

# A Semidefinite Programming Method for Moment Approximation in Stochastic Differential Algebraic Systems

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**Abstract**—This paper presents a continuous-time semidefinite-programming method for bounding statistics of stochastic processes governed by stochastic differential-algebraic equations with trigonometric and polynomial nonlinearities. Upper and lower bounds on the moments are then computed by solving linear optimal control problems for an auxiliary linear control system in which the states and inputs are systematically constructed vectors of mixed algebraic-trigonometric moments. Numerical simulations demonstrate how the method can be applied to solve moment-closure problems in representative systems described by stochastic differential algebraic equation models.

## I. INTRODUCTION

Stochastic differential equations can represent phenomena from domains such as finance [1], biology [2], [3], and physics [2]. Important insights into the dynamics of a stochastic system can be obtained from statistical quantities such as the mean and variance. However, aside from linear systems, such quantities typically cannot be computed exactly. For more complex systems, it is desirable to find methods that can yield approximate values of desired statistical quantities with provable guarantees about approximation quality. This paper presents an optimization-based method to bound the statistics of stochastic processes governed by a particular class of stochastic differential-algebraic equations involving trigonometric and polynomial nonlinearities.

### A. Contribution

This paper extends the analysis results from [4] and its precursor [5]. In particular, the work in [4] developed a method to compute upper and lower bounds on the moments of stochastic processes by solving auxiliary linear optimal control problems. When the gap between the upper and lower bounds is small, the method guarantees that the true value lies in a small interval. That method, however, is limited to problems whose dynamics are defined by polynomials. Thus, the number of compelling problems that it can address is limited. This work extends those results by: 1) encompassing stochastic differential-algebraic equations, and 2) allowing for mixtures of algebraic and trigonometric polynomials. In this work, we show how the extended method can be used to analyze stochastic volatility models from finance, a pendulum with noise, and a single-machine infinite-bus power system model. Work still remains to improve the scalability of the method, but initial results indicate that the

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technique often converges to tight upper and lower bounds on the statistical quantity of interest.

### B. Related Work

Most closely related to this work are the results of [6], which gives a method for bounding stationary moments of stochastic processes defined by polynomials. The main differences between this work and the current paper are that 1) this work focuses on transient dynamics of moments, and 2) it encompasses cases beyond polynomials.

Other closely related work includes the occupation measure framework for analysis and control of stochastic processes [7]–[9]. The key difference is that the method here examines stochastic processes pointwise in time, while occupation measures examines them across time. This difference leads to quite different optimization problems. Another related method based on barrier certificates is given in [10].

This work is also related to work on moment closure [11]–[14]. As in this work, moment closure methods study the dynamics of moments to estimate statistics of a stochastic process. The difference between moment closure methods and this method is that moment closure methods compute point estimates of moments, while this method produces upper and lower bounds.

### C. Outline

The paper is organized as follows. Section II introduces the class of problems as well as a collection of running examples. Section III presents the optimization method for stochastic analysis. Section IV applies the method to the running examples. Finally, conclusions and future work are discussed in Section V.

## II. FUNDAMENTALS

In this section, we first outline pertinent notation. Following this, we describe the stochastic differential algebraic equation model, and formalize the notion of mixed algebraic-trigonometric polynomials.

### A. Notation

Stochastic processes are denoted by bold symbols  $\mathbf{x}(t)$ . The notation  $d\mathbf{x}(t)$  denotes the increment:  $d\mathbf{x}(t) = \mathbf{x}(t + dt) - \mathbf{x}(t)$ . The process  $\mathbf{w}(t)$  is a Brownian motion with mean 0 and covariance  $tI$ . ( $I$  is the identity matrix.) The expectation of a random variable,  $\mathbf{x}$ , is denoted by  $\mathbb{E}[\mathbf{x}]$ . The set of real numbers is denoted by  $\mathbb{R}$  while the set of non-negative integers is denoted by  $\mathbb{N}$ . For a matrix  $M$ ,  $M \succeq 0$  denotes that  $M$  is symmetric and positive semidefinite.

## B. Stochastic Differential Algebraic Equations

In this paper we will develop methods to analyze statistical properties of stochastic differential-algebraic equations (SDAEs) of the form:

$$dx(t) = f(x(t), y(t))dt + g(x(t), y(t))d\mathbf{w}(t), \quad (1a)$$

$$0 = a(x(t), y(t)). \quad (1b)$$

Assume that  $x(t) \in \mathbb{R}^{D_x}$  and  $y(t) \in \mathbb{R}^{D_y}$ . In order for the equality constraint to be solvable, we will assume that  $a(x(t), y(t)) \in \mathbb{R}^{D_y}$  and that  $\frac{\partial a(x(t), y(t))}{\partial y}$  is invertible along the trajectories of the system.

