$

5 Since  $e^x|x - 1 + e^{-x}| \leq C$  on  $[-\epsilon^{-1}, 0]$ , we obtain  
 6

$$7 |T_3(x)| \leq C\epsilon^2 \|z_1 - z_2\|. \\ 8$$

9 It follows that  
 10

$$11 \|F_{\epsilon, \eta}^1(z_1) - F_{\epsilon, \eta}^1(z_2)\| \leq \int_{-\epsilon^{-1}}^0 (|T_1(x)| + |T_2(x)| + |T_3(x)|) dx \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \leq C\epsilon \|z_1 - z_2\|.$$

17 By (3.4), we have  
 18

$$19 \|\alpha(\epsilon, \eta, z_1)e^{x/2} - \alpha(\epsilon, \eta, z_2)e^{x/2}\| = \|F_{\epsilon, \eta}^1(z_1) - F_{\epsilon, \eta}^1(z_2)\| \leq C\epsilon \|z_1 - z_2\|. \\ 20 \\ 21$$

22 Thus  $F_{\epsilon, \eta}$  is a contraction mapping on  $\Omega$  provided that  $\epsilon \in (0, \epsilon^0]$  and  $\eta \in J$  with  $\epsilon^0 = \\ 23 \min\{\epsilon_0, (3C)^{-1}\}$ .  
 24

25 Summarizing, we have proved the following result.  
 26

27 **Lemma 3.2.** *There exists  $\epsilon^0 \in (0, \epsilon_0]$  such that  $F_{\epsilon, \eta}$  is a contraction mapping on  $\Omega$  for every  
 28  $\epsilon \in (0, \epsilon^0]$  and  $\eta \in J$ .*  
 29

30 Applying Banach's fixed point theorem, we obtain  
 31

32 **Proposition 3.3.** *For each  $\epsilon \in (0, \epsilon^0]$  and  $\eta \in J$ ,  $F_{\epsilon, \eta}$  has a unique fixed point  $z_{\epsilon, \eta} \in \Omega$ .*  
 33

34 **Remark 3.4.** Since  $F_{\epsilon, \eta}$  depends continuously on  $(\epsilon, \eta)$ , the uniqueness of the fixed point implies  
 35 that  $z_{\epsilon, \eta}$  also depends continuously on  $(\epsilon, \eta)$ .  
 36

### 3.4. The reduced equation

38 **Proposition 3.5.** *Suppose  $\epsilon \in (0, \epsilon^0]$ . If  $\eta \in J$  satisfies*  
 39

$$40 \lambda(\epsilon, \eta, z_{\epsilon, \eta}) = 0, \tag{3.9} \\ 41$$

42 then  
 43

$$44 u_{\epsilon, \eta} = \epsilon^{-1} \eta e^{x/2} + z_{\epsilon, \eta}(x) \\ 45$$

46 is a positive solution to (3.1).  
 47

1 **Proof.** Since  $F_{\epsilon,\eta} z_{\epsilon,\eta} = z_{\epsilon,\eta}$  and  $\lambda(\epsilon, \eta, z_{\epsilon,\eta}) = 0$ , from the definition of  $F_{\epsilon,\eta}$  we find that  $u =$   
 2  $u_{\epsilon,\eta}$  satisfies, for  $x \in (-\epsilon^{-1}, 0)$ ,

$$4 -u_{xx} + \frac{1}{4}u = -(z_{\epsilon,\eta})_{xx} + \frac{1}{4}z_{\epsilon,\eta} = f(x; \epsilon, \eta, z_{\epsilon,\eta}).$$

$$5$$

6 Using the definition of  $f$  we have

$$8 -u_{xx} + \left(\frac{1}{4} + \epsilon^2 d\right)u = \epsilon^2 g(\tilde{I})u^+ \geq 0 \quad \text{in } (-\epsilon^{-1}, 0), \quad (3.10)$$

$$9$$

$$10$$

11 with

$$13 \tilde{I} = \exp\left(-k_0 - \epsilon k_0 x - \epsilon k \int_{-\epsilon^{-1}}^x u(y)e^{y/2} dy\right).$$

$$14$$

$$15$$

$$16$$

17 Clearly  $u$  also satisfies the boundary condition of (3.1). Hence, due to  $\epsilon^2 d > 0$ , we can apply the  
 18 maximum principle to (3.10) to conclude that  $u \geq 0$ . Thus  $u^+ = u$  and  $u$  solves (3.1).

