GLOBALLY IRREDUCIBLE WEYL MODULES

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Dedicated to Benedict Gross

ABSTRACT. In the representation theory of split reductive algebraic groups, it is well known that every Weyl module with minuscule highest weight is irreducible over every field. Also, the adjoint representation of E_8 is irreducible over every field. In this paper, we prove a converse to these statements, as conjectured by Gross: if a Weyl module is irreducible over every field, it must be either one of these, or trivially constructed from one of these. We also prove a related result on non-degeneracy of the reduced Killing form.

1. INTRODUCTION

Split semisimple linear algebraic groups over arbitrary fields can be viewed as a generalization of semisimple Lie algebras over the complex numbers, or even compact real Lie groups. As with Lie algebras, such algebraic groups are classified up to isogeny by their root system. Moreover, the set of irreducible representations of such a group is in bijection with the cone of dominant weights for the root system and the representation ring — i.e., K_0 of the category of finite-dimensional representations — is a polynomial ring with generators corresponding to a basis of the cone.

One way in which this analogy breaks down is that, for an algebraic group G over a field k of *prime* characteristic, in addition to the irreducible representation $L(\lambda)$ corresponding to a dominant weight λ, there are three other representations naturally associated with λ, namely the standard module $H^0(\lambda)$, the Weyl module $V(\lambda)$, and the tilting module $T(\lambda)$.[1] The definition of $H^0(\lambda)$ is particularly simple: view k as a one-dimensional representation of a Borel subgroup B of G where B acts via the character λ, then define $H^0(\lambda) := \mathrm{ind}_B^G \lambda$ to be the induced G-module. The *Weyl module* $V(\lambda)$ is the dual of $H^0(-w_0\lambda)$ for w_0 the longest element of the Weyl group and has head $L(\lambda)$. Typical examples of Weyl modules are $\mathrm{Lie}(G)$ for G semisimple simply connected ($V(\lambda)$ for λ the highest root) and the natural module of SO_n. See [Jan03] for general background on these three families of representations.

It turns out that if any two of the four representations $L(\lambda)$, $H^0(\lambda)$, $V(\lambda)$, $T(\lambda)$ are isomorphic over a given field k, then all four are. Our focus is on the question: for which λ are all four isomorphic for *every* field k?

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[1] The definitions of these three modules make sense also when $\mathrm{char}\, k = 0$, and in that case all four modules are isomorphic.

This can be interpreted as a question about representations of split reductive group schemes over $\mathbb{Z}$. Recall that isomorphism classes of such groups are in bijection with (reduced) root data as described in [DG11, XXIII.5.2]. A root datum for a group G includes a character lattice $X(T)$ of a split maximal torus T and the set $R \subset X(T)$ of roots of G with respect to T. Picking an ordering on R specifies a cone of dominant weights $X(T)_+$ in $X(T)$. For each $\lambda \in X(T)_+$, there is a representation $V(\lambda)$ for G, defined over $\mathbb{Z}$, that is generated by a highest weight vector with weight λ such that $V(\lambda) \otimes \mathbb{C}$ is the irreducible representation with highest weight λ of the complex reductive group $G \times \mathbb{C}$ and for every field k, $V(\lambda) \otimes k$ is the Weyl module of $G \times k$ mentioned above, see [Jan03, II.8.3] or [Ste68, p. 212]. Consequently, the question in the preceding paragraph is the same as asking: *For which G and λ is it true that $V(\lambda) \otimes k$ is an irreducible representation of $G \times k$ for every field k?* Because G is split, $V(\lambda) \otimes k$ is irreducible if and only if $V(\lambda) \otimes P$ is irreducible where P is the prime field of k[2], it is natural to call such $V(\lambda)$ *globally irreducible*.

There is a well known and elementary sufficient criterion:

$$\textit{If } \lambda \textit{ is minuscule, then } V(\lambda) \otimes k \textit{ is irreducible for every field } k. \tag{1}$$

See §2 for the definition of minuscule. This provides an important family of examples, because representations occurring in this way include $\Lambda^r(V)$ for $1 \leq r < n$ where V is the natural module for SL_n; the natural modules for SO_{2n}, Sp_{2n}, E_6 and E_7; and the (half) spin representations of Spin_n.

While these representations play an outsized role, it is nevertheless true that in any reasonable sense they are a set of measure zero among the list of irreducible representations. Therefore, we were surprised when Benedict Gross proposed to us that the sufficient condition (1) is quite close to also being a *necessary* condition, i.e., that there is only one other example. The purpose of this paper is to prove his claim, which is the following theorem.