For the system (1), we will develop optimization-based methods for approximating expectations of the form:

$$\mathbb{E} \left[ h(x(T), y(T)) + \int_0^T c(x(t), y(t))dt \right]. \quad (2)$$

In particular, the main contribution of this paper is a method for generating a sequence of lower bounds  $\mathfrak{L}_1 \leq \mathfrak{L}_2 \leq \dots$  and upper bounds  $\dots \leq \mathfrak{U}_2 \leq \mathfrak{U}_1$  such that:

$$\mathfrak{L}_i \leq \mathbb{E} \left[ h(x(T), y(T)) + \int_0^T c(x(t), y(t))dt \right] \leq \mathfrak{U}_i. \quad (3)$$

This paper extends the settings to which the methodology in [4] can be applied. While [4] considered controlled jump diffusions in which all of the functions are polynomial, here:

- An algebraic constraint is included so that the method can handle differential algebraic systems.
- The functions  $f$ ,  $g$ ,  $a$ ,  $c$ , and  $h$  can include a mixture of trigonometric and polynomial functions.

For simplicity, we do not consider jumps or external control inputs, but these could also be incorporated.

## C. Algebraic-trigonometric Polynomials

Formally, the functions  $f$ ,  $g$ ,  $a$ ,  $c$ , and  $h$  must belong to the class of mixed algebraic-trigonometric polynomials over  $z = [x, y]^\top \in \mathbb{R}^{D_z}$ . A mixed algebraic-trigonometric polynomial,  $\psi(z)$ , is a function that can be expanded as

$$\begin{aligned} \psi(z) = & \sum_{m,n} \psi_{m,n}^c \prod_{i=1}^{D_z} (z_i^{m_i} \cos(n_i z_i)) \\ & + \sum_{m,n} \psi_{m,n}^s \prod_{i=1}^{D_z} (z_i^{m_i} \sin(n_i z_i)), \end{aligned} \quad (4)$$

where  $m$  and  $n$  are vectors in  $\mathbb{N}^{D_z}$ , and only finitely many of the coefficients  $\psi_{m,n}^c$  and  $\psi_{m,n}^s$  are non-zero. If the function is vector/matrix valued, then the coefficients  $\psi_{m,n}^c$  and  $\psi_{m,n}^s$  are constant vectors/matrices of appropriate dimension.

The natural *basis functions* are given by the functions  $\prod_{i=1}^{D_z} (z_i^{m_i} \cos(n_i z_i))$  and  $\prod_{i=1}^{D_z} (z_i^{m_i} \sin(n_i z_i))$ .

Several interesting systems can be posed as special cases of (1) with functions defined according to (4). Below, three running examples are described.

### Example 1 (Heston's Stochastic Volatility Model):

Stock prices are commonly modeled as a geometric Brownian motion. The variance parameter is known as the

*volatility*. Heston's stochastic volatility model, [15], captures the setting where the volatility itself can be random:

$$dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)d\mathbf{w}_1(t), \quad (5a)$$

$$d\nu(t) = \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}d\mathbf{w}_2(t). \quad (5b)$$

Here,  $S(t)$  represents the stock price,  $\nu(t)$  represents the volatility, and  $\mu$ ,  $\kappa$ ,  $\theta$ , and  $\xi$  are constant parameters. Heston's model (5), can be cast in the form (1) (i.e., using only polynomial variables) by representing  $\sqrt{\nu(t)}$  with an algebraic equality constraint:

$$dS(t) = \mu S(t)dt + y(t)S(t)d\mathbf{w}_1(t), \quad (6a)$$

$$d\nu(t) = \kappa(\theta - \nu(t))dt + \xi y(t)d\mathbf{w}_2(t), \quad (6b)$$

$$0 = y(t)^2 - \nu(t). \quad (6c)$$

It can be shown that if  $\nu(0) > 0$ , then  $\nu(t) > 0$ ,  $\forall t \geq 0$  whenever  $2\kappa\theta \geq \xi$ , [16]. In this case, the solvability condition  $\frac{\partial}{\partial y}(y(t)^2 - \nu(t)) \neq 0$  holds  $\forall t \geq 0$ .