19 To show  $u$  is a positive solution, by the strong maximum principle, it suffices to show that  
 20  $u \not\equiv 0$ . But this follows trivially from

$$22 \int_{-\epsilon^{-1}}^0 u(x)e^{x/2} dx = \int_{-\epsilon^{-1}}^0 \epsilon^{-1} \eta e^x dx > 0. \quad \square$$

$$23$$

$$24$$

$$25$$

$$26$$

27 We next examine closely the reduced equation  $\lambda(\epsilon, \eta, z_{\epsilon,\eta}) = 0$ , that is

$$29 \int_{\epsilon^{-1}}^0 e^{x/2} \{g(I_{\epsilon,\eta})[\epsilon^{-1} \eta e^{x/2} + z_{\epsilon,\eta}(x)]^+ - d[\epsilon^{-1} \eta e^{x/2} + z_{\epsilon,\eta}(x)]\} dx = 0, \quad (3.11)$$

$$30$$

$$31$$

$$32$$

33 with

$$34 I_{\epsilon,\eta}(x) = \exp\left(-k_0 - \epsilon k_0 x - \epsilon k \int_{-\epsilon^{-1}}^x [\epsilon^{-1} \eta e^{y/2} + z_{\epsilon,\eta}(y)] e^{y/2} dy\right).$$

$$35$$

$$36$$

$$37$$

$$38$$

39 Multiplying (3.11) by  $\epsilon \eta^{-1}$  we obtain

$$41 0 = \int_{\epsilon^{-1}}^0 e^{x/2} \{g(I_{\epsilon,\eta})[e^{x/2} + \epsilon \eta^{-1} z_{\epsilon,\eta}(x)]^+ - d[e^{x/2} + \epsilon \eta^{-1} z_{\epsilon,\eta}(x)]\} dx$$

$$42$$

$$43$$

$$44$$

$$45$$

$$46$$

$$47$$

$$= \int_{\epsilon^{-1}}^0 e^x [g(I_{\epsilon,\eta}) - d] dx + \delta_1,$$

1 with

$$\begin{aligned}
 |\delta_1| &= \left| \int_{\epsilon^{-1}}^0 e^{x/2} g(I_{\epsilon, \eta}) \{ [e^{x/2} + \epsilon \eta^{-1} z_{\epsilon, \eta}(x)]^+ - e^{x/2} \} dx \right| \\
 &\leq C\epsilon \int_{-\epsilon^{-1}}^0 |z_{\epsilon, \eta}(x)| e^{x/2} dx \\
 &\leq C\epsilon.
 \end{aligned}$$

12 We may write

13

14

15

16 with

17

18

19

20

21

22 uniformly for  $x \in [-\epsilon^{-1}, 0]$  and  $\eta \in J$  as  $\epsilon \rightarrow 0$ . Therefore

23

24

25

26 with

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30 It follows that

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37 where

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39

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41

42

43 We thus have

44

45

46

47

1 and

$$2$$

$$3 d = \int_{-\epsilon^{-1}}^0 g(\xi_0 e^{-\epsilon k_0 x - k \eta e^x}) e^x dx + \delta_1 + \delta_2 + \delta_3$$

$$4$$

$$5$$

$$6$$

7 with

8

9

10

11

12 Finally we have

$$13$$

$$14 \int_{-\epsilon^{-1}}^0 g(\xi_0 e^{-\epsilon k_0 x - k \eta e^x}) e^x dx = \int_{-\epsilon^{-1}}^0 g(\xi_0 e^{-\epsilon k_0 x - k \eta e^x}) (\epsilon k_0 x + k \eta e^x)' \frac{1}{k \eta} dx$$