Theorem 1.1. *Let G be a split, simple algebraic group over $\mathbb{Z}$ with split maximal torus T and fix $\lambda \in X(T)_+$. In the following cases, $V(\lambda) \otimes k$ is irreducible for every field k:*

- (a) *λ is a minuscule dominant weight, or*
- (b) *G is a group of type E_8 and λ is the highest root (i.e., $V(\lambda)$ is the adjoint representation for E_8);*

Otherwise, there is a prime $p \leq 2(\operatorname{rank} G) + 1$ such that $V(\lambda) \otimes k$ is a reducible representation of G for every field k of characteristic p.

The bound $2(\operatorname{rank} G) + 1$ is sharp by Theorem 5.1 below. The case where G is simple and simply connected (as in Theorem 1.1) is the main case. We have stated the theorem with these simplified hypotheses for the sake of clarity. See §2 for a discussion of the more general version where G is assumed merely to be reductive.

One surprising feature of our proof is the method we use to address a particular Weyl module of type B in §5, which we settle by appealing to modular representation theory of finite groups.

The literature contains some results complementary to Theorem 1.1, although we do not use them in our proof. For G of type A, Jantzen gave in [Jan03, II.8.21] a necessary

[2] See [Jan03, II.2.9]. For a detailed study of how this fails when G is not split, see [Tit71].

and sufficient condition for the Weyl module $V(\lambda)$ to be irreducible over fields of characteristic p. McNinch [McN98] (extending Jantzen [Jan96]) showed that for simple G and for $\dim V(\lambda) \leq (\operatorname{char} k) \cdot (\operatorname{rank} G)$, $V(\lambda)$ is irreducible.

We remark that John Thompson asked in [Tho76] an analogous question where G is finite: for which $\mathbb{Z}[G]$-lattices L is L/pL irreducible for every prime p? This was extended by Gross to the notion of globally irreducible representations, see [Gro90] and [Tie97]. Our results demonstrate that F_4 and G_2 are the only groups that do not admit globally irreducible representations other than the trivial representation.

In an appendix, we prove another result that is similar in flavor to Theorem 1.1: we determine the split simple G over $\mathbb{Z}$ such that the reduced Killing form on $\operatorname{Lie}(G) \otimes k$ is nondegenerate for every field k. This is done by calculating the determinant of the form on $\operatorname{Lie}(G)$, completing the calculation for G simply connected in [SS70].

Quasi-minuscule representations. The representations appearing in (a) and (b) of Theorem 1.1 are *quasi-minuscule* (called "basic" in [Mat69]), meaning that the non-zero weights are a single orbit under the Weyl group. For G simple, the quasi-minuscule Weyl modules are the $V(\lambda)$ with λ minuscule or equal to the highest short root α_0.

It is not hard to see that $V(\alpha_0) \otimes k$ is reducible for some k when G is not of type E_8. If G has type A, D, E_6, or E_7, then $V(\alpha_0)$ is the action of G on the Lie algebra of its simply connected cover $\widetilde{G}$, and the Lie algebra of the center Z of $\widetilde{G}$ is a nonzero invariant submodule when $\operatorname{char} k$ divides the exponent of Z. The case where G has type B or C is discussed in §4. If G has type G_2 or F_4, then $V(\alpha_0)$ is the space of trace zero elements in an octonion or Albert algebra, and the identity element generates an invariant subspace if $\operatorname{char} k = 2$ or 3 respectively.

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2. Definitions and notation

We will follow the notation and conventions presented in [Jan03]. When we refer to an algebraic group G, we mean a smooth affine group scheme of finite type as in [DG11] or [KMRT98, Ch. VI], as opposed to its (abstract) group of k-points, which we denote by $G(k)$. An example of this difference is that the natural map $\mathrm{SL}_p \to \mathrm{PGL}_p$ has nontrivial kernel the group scheme μ_p, yet for k a field of prime characteristic p, the map $\mathrm{SL}_p(k) \to \mathrm{PGL}_p(k)$ is injective.

Let G be a simple simply connected algebraic group, T be a maximal split torus of G and Φ be the root system associated to (G, T). Fix a choice of simple roots Δ. Let B be a Borel subgroup containing T corresponding to the negative roots and let U denote the unipotent radical of B.

One can naturally view Φ as contained in a Euclidean space $\mathbb{E}$ with inner product $\langle\ ,\ \rangle$. Let $X(T)$ be the integral weight lattice obtained from Φ. The set $X(T)$ has a partial ordering defined as follows. If $\lambda, \mu \in X(T)$, then $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$.

