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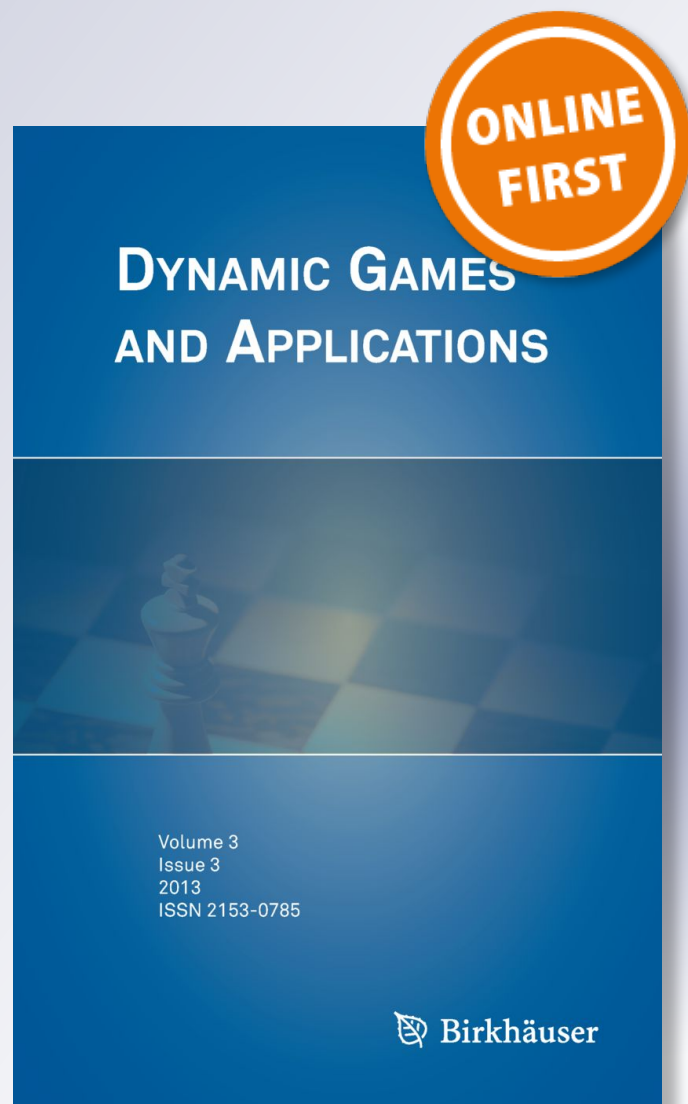
René Carmona & Christy V. Graves

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Jet Lag Recovery: Synchronization of Circadian Oscillators as a Mean Field Game

René Carmona¹ · Christy V. Graves¹ 

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Abstract

A mean field game is proposed for the synchronization of oscillators facing conflicting objectives. Our motivation is to offer an alternative to recent attempts to use dynamical systems to illustrate some of the idiosyncrasies of jet lag recovery. Our analysis is driven by two goals: (1) to understand the long time behavior of the oscillators when an individual remains in the same time zone, and (2) to quantify the costs from jet lag recovery when the individual has traveled across time zones. Finite difference schemes are used to find numerical approximations to the mean field game solutions. They are benchmarked against explicit solutions derived for a special case. Numerical results are presented and conjectures are formulated. The numerics suggest that the cost the oscillators accrue while recovering is larger for eastward travel which is consistent with the widely admitted wisdom that *jet lag is worse after traveling east than west*.

Keywords Mean field game · Mean field control · Jet lag · Synchronization · Oscillators · Partial differential equations · Explicit solutions · Perturbation analysis · Numerical results · Ergodic

1 Introduction

Circadian rhythm refers to the oscillatory behavior of certain biological processes occurring with a period close to 24 h. Recent interest in these biological processes has spawned from the Nobel Prize winning work of Hall, Rosbash, and Young, who discovered molecular mechanisms for controlling the circadian rhythm in fruit flies [20]. Examples of circadian rhythms in animals include sleep/wake patterns, eating schedules, bodily temperatures, hormone production, and brain activity. These oscillations can be entrained to the 24 h cycle of

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✉ Christy V. Graves
cjvaughn@princeton.edu

René Carmona
rcarmona@princeton.edu

¹ ORFE & PACM, Princeton University, Princeton, NJ 08544, USA

sunlight exposure. Abrupt disruptions of such circadian rhythms can occur, such as when an individual travels across time zones, resulting in jet lag.

The suprachiasmatic nucleus (SCN) is a region in the brain that is responsible for controlling circadian rhythms [13,19]. The SCN contains on the order of 10^4 neuronal oscillator cells, each of which has a preferred frequency corresponding to a period slightly longer than 24 h [21]. The sheer size of the SCN suggests a mean field model to work with effective equations capturing macroscopic features of the system. We model the oscillators as rational agents attempting to synchronize with each other as well as the natural 24-h sunlight cycle, while at the same time, minimizing their effort. It is clearly unrealistic to assume that the SCN oscillators are rational agents. Still, the merit of this approach is that the equilibrium dynamics of the oscillators arise endogenously from simple basic principles. This is in contrast with many *agent-based* models who essentially specify the equilibrium dynamics directly.

While we will focus on the case of the search for a distributed Nash equilibrium, the case of cooperative oscillators implementing the control prescribed by a central planner, leading to a mean field control problem, is almost equivalent. Indeed, since the cost function we use is in the form of a potential (see section 6.7.2 in [11]) the solutions of the two problems are closely related. This point is the object of Proposition 1.

Quite naturally, we use the mean field game (MFG) paradigm as originally introduced independently by Lasry and Lions [16] and Caines et al. [14]. Working with frequencies, our model is naturally set on a torus. In such a case, existence of classical solutions is shown in [16], with more detail in Bardi and Feleqi [6] and Cardaliaguet's notes based on Lions' lectures [8]. These proofs rely on compactness arguments and Schauder's fixed point theorem. Uniqueness, on the other hand, is hard to prove for mean field games. Typically, uniqueness arguments are based on the so-called Lasry–Lions monotonicity condition [16]. The rationale for our model also relies on the connection between finite horizon and infinite horizon models as studied, for example, in the papers of Cardaliaguet, Lasry, Lions, and Porretta, in which they prove the convergence of the solution to the former to the solution of the latter as the horizon tends to infinity [9,10]. Finally, we mention that finite difference schemes for numerically solving the partial differential equation (PDE) formulations of mean field games are discussed in the works of Achdou and collaborators [1–4].

Here, our goal is to model stylized facts commonly known about jet lag recovery. To this end, we concentrate on: (1) the long time behavior of the neuronal oscillator cells for an individual that is entrained to the 24-h light/dark rhythm, and (2) how the oscillators resynchronize to a shifted 24-h light/dark rhythm after travel across time zones.

This project was inspired by [17] which proposes a model of jet lag recovery for SCN oscillators using the classical Kuramoto model. In this model, each oscillator has a random preferred frequency, and their phases evolve forward in time according to deterministic coupled ordinary differential equations (ODEs). By making an ansatz and turning to the limit as the number of oscillators tends to infinity, the authors reduce the dynamics of the system to an ODE for a complex order parameter. From there, they argue for a larger recovery time for eastward travel. Our model differs from theirs in several ways. In particular, the system is modeled as a stochastic differential game and the macroscopic behavior of the system occurs endogenously. The thrust of our analysis is to identify and compute the Nash equilibria of that model.

We know of one instance in which the synchronization of oscillators was approached as a mean field game. It is in the work of Yin et al. [23] in which the oscillators choose their control to minimize a cost objective which encourages synchronization of the oscillators with each other. Our model is different because it includes an external forcing term. While we also wish to see the oscillators synchronize with each other, they also need to synchronize

with the natural 24-h light/dark cycle which is crucial for SCN oscillators. When interpreted in terms of jet lag recovery, our main finding is that there is a larger cost associated with recovery from jet lag after traveling east.

The paper is organized as follows: Our problem formulation is described in Sect. 2. Our model of long time behavior of the oscillators is provided in Sect. 3 where we also derive an explicit solution for a special case of the model parameters. Section 4 details our model for resynchronization after travel. Jet lag recovery is described in Sect. 4.2 where we provide a few notions of jet lag recovery time and jet lag recovery cost. We use finite difference schemes to find numerical approximations to the solutions. Numerical results and conjectures are presented in Sect. 5. Section 6 concludes the paper.