$$15$$

$$16 \quad - \frac{\epsilon k_0}{k \eta} \int_{-\epsilon^{-1}}^0 g(\xi_0 e^{-\epsilon k_0 x - k \eta e^x}) dx$$

$$17$$

$$18 \quad = \frac{1}{k \eta} \int_{-k_0 + k \eta \exp(-\epsilon^{-1})}^{k \eta} g(\xi_0 e^{-\mu}) d\mu$$

$$19$$

$$20 \quad - \frac{1}{k \eta} \int_{-k_0}^0 g(\xi_0 e^{-\mu - k \eta \exp(\frac{\mu}{\epsilon k_0})}) d\mu$$

$$21$$

$$22 \quad = \frac{1}{k \eta} \int_{-k_0}^{k \eta} g(\xi_0 e^{-\mu}) d\mu + o(1)$$

$$23$$

$$24 \quad - \frac{1}{k \eta} \int_{-k_0}^0 g(\xi_0 e^{-\mu}) d\mu + o(1)$$

$$25$$

$$26 \quad = \frac{1}{k \eta} \int_0^{k \eta} g(\xi_0 e^{-\mu}) d\mu + o(1).$$

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42 Therefore the reduced equation can be written as

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1 where  $o(1) \rightarrow 0$  uniformly in  $\eta \in J$  as  $\epsilon \rightarrow 0$ . Recalling the definition of  $G(\mu)$  we find that the  
 2 reduced equation can be written as  
 3

$$4 \quad d = G(k\eta) + o(1). \quad (3.12) \quad 5$$

6 Fix  $\eta_1, \eta_2$  and  $\eta_3$  in  $J$  such that  
 7

$$8 \quad \eta_1 \in \left( \frac{\mu_1}{2k}, \frac{\mu_1}{k} \right), \quad \eta_2 \in \left( \frac{\mu_1}{k}, \frac{\mu_2}{k} \right), \quad \eta_3 \in \left( \frac{\mu_2}{k}, \frac{2\mu_2}{k} \right). \quad 9 \quad 10$$

11 Then from (3.8) we find that  
 12

$$13 \quad d > G(k\eta_1), \quad d < G(k\eta_2), \quad d > G(k\eta_3). \quad 14 \quad 15$$

16 Since  $\lambda(\epsilon, \eta, z_{\epsilon, \eta})$  is a continuous function in  $(\epsilon, \eta)$ , the term  $o(1)$  in (3.12) can be written as  
 17  $o(\epsilon, \eta)$  which is continuous and satisfies  $o(\epsilon, \eta) \rightarrow 0$  uniformly in  $\eta \in J$  as  $\epsilon \rightarrow 0$ . Therefore,  
 18 for all small  $\epsilon > 0$ , say  $\epsilon \in (0, \epsilon^*]$ , with some  $\epsilon^* \in (0, \epsilon^0)$ , the equation  
 19

$$20 \quad d = G(k\eta) + o(\epsilon, \eta) \quad 21$$

22 has at least two solutions in  $J$ :  $\eta_1^\epsilon \in (\eta_1, \eta_2)$  and  $\eta_2^\epsilon \in (\eta_2, \eta_3)$ . Moreover,  
 23

$$24 \quad \lim_{\epsilon \rightarrow 0} \eta_1^\epsilon = \frac{\mu_1}{k}, \quad \lim_{\epsilon \rightarrow 0} \eta_2^\epsilon = \frac{\mu_2}{k} \quad \text{and} \quad G'(\eta_1^\epsilon) > 0 > G'(\eta_2^\epsilon). \quad 25 \quad 26$$

27 We have thus proved the following result.  
 28

29 **Theorem 3.6.** Suppose that (3.7) holds. Then there exists  $\epsilon^* > 0$  such that for each  $\epsilon \in (0, \epsilon^*]$ ,  
 30 (3.1) has two positive solutions of the form  
 31

$$32 \quad u_1 = \epsilon^{-1}(\eta_1^\epsilon e^{x/2} + z_1^\epsilon), \quad u_2 = \epsilon^{-1}(\eta_2^\epsilon e^{x/2} + z_2^\epsilon) \quad 33 \quad 34$$