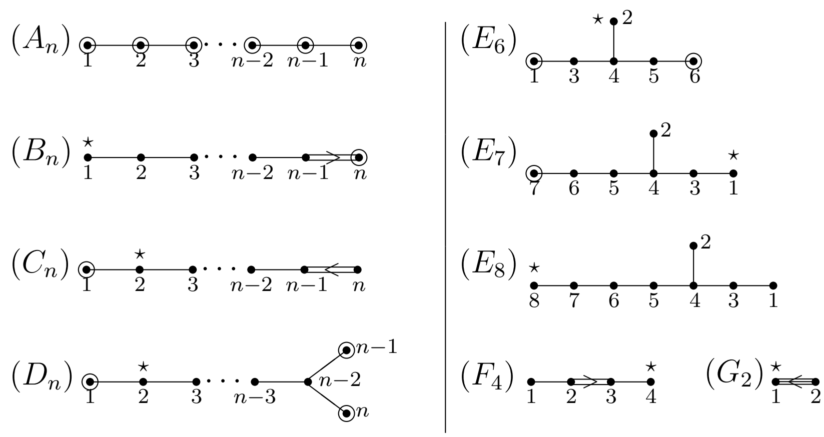

TABLE 1. Dynkin diagrams of simple root systems, with simple roots numbered. A circle around vertex i indicates that the fundamental weight ω_i is minuscule. A $\star$ indicates that ω_i is the highest short root α_0. The highest short root of A_n is $\omega_1 + \omega_n$.

For $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ the coroot corresponding to $\alpha \in \Phi$, the set of dominant integral weights is defined by

$$X(T)_+ := \{\lambda \in X(T) :\ 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Delta\}.$$

The fundamental weights ω_j for $j = 1, 2, \ldots, n$ are the dual basis to the simple coroots. That is, if $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ then $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We call the weights in $X(T)_+$ that are minimal with respect to the partial ordering *minuscule* weights. Note that the zero weight is minuscule by this definition (in some references this is not the case). Every nonzero minuscule weight is a fundamental dominant weight (one of the ω_i's), and we have marked them in Table 1. We remark that there is a unique minuscule weight in each coset of the root lattice $\mathbb{Z}\Phi$ in the weight lattice $X(T)$ by [Bou02, §VI.2, Exercise 5a] or [Hum80, §13, Exercise 13]; this can be an aid for remembering the number of minuscule weights for each type and for determining which minuscule weight lies below a given dominant weight.

Generalization of Theorem 1.1 to split reductive groups. Suppose now that G is a split reductive group over a field k. Then there is a unique split reductive group scheme over $\mathbb{Z}$ whose base change to k is G, which we denote also by G; it is the split reductive group scheme over $\mathbb{Z}$ with the same root datum as G. Moreover, there is a split reductive group scheme G' over $\mathbb{Z}$ with a central isogeny $G' \to G$ where $G' = \prod_{i=0}^r G_i$ for G_0 a torus and G_i simple and simply connected for $i \neq 0$, cf. [DG11, XXI.6.5.10]. A Weyl module $V(\lambda)$ for G restricts to a Weyl module $V(\sum \lambda_i)$ for G', where λ_i denotes the restriction of λ to a maximal torus in G_i, and as in [Jan03, Lemma I.3.8] we have $V(\sum \lambda_i) \cong \otimes_{i=0}^r V(\lambda_i)$ where $V(\lambda_0)$ is one-dimensional. Therefore, $V(\lambda) \otimes k$ is an irreducible G-module for every field k if and only if $V(\lambda_i) \otimes k$ is an irreducible G_i-module for every k, i.e., if and only if $(G_i, V(\lambda_i))$ satisfies condition (a) or (b) of Theorem 1.1 for all $i \neq 0$.

3. Restriction to Levi subgroups

For $J \subseteq \Delta$, let L_J be the Levi subgroup of G generated by the maximal torus T and the root subgroups corresponding to roots that are linear combinations of elements of J. Set

$$X_J(T)_+ := \{\lambda \in X(T) \ : \ 0 \le \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in J\}.$$

For $\lambda \in X_J(T)_+$, we can construct an induced module $H^0_J(\lambda) := \mathrm{ind}_{L_J \cap B}^{L_J} \lambda$ with simple L_J-socle $L_J(\lambda)$, and dually a Weyl module $V_J(\lambda)$ with head $L_J(\lambda)$.