**Example 2 (Noisy Pendulum):** A simple model with trigonometric functions is given by a pendulum with noisy torque:

$$d\theta(t) = v(t)dt \quad (7a)$$

$$dv(t) = -(\alpha \sin(\theta(t)) + \beta v(t))dt + \sigma d\mathbf{w}(t) \quad (7b)$$

Here  $\theta(t)$  is the angle of the pendulum,  $v(t)$  is the angular velocity, and  $\alpha$ ,  $\sigma$ , and  $\beta$  are constants.

**Example 3 (Single-Machine Infinite-Bus):** This example studies a popular power system model known as the single-machine infinite-bus. This system models the interconnection between a single generator and an infinitely large power grid. Stochasticity arises from variable loads connected to the generator. The model is given by:

$$d\delta(t) = \omega_s(\omega(t) - 1)dt \quad (8a)$$

$$\begin{aligned} d\omega(t) = & \frac{1}{M} \left( P_M - \frac{EV(t)}{X_G} \sin(\delta(t) - \theta(t)) \right. \\ & \left. - D(\omega(t) - 1) \right) dt \end{aligned} \quad (8b)$$

$$dP_L(t) = a_P(P_{L0} - P_L(t)) + \sigma d\mathbf{w}_1(t) \quad (8c)$$

$$dQ_L(t) = a_Q(Q_{L0} - Q_L(t)) + \sigma d\mathbf{w}_2(t) \quad (8d)$$

$$0 = P_L(t) - \frac{EV}{X_G} \sin(\delta(t) - \theta(t)) + \frac{V_s V(t)}{X_L} \sin(\theta(t)) \quad (8e)$$

$$0 = Q_L(t) + \frac{X_G X_L}{X_G + X_L} V(t)^2 \quad (8f)$$

$$- \frac{EV(t)}{X_G} \cos(\delta(t) - \theta(t)) - \frac{V_s V(t)}{X_L} \cos(\theta(t)). \quad (8g)$$

Here  $\delta(t)$  is the generator phase angle and  $\omega(t)$  is normalized generator frequency, relative to the nominal normalized value of 1. The stochastic real and reactive power injections are given by  $P_L(t)$  and  $Q_L(t)$ , respectively. For a deeper discussion of this model, see [17], and for stochastic extensions see [18], [19]. While the model is similar to that from [18], we model the loads as Ornstein-Uhlenbeck processes, as suggested in [20].

### III. CONTROL OF THE MOMENTS

This section shows how the statistics of the states in (1) can be bounded by solving an optimal control problem with respect to its moments. Subsections III-A and III-B give background results for the construction of the auxiliary control problem. Subsection III-C describes the control system, and Subsection III-D describes the auxiliary states of the running examples. Finally, Section III-E describes the corresponding optimal control problem.

#### A. Closure Properties

The following result establishes that the mixed algebraic-trigonometric polynomials introduced in (4) are closed under the operations of addition, multiplication, and differentiation. The proof is an application of standard trigonometric identities, and is omitted.

**Lemma 1:** *Let  $\phi(z)$  and  $\psi(z)$  be scalar-valued mixed algebraic-trigonometric polynomials with finite expansions of the form in (4). Then, the following functions also have finite expansions of the form in (4):*

$$\phi(z) + \psi(z); \quad \phi(z)\psi(z); \quad \frac{\partial\phi(z)}{\partial z_i}, \quad \forall i = 1, \dots, D_Z. \quad (9)$$

#### B. Generators

For any smooth function  $\phi : \mathbb{R}^{D_X} \rightarrow \mathbb{R}$ , a classical result in stochastic differential equations, [1], shows that the dynamics of  $\mathbb{E}[\phi(\mathbf{x}(t))]$  is given by:

$$\frac{d}{dt} \mathbb{E}[\phi(\mathbf{x}(t))] = \mathbb{E}[L\phi(\mathbf{x}(t), \mathbf{y}(t))], \quad (10)$$

where  $L\phi$  is the *generator* of  $\phi$ , and is defined by:

$$\begin{aligned} L\phi(x, y) &= \frac{\partial\phi(x)}{\partial x} f(x, y) \\ &+ \frac{1}{2} \text{Tr} \left( \frac{\partial^2\phi(x)}{\partial x^2} g(x, y) g(x, y)^\top \right). \end{aligned} \quad (11)$$

If  $\phi(x)$ ,  $f(x, y)$ , and  $g(x, y)$  are all mixed algebraic-trigonometric polynomials, then the closure properties of Lemma 1 imply that so is the generator  $L\phi(x, y)$ .