2 Model Formulation

2.1 N Player Game Formulation

We model the SCN as N coupled oscillators, $i \in \{1, \dots, N\}$, with at time t , phases Θ_t^i in the one-dimensional torus $\mathbb{T} = [0, 2\pi)$. Without any external stimuli, SCN oscillators are believed to have an intrinsic frequency ω^i corresponding to a period slightly longer than 24 h [21]. We will take this frequency to be the same for all of the oscillators: $\omega^i = \omega_0 \in \mathbb{R}^+$, $\forall i \in \{1, \dots, N\}$. Note that, this differs from the formulation in [17] where the ω^i needed to be independent and identically distributed random variable with a non-trivial common distribution. We assume that for each oscillator i , the phase Θ_t^i evolves according to the equation:

$$d\Theta_t^i = (\omega_0 + \alpha_t^i)dt + \sigma dB_t^i, \quad (1)$$

where the B_t^i 's are independent standard Wiener processes, $\sigma > 0$ is a constant independent of i , and the processes $\alpha^i = (\alpha_t^i)_{t \geq 0} \in \mathbb{A}^i$ are expected to provide a Nash equilibrium for the long run average (LRA) costs:

$$\tilde{J}^i(\alpha) := \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \left(\frac{1}{2} (\alpha_t^i)^2 + K \bar{c}(\Theta_t^i, \mu^N(t, \cdot)) + F \tilde{c}_{\text{sun}}(t, \Theta_t^i, \rho(t)) \right) dt \right], \quad (2)$$

where we use the notation $\alpha = (\alpha^1, \dots, \alpha^N)$ and $\mathbb{A}^i$ is the set of locally square integrable stochastic processes that are adapted to B_t^i for $i = 1, \dots, N$. Also, $\mu^N(t, \cdot) = \frac{1}{N} \sum_{j=1}^N \delta_{\Theta_t^j}$, is the empirical measure of the phases of the N oscillators at time $t \geq 0$.

The aggregate cost over the time interval $[0, T]$ appearing in (2) has three components which we now explain. The term $(\alpha_t^i)^2/2$ penalizes the efforts to change the dynamics (1) of the phase through the control of its drift. The goal of the second term is to encourage synchronization of the oscillators with each other through their empirical measure. To be specific, we use the cost function $\bar{c} : [0, 2\pi) \times \mathcal{P}([0, 2\pi)) \ni (\theta, \nu) \mapsto \bar{c}(\theta, \nu) \in \mathbb{R}$ given by:

$$\bar{c}(\theta, \nu) := \frac{1}{2} \int_0^{2\pi} \sin^2 \left(\frac{\theta' - \theta}{2} \right) d\nu(\theta'). \quad (3)$$

Here, $\mathcal{P}([0, 2\pi])$ denotes the space of probability measures on $[0, 2\pi]$. Thus, we have:

$$\bar{c}(\Theta_t^i, \mu^N(t, \cdot)) := \frac{1}{2N} \sum_{j=1}^N \sin^2 \left(\frac{\Theta_t^j - \Theta_t^i}{2} \right).$$

In other words, oscillator i has a cost associated with the difference between its phase and the phase of all of the other oscillators. Finally, the third term is intended to encourage alignment of the oscillators' phases with the natural 24-h sunlight cycle. It is given in terms of the cost function $\tilde{c}_{\text{sun}} : \mathbb{R}^+ \times [0, 2\pi]^2 \rightarrow \mathbb{R}$ defined as:

$$\tilde{c}_{\text{sun}}(t, \theta, \rho(t)) := \frac{1}{2} \sin^2 \left(\frac{\omega_S t + \rho(t) - \theta}{2} \right),$$

where $\omega_S = 2\pi/24$ radians per hour is the frequency of the 24-h sunlight cycle, and $\rho : \mathbb{R}^+ \rightarrow [-\pi, \pi]$ is a phase shift accounting for the longitude (in radians) of the traveler's location at time t . $\rho(t) \equiv p \in [-\pi, \pi]$ models an individual that stays in their time zone forever, whereas ρ increases for eastward travel and decreases for westward travel. For example, if an individual travels from longitude 0 to longitude $p > 0$, they have traveled roughly p/ω_S time zones east. The quantity $\omega_S t + \rho(t)$ represents the phase of the sun at time t in longitude $\rho(t)$. In other words, oscillator i incurs a cost for the difference between its phase and the phase $\omega_S t + \rho(t)$. The constants, $K \geq 0$ and $F \geq 0$, are used to weigh the three components of the cost.

2.2 Mean Field Game Formulation

By considering the limit $N \rightarrow \infty$, we reformulate the problem as a mean field game:

- (1) Fix a deterministic flow of measures, $\mu = (\mu(t, \cdot))_{t \geq 0}$.
- (2) Solve the standard control problem of finding $\alpha^\mu = (\alpha_t^\mu)_{t \geq 0} \in \mathbb{A}$ to minimize the LRA cost:

$$\tilde{J}^\mu(\alpha) = \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \alpha_t^2 + K \bar{c}(\Theta_t^\alpha, \mu(t, \cdot)) + F \tilde{c}_{\text{sun}}(t, \Theta_t, \rho(t)) \right) dt \right],$$

subject to the dynamical constraint:

$$d\Theta_t^\alpha = (\omega_0 + \alpha_t)dt + \sigma dB_t,$$

where B_t is a Wiener process.

- (3) Find a flow $\mu = (\mu(t, \cdot))_{t \geq 0}$ such that for every $t > 0$, $\mu(t, \cdot) = \mathcal{L}(\Theta_t^{\alpha^\mu})$.

In the fixed point step of bullet point (3) above, we use the notation $\mathcal{L}(\cdot)$ for the law of a random variable. The convergence of the N player game to the mean field game as formulated above is the major underpinning of the theory of mean field games. It will not be addressed here, but for the remainder of the paper, we will work with the mean field game.

2.3 Change of Variables

For now, $\rho(t) \equiv p$ will be constant. Note that, the time dependency of $\tilde{c}_{\text{sun}}$ can be removed if we make the change of variables: $\Phi_t = \Theta_t - \omega_S t$, where Φ_t is also in the periodic domain $[0, 2\pi]$. Now the optimal control part of the mean field game is the following. The phase of

a generic oscillator evolves according to:

$$d\phi_t = (\omega_0 - \omega_S + \alpha_t)dt + \sigma dB_t, \quad (4)$$

where $\alpha \in \mathbb{A}$ is chosen to minimize the LRA cost:

$$J^\mu(\alpha) = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\frac{1}{2} \alpha_t^2 + K \bar{c}(\Phi_t, \mu(t, \cdot)) + F c_{\text{sun}}(\Phi_t, p) \right) dt, \quad (5)$$

where $\bar{c}$ is as defined in Eq. (3), and:

$$c_{\text{sun}}(\phi, p) := \frac{1}{2} \sin^2 \left(\frac{p - \phi}{2} \right). \quad (6)$$

Now that, the cost functions no longer depend on t , we can address our first problem of interest: to understand the long time behavior of the oscillators when an individual remains in the same time zone. In fact, this formulation will also help us with the second problem of interest: to understand how the oscillators recover when the individual has traveled across time zones. Thus, for the remainder of the paper, we will refer to Φ_t as the *phase* of a representative oscillator at time t .

3 Long Time Behavior in One Time Zone

When the longitude p of an individual does not change for a long time, we expect the distribution of their oscillators to offer a stationary solution to the mean field game posed in Eqs. (4)–(6). Using the analytic (PDE) approach to solving mean field games, this stationary solution is given by $(\mu_p^*, V_p^*, \lambda_p^*)$ solving the coupled ergodic Hamilton–Jacobi–Bellman (HJB) and Poisson equations [16]:

$$\begin{cases} (\omega_0 - \omega_S) \partial_\phi V_p^* - \frac{1}{2} (\partial_\phi V_p^*)^2 + \frac{\sigma^2}{2} \partial_{\phi\phi}^2 V_p^* &= \lambda_p^* - K \bar{c}(\phi, \mu_p^*(\cdot)) - F c_{\text{sun}}(\phi, p), \\ (\omega_0 - \omega_S) \partial_\phi \mu_p^* - \partial_\phi [\mu_p^* (\partial_\phi V_p^*)] - \frac{\sigma^2}{2} \partial_{\phi\phi}^2 \mu_p^* &= 0, \end{cases}$$

over the periodic domain $[0, 2\pi)$. Note that, μ^* plays the role of the distribution of the phases of the oscillators, while V_p^* plays the role of the value function giving the expected cost that an oscillator at a given position will accumulate during a unit of time. Since V_p^* is defined up to a constant, without loss of generality, we add the constraint $\int_0^{2\pi} V_p^*(\phi) d\phi = 0$. Also, μ_p^* should be a probability measure, so we require $\mu_p^*(\phi) \geq 0$ for all ϕ and $\int_0^{2\pi} d\mu_p^*(\phi) = 1$. Note that, λ_p^* is given by the ergodic average cost, i.e.,

$$\lambda_p^* = \int_0^{2\pi} \left[\frac{1}{2} (\partial_\phi V_p^*)^2 + K \bar{c}(\phi, \mu_p^*(\cdot)) + F c_{\text{sun}}(\phi, p) \right] d\mu_p^*(\phi).$$

If we denote by (μ^*, V^*, λ^*) without subscripts the solution to the above PDE system when $p = 0$, it is easy to check that for $p \neq 0$, $(\mu_p^*(\cdot), V_p^*(\cdot), \lambda_p^*) := (\mu^*(\cdot - p), V^*(\cdot - p), \lambda^*)$ is a solution to the above PDE system. Therefore, if we can find the solution of this system for $p = 0$, then we have simultaneously solved for the long time behavior in any longitude p . Thus, without loss of generality, let $p = 0$ and pose the *Ergodic Mean Field Game (MFG) Problem*.