35 with  $z_1^\epsilon, z_2^\epsilon \in H$  satisfying  $\|z_i^\epsilon\| \leq C\epsilon$ , and  
 36

$$37 \quad \lim_{\epsilon \rightarrow 0} \eta_1^\epsilon = \frac{\mu_1}{k}, \quad \lim_{\epsilon \rightarrow 0} \eta_2^\epsilon = \frac{\mu_2}{k}, \quad G'(\mu_1) > 0 > G'(\mu_2). \quad 38 \quad 39$$

40 Since, for  $i = 1, 2$ ,  
 41

$$42 \quad 43 \quad u_i = \epsilon^{-1}(\eta_i^\epsilon e^{x/2} + z_i^\epsilon) = \epsilon^{-1}\left(\frac{\mu_i}{k}e^{x/2} + \tilde{z}_i^\epsilon\right) \quad 44 \quad 45$$

46 with  $\tilde{z}_i^\epsilon := z_i^\epsilon + o(1)e^{x/2}$ , we find that  $\|\tilde{z}_i^\epsilon\| = o(1)$ , and part (ii) of Theorem 1.2 now follows  
 47 directly from Theorem 3.6.

#### 1 4. Stability analysis

3 In this section, we consider the linearized stability of the two solutions  $u_1$  and  $u_2$  in [Theorem 3.6](#). Let  $u^* = u_\epsilon^*$  denote either  $u_1$  or  $u_2$ . The linearized eigenvalue problem of [\(3.1\)](#) at  $u = u^*$  is given by

$$7 \quad \begin{cases} \phi_{xx} - \frac{1}{4}\phi + \epsilon^2[g(I_\epsilon) - d]\phi \\ 8 \quad - \epsilon^2 u^* g'(I_\epsilon) I_\epsilon \epsilon k \int_{-1/\epsilon}^x \phi(s) e^{s/2} ds + \lambda\phi = 0, \quad x \in (-1/\epsilon, 0), \\ 9 \quad \phi_x - \frac{1}{2}\phi = 0, \quad x \in \{-1/\epsilon, 0\}, \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{cases} \quad (4.1)$$

15 where

$$16 \quad I_\epsilon(x) = e^{-k_0 - \epsilon k_0 x - k\epsilon \int_{-1/\epsilon}^x u^*(s) e^{s/2} ds}.$$

19 We will show that, for every small  $\epsilon > 0$ , this problem has an eigenpair  $(\lambda, \phi)$  of the form

$$21 \quad \lambda = \epsilon^2 \mu_\epsilon, \quad \phi(x) = e^{x/2} + \epsilon w_\epsilon(x), \quad w_\epsilon \in H,$$

24 with  $\mu_\epsilon \rightarrow \mu_0 \neq 0$  as  $\epsilon \rightarrow 0$  and  $\|w_\epsilon\| \leq C$  for all  $\epsilon > 0$  small. Here  $H$  is the Banach space  
25 defined in the previous section. Moreover, we will show that  $\mu_0 < 0$  when  $u^* = u_1$ , and  $\mu_0 > 0$   
26 when  $u^* = u_2$ . This implies that  $u_1$  is linearly unstable. Although this does not prove that  $u_2$  is  
27 linearly stable, but as explained below, it strongly suggests the validity of such a conclusion.

28 The difficulty in proving the linearized stability of  $u_2$  is due to the fact that [\(4.1\)](#) is a nonlocal  
29 eigenvalue problem, and the corresponding linear operator to this problem is not self-adjoint, nor  
30 order-preserving. Therefore it is difficult to use variational characterization or the well-known  
31 Krein–Rutman theorem to know the relationship of the eigenvalue obtained above to the rest of  
32 the spectrum. However, since  $\phi = e^{x/2} + \epsilon w_\epsilon$  is a small perturbation of a positive function, it is  
33 reasonable to believe that  $\lambda = \epsilon \mu_\epsilon$  behaves like a principle eigenvalue, with all other eigenvalues  
34 having real parts strictly greater than  $\lambda$ , which would imply the linearized stability of  $u_2$ .