#### C. Auxilliary Linear Control System

In this section we will see how (10) can be used to define an auxilliary deterministic linear control system. The states of this auxilliary system will be means of so-called *test functions*. Recalling that  $x \in \mathbb{R}^{D_X}$ , we define the auxiliary state,  $\mathcal{X}(t)$  by:

$$\mathcal{X}(t) = \mathbb{E} \begin{bmatrix} 1 \\ \phi_1(\mathbf{x}(t)) \\ \vdots \\ \phi_P(\mathbf{x}(t)) \end{bmatrix}, \quad (12)$$

where the *test-functions* have the form:

$$\phi_j(x) = \begin{cases} \prod_{i=1}^{D_X} x_i^{m_i^j} \cos(n_i^j x_i) \\ \prod_{i=1}^{D_X} x_i^{m_i^j} \sin(n_i^j x_i) \end{cases}. \quad (13)$$

Recalling that  $z = [x, y]^\top \in \mathbb{R}^{D_Z}$ , the input for the auxiliary system,  $\mathcal{U}(t)$  takes the form:

$$\mathcal{U}(t) = \mathbb{E} \begin{bmatrix} \psi_1(z(t)) \\ \vdots \\ \psi_Q(z(t)) \end{bmatrix}, \quad (14)$$

where the functions  $\psi_j(z)$  have the form:

$$\psi_j(z) = \begin{cases} \prod_{i=1}^{D_Z} z_i^{m_i^j} \cos(n_i^j z_i) \\ \prod_{i=1}^{D_Z} z_i^{m_i^j} \sin(n_i^j z_i) \end{cases}. \quad (15)$$

The moments  $\mathcal{X}(t)$  and  $\mathcal{U}(t)$  are chosen so that Lemmas 2, 3, and 4 below hold. More detailed discussion of the choice of  $\mathcal{X}(t)$  and  $\mathcal{U}(t)$  is given at the end of Section III-E. For the special case of pure polynomial systems with no algebraic equality constraints, versions of the lemmas are presented in [4]. The proof of Lemma 2 is sketched, while the other proofs are omitted for space.

The first of the lemmas shows how to construct a deterministic linear DAE from the moments of the original stochastic DAE (1). For the lemma, recall that the equality constraint function  $a(x, y)$  takes values in  $\mathbb{R}^{D_Y}$ .

**Lemma 2:** *Consider the dynamics from (1). Let  $\mathcal{X}(t)$  be the vector of moments defined in (12), let  $\mathcal{U}(t)$  be the auxilliary control process from (14), and let  $R(x, y)$  be a mixed algebraic-trigonometric polynomial matrix with values in  $S \times D_Y$ , for some  $S \geq 1$ . Then, there are corresponding constant matrices  $A \in \mathbb{R}^{(P+1) \times (P+1)}$ ,  $B \in \mathbb{R}^{(P+1) \times Q}$ ,  $C \in \mathbb{R}^{S \times (P+1)}$ , and  $D \in \mathbb{R}^{S \times Q}$  such that  $\forall t \geq 0$*

$$\dot{\mathcal{X}}(t) = A\mathcal{X}(t) + BU(t) \quad (16a)$$

$$0 = C\mathcal{X}(t) + DU(t). \quad (16b)$$

*Proof Sketch:* Since  $\phi_j(x)$ ,  $f(z)$ , and  $g(z)$  are all mixed algebraic-trigonometric polynomials, as in (4), the generator  $L\phi_j(z)$  has an expansion of the form (4). Thus, (10) implies that the matrices  $A$  and  $B$  exist as long as  $\mathcal{U}(t)$  contains all the terms arising from the generators  $L\phi_j(z)$  that are not moments in  $\mathcal{X}(t)$ .

Since  $a(\mathbf{x}(t), \mathbf{y}(t)) = 0$  for all  $t \geq 0$ , we must have that  $R(\mathbf{x}(t), \mathbf{y}(t))a(\mathbf{x}(t), \mathbf{y}(t)) = 0$  as well. Thus, there are matrices  $C$  and  $D$  such that

$$\mathbb{E}[R(\mathbf{x}(t), \mathbf{y}(t))a(\mathbf{x}(t), \mathbf{y}(t))] = C\mathcal{X}(t) + DU(t) = 0.$$

As above, we assume that  $\mathcal{U}(t)$  contains the required terms. ■

The next lemma shows that the cost can be cast as a linear function of the moments in  $\mathcal{X}(t)$  and  $\mathcal{U}(t)$ .