Definition 1 *Ergodic MFG Problem:* Find (μ^*, V^*, λ^*) solving

$$(\omega_0 - \omega_S) \partial_\phi V^* - \frac{1}{2} (\partial_\phi V^*)^2 + \frac{\sigma^2}{2} \partial_{\phi\phi}^2 V^* = \lambda^* - K \bar{c}(\phi, \mu^*) - F c_{\text{sun}}(\phi, 0), \quad (7)$$

$$(\omega_0 - \omega_S) \partial_\phi \mu^* - \partial_\phi [\mu^* (\partial_\phi V^*)] - \frac{\sigma^2}{2} \partial_{\phi\phi}^2 \mu^* = 0, \quad (8)$$

$$\int_0^{2\pi} V^*(\phi) d\phi = 0, \quad (9)$$

$$\mu^*(\phi) \geq 0, \quad \int_0^{2\pi} d\mu^*(\phi) = 1, \quad (10)$$

on the periodic domain $\phi \in [0, 2\pi)$, where $\bar{c}$ is defined in Eq. (3) and c_{sun} in (6).

We now pause to elaborate on a comment made in the introduction. We have thus far modeled the SCN oscillators as selfish players, which leads to the *Ergodic MFG Problem* posed above. If we instead considered cooperative players that are willing to follow the instructions of a central planner, and replace the weight K by a new parameter κ , then the central planner would consider the following optimal control problem of McKean–Vlasov type:

Find $\alpha : [0, 2\pi) \rightarrow \mathbb{R}$ minimizing:

$$J(\alpha) = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\frac{1}{2} \alpha_t^2 + \kappa \bar{c}(\Phi_t, \mathcal{L}(\Phi_t)) + F c_{\text{sun}}(\Phi_t, 0) \right) dt,$$

under the dynamical constraint (4).

In the introduction, we claimed that the two modeling formulations are almost equivalent. We make the connection more precise in the following proposition:

Proposition 1 *The optimal control for the Ergodic MFG Problem is an optimal control for the McKean–Vlasov control problem posed above when $\kappa = \frac{K}{2}$. The converse also holds.*

Proof Using the analytic (PDE) approach to optimal control of McKean–Vlasov type, we recover exactly the system (7)–(10) when $\kappa = \frac{K}{2}$. See section 6.7.2 in [11]. $\square$

Theorem 1 *For any solution (μ^*, V^*, λ^*) to the Ergodic MFG Problem, μ^* has a density given by:*

$$\mu^*(\phi) = \frac{1}{z \zeta(\phi)} \left[\int_0^{2\pi} \zeta(\phi') d\phi' + \left(e^{-\frac{4\pi}{\sigma^2}(\omega_0 - \omega_S)} - 1 \right) \int_0^\phi \zeta(\phi') d\phi' \right], \quad \forall \phi \in [0, 2\pi),$$

where $\zeta(\phi) := e^{-\frac{2}{\sigma^2}[(\omega_0 - \omega_S)\phi - V^*(\phi)]}$ and $z \in \mathbb{R}^+$ is a constant such that μ^* integrates to 1.

Proof Given V^* , there is a unique solution to Eq. (8). One can check that the above density satisfies Eq. (8) with periodic boundary. $\square$

Remark 1 As a consequence of Theorem 1, we have the below Corollary 1, which states that the search for a solution to the coupled system defined by the *Ergodic MFG Problem* is equivalent to the search for a solution to a single integro-differential equation. Although this corollary is not used in the sequel, we note it here because this type of result seems to be uncommon in the mean field games literature.

Corollary 1 For any solution (μ^*, V^*, λ^*) to the Ergodic MFG Problem, we can write $V^*(\phi) = (\omega_0 - \omega_S)\phi + \frac{\sigma^2}{2} \log(\zeta(\phi))$ and $\mu^*(\phi) = v(\phi, \zeta)$, where $\zeta : [0, 2\pi) \rightarrow \mathbb{R}$ is a solution to the following integro-differential equation:

$$\zeta''\zeta - \frac{3}{2}(\zeta')^2 = \frac{4}{\sigma^4}\zeta^2 \left[\lambda^* - K\bar{c}(\phi, v(\phi, \zeta)) - Fc_{\text{sun}}(\phi, 0) - \frac{1}{2}(\omega_0 - \omega_S)^2 \right], \quad \phi \in [0, 2\pi) \quad (11)$$

with boundary constraint:

$$\zeta(2\pi) = e^{-\frac{4\pi}{\sigma^2}(\omega_0 - \omega_S)}\zeta(0), \quad \text{and} \quad \zeta'(2\pi) = e^{-\frac{4\pi}{\sigma^2}(\omega_0 - \omega_S)}\zeta'(0),$$

and with, without loss of generality, the normalization constraint:

$$\int_0^{2\pi} (\omega_0 - \omega_S)\phi + \frac{\sigma^2}{2} \log(\zeta(\phi)) d\phi = 0,$$

and where:

$$v(\phi, \zeta) := \frac{\frac{1}{\zeta(\phi)} \left[\int_0^{2\pi} \zeta(\phi') d\phi' + \left(e^{-\frac{4\pi}{\sigma^2}(\omega_0 - \omega_S)} - 1 \right) \int_0^\phi \zeta(\phi') d\phi' \right]}{\int_0^{2\pi} \frac{1}{\zeta(\phi)} \left[\int_0^{2\pi} \zeta(\phi') d\phi' + \left(e^{-\frac{4\pi}{\sigma^2}(\omega_0 - \omega_S)} - 1 \right) \int_0^\phi \zeta(\phi') d\phi' \right] d\phi}, \quad \forall \phi \in [0, 2\pi).$$

Conversely, if (ζ, λ^*) is a solution to (11) satisfying $\zeta(\phi) > 0$ for all $\phi \in [0, 2\pi)$, then for $V^*(\phi) := (\omega_0 - \omega_S)\phi + \frac{\sigma^2}{2} \log(\zeta(\phi))$ and $\mu^*(\phi) := v(\phi, \zeta)$, (μ^*, V^*, λ^*) is a solution to the Ergodic MFG Problem.

The following definition of *entrainment* will help us formulate the problem of the recovery from jet lag.

Definition 2 An individual is *entrained* to the longitude p at time t if the phases of the oscillators are distributed according to $\mu^*(\cdot - p)$ for all $s \geq t$, where μ^* solves the Ergodic MFG Problem.

In other words, $\mathcal{L}(\Phi_s) = \mu^*(\cdot - p)$ for all $s \geq t$. Note that, the optimal control is given by the feedback function [16]:

$$\alpha^*(\cdot) := -\partial_\phi V^*(\cdot). \quad (12)$$

Theorem 2 There exists a classical solution to the Ergodic MFG Problem defined by equations (7)–(10). If $K = 0$, the solution is unique. If $K > 0$, the Lasry–Lions monotonicity condition does not hold. If $K = 1$, $\omega_0 = \omega_S$, and $F = 0$, uniqueness does not hold.

Proof Existence follows from the original result of Lasry and Lions [16]. A detailed proof can be found in Bardi and Feleqi [6]. See Theorem 2.1 therein. When $K = 0$, i.e., no weight is given to the cost c_{osc} , there is no interaction between the oscillators, and it is easy to see that the Lasry–Lions monotonicity condition holds. If $K > 0$, the Lasry–Lions monotonicity does not hold due to the cost $\bar{c}(\phi, \mu^*(\cdot))$. Indeed, the following provides a counterexample: if $\mu_1 = \delta_{x_1}$ and $\mu_2 = \delta_{x_2}$ for $x_1 \neq x_2$, then $\int_0^{2\pi} (\bar{c}(\phi, \mu_1) - \bar{c}(\phi, \mu_2))(\mu_1 - \mu_2) d\phi = -\sin^2(\frac{x_1 - x_2}{2}) < 0$. Finally, if $K = 1$, $\omega_0 = \omega_S$, and $F = 0$, we recover the model used in [23] (where the parameter ω is given by $\omega := \omega_0 - \omega_S = 0$), and for which the authors identify multiple solutions. $\square$

Theorem 3 For a given solution (μ^*, V^*, λ^*) to the Ergodic MFG Problem, the state dynamics:

$$d\phi_t = (\omega_0 - \omega_S + \alpha^*(\Phi_t))dt + \sigma dB_t, \quad (13)$$

where α^* , defined by Eq. (12), is ergodic and $\mathcal{L}(\Phi_t) \rightarrow \mu^*(\cdot)$ as $t \rightarrow \infty$ at an exponential rate.

Proof Because of the ellipticity of Eq. (13), recall $\sigma > 0$, the transition probabilities have strictly positive densities (see, for example, the discussion on page 28 in [5] which refers to [7]). Thus, since the state space of our process is compact, the process is ergodic and the invariant measure is unique. See [15] for a proof and a discussion of the Doeblin stronger form of ergodicity and exponential convergence. $\square$

3.1 A Special Case

In this section, we consider the special case $K = 0$ and $\omega_0 = \omega_S$ for which we can compute an explicit solution. But first we introduce some notation related to Mathieu's differential equation:

$$\partial_{xx}^2 f + [a - 2q \cos(2x)]f = 0 \quad (14)$$

where $a, q \in \mathbb{R}$. Let $M(a, q, x)$ denote the unique even solution to Eq. (14) with the normalization constraint $M(a, q, 0) = 1$, and let $\mathcal{C}(q)$ be the zeroth order characteristic value of q . Note that, $M(\mathcal{C}(q), q, x)$ is nonnegative and periodic with period π [18]. We have the following result.