Theorem 3.1. *Let G be a simple simply connected algebraic group and $J \subseteq \Delta$. If $V(\lambda) \otimes k$ is an irreducible G-module, then $V_J(\lambda) \otimes k$ is an irreducible L_J-module.*

Proof. For k of characteristic 0, $V_J(\lambda)$ is just the set of fixed points of Q_J on $V(\lambda)$ (the unipotent radical of the parabolic $P_J = L_J Q_J$); this is part of [Smi82]. Taking a $\mathbb{Z}$-form and reducing modulo p, we see that the dimension of the space of fixed points of Q_J on $V(\lambda)$ can only go up in characteristic p.

So if $V(\lambda) = L(\lambda)$, then again by [Smi82], the fixed points of Q_J on this module is $L_J(\lambda)$ but has dimension at least $V_J(\lambda)$. The other inequality is clear since $L_J(\lambda)$ is a quotient of $V_J(\lambda)$, so $L_J(\lambda) = V_J(\lambda)$. □

Remark 3.2. Given a group G and a particular prime p, there are few known necessary and sufficient conditions in terms of λ for the Weyl module $V(\lambda) \otimes k$ to be irreducible over every field k of characteristic p. There is an easy-to-apply statement for $G = \mathrm{SL}_2$. For $G = \mathrm{SL}_n$, Jantzen gives a necessary and sufficient condition, but it is less easy to apply. There are also sporadic results in one direction or another, such as consequences of the Linkage Principle like [Jan03, II.6.24] or irreducibility when λ is restricted and $\dim V(\lambda)$ is small. Theorem 3.1 provides an easy way to get necessary conditions on λ by taking various small J. Writing $\lambda = \sum c_i \omega_i$ and taking $J = \{\alpha_i\}$ one can apply the SL_2 criterion to constrain the possible values of c_i. Taking J to be pairs of adjacent roots of the same length allows one to reduce to the case of A_2, for which a lot is known, see [Jan03, II.8.20].

We mention the following related result that includes the case where $V(\lambda) \otimes k$ is reducible.

Proposition 3.3. *For every $\lambda \in X(T)_+$, every $J \subseteq \Delta$, and every field k, the irreducible representation $L_J(\lambda)$ of L_J is a direct summand of $L(\lambda)|_{L_J}$.*

Proof. For the sake of completeness we describe the analysis given in [CN11, Section 8] which follows [Smi82] and [Jan03, II.5.21]. There exists a weight space decomposition for the induced module given by

$$H^0(\lambda) = \left(\bigoplus_{\nu \in \mathbb{Z}J} H^0(\lambda)_{\lambda - \nu} \right) \oplus M.$$

where M is the direct sum of all weight spaces $H^0(\lambda)_\sigma$ where $\sigma \neq \lambda - \nu$ for any $\nu \in \mathbb{Z}J$. Furthermore, $H^0_J(\lambda) = \oplus_{\nu \in \mathbb{Z}J} H^0(\lambda)_{\lambda-\nu}$ with the aforementioned decomposition being L_J-stable. This allows us to identify an L_J-direct summand

$$H^0(\lambda)|_{L_J} \cong H^0_J(\lambda) \oplus M. \tag{2}$$

By definition $L(\lambda) = \mathrm{soc}_G(H^0(\lambda))$. This implies that $\mathrm{soc}_{L_J} L(\lambda) \subseteq \mathrm{soc}_{L_J}(H^0(\lambda))$. Note that

$$L_J(\lambda) = \mathrm{soc}_{L_J}(H^0_J(\lambda)) \subseteq \mathrm{soc}_{L_J}(H^0(\lambda)). \tag{3}$$

Now $L_J(\lambda)$ appears as an L_J-composition factor of $L(\lambda)$ and $H^0(\lambda)$ with multiplicity one. Consequently, $L_J(\lambda)$ must occur in $\mathrm{soc}_{L_J} L(\lambda)$.

One can also apply the same argument for Weyl modules and see that

$$V(\lambda)|_{L_J} \cong V_J(\lambda) \oplus M'. \tag{4}$$

for some L_J-module M'. By an argument dual to the one in the preceding paragraph, we deduce that $L_J(\lambda)$ appears in the head of $L(\lambda)|_{L_J}$. The fact that $L_J(\lambda)$ has multiplicity one in $L(\lambda)$ now shows that $L_J(\lambda)$ is an L_J-direct summand of $L(\lambda)$. □

4. The case of fundamental weights

We now verify Theorem 1.1 for every fundamental weight. We abuse notation and write $V(\lambda)$ instead of $V(\lambda) \otimes k$.

Type A_n ($n \geq 1$). In this case, all the fundamental weights are minuscule, so $V(\lambda) = L(\lambda)$ for all $\lambda = \omega_j$, $j = 1, 2, \ldots, n$.