**Lemma 3:** *There exist constant row vectors  $F \in \mathbb{R}^{1 \times P+1}$ ,  $G \in \mathbb{R}^{1 \times Q}$ ,  $H \in \mathbb{R}^{1 \times P+1}$ , and  $J \in \mathbb{R}^{1 \times P+1}$  such that*

$$\begin{aligned} \mathbb{E} \left[ h(\mathbf{x}(T), \mathbf{y}(T)) + \int_0^T c(\mathbf{x}(t), \mathbf{y}(t)) dt \right] &= \\ H\mathcal{X}(T) + J\mathcal{U}(T) + \int_0^T (F\mathcal{X}(t) + GU(t)) dt. \end{aligned} \quad (17)$$

This final lemma establishes a linear matrix inequality that must be satisfied by the moments of the SDAE, (1).

**Lemma 4:** Let  $v_1(\mathbf{x}(t), \mathbf{y}(t)), \dots, v_m(\mathbf{x}(t), \mathbf{y}(t))$  be any collection of mixed algebraic-trigonometric polynomials. There is an affine matrix-valued function  $M$  such that the following holds:

$$\left\langle \begin{bmatrix} v_1(\mathbf{x}(t), \mathbf{y}(t)) \\ \vdots \\ v_m(\mathbf{x}(t), \mathbf{y}(t)) \end{bmatrix} \begin{bmatrix} v_1(\mathbf{x}(t), \mathbf{y}(t)) \\ \vdots \\ v_m(\mathbf{x}(t), \mathbf{y}(t)) \end{bmatrix}^T \right\rangle = M(\mathcal{X}(t), \mathcal{U}(t)) \succeq 0. \quad (18)$$

#### D. Test Functions for Running Examples

Recall that the auxiliary state  $\mathcal{X}(t)$  is composed of expected values of a collection of test functions. In this subsection, we will describe the test functions of the running examples as well as their generators. We will see that in each case, the dynamics of the test functions are not *closed*. This means the dynamics cannot be described by a finite set of differential equations of other test functions.

##### Example 4 (Heston's Stochastic Volatility Model):

For this model, our test functions will take the form  $\phi(S, \nu) = S^m \nu^n$  where  $m$  and  $n$  are non-negative integers. Combining (5) with (11) shows that:

$$\begin{aligned} L(S^m \nu^n) &= m\mu S^m \nu^n + nS^m \nu^{n-1} \kappa(\theta - \nu) \\ &+ \frac{1}{2}m(m-1)S^m \nu^{n+1} + \frac{1}{2}n(n-1)\xi^2 S^m \nu^{n-1}. \end{aligned} \quad (19)$$

Note that when  $m \geq 2$ ,  $L(S^m \nu^n)$  depends on the monomial  $S^m \nu^{n+1}$ . In this case, the moments are not closed because (10) implies the dynamics of  $S(t)^m \nu(t)^n$  will always depend on a higher-order moment.

**Example 5 (Noisy Pendulum):** In the case of the noisy pendulum, our test functions will take the form  $\phi(\theta, v) = \cos(m\theta)v^n$  and  $\phi(\theta, v) = \sin(m\theta)v^n$ . Combining (7) with (11) shows that:

$$\begin{aligned} L(\cos(m\theta)v^n) &= -m \sin(m\theta)v^{n+1} \\ &- n \cos(m\theta)v^{n-1}(\alpha \sin(\theta) + \beta v) \\ &+ \frac{1}{2}n(n-1)\sigma^2 \cos(m\theta)v^{n-2}. \end{aligned} \quad (20)$$

The case of  $\sin(m\theta)v^n$  is similar. Thus, if  $m \geq 1$  and  $n \geq 0$ , the test function will always depend on a function of the form  $\cos(m\theta)v^{n+1}$  or  $\sin(m\theta)v^{n+1}$ . So, again the dynamics are not closed.