Theorem 4 In the special case $K = 0$ and $\omega_0 = \omega_S$, the unique solution to the Ergodic MFG Problem, denoted $(\mu^{*,0}, V^{*,0}, \lambda^{*,0})$ in this case, is given by:

$$\begin{aligned} \partial_\phi V^{*,0}(\phi) &= -\sigma^2 \partial_\phi \left[\log \left(M \left(\mathcal{C} \left(-\frac{F}{\sigma^4} \right), -\frac{F}{\sigma^4}, \frac{\phi}{2} \right) \right) \right], \\ \lambda^{*,0} &= \frac{F}{4} + \frac{\sigma^4}{8} \mathcal{C} \left(-\frac{F}{\sigma^4} \right), \\ \mu^{*,0}(\phi) &= \frac{1}{c_1} \left[M \left(\mathcal{C} \left(-\frac{F}{\sigma^4} \right), -\frac{F}{\sigma^4}, \frac{\phi}{2} \right) \right]^2, \end{aligned}$$

where c_1 is the normalization constant:

$$c_1 := \int_0^{2\pi} \left[M \left(\mathcal{C} \left(-\frac{F}{\sigma^4} \right), -\frac{F}{\sigma^4}, \frac{\phi}{2} \right) \right]^2 d\phi.$$

Proof The ergodic HJB can be linearized through the Cole-Hopf transformation:

$$f(x) = e^{-V^{*,0}(2x)/\sigma^2}. \quad (15)$$

Thus, when $K = 0$ and $\omega_0 = \omega_S$, Eq. (7) becomes:

$$\partial_{xx}^2 f + \left[\frac{8}{\sigma^4} \left(\lambda^{*,0} - \frac{F}{4} \right) + 2 \cdot \frac{F}{\sigma^4} \cos(2x) \right] f = 0. \quad (16)$$

Since $V^{*,0}$ is periodic with period 2π , and with Eq. (15), we want a solution to Eq. (16) which is nonnegative with period π . This is given by:

$$f(x) = c_2 M \left(\mathcal{C} \left(-\frac{F}{\sigma^4} \right), -\frac{F}{\sigma^4}, x \right),$$

where $\lambda^{*,0} = \frac{F}{4} + \frac{\sigma^4}{8} \mathcal{C} \left(-\frac{F}{\sigma^4} \right)$.

The constant $c_2 > 0$ can be chosen to satisfy the normalization constraint (9) and does not affect $\partial_\phi V^{*,0}$. Finally, it can be verified that $\mu^{*,0}(\phi) = \frac{f(\phi/2)^2}{\int_0^{2\pi} f(\phi'/2)^2 d\phi'}$ solves Eqs. (8) and (10). $\square$

Remark 2 After completing the first version of this paper, the authors were made aware of the paper of Yin et al. [22], where a similar connection is made between the solution to a mean field game and an eigenvalue problem, analogous to finding the appropriate characteristic value of a in Eq. (14). However, [22] does not directly make the connection with Mathieu's equation.

3.2 Perturbation of the Special Case: Small $K > 0$

In this section, we consider the case $\omega_0 = \omega_S$ and small $K > 0$, which corresponds to weak interaction between the oscillators. Similar to the methodology of Chan and Sircar [12], we expand our solution (μ^*, V^*, λ^*) to the *Ergodic MFG Problem* by the parameter K , i.e., we search for a first-order correction, $(\mu^{*,1}, V^{*,1}, \lambda^{*,1})$, independent of K , such that:

$$\begin{aligned} \mu^*(\phi) &= \mu^{*,0}(\phi) + K\mu^{*,1}(\phi) + \mathcal{O}(K^2), \\ V^*(\phi) &= V^{*,0}(\phi) + KV^{*,1}(\phi) + \mathcal{O}(K^2), \\ \lambda^* &= \lambda^{*,0} + K\lambda^{*,1} + \mathcal{O}(K^2). \end{aligned} \quad (17)$$

We have the following result.

Theorem 5 *The quantities $(\mu^{*,1}, V^{*,1}, \lambda^{*,1})$ such that Eq. (17) holds for all $K \geq 0$, $\phi \in [0, 2\pi)$ are given by:*

$$\partial_\phi V^{*,1}(\phi) = \frac{2}{\sigma^2 \mu^{*,0}(\phi)} \left[c_3 + \int_0^\phi \mu^{*,0}(\phi') \left(\lambda^{*,1} - \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi') \right) d\phi' \right], \quad (18)$$

where

$$c_3 = - \frac{\int_0^{2\pi} \int_0^\phi \frac{\mu^{*,0}(\phi')}{\mu^{*,0}(\phi)} \left(\lambda^{*,1} - \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi') \right) d\phi' d\phi}{\int_0^{2\pi} \frac{1}{\mu^{*,0}(\phi)} d\phi}, \quad (19)$$

and

$$\lambda^{*,1} = \int_0^{2\pi} \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi) d\phi, \quad (20)$$

where $*$ denotes convolution. Finally, we have $\mu^{*,1}(\phi) = \Gamma(\phi)W(\phi)$ where

$$W(\phi) = M \left(\mathcal{C} \left(-\frac{2F}{\sigma^4} \right), -\frac{2F}{\sigma^4}, \phi/2 \right)$$

and where Γ solves:

$$\frac{\sigma^2}{2} \left(\Gamma \partial_{\phi\phi}^2 W - W \partial_{\phi\phi}^2 \Gamma \right) = \frac{2}{\sigma^2} \mu^{*,0} \left(\lambda^{*,1} - \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi) \right). \quad (21)$$

Proof After substituting the expansion from Eq. (17) into the ergodic HJB in Eq. (7), and collecting terms of order K , we arrive at:

$$\frac{\sigma^2}{2} \partial_{\phi\phi}^2 V^{*,1} - \partial_{\phi} V^{*,0} \partial_{\phi} V^{*,1} = \lambda^{*,0} - \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi).$$

Using the integration factor method, we arrive at Eq. (18) for some constant c_3 . Since $V^{*,1}$ is periodic, we must have $\int_0^{2\pi} \partial_{\phi} V^{*,1} d\phi = 0$, which gives Eq. (19) for c_3 .

Next, we substitute the expansion from Eq. (17) into the Poisson Eq. (8). After collecting terms of order K , we arrive at:

$$\partial_{\phi} [\mu^{*,0} \partial_{\phi} V^{*,1} + \mu^{*,1} \partial_{\phi} V^{*,0}] + \frac{\sigma^2}{2} \partial_{\phi\phi}^2 \mu^{*,1} = 0.$$

Next, we use the fact that $\partial_{\phi} V^{*,0} = -\sigma^2 \frac{\partial_{\phi} W}{W}$ and we make the substitution $\mu^{*,1}(\phi) = \Gamma(\phi)W(\phi)$, which leads to:

$$\frac{\sigma^2}{2} \left(\Gamma \partial_{\phi\phi}^2 W - W \partial_{\phi\phi}^2 \Gamma \right) = \partial_{\phi} [\mu^{*,0} \partial_{\phi} V^{*,1}].$$

Substituting Eq. (18), we have Eq. (21). Since Γ , W , and their derivatives are periodic,

$$\int_0^{2\pi} \Gamma \partial_{\phi\phi}^2 W d\phi = \int_0^{2\pi} W \partial_{\phi\phi}^2 \Gamma d\phi,$$

and thus with Eq. (21), we have:

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{2}{\sigma^2} \mu^{*,0}(\phi) \left(\lambda^{*,1} - \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi) \right) d\phi, \\ &= \lambda^{*,1} - \int_0^{2\pi} \left(\frac{1}{2} \sin^2 \left(\frac{\cdot}{2} \right) * \mu^{*,0} \right) (\phi) d\phi, \end{aligned}$$

which gives Eq. (20). $\square$

Remark 3 The reader may notice that Theorem 2 suggests potential non-uniqueness of μ^* , the solution to the *Ergodic MFG Problem* when $K > 0$. Suppose $\omega_0 = \omega_S$ and for a given $K > 0$ there are two solutions, μ_1^* and μ_2^* . Theorems 4 and 5, together with Eq. (17), say the difference between μ_1^* and μ_2^* is at most order of K^2 .

Since we know that uniqueness does not hold when $F = 0$, we will require $F > 0$. From now on, we assume uniqueness holds in order to have clear definitions for jet lag recovery.

4 Jet Lag Recovery

In Section 3, we posed the *Ergodic MFG Problem*, which models the long time behavior of oscillators when an individual remains in their time zone for a long period of time. In other words, this problem models oscillators which are entrained to their local time zone. The second goal of this study is to model the resynchronization of oscillators after travel to a new time zone (i.e., how the oscillators return to the ergodic solution after switching time zones).