Type B_n ($n \geq 2$). For B_n, we claim that $V(\omega_i)$ *is reducible for* $1 \leq i < n$ *and* $\mathrm{char}\, k = 2$.

The split adjoint group of type B_n is $\mathrm{SO}(q)$ for a quadratic form q on a vector space X of dimension $2n+1$ where the tautological action on X is $V(\omega_1)$, see [KMRT98] or [Bor91, §23]. As $\mathrm{char}\, k = 2$, the bilinear form b_q deduced from q by the formula $b_q(x, y) := q(x + y) - q(x) - q(y)$ is necessarily degenerate with 1-dimensional radical, providing an $\mathrm{SO}(q)$-invariant line, call it S.

For $2 \leq i < n$, we restrict to the Levi subgroup of type B_{n-i+1} corresponding to $J = \{\alpha_i, \alpha_{i+1}, \ldots, \alpha_n\}$. By the previous paragraph, $V_J(\omega_i)$ is reducible in characteristic 2, hence $V(\omega_i)$ is by Theorem 3.1.

Alternatively, one can see the reducibility concretely by noticing that $V(\omega_i)$ has the same character and dimension as $\Lambda^i(V(\omega_1))$, because this is so in case $k = \mathbb{C}$. In particular, $\Lambda^i(V(\omega_1))$ has a unique maximal weight, the highest weight of $V(\omega_i)$, and there is a nonzero $\mathrm{SO}(q)$-equivariant map $\phi\colon V(\omega_i) \to \Lambda^i(V(\omega_1))$. As $S \wedge \Lambda^{i-1}(V(\omega_1))$ is a proper and $\mathrm{SO}(q)$-invariant subspace of $\Lambda^i(V(\omega_1))$, it follows that $V(\omega_i)$ is reducible.

Type D_n ($n \geq 4$). For type D_n, we claim that $V(\omega_i)$ *is reducible for* $2 \leq i \leq n-2$ *and* $\mathrm{char}\, k = 2$.

The representation $V(\omega_2)$ has the same character and dimension as $\Lambda^2(V(\omega_1))$. The alternating bilinear form b_q deduced from the invariant quadratic form q on $V(\omega_1)$ gives an invariant line in $\Lambda^2(V(\omega_1))$ — i.e., D_n maps into C_n, which is already reducible on $\Lambda^2(V(\omega_1))$ — proving the claim for $i = 2$.

Alternatively, $V(\omega_2)$ is the adjoint action on the Lie algebra of Spin_{2n} (when $\mathrm{char}\, k = 2$, this is distinct from $\mathrm{Lie}(\mathrm{SO}_{2n})$), and the center S is a proper submodule, namely $\mathrm{Lie}(\mu_2 \times \mu_2)$ (if n is even) or $\mathrm{Lie}(\mu_4)$ (if n is odd).

For $2 < i \leq n-2$, we may use either of the arguments employed in the B_n case.

Type C_n ($n \geq 3$). For type C_n with $n \geq 3$, [PS83, Th. 2(iv)] gives that $V(\omega_2)$ is reducible when $\mathrm{char}\, k = p$ if and only if the prime p divides n, compare [Jan03, p. 287]. For ω_i with $2 < i < n$, restricting to the Levi of type C_{n-i+2} corresponding to $J = \{\alpha_{i-1}, \alpha_i, \ldots, \alpha_n\}$ shows that $V_J(\omega_i)$ is reducible if p divides $n - i + 2$. For $i = n$, we restrict to the Levi subgroup of type $C_2 = B_2$ corresponding to $J = \{\alpha_{n-1}, \alpha_n\}$ to find that $V_J(\omega_n)$ is the 5-dimensional natural module for B_2, which is reducible in characteristic 2.

Exceptional types. For exceptional types, tables of which fundamental weights ω have $V(\omega)$ reducible in which characteristics can be found in [Jan91, p. 299] or, for smaller dimensions, in [Lüb01]. These confirm our main theorem, and, in case $V(\omega) \otimes k$ is reducible for some k, it is so for a k with $\mathrm{char}\, k = 2$ or 3.

We remark that for the representations $V(\omega_i)$ of E_8 for $i \neq 8$, one can verify Theorem 1.1 by restricting to a Levi and using induction, instead of referring to [Jan91] or [Lüb01] directly.