**Example 6 (Single-Machine Infinite Bus):** For the single-machine infinite-bus model, we take our test functions to be of the form:

$$\phi(x) = \begin{cases} \cos(i\delta)\omega^j P_L^k Q_L^l \\ \sin(i\delta)\omega^j P_L^k Q_L^l. \end{cases} \quad (21)$$

The dynamics of this model, (8), contain a variant of the pendulum model in the first two dynamic equations. As such, the dynamics of the test functions are not closed.

According to the basis conventions described in the paper, the terms  $\cos(\delta - \theta)$  and  $\sin(\delta - \theta)$  are interpreted as “second-order” polynomials. This is because representing them using expansions of the form (4) requires products. For example:

$$\sin(\delta - \theta) = \sin(\delta) \cos(\theta) - \cos(\delta) \sin(\theta).$$

This means that the terms  $\mathbf{V}(t) \sin(\delta(t) - \theta(t))$  and  $\mathbf{V}(t) \cos(\delta(t) - \theta(t))$  are considered third-order. When computing the approximate dynamics for this problem we get simpler representations by introducing extra variables with equality constraints to reduce the degree of the problem:

$$c(t) = \cos(\delta(t) - \theta(t)), \quad s(t) = \sin(\delta(t) - \theta(t)). \quad (22)$$

Note that these variables enable us to represent the whole system using basis elements of order 2 or less.

In order to construct the auxiliary inputs, we consider basis functions that are products of the form  $\phi(x)\psi(y)$ , where  $\phi(x)$  has the form from (21) and  $\psi(y)$  has the form:

$$\psi(y) = \begin{cases} \cos(i\theta)V^j c^k s^l \\ \sin(i\theta)V^j c^k s^l. \end{cases}$$

#### E. The Auxilliary Optimal Control Problem

The following is the main theoretical result of the paper. It is analogous to the main result of [4], which applies to stochastic control problems defined by polynomials. Formally, the main difference is that this problem has algebraic equality constraints corresponding to  $0 = a(\mathbf{x}(t), \mathbf{y}(t))$ .

**Theorem 1:** Let  $A, B, C, D, F, G, H, J$ , and  $M$  be the terms defined in Lemmas 2 – 4. Consider the corresponding continuous-time semidefinite program:

$$\underset{X(t), U(t)}{\text{minimize}} \quad HX(T) + JU(T) + \int_0^T (FX(t) + GU(t)) dt \quad (23a)$$

$$\text{subject to} \quad \dot{X}(t) = AX(t) + BU(t) \quad (23b)$$

$$0 = CX(t) + DU(t) \quad (23c)$$

$$X(0) = \mathcal{X}(0) \quad (23d)$$

$$U(0) = \mathcal{U}(0) \quad (23e)$$

$$M(X(t), U(t)) \succeq 0 \quad \forall t \in [0, T]. \quad (23f)$$

The optimal value for this problem is always a lower bound on the expected value from (2), provided that the corresponding moments exist. If the number of constraints in the linear problem, (23), is increased (either by: i) adding more test functions to  $\mathcal{X}(t)$ , ii) increasing the number of equality constraints from (23c), or iii) adding more semidefinite constraints), the value of (23) cannot decrease.

*Proof Sketch:* Lemmas 2 and 4 imply that the true expected values in  $\mathcal{X}(t)$  and  $\mathcal{U}(t)$  must satisfy all of the constraints in the semidefinite program, and Lemma 3 implies that the desired expected value from (2) is calculated via the objective function. Thus, the minimizing value must be a lower bound on the true expected value. The lower bound cannot decrease with more constraints because the set of feasible solutions becomes more constrained. ■

By changing the objective to a maximization, we get another continuous-time semidefinite program to compute an upper bound on the desired expectation, (2). With the maximization and minimization problems at hand, we can obtain upper and lower bounds as described in (3).

The optimal control problem, (23), has a large amount of flexibility in how it is posed. In our numerical examples, we use the following recipe, which can be automated:

- Fix some order  $k$  and let  $v(x, y)$  be a vector of basis functions up to order  $k$ .
- Construct the outer product  $v(x, y)v(x, y)^\top$ , which will contain basis functions up to order  $2k$ .
- Construct  $\mathcal{X}(t)$  by using all test functions  $\phi(x)$  such that the basis functions of the generators  $L\phi(x, y)$  are contained in the outer product matrix  $v(x, y)v(x, y)^\top$ .
- For each equality constraint function  $a_i(x, y)$ , find all of the basis functions  $b(x, y)$  such that basis functions of  $b(x, y)a_i(x, y)$  are contained in the outer product matrix,  $v(x, y)v(x, y)^\top$ . Let  $R(x, y)a(x, y)$  be the vector with rows given by the  $b(x, y)a_i(x, y)$  terms.