35 We now look for an eigenpair of [\(4.1\)](#) of the form

$$37 \quad \lambda = \epsilon^2 \mu, \quad \phi = e^{x/2} + \epsilon w, \quad w \in H.$$

39 Substituting these into [\(4.1\)](#) we obtain

$$41 \quad \begin{cases} 42 \quad w_{xx} - \frac{1}{4}w + \epsilon[g(I_\epsilon) - d](e^{x/2} + \epsilon w) \\ 43 \quad - \epsilon u^* g'(I_\epsilon) I_\epsilon \epsilon k \int_{-1/\epsilon}^x [e^s + \epsilon w e^{s/2}] ds + \epsilon \mu(e^{x/2} + \epsilon w) = 0, \quad x \in (-1/\epsilon, 0), \\ 44 \\ 45 \\ 46 \\ 47 \quad w_x - \frac{1}{2}w = 0, \quad x \in \{-1/\epsilon, 0\}. \end{cases} \quad (4.2)$$

1 Multiplying the first equation in (4.2) by  $\epsilon^{-1}e^{x/2}$ , and integrating over  $(-1/\epsilon, 0)$ , we obtain  
 2

$$\begin{aligned}
 3 & \int_{-1/\epsilon}^0 [g(I_\epsilon) - d](e^x + \epsilon w e^{x/2}) dx - \int_{-1/\epsilon}^0 e^{x/2} u^* g'(I_\epsilon) I_\epsilon \epsilon k \int_{-1/\epsilon}^x [e^s + \epsilon w e^{s/2}] ds dx \\
 4 & = -\mu \int_{-1/\epsilon}^0 e^x dx = -\mu(1 - e^{-1/\epsilon}). \\
 5 & \\
 6 & \\
 7 & \\
 8 & \\
 9 & \\
 10 & 
 \end{aligned} \tag{4.3}$$

11 This defines  $\mu$  as a functional of  $w$  and we may write  
 12

$$13 \quad 14 \quad \mu = \mu(w, \epsilon).$$

15 Much as in the previous section, from (4.2) we obtain  
 16

$$\begin{aligned}
 17 & w(x) = \epsilon \int_x^0 [g(I_\epsilon) - d](e^{y/2} + \epsilon w) 2 \sinh\left(\frac{x-y}{2}\right) dy \\
 18 & - \epsilon \int_x^0 u^* g'(I_\epsilon) I_\epsilon \epsilon k \left( \int_{-1/\epsilon}^y [e^s + \epsilon w e^{s/2}] ds \right) 2 \sinh\left(\frac{x-y}{2}\right) dy \\
 19 & + \epsilon \mu \int_x^0 (e^{y/2} + \epsilon w) 2 \sinh\left(\frac{x-y}{2}\right) dy + \gamma e^{x/2}, \\
 20 & \\
 21 & \\
 22 & \\
 23 & \\
 24 & \\
 25 & \\
 26 & \\
 27 & \\
 28 & \\
 29 & 
 \end{aligned}$$

30 with  $\gamma \in \mathbb{R}^1$  chosen such that  $\int_{-1/\epsilon}^0 e^{x/2} w(x) dx = 0$ .  
 31

32 For  $w \in H$ , we now define the operator

$$33 \quad 34 \quad L_\epsilon(w)(x) := L_\epsilon^1(w)(x) + L_\epsilon^2(w)(x) + L_\epsilon^3(w)(x) + \gamma(w, \epsilon) e^{x/2},$$

35 with  
 36

$$\begin{aligned}
 37 & L_\epsilon^1(w)(x) := \epsilon \int_x^0 [g(I_\epsilon) - d](e^{y/2} + \epsilon w) 2 \sinh\left(\frac{x-y}{2}\right) dy, \\
 38 & \\
 39 & \\
 40 & \\
 41 & L_\epsilon^2(w)(x) := -\epsilon \int_x^0 u^* g'(I_\epsilon) I_\epsilon \epsilon k \left( \int_{-1/\epsilon}^y [e^s + \epsilon w e^{s/2}] ds \right) 2 \sinh\left(\frac{x-y}{2}\right) dy, \\
 42 & \\
 43 & \\
 44 & \\
 45 & L_\epsilon^3(w)(x) := \epsilon \mu(w, \epsilon) \int_x^0 (e^{y/2} + \epsilon w) 2 \sinh\left(\frac{x-y}{2}\right) dy, \\
 46 & \\
 47 & 
 \end{aligned}$$