Because it is such an important example, we mention specifically that the adjoint representation $V(\omega_8)$ of E_8 is irreducible because $\mathrm{Lie}(E_8)$ is simple for every field, see [Ste61] or [Hog82]. Here is an alternative argument provided to us by Gross: As E_8 is simply-laced, the Weyl group acts transitively on the roots, so the normalizer $N_{E_8}(T)$ of a split torus has an irreducible submodule in the adjoint representation $\mathrm{Lie}(E_8)$ given by the sum of all the root spaces. The miracle that is special to E_8 is that the Weyl group acts irreducibly on the submodule $\mathrm{Lie}(T)$, which is the E_8-lattice mod $\mathrm{char}\, k$.[3] Then the restriction of the representation $\mathrm{Lie}(E_8)$ to $N_{E_8}(T)$ is the direct sum of two irreducible representations, one of dimension 240 and the other of dimension 8. Since E_8 has no nontrivial map into SL_8, it does not preserve either submodule, and so acts irreducibly on $\mathrm{Lie}(E_8)$.

This is in contrast to the case where G is simple of type other than E_8, where $N_G(T)$ acts reducibly on $\mathrm{Lie}(T)$ for some characteristic (2 for types B, C, D, E_7 and F_4; 3 for E_6 and G_2; and dividing n for type A_{n-1}). And of course if G has roots of different lengths and is simply connected, then for $\mathrm{char}\, k = 2$ or 3, the short roots generate a subalgebra of $\mathrm{Lie}(G)$ invariant under G, see e.g. [His84, Lemma 3.2] or [Ste61, p. 1121].

Here is yet another argument to see that $\mathrm{Lie}(E_8)$ is an irreducible representation for every field k. Namely, it is a special case of the following observation: *If G is simple and simply connected over a field k, the center Z of G is étale*[4]*, and all the roots of G have the same length, then $\mathrm{Lie}(G)$ is an irreducible representation of G.* To prove this general statement, note that the natural map $\mathrm{Lie}(G) \to \mathrm{Lie}(G/Z)$ has kernel $\mathrm{Lie}(Z) = 0$, so is an isomorphism by dimension count. But the domain is the Weyl module $V(\widetilde{\alpha})$ and the codomain is its dual $V(\widetilde{\alpha})^* = H^0(\widetilde{\alpha})$ because of the assumption on the roots [Gar09, 3.5]. Since $V(\widetilde{\alpha}) \cong H^0(\widetilde{\alpha})$, they are irreducible G-modules.

5. TYPE B_n, WEIGHT $\omega_1 + \omega_n$

Let k be an algebraically closed field of characteristic $p \geq 0$. Let $G = \mathrm{Spin}_{2n+1}(k)$ for $n \geq 2$. The irreducible G-module $L(\omega_1)$ has dimension $2n+1$ if $\mathrm{char}\, k \neq 2$ and

[3] This is an illustration of a specific case, for G the Weyl group of E_8, of Thompson's question mentioned in the introduction.

[4] For example, this is true if $\mathrm{char}\, k$ is "very good" for G.

dimension $2n$ if $\operatorname{char} k = 2$. Moreover, the irreducible G-module $L(\omega_n)$ is the spin module for G of dimension 2^n. In this section we show the following, which amounts to a specific case of Theorem 1.1. Although a different proof can be found in Lemmas 2.3.4 and 2.2.7 of [BGT16], we include the proof below because it is a nice illustration of the use of finite group theory to prove a result about connected algebraic groups.

Theorem 5.1. *Let $G = \operatorname{Spin}_{2n+1}(k)$ with $n \geq 2$. Then*

$$\dim L(\omega_1 + \omega_n) = \begin{cases} 2^n \cdot 2n & \text{if } \operatorname{char} k \text{ does not divide } 2n+1; \\ 2^n \cdot (2n-1) & \text{if } \operatorname{char} k \text{ does divide } 2n+1. \end{cases}$$

The proof will appear at the end of the section. The analysis will entail the restriction of modules to a monomial subgroup of SO_{2n+1} via its lift to $\operatorname{Spin}_{2n+1}$ and the use of permutation modules for the alternating group.

Let $U := L(\omega_1) \otimes L(\omega_n)$. If $p = 0$ (and so also for all but finitely many p), this is a direct sum of two composition factors $L(\omega_1 + \omega_n)$ and $L(\omega_n)$. In particular, the Weyl module for the dominant weight $\omega_1 + \omega_n$ has dimension $2n \cdot 2^n$.

If $p = 2$ then as in [Ste63], U is $L(\omega_1 + \omega_n)$, verifying the theorem. We assume for the rest of the section that p is odd.

Note that in $G/Z(G) = \mathrm{SO}_{2n+1}(k)$, there is a finite subgroup X isomorphic to $A.A_{2n+1}$ where A is an elementary abelian 2-group of rank $2n$ and A_{2n+1} denotes the alternating group on $2n+1$ symbols. The group X is the derived subgroup of the group of orthogonal transformations preserving an orthogonal set of $2n+1$ lines.