4.1 Travel Formulation

From our model formulation, provided by Eqs. (4)–(6), we notice that the LRA form of the cost is not amenable to modeling the transient period of jet lag recovery. To be more specific, the cost in Eq. (5) will have the same value, regardless of the behavior of the oscillators over a fixed finite time interval. Thus, the LRA form of the cost does not penalize the oscillators for taking an arbitrarily large amount of time to resynchronize, which is undesirable.

Since the oscillators have already learned the optimal feedback control, $\alpha^*(\phi)$, for long time behavior at longitude 0, they have simultaneously learned the optimal feedback control, $\alpha^*(\phi - p)$, for long time behavior at the longitude p . By ergodicity, they could use the feedback control $\alpha^*(\phi - p)$ after traveling p/ω_S time zones, in which case the distribution of the oscillators would relax to $\mu^*(\phi - p)$. Thus, the oscillators could shift by p the control they have already learned to recover from traveling p/ω_S time zones. We would like to compute the distribution, denoted $\mu_p(t, \cdot)$, of the oscillators at time t as they recover from traveling p/ω_S time zones. We make the following assumptions:

(A1) Travel is immediate. Without loss of generality,

$$\begin{aligned}\rho(0) &= 0, \\ \rho(t) &= p \in [-\pi, \pi), \quad \forall t > 0.\end{aligned}$$

(A2) The individual is entrained to their local time zone before travel begins:

$$\mu_p(0, \cdot) = \mu^*(\cdot),$$

where μ^* is a solution to the *Ergodic MFG Problem* (definition 1).

(A3) The oscillators adopt the feedback control:

$$\alpha_p(t, \phi) := \alpha^*(\phi - p), \quad \forall t > 0,$$

where α^* , defined by Eq. (12), is computed from the solution to the *Ergodic MFG Problem*.

Under these assumptions, the law of the phases of the oscillators is given by solving the Kolmogorov/Fokker–Planck equation forward in time. We now pose the *Ergodic Recovery*.

Definition 3 *Ergodic Recovery*: Find $\mu_p(t, \phi)$ solving

$$\begin{aligned}\partial_t \mu_p + (\omega_0 - \omega_S) \partial_\phi \mu_p + \partial_\phi [\mu_p \alpha_p] - \frac{\sigma^2}{2} \partial_\phi^2 \mu_p &= 0, \\ \mu_p(0, \phi) &= \mu^*(\phi),\end{aligned}\tag{22}$$

on the domain $t \in [0, +\infty)$, and $\phi \in [0, 2\pi)$ with periodic boundary.

Note that, since the initial condition, $\mu^*(\phi)$, is a probability measure, a solution $\mu_p(t, \phi)$ to Eq. (22) is a probability measure for all $t \geq 0$. Furthermore, given a solution to the *Ergodic MFG Problem*, existence and uniqueness for the *Ergodic Recovery* are clear.

To summarize the proposed models, in Sect. 3, we posed the *Ergodic MFG Problem* to describe the long time behavior of the oscillators for an individual who remains in one time zone. In the *Ergodic Recovery* problem, which models jet lag recovery, we assume that travel is immediate, the oscillators are synchronized with their time zone before travel, and the oscillators attempt to resynchronize to the new time zone by adopting the control from the *Ergodic MFG Problem* for the new time zone. In the next subsection, we provide some definitions for quantifying jet lag recovery.

4.2 Quantifying Jet Lag Recovery

To quantify jet lag recovery, we measure the time it takes to recover from jet lag, and the cost the oscillators accrue while recovering from jet lag. First, we need to define what it means to have recovered from jet lag. Since we are assuming (μ^*, V^*, λ^*) solving the *Ergodic MFG Problem* is unique, we define jet lag recovery as the following.

Definition 4 For a given $p \in [-\pi, \pi)$ and $\epsilon_W > 0$, we say that the oscillators have ϵ_W -recovered at time $t \geq 0$ from jet lag after traveling p/ω_S time zones away if:

$$\mathcal{W}_2(\mu_p(s, \cdot), \mu^*(\cdot - p)) < \epsilon_W, \quad \forall s \geq t$$

where $\mathcal{W}_2$ denotes the 2-Wasserstein distance, $\mu^*(\cdot)$ denotes the solution to the *Ergodic MFG Problem*, and $\mu_p(t, \cdot)$ denotes the distribution of the oscillators' phases t units of time after the instantaneous travel of p/ω_S time zones.

In other words, the oscillators are within ϵ_W of entrainment after traveling from longitude 0 to longitude p . We define the recovery time as:

$$\tau_p^W := \inf\{t > 0 : \mathcal{W}_2(\mu_p(s, \cdot), \mu^*(\cdot - p)) < \epsilon_W, \quad \forall s \geq t\}.$$

A second notion of jet lag recovery time is based on the analog of the order parameter introduced in [17]. Let $z^* = \int_0^{2\pi} e^{i\phi} d\mu^*(\phi)$. Similarly, let $z_p(t) = \int_0^{2\pi} e^{i\phi} d\mu_p(t, \phi)$. We define a second notion of jet lag recovery time:

$$\tau_p^z := \inf\{t > 0 : |z_p(s) - e^{ip} z^*| < \epsilon_z, \quad \forall s \geq t\}.$$

We also consider the cost accrued while recovering from jet lag using the values of the cost functions that the oscillators wish to minimize. We define the instantaneous contributions to the overall cost by:

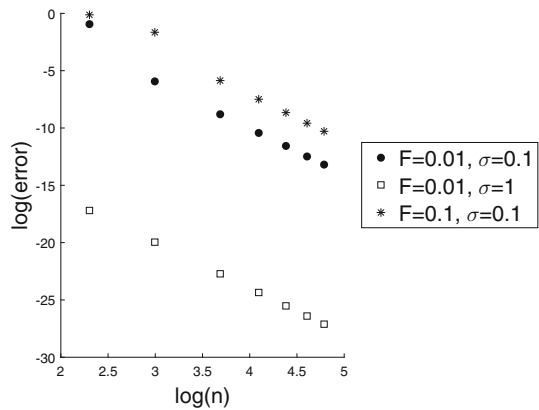
$$\begin{aligned} f_\alpha(t) &:= \int_0^{2\pi} \frac{1}{2} \alpha_p(t, \phi)^2 d\mu_p(t, \phi) = \int_0^{2\pi} \frac{1}{2} \alpha^*(\phi - p)^2 d\mu_p(t, \phi), \\ f_{\text{osc}}(t) &:= \int_0^{2\pi} \bar{c}(\phi, \mu_p(t, \cdot)) d\mu_p(t, \phi), \\ f_{\text{sun}}(t) &:= \int_0^{2\pi} c_{\text{sun}}(\phi, p) d\mu_p(t, \phi), \\ f(t) &:= \frac{f_\alpha(t) + K \cdot f_{\text{osc}}(t) + F \cdot f_{\text{sun}}(t)}{1 + K + F}. \end{aligned}$$

To simplify the presentations of the numerical results, we compare the cost accrued over the first 10 days after a time zone change, since the oscillators recover within 10 days for almost all of the cases we consider. With an abuse of notation, we compute $f_{p,\alpha} := \int_0^{240} f_{p,\alpha}(t) dt$, $f_{p,\text{osc}} := \int_0^{240} f_{p,\text{osc}}(t) dt$, $f_{p,\text{sun}} := \int_0^{240} f_{p,\text{sun}}(t) dt$, and $f_p := \int_0^{240} f_p(t) dt$.

5 Numerical Results

We have two problems to solve numerically: the *Ergodic MFG Problem* and the *Ergodic Recovery*. Although Corollary 1 says that we can solve an integro-differential equation in place of the coupled system in the *Ergodic MFG Problem*, numerical methods for the latter are

Fig. 1 Log-log plot of the error between the numerical solution and the true solution for the *Ergodic MFG Problem* when $K = 0$, $\omega_0 = \omega_S$, as a function of the number of discretization points



well studied, so we will stick to this formulation for our numerical results. For both problems, our numerical approach is to use finite differences and Picard iterations, where we solve iteratively each equation. The main numerical complications are (1) the coupling between the ergodic HJB and the Poisson equation, which is tackled through Picard iterations, and (2) the nonlinearity in the HJB equation, which we avoid by using previous Picard iterations. Note that, the *Ergodic MFG Problem* needs to be solved first, since its solution is used as the initial condition for the *Ergodic Recovery*.

For our numerical results, we fix $\omega_S = \frac{2\pi}{24}$ radians per hour. The number of grid points in ϕ is also fixed at $n = 120$. The parameters of interest are thus p , ω_0 , σ , K , and F . Results are presented for a reference set of values as well as unilateral changes of the parameters. For our reference set, we let $p = \pm 9\omega_S$ (travel east or west by 9 time zones), $\omega_0 = \frac{2\pi}{24.5}$ radians per hour, $\sigma = 0.1$, $K = 0.01$, and $F = 0.01$.

5.1 Verification of Numerical Methods

In Section 3.1, we derived an explicit solution to the *Ergodic MFG Problem* when $K = 0$ and $\omega_0 = \omega_S$. By using the Python package `scipy.special` to compute values of Mathieu's function, we verify that our numerical methods are accurate. Figure 1 shows that the error between our numerical solution and the true solution decreases as we increase n , the number of discretization points of the spatial domain $[0, 2\pi)$.