1 where  $\mu(w, \epsilon)$  is determined by (4.3), and

$$2 \\ 3 \quad \gamma(w, \epsilon) = - \left[ \int_{-1/\epsilon}^0 e^x dx \right]^{-1} \int_{-1/\epsilon}^0 e^{x/2} [L_\epsilon^1(w)(x) + L_\epsilon^2(w)(x) + L_\epsilon^3(w)(x)] dx. \\ 4 \\ 5 \\ 6$$

7 We are going to show that there exists  $C > 0$  such that for every small  $\epsilon > 0$ ,  $L_\epsilon$  maps  $B :=$   
8  $\{w \in H : \|w\| \leq 2C\}$  into itself, and is a contraction mapping. Therefore  $L_\epsilon$  has a unique fixed  
9 point  $w_\epsilon$  in  $B$ :  $L_\epsilon(w_\epsilon) = w_\epsilon$ . Clearly this gives an eigenpair to (4.1):  
10

$$11 \quad (\lambda, \phi) = (\epsilon^2 \mu(w_\epsilon, \epsilon), e^{x/2} + \epsilon w_\epsilon). \\ 12$$

13 From (4.3), we easily obtain

$$14 \quad |\mu(w, \epsilon)| \leq C(1 + \epsilon \|w\|) \quad \text{for all small } \epsilon > 0 \text{ and some } C > 0. \\ 15 \\ 16$$

17 Clearly

$$18 \quad e^{x/2} \left| \int_x^0 (e^{y/2} + \epsilon w) 2 \sinh \left( \frac{x-y}{2} \right) dy \right| \leq \int_x^0 [e^y + \epsilon |w| e^{y/2}] |e^{x-y} - 1| dy \leq C(1 + \epsilon \|w\|). \\ 19 \\ 20 \\ 21 \\ 22 \\ 23$$

24 It follows that

$$25 \quad \|L_\epsilon^3(w)\| = \int_{-1/\epsilon}^0 e^{x/2} |L_\epsilon^3(w)(x)| dx \leq C(1 + \epsilon \|w\|)^2. \\ 26 \\ 27 \\ 28 \\ 29$$

30 Using

$$31 \quad e^{x/2} |L_\epsilon^1(w)(x)| = \epsilon e^{x/2} \left| \int_x^0 [g(I_\epsilon) - d] (e^{y/2} + \epsilon w) (e^{\frac{x-y}{2}} - e^{\frac{y-x}{2}}) dy \right| \\ 32 \\ 33 \quad \leq \epsilon \int_x^0 [g(I_\epsilon) + d] (e^y + \epsilon |w| e^{y/2}) |e^{x-y} - 1| dy \\ 34 \\ 35 \\ 36 \\ 37 \\ 38 \\ 39 \\ 40 \\ 41 \\ 42 \\ 43$$

$$44 \quad \leq \epsilon C \int_{-1/\epsilon}^0 (e^y + \epsilon |w| e^{y/2}) dy \leq \epsilon C(1 + \epsilon \|w\|),$$

45 we deduce

$$46 \quad \|L_\epsilon^1(w)\| \leq C(1 + \epsilon \|w\|). \\ 47$$

1 To estimate  $L_\epsilon^2(w)$  we notice that

$$3 \quad \left| \int_{-1/\epsilon}^y [e^s + \epsilon w e^{s/2}] ds \right| \leq C(1 + \epsilon \|w\|) \quad \text{for all } y \in (-1/\epsilon, 0),$$

$$7 \quad |u^* g'(I_\epsilon) I_\epsilon \epsilon k| \leq C(e^{y/2} + \epsilon |z_i^\epsilon(y)|),$$