Let H denote the lift of X to G. Let E be the lift of A to G. First we note:

Lemma 5.2. *E is extraspecial of order 2^{1+2n}.*

Proof. Since X acts irreducibly on $E/Z(G)$, E is either elementary abelian or extraspecial of the given order. By induction it suffices to see that E is nonabelian in the case $n = 2$ (actually we could start with $n = 1$). This is clear since $\operatorname{Spin}_5(k) \cong \mathrm{Sp}_4(k)$ and so contains no rank 5 elementary abelian 2-groups. □

Note that $H/E \cong A_{2n+1}$. Let H_1 be a subgroup of H containing E with $H_1/E \cong A_{2n}$.

The group E has a unique faithful irreducible module over k of dimension 2^n that is the restriction of $L(\omega_n)$. (It is a tensor product of n 2-dimensional representations of the central factors of E, cf. [Gor80, 5.5.4, 5.5.5].) Since $Z(G) = Z(E)$ acts nontrivially on U, every composition factor for E on U is isomorphic to $L(\omega_n)$. It follows immediately that $L(\omega_1)$ and $L(\omega_n)$ are each irreducible for H. Note also that $L(\omega_1)$ is induced from a linear character ϕ of H_1. Thus, as an H-module, $U \cong L(\omega_n) \otimes \phi_{H_1}^H$. In fact, we see that we can replace ϕ by the trivial character of H_1:

Lemma 5.3. *$U \cong L(\omega_n) \otimes k_{H_1}^H$ as an H-module.*

Proof. It suffices to show that $L(\omega_n) \otimes \phi_{H_1} \cong L(\omega_n) \otimes k_{H_1}$ as H_1-modules. Note that they are both irreducible since they are irreducible E-modules. If $n = 2$, the result is easy to see (alternatively, one can modify the argument below). So assume that $n > 2$.

In fact, we observe that any H_1-module V_1 that is isomorphic to $L(\omega_n)$ as an E-module is isomorphic to $L(\omega_n)$ as an H_1-module. This follows by noting that $\operatorname{Hom}_E(L(\omega_n), V_1)$ is 1-dimensional and since H_1/E is perfect, H_1/E acts trivially on this 1-dimensional space, whence $\mathrm{Hom}_{H_1}(L(\omega_n), V_1)$ is also 1-dimensional. Since the two modules are irreducible, this shows they are isomorphic. □

Lemma 5.4. $\dim \mathrm{Hom}_H(U, U) = 2$.

Proof. This follows by Lemma 5.3 and Frobenius reciprocity. □

Let V be the unique nontrivial composition factor of $k_{A_{2n}}^{A_{2n+1}}$ (for $n > 1$). This has dimension $2n$ if p does not divide $2n+1$ and dimension $2n-1$ if p does divide $2n+1$.

By [Dad80] or [Nav98, Cor. 8.19], we know:

Lemma 5.5. *Viewing V as an H-module (that is trivial on E), $L(\omega_n) \otimes_k V$ is irreducible.* □

Proof of Theorem 5.1. From the above, we see that U has two H-composition factors if p does not divide $2n+1$ and three composition factors if p does divide $2n+1$. This immediately implies that if p does not divide $2n+1$, then $L(\omega_1+\omega_n)$ is irreducible for H and has dimension $2n \cdot 2^n$ (whence also for G).

Now assume that p does divide $2n+1$. For sake of contradiction, suppose that $L(\omega_1 + \omega_n)$ has the same dimension as the Weyl module $V(\omega_1+\omega_n)$ for G, so U has precisely two nonisomorphic composition factors as a G-module, $L(\omega_1+\omega_n)$ and $L(\omega_n)$. Since U is self-dual it would be a direct sum of the two modules.

Recall U has three H-composition factors (two isomorphic to $L(\omega_n)$). Thus, the G-submodule $L(\omega_1+\omega_n)$ must have two nonisomorphic H-composition factors. Again, since $L(\omega_1+\omega_n)$ is self dual, this implies that U is a direct sum of three simple H-modules. This contradicts Lemma 5.4 and completes the proof of Theorem 5.1. □

Our analysis shows that when $p \mid 2n+1$ the Weyl module $V(\omega_1+\omega_n)$ has two composition factors: $L(\omega_1+\omega_n)$, $L(\omega_n)$. Therefore, one can apply [Jan03, II.2.14] to determine Ext^1 between these simple modules.