This recipe ensures that for a fixed outer product matrix,  $v(x, y)v(x, y)^\top$ , maximal sets of dynamics constraints and equality constraints are used.

#### IV. NUMERICAL SIMULATIONS

Here we show numerical results for the running examples.

**Example 7 (Heston's Stochastic Volatility Model):** The variance of prices is commonly desired. The variance of the price is given by  $\text{var}(\mathbf{S}(t)) = \mathbb{E}[\mathbf{S}(t)^2] - \mathbb{E}[\mathbf{S}(t)]^2$ . The mean of the price can be computed analytically since

$$\frac{d}{dt}\mathbb{E}[\mathbf{S}(t)] = \mu\mathbb{E}[\mathbf{S}(t)] \implies \mathbb{E}[\mathbf{S}(t)] = e^{\mu t}\mathbb{E}[\mathbf{S}(0)].$$

So, in order to estimate the variance, it suffices to estimate the second moment  $\mathbb{E}[\mathbf{S}(t)^2]$ . However, as discussed in Example 4, there is no finite set of differential equations for the second moment.

To get an estimate of the second moment, we used the method of this paper to compute bounds on the quantity:  $\mathbb{E}\left[\int_0^T \mathbf{S}(t)^2 dt\right]$ . See Fig. 1. Despite the lack of closed moments, the predicted upper and lower bounds are nearly exact when moments up to degree 6 are used.

Simulations were performed using an Euler-Maryuama scheme. The linear dynamics, (23b), were discretized via a pseudospectral method [21]. The optimization was performed via CVXPY [22] using the SDP solver SCS [23].

**Example 8 (Noisy Pendulum):** In the noisy pendulum, the horizontal position of the pendulum end is given by  $\sin(\theta(t))$  while the vertical position is given by  $\cos(\theta(t))$ . As described in Example 5, no finite set of test function differential equations can describe the dynamics of  $\mathbb{E}[\sin(\theta(t))]$  and  $\mathbb{E}[\cos(\theta(t))]$ . Figure 2 depicts the predictions. The predictions were found by bounding the following values:

$$\begin{aligned} \mathbb{E}\left[\sin(\theta(T)) + \int_0^T \sin(\theta(t)) dt\right], \\ \mathbb{E}\left[\cos(\theta(T)) + \int_0^T \cos(\theta(t)) dt\right]. \end{aligned}$$

Similar to Example 7, the simulation is via Euler-Maryuama, optimal control discretization via the pseudospectral method, and the optimization performed with CVXPY and SCS.

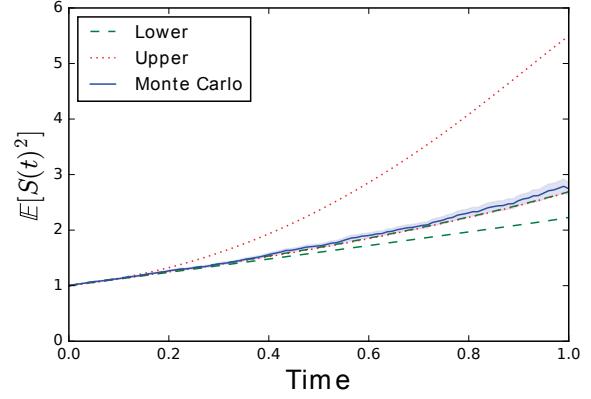


Fig. 1: Second moment of price for Heston's Stochastic Volatility model. Upper and lower bounds were computed using moments up to degree 4 and 6, respectively. The green dashed lines give the lower bounds and the red dotted lines give the upper bounds. When moments of degree 6 are used, the predictions are nearly identical, and predict the average trajectory closely. The blue line depicts the average of  $10^5$  simulations, while the filled boundary represents  $\pm$  one standard error of the mean. The model parameters were  $\mu = 0.1$ ,  $\kappa = 1$ ,  $\theta = 0.5$ , and  $\xi = 0.5$  and the initial conditions were  $\mathbf{S}(0) = 1$  and  $\nu(0) = 1$ .