8 and hence

$$10 \quad e^{x/2} |L_\epsilon^2(w)(x)| \leq \epsilon C(1 + \epsilon \|w\|) \int_x^0 (e^y + \epsilon |z_i^\epsilon(y)| e^{y/2}) |e^{x-y} - 1| dy \leq \epsilon C(1 + \epsilon \|w\|).$$

14 It follows that

$$16 \quad \|L_\epsilon^2(w)\| \leq C(1 + \epsilon \|w\|).$$

18 From the definition of  $\gamma(w, \epsilon)$  we obtain

$$20 \quad \|\gamma(w, \epsilon) e^{x/2}\| \leq \|L_\epsilon^1(w) + L_\epsilon^2(w) + L_\epsilon^3(w)\| \leq C(1 + \epsilon \|w\|)^2.$$

23 Therefore

$$24 \quad \|L_\epsilon(w)\| \leq C(1 + \epsilon \|w\|)^2.$$

27 If  $\|w\| \leq 2C$ , we obtain

$$28 \quad \|L_\epsilon(w)\| \leq C(1 + 2\epsilon C)^2 \leq 2C$$

31 provided that  $\epsilon \in (0, \epsilon_0]$ , with  $\epsilon_0$  determined by  $(1 + 2\epsilon_0 C)^2 = 2$ .

32 Next we show that  $L_\epsilon$  is a contraction mapping on  $B \subset H$  for all small  $\epsilon > 0$ . Let  $w_1, w_2 \in B$ .  
33 Then

$$34 \quad 35 \quad 36 \quad 37 \quad L_\epsilon^1(w_1) - L_\epsilon^1(w_2) = \epsilon \int_x^0 [g(I_\epsilon) - d] \epsilon (w_1 - w_2) 2 \sinh\left(\frac{x-y}{2}\right) dy.$$

38 It follows easily that

$$40 \quad 41 \quad \|L_\epsilon^1(w_1) - L_\epsilon^1(w_2)\| \leq \epsilon C \|w_1 - w_2\|.$$

43 Similarly

$$44 \quad 45 \quad 46 \quad 47 \quad L_\epsilon^2(w_1) - L_\epsilon^2(w_2) = -\epsilon \int_x^0 u^* g'(I_\epsilon) I_\epsilon \epsilon k \left( \int_{-1/\epsilon}^y \epsilon (w_1 - w_2) e^{s/2} ds \right) 2 \sinh\left(\frac{x-y}{2}\right) dy,$$

1 which gives  
2

$$3 \|L_\epsilon^2(w_1) - L_\epsilon^2(w_2)\| \leq \epsilon C \|w_1 - w_2\|. \\ 4$$

5 Finally  
6

$$7 \\ 8 L_\epsilon^3(w_1) - L_\epsilon^3(w_2) = \epsilon \mu(w_1, \epsilon) \int_x^0 \epsilon(w_1 - w_2) 2 \sinh\left(\frac{x-y}{2}\right) dy \\ 9 \\ 10 + \epsilon [\mu(w_1, \epsilon) - \mu(w_2, \epsilon)] \int_x^0 (e^{y/2} + \epsilon w_2) 2 \sinh\left(\frac{x-y}{2}\right) dy. \\ 11 \\ 12$$

13 By (4.3) we easily see that  
14

$$15 \\ 16 |\mu(w_2, \epsilon) - \mu(w_1, \epsilon)| \leq \epsilon C \|w_1 - w_2\|. \\ 17$$

18 It then follows easily that  
19

$$20 \\ 21 \|L_\epsilon^3(w_1) - L_\epsilon^3(w_2)\| \leq \epsilon C \|w_1 - w_2\|. \\ 22$$

23 We now obtain  
24

$$25 \\ 26 \|\gamma(w_1, \epsilon) e^{x/2} - \gamma(w_2, \epsilon) e^{x/2}\| \\ 27 \\ 28 \leq \|L_\epsilon^1(w_1) - L_\epsilon^1(w_2)\| + \|L_\epsilon^1(w_1) - L_\epsilon^1(w_2)\| + \|L_\epsilon^3(w_1) - L_\epsilon^3(w_2)\| \\ 29 \\ 30 \leq \epsilon C \|w_1 - w_2\|. \\ 31 \\ 32$$