5.2 Ergodic MFG Problem Numerical Solution

For our reference set of parameters, the solution to the *Ergodic MFG Problem* is shown in Fig. 2. Note that, the measure is concentrated near $p = 0$. (Recall that the domain is periodic.) The stationary solution is used to calculate the optimal feedback control α^* . The drift of an oscillator using the optimal control is a function of ϕ , shown in Fig. 2c. This plot clearly illustrates the mean reversion, the main reason for strong ergodicity. Note that, near $\phi = 0$, if $\phi < 0$, meaning the oscillator is lagging behind the phase of the natural 24-h cycle, then the drift is positive. This allows the oscillator to advance its phase forward to try to 'catch up.' Similarly, if $\phi \in (0, \pi]$, the drift is negative, meaning that the oscillator delays its phase. We restate this observation more precisely in the following conjecture: For the reference set of parameters, there exists $l > 0$ and $r > 0$ such that $\omega_0 - \omega_S + \alpha^*(\phi) > 0$ for $\phi \in [-l, 0)$, and

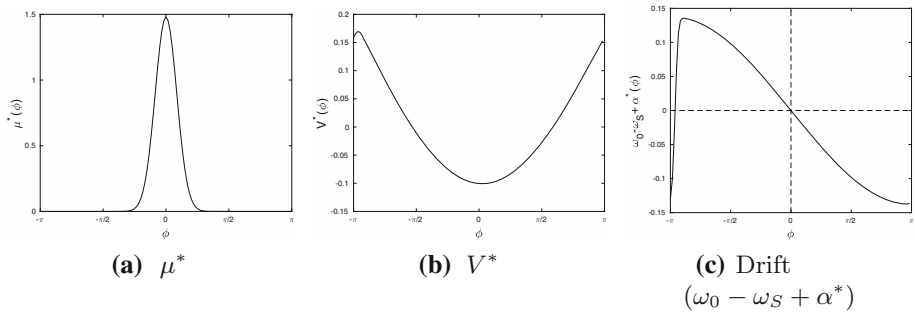


Fig. 2 Solution to *Ergodic MFG Problem* for the reference set of parameters

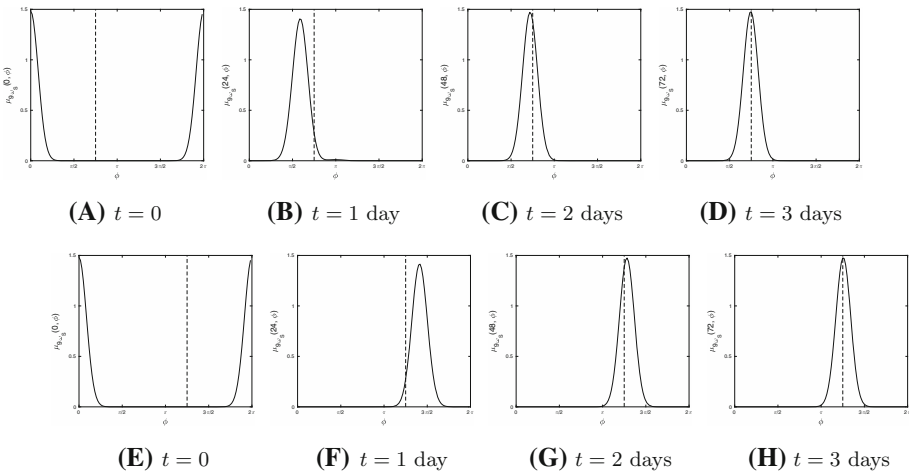


Fig. 3 *Ergodic recovery* Jet lag recovery after eastern travel (a–d) and western travel (e–h) by 9 time zones for the reference set of parameters

$\omega_0 - \omega_S + \alpha^*(\phi) < 0$ for $\phi \in (0, r]$, and $\omega_0 - \omega_S + \alpha^*(0) = 0$. These qualitative properties of the solution shown in Fig. 2 mirror the plots shown in Figs. 4 and 6 in [23].

5.3 Ergodic Recovery Numerical Solution

Using the numerical solution to the *Ergodic MFG Problem* as the initial condition and control, the solution to the *Ergodic Recovery* is shown in Figs. 3a–h for traveling 9 time zones east and west, respectively. The solution is shown at time 0, after 1 day, after 2 days, and after 3 days to illustrate the gradual adjustment to the new longitude p , which is plotted as a vertical line.

As shown in the plots, after a few days, the distribution of the oscillators has adjusted itself to align with the new value of p . Now we compare our measures of jet lag recovery between east and west travel to test the claim that jet lag is worse when traveling east.

Figure 4 shows the distance between the distribution of the oscillators over time as they recover back to their stationary solution, in both the 2-Wasserstein distance and the distance between the order parameters $z_p(t)$ and z^* , the latter plot being analogous to Fig. 3 in

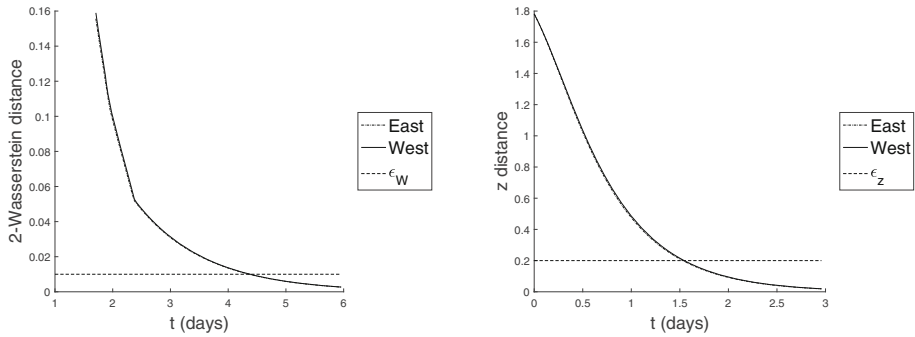


Fig. 4 Ergodic Recovery Distance from the stationary solution while recovering from jet lag after traveling 9 time zones east and west, as measured using 2-Wasserstein distance (left) and distance between the order parameters $z_p(t)$ and z^* (right)

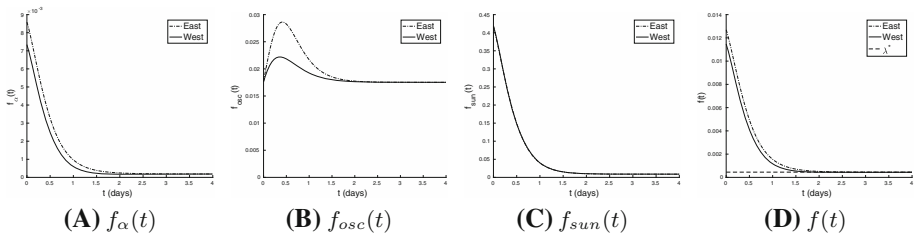
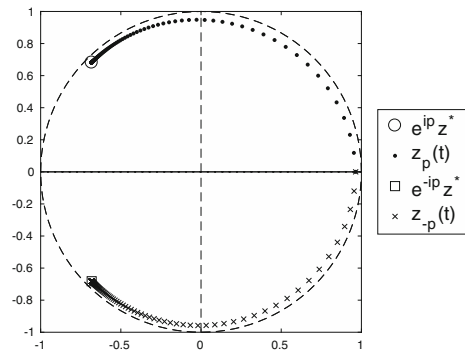


Fig. 5 Ergodic Recovery Jet lag recovery costs for travel by 9 time zones for the reference set of parameters

[17]. Recall that the jet lag recovery times are when these distances stay below a threshold. The jet lag recovery times based on the 2-Wasserstein distance are $\tau_{9\omega_S}^W = 4.38$ days and $\tau_{-9\omega_S}^W = 4.42$ days, where we take $\epsilon_W = 0.01$. The jet lag recovery times based on the order parameters $z_p(t)$ and z^* are $\tau_{9\omega_S}^z = 1.54$ days and $\tau_{-9\omega_S}^z = 1.58$ days, where we take the same value of $\epsilon_z = 0.2$ as in [17]. Thus, we do not find a significant difference in jet lag recovery time between east versus west travel for the reference set of parameters. However, we do see a difference in the jet lag recovery costs, which are shown in Fig. 5. The cost from the controls, $f_\alpha(t)$, and the cost of synchronizing with the other oscillators, $f_{osc}(t)$, are larger for the eastward trip than the westward trip, while the cost for the 24 h cycle, $f_{sun}(t)$, is the same for east and west. Thus, the total recovery cost is larger when recovering from traveling east. To summarize, we conjecture that if $\omega_0 < \omega_S$, there is a larger recovery cost associated with traveling east. And similarly, if $\omega_0 > \omega_S$, there is a larger recovery cost for the westward trip.

For comparison with [17], Fig. 6 shows the paths $z_p(t)$ for $p = \pm 9\omega_S$. Recall that a larger $|z_p(t)|$ means that the oscillators are more synchronized with each other. The results are similar to Fig. 2a in [17], except the stationary solution z^* for our approach is closer to $(1, 0)$ on the complex plane. Note that, the presentation is slightly different in that we take $\rho(0) = 0$ and $\rho(t) = p$ for $t > 0$ whereas in [17], they take $\rho(0) = p$ and $\rho(t) = 0$ for $t > 0$.