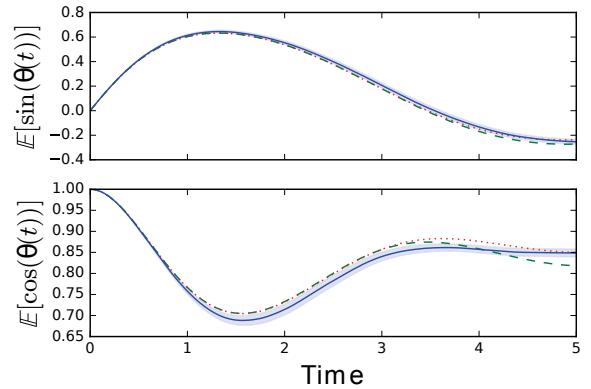


Fig. 2: Expected values of the horizontal position,  $\sin(\theta(t))$ , and height,  $\cos(\theta(t))$ , of the pendulum were computed using mixed moments up to degree 6. The red dotted line is the predicted upper bound, the green dashed line is the predicted lower bound. The solid blue line is the mean of 500 simulations and the filled space represents  $\pm$  one standard error of the mean. The model parameters were  $\alpha = 1$ ,  $\beta = 0.5$ , and  $\sigma = 0.5$  and the initial conditions were  $\theta(0) = 0$  and  $v(0) = 1$ .

**Example 9 (Single-Machine Infinite-Bus):** Now we describe how the method from Subsection III-E can be applied to the single-machine infinite-bus system. For this problem, we wish to examine the fluctuations of generator frequency around its nominal normalized value of 1.

We construct an outer product matrix that contains all of the basis functions described in Example 6 up to second

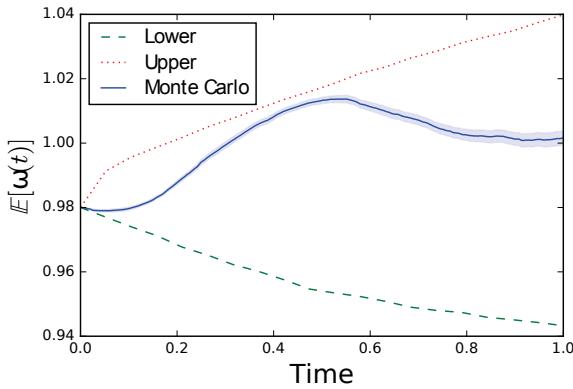


Fig. 3: This figure shows predicted bounds on the mean of the generator frequency,  $\mathbb{E}[\omega(t)]$ . The green dashed line shows the predicted lower bound and the red dotted line shows the predicted upper bound. The solid blue line gives the mean of 100 simulations and the filled space represents  $\pm$  one standard error of the mean. For this problem, even the second-order basis functions lead to a large optimization problem. At this low order, only fairly crude bounds can be obtained. The simulation is initialized with  $\delta(0) = 1$ ,  $\omega(0) = 0.98$ ,  $\mathbf{P}_L(0) = \mathbf{P}_{L0}$ ,  $\mathbf{Q}_L(0) = \mathbf{Q}_{L0}$ , and the algebraic variables are initialized by root finding. The parameters are set as  $\omega_s = 377$ ,  $D = 2$ ,  $M = 7$ ,  $E = 1.1$ ,  $P_m = 0.8$ ,  $X_G = 0.45$ ,  $X_L = 0.5$ ,  $V_s = 1$ ,  $P_{L0} = 1$ ,  $Q_{L0} = 0.5$ ,  $a_P = a_Q = 0.01$ , and  $\sigma = 1$ .

order. Employing the new variables  $c(t)$  and  $s(t)$  introduced in (22) ensures that the generator only uses terms up to second order. Constraints were found via the recipe in Subsection III-E. The results are shown in Fig. 3.

The stochastic simulations for this problem were computed via the method from [20]. The linear dynamics, (23b), were discretized via the Euler method. The optimization was performed via a large-scale semidefinite programming method under development by the first author. These methods were used because pseudospectral methods and SCS showed slow convergence.

## V. CONCLUSION

This paper presented a method for bounding statistics of stochastic processes via continuous-time semidefinite programming. The main contribution over previous work is the ability to handle stochastic differential-algebraic equations defined by mixed algebraic-trigonometric polynomials. These extensions make the method applicable to a rich set of stochastic process models of theoretical and practical interest. For systems with few variables and low degrees, the method gives accurate predictions of statistical quantities. However, for larger systems, the algorithms currently do not scale well and only crude bounds can be obtained. Future work will focus on analyzing the discretization schemes and improving the scaling of the method.

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