33 Thus we have  
34

$$35 \\ 36 \|L_\epsilon(w_1) - L_\epsilon(w_2)\| \leq \epsilon C \|w_1 - w_2\| \leq \frac{1}{2} \|w_1 - w_2\| \quad \text{for all } w_1, w_2 \in B \\ 37$$

38 provided that  $\epsilon > 0$  is small enough, say  $0 < \epsilon \leq \epsilon_1 \leq \epsilon_0$ .  
39

40 We may now use the contraction mapping theorem to conclude that for every small  $\epsilon > 0$ ,  
41 say  $\epsilon \in (0, \epsilon_1]$ ,  $L_\epsilon$  has a unique fixed point  $w_\epsilon$  in  $B$ :  $L_\epsilon(w_\epsilon) = w_\epsilon$ . It follows that, for such  $\epsilon$ ,  
(4.1) has an eigenpair of the form  $(\lambda, \phi) = (\epsilon^2 \mu(w_\epsilon, \epsilon), e^{x/2} + \epsilon w_\epsilon)$ .

42 Let us now determine the sign of  $\mu(w_\epsilon, \epsilon)$  for  $u^* = u_1$  and  $u^* = u_2$ , respectively. We will  
43 denote  $\mu(w_\epsilon, \epsilon)$  by  $\mu_\epsilon^i$  for  $u^* = u_i$ ,  $i = 1, 2$ . We define  $I_\epsilon^i(x)$  from  $I_\epsilon(x)$  analogously. From its  
44 definition, we easily see that  
45

$$46 \\ 47 \lim_{\epsilon \rightarrow 0} I_\epsilon^i(x) = e^{-k_0 - \int_{-\infty}^x k \eta_i e^s ds} = \xi_0 e^{-\mu_i e^x} \text{ uniformly in compact subsets of } (-\infty, 0].$$

1 Thus we can use (4.3) to obtain  
 2  
 3

$$\begin{aligned}
 -\lim_{\epsilon \rightarrow 0} \mu_\epsilon^i &= \int_{-\infty}^0 [g(\xi_0 e^{-\mu_i e^x}) - d] e^x dx - \int_{-\infty}^0 \eta_i k e^x g'(\xi_0 e^{-\mu_i e^x}) \xi_0 e^{-\mu_i e^x} e^x dx \\
 &= \int_0^1 g(\xi_0 e^{-\mu_i s}) ds - d - \int_0^1 \xi_0 \mu_i s e^{-\mu_i s} g'(\xi_0 e^{-\mu_i s}) ds \\
 &= G(\mu_i) - d + \mu_i G'(\mu_i) = \mu_i G'(\mu_i).
 \end{aligned}$$

12 Therefore  
 13

$$\mu_\epsilon^1 = -\mu_1 G'(\mu_1) + o_\epsilon(1) < 0, \quad \mu_\epsilon^2 = -\mu_2 G'(\mu_2) + o_\epsilon(1) > 0.$$

16 Summarizing, we have proved the following result:  
 17

19 **Theorem 4.1.** Let  $u_1$  and  $u_2$  be given by Theorem 3.6. Then for each small  $\epsilon > 0$ , the linearized  
 20 eigenvalue problem (4.1) with  $u^* = u_i$  has an eigenpair of the form  
 21

$$(\lambda, \phi) = (\epsilon^2 \mu_\epsilon^i, e^{x/2} + \epsilon w_\epsilon^i), \quad i = 1, 2,$$

24 with  $w_\epsilon^i \in H$  having a uniform bound independent of  $\epsilon$ , and  
 25

$$\mu_\epsilon^1 = -\mu_1 G'(\mu_1) + o_\epsilon(1) < 0, \quad \mu_\epsilon^2 = -\mu_2 G'(\mu_2) + o_\epsilon(1) > 0.$$

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**4Q8** Australian Research Council, *country*=Australia, *grants*=

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