Fig. 6 Ergodic Recovery Path $z_p(t)$ while recovering from jet lag after traveling 9 time zones east (dots) and west (x's) for the reference set of parameters. A point is plotted every hour



5.4 Parameter Sensitivity Analysis for the Ergodic Recovery

Now we explore how the results change with the five parameters of interest: p , ω_0 , σ , K , and F . Unless otherwise specified, parameters remain at their reference values of $p = \pm 9\omega_S$, $\omega_0 = 2\pi/24.5$, $\sigma = 0.1$, $K = 0.01$, and $F = 0.01$. We are somewhat restricted because we can only obtain results for the *Ergodic Recovery* when the Picard iterations of our numerical algorithm for solving the *Ergodic MFG Problem* converge. (Otherwise, we do not have an initial condition and a control to feed into the *Ergodic Recovery*.) In particular, we found that the values of K and F need to be sufficiently small for the Picard iterations to converge (hence, our choice of $K = 0.01$, and $F = 0.01$ in the reference set). In addition, we should not consider values of F which are too small, since when $F = 0$, it is known that uniqueness does not hold [23]. Despite these limitations, we still get a picture of the behavior of the solutions as we vary each parameter.

Figure 7a shows the two jet lag recovery times, τ_p^W and τ_p^Z , as we vary p/ω_S , the number of time zones. The east and west recovery times almost completely coincide, except for deviations at $|p| \geq 9\omega_S$. The costs accrued over the first 10 days are summarized in Fig. 7b, which shows a larger cost for eastward trips with an increasing disparity between east and west as $|p|$ increases from $1\omega_S$ through $11\omega_S$. Note that, in our model, a trip of 12 time zones is the same if viewed as an eastward or a westward trip, and the results for $p = \pm 12\omega_S$ coincide.

The results of changing ω_0 , the preferred frequency of the oscillators, are shown in Fig. 8. As in the previous results, we see that the recovery time is about the same for east and west travels, but the cost is larger for east travels when $\omega_0 < \omega_S$. By symmetry, we expect that if $\omega_0 - \omega_S = \Omega$ and $p = +j\omega_S$, the results should be the same as when $\omega_0 - \omega_S = -\Omega$ and $p = -j\omega_S$. This is confirmed in Fig. 8.

Returning to our analysis of Fig. 8a, we note that the recovery time is fairly stable as we change ω_0 to different values near ω_S , and recovery times increase for values of ω_0 far from ω_S . Figure 8b shows that the jet lag recovery costs increase as ω_0 goes further away from ω_S .

To understand why it takes so much longer to recover for the extremal values of $\omega_0 - \omega_S$, Fig. 9 shows the paths $z_p(t)$ for $p = \pm 9\omega_S$ for different values of ω_0 . For values of ω_0 closer to ω_S , as in Fig. 9a and b, the oscillators phase advance after traveling east and phase delay after traveling west. If ω_0 is much smaller than ω_S , as in Fig. 9c and d, the oscillators recover by phase delaying for both east and west travels. Clearly by symmetry, if ω_0 is much larger than ω_S , then the oscillators recover by phase advancing for both east and west travels. It

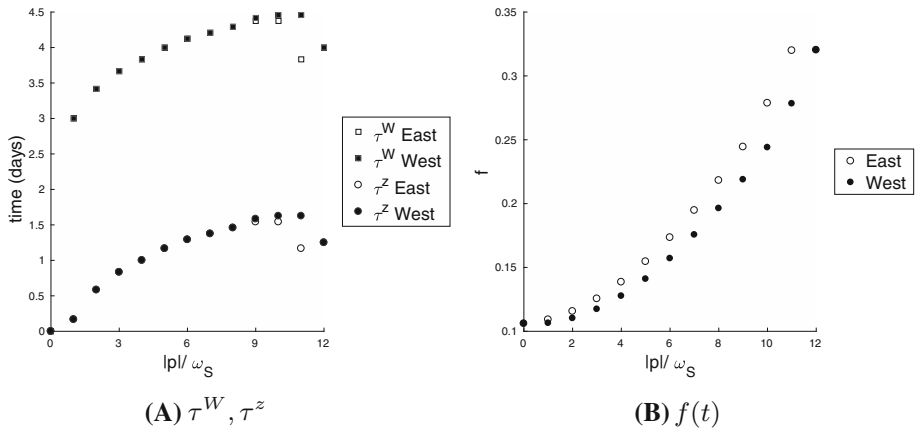


Fig. 7 Ergodic Recovery Jet lag recovery times and costs as a function of p . Note that, a trip of 12 time zones is the same when viewed as an eastward or westward trip

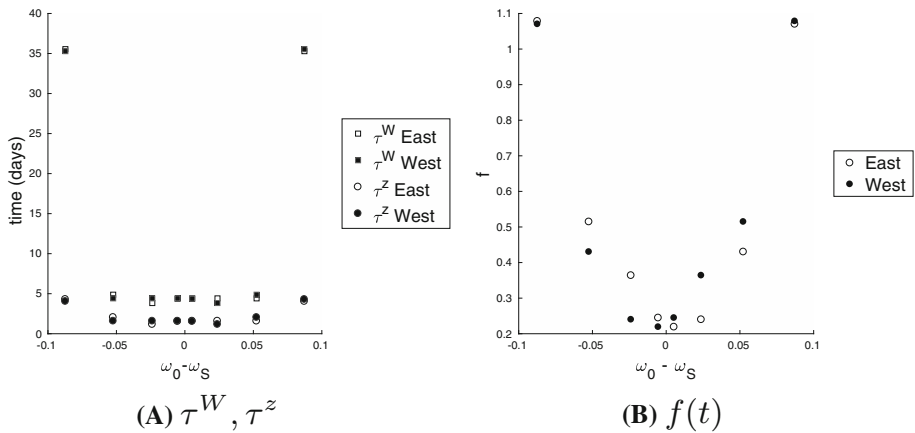


Fig. 8 Ergodic Recovery Jet lag recovery times and costs as a function of $\omega_0 - \omega_S$

is interesting to note that in Fig. 9d the path for recovery is not direct as in the other cases, which is why it takes so much longer to recover.

The results of changing σ , the strength of the noise, are shown in Fig. 10. At first glance, it is surprising that the recovery times decrease as σ increases, as one might think that it would be harder to recover from jet lag when there is more noise. However, the solution to the *Ergodic MFG Problem* as σ increases becomes closer and closer toward a uniform distribution, and thus $\mu^*(\phi)$ and $\mu^*(\phi - p)$ become closer to each other. In fact, when $|z^*|$ is close enough to 0, then $|z_p(0) - e^{ip}z^*| = |z^* - e^{ip}z^*| < \epsilon_z$ and $\tau_p^z = 0$. This is the case when $\sigma = 0.5$ and $\sigma = 1$ as shown in Fig. 10a. Since $\mu^*(\phi)$ becomes closer to a uniform distribution with larger σ , it makes sense that the cost for synchronization with the other oscillators, f_{osc} , and the cost for synchronization with the natural 24-h cycle, f_{sun} , both increase with σ . The reason for the increase and then decrease of f_α is less clear.

Figure 11 shows the results when changing K , which encourages synchronization of the oscillators with each other. The recovery times decrease with K . Since a larger K puts more

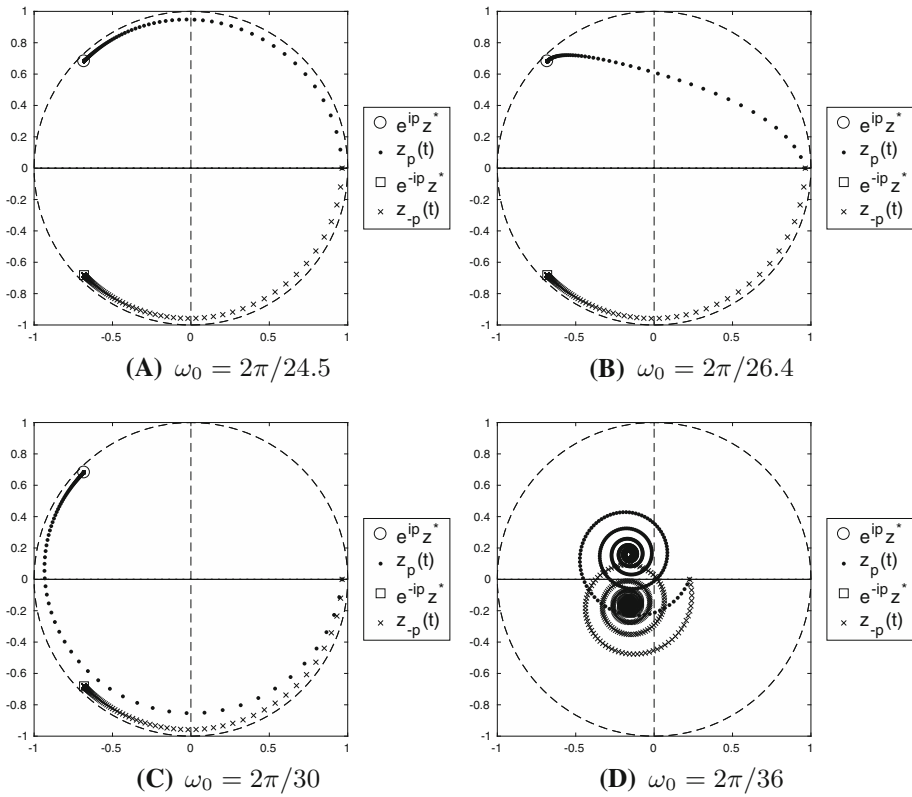


Fig. 9 Ergodic Recovery Path $z_p(t)$ while recovering from jet lag after traveling 9 time zones east (dots) and west (x's) for various values of ω_0 . A point is plotted every hour

weight on synchronization, it is unsurprising that the f_{osc} and f_{sun} decrease with K . Since more synchronization requires more effort, it is also unsurprising that f_{α} increases with K .

The results for changing F , which encourages alignment with the 24-h sunlight cycle, are shown in Fig. 12. The recovery times decrease as F increases, which is intuitive. The recovery costs are qualitatively the same as for changing K , with similar justifications. As we increase F , we put more weight on synchronization with the 24-h light/dark cycle, which will decrease f_{sun} . As a result, the oscillators will be more synchronized with each other as well, which will decrease f_{osc} . To achieve a larger degree of synchronization will require a larger control, so f_{α} increases.

In summary, when the rest of the parameters remain at the reference values, we conjecture the following behavior for unilateral changes of the parameters:

- (C1) There is a threshold Ω^* such that if $|\omega_0 - \omega_S| < \Omega^*$, oscillators will phase advance after traveling east and phase delay after traveling west. If $|\omega_0 - \omega_S| > \Omega^*$, the oscillators will phase delay if $\omega_0 < \omega_S$ and phase advance if $\omega_0 > \omega_S$.
- (C2) τ^W and τ^z decrease with K and F , and $\tau_p^W \rightarrow 0$ and $\tau_p^z \rightarrow 0$ as $\sigma \rightarrow \infty$.
- (C3) For a given $p \in [0, \pi)$, f is larger when traveling to p (i.e., traveling east) than traveling to $-p$ (i.e., traveling west).
- (C4) f increases with $|p|$, $|\omega_0 - \omega_S|$, σ , K , and F .

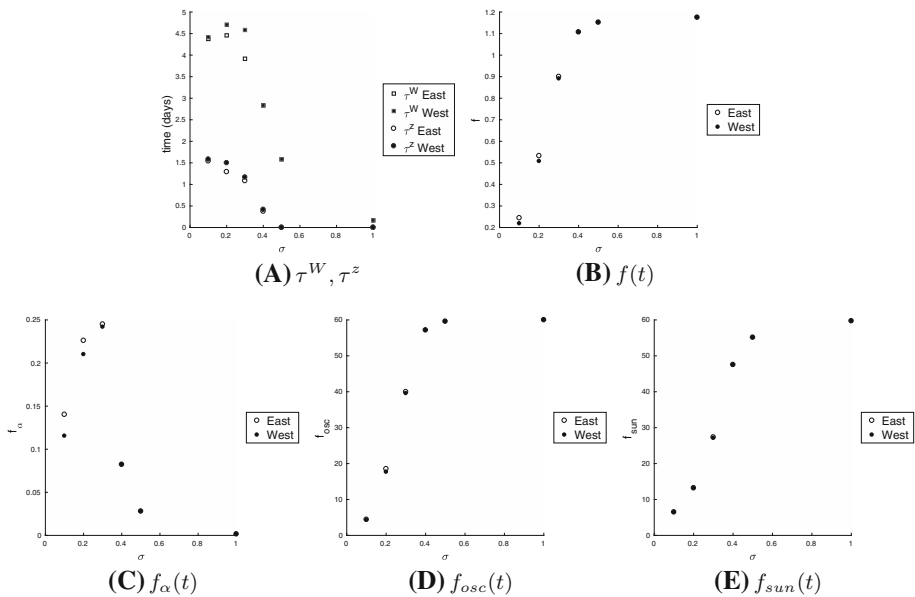


Fig. 10 Ergodic Recovery Jet lag recovery times and costs as a function of σ

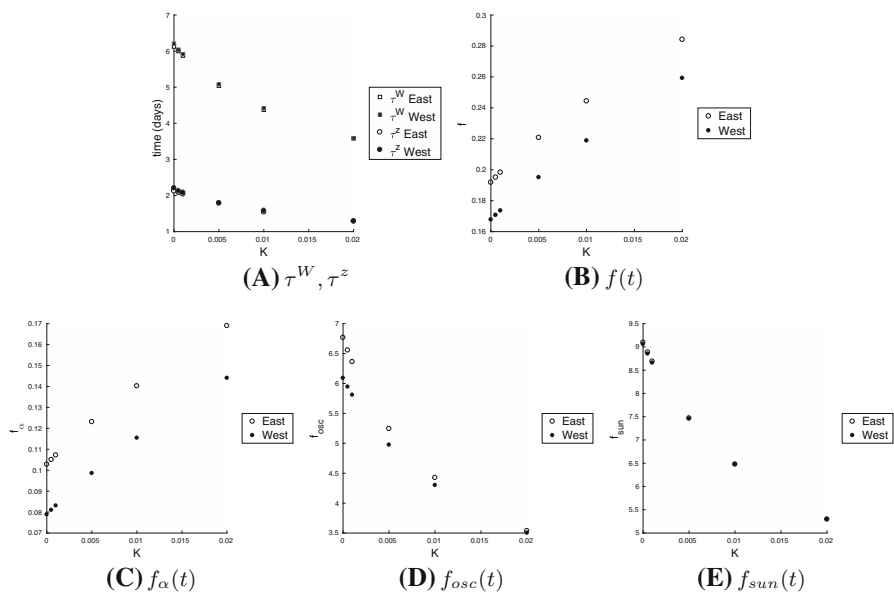


Fig. 11 Ergodic Recovery Jet lag recovery times and costs as a function of K

6 Summary/Conclusion

We provided a mean field game formulation for the synchronization of SCN circadian oscillators. The benefit of this approach is that the equilibrium dynamics of the oscillators arise endogenously, a characteristic shared by a mean field control approach. We stress that the

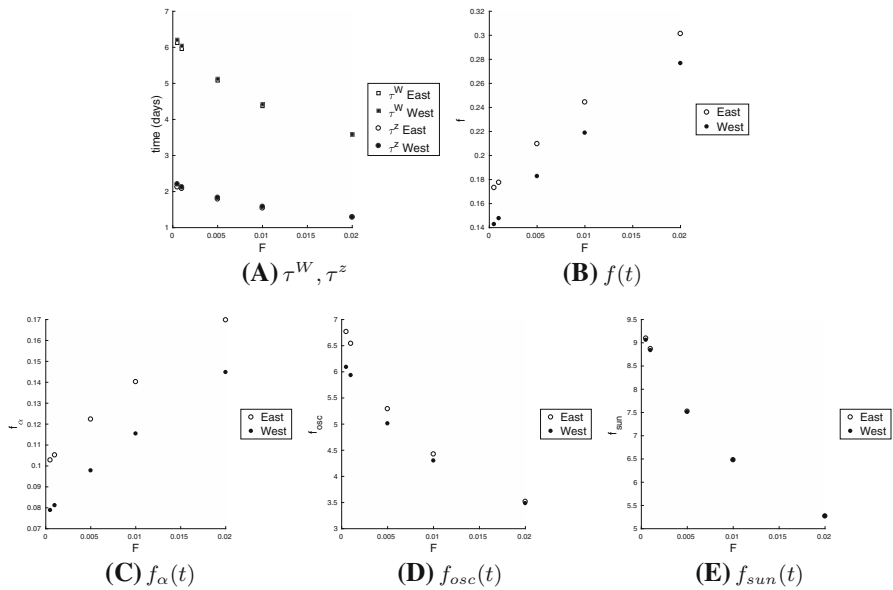


Fig. 12 Ergodic Recovery Jet lag recovery times and costs as a function of F

two formulations are equivalent, up to a rescaling of one parameter. The long time behavior of the oscillators is described by the solution of an *Ergodic Mean Field Game Problem* in which the oscillators optimize an ergodic cost. We show that the equilibrium distribution of the oscillators' phases has a density which can be written explicitly in terms of the ergodic value function, and as a result, the system of equations describing the system can be replaced by a single integro-differential equation. Furthermore, for a special case of the parameters we find an explicit solution in terms of the Mathieu functions, and a first-order perturbation around this solution is analyzed.

In order to study jet lag recovery from travel, we assume that travel is immediate, the oscillators are entrained to their time zone before travel, and that the oscillators use the control they have already learned to resynchronize to a new time zone. A finite differences approach was implemented to solve the above problems numerically. The numerics suggest that the cost accrued while recovering from jet lag is larger for eastward travels. This is consistent with the experience of frequent travelers who claim that it is harder to recover from jet lag after traveling eastward.

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