Corollary 5.6. *Let $G = \mathrm{Spin}_{2n+1}$. Then*

$$\dim \mathrm{Ext}^1_G(L(\omega_1+\omega_n), L(\omega_n)) = \begin{cases} 0 & \textit{if } \mathrm{char}\, k \textit{ does not divide } 2n+1; \\ 1 & \textit{if } \mathrm{char}\, k \textit{ does divide } 2n+1. \end{cases}$$

6. Proof of Theorem 1.1

We now prove Theorem 1.1 by induction on the Lie rank of G.

Type A_1. In case of rank 1, G is SL_2 and $V(d) \otimes k$ is irreducible if and only if, for $p = \mathrm{char}\, k$, $d+1 = cp^e$ for some $0 < c < p$ and $e \geq 0$ [Win77, pp. 239, 240]. (This can be seen by comparing dimensions: Write out d in base p as $d = \sum_i c_i p^i$. Then $\dim V(d) = d+1$ whereas the irreducible module $L(d)$ over k has dimension $\prod_i (c_i+1)$ by Steinberg's tensor product theorem.) As a consequence, for $d \geq 2$, it is impossible for $V(d) \otimes \mathbb{F}_p$ to be irreducible for both $p = 2$ and 3.

An alternate argument (as noted by Andersen) can be provided if one does not require $p \leq 3$. Choose a prime p dividing d. Now $\dim V(d) = d+1$ with $V(d) \twoheadrightarrow L(d) \cong L(d/p)^{(1)}$. But, $\dim L(d/p)^{(1)} \leq \frac{d}{p} + 1 < d+1 = \dim V(d)$. So $V(d)$ is reducible in characteristic p.

Reductions. So suppose $\operatorname{rank} G \geq 2$ and Theorem 1.1 holds for all groups of lower rank.

Write $\lambda = \sum c_i \omega_i$ with every $c_i \geq 0$. If some $c_i > 1$, then taking $J = \{\alpha_i\}$, the Levi subgroup L_J has semisimple type A_1 and the restriction of $V_J(\lambda)$ to L_J is reducible when $\operatorname{char} k$ is 2 or 3 by the argument for type A_1. Therefore, by Theorem 3.1 we may assume that $c_i \in \{0, 1\}$ for all i.

If $\lambda = 0$ or $\lambda = \omega_i$ for some i, then we are done by §4. Hence, we may assume that at least two of the c_i's are nonzero.

If there is a connected and proper subset J of Δ such that $c_i \neq 0$ for at least two indexes i with $\alpha_i \in J$, then we are done by induction and Theorem 3.1.

Sums of extreme weights. The remaining case is when the Dynkin diagram has no branches (i.e., G has type A, B, C, F_4, or G_2) and $\lambda = \omega_1 + \omega_n$ is the sum of dominant weights corresponding to the simple roots at the two ends of the diagram. For type A_n, $G = \mathrm{SL}_{n+1}$ and $V(\omega_1 + \omega_n)$ is the natural action on $\operatorname{Lie}(\mathrm{SL}_{n+1})$, the trace zero matrices. If p divides $n + 1$, then the scalar matrices are a G-invariant subspace. Type B was handled in Theorem 5.1.

For type C_n with $n \geq 3$, we restrict to the Levi subgroup of type C_2 and find that $V_J(\omega_1 + \omega_n)$ has dimension 5 and is reducible in characteristic 2. Alternatively, as in [Ste63, §11], in characteristic 2 one finds $L(\omega_1 + \omega_n) \cong L(\omega_1) \otimes L(\omega_n)$, which has dimension $n2^{n+1}$, whereas by the Weyl dimension formula,

$$\dim V(\omega_1 + \omega_n) = 7 \cdot 2^{n-1} \cdot \frac{n(2n+1)}{n+3} \cdot \prod_{i=6}^{n+1} \frac{2i-3}{i} \quad \text{for } n \geq 4.$$

In the case of exceptional groups, for type F_4, $V(\omega_1+\omega_4)$ is reducible in characteristic 2 because it has dimension 1053, yet by [Ste63] $L(\omega_1+\omega_4) \cong L(\omega_1) \otimes L(\omega_4)$ has dimension $26^2 = 676$. For type G_2, $\dim V(\omega_1 + \omega_2) = 64$ yet by Steinberg in characteristic 3 $L(\omega_1 + \omega_2)$ has dimension $7^2 = 49$. Alternatively, one can refer to [Lüb01, Tables A.49, A.50]. This completes the proof of Theorem 1.1. □

7. Complements to Theorem 1.1

Invariant bilinear forms. Let G be a split reductive group over $\mathbb{Z}$ with a split maximal torus T and $\lambda \in X(T)_+$. A G-invariant bilinear form b on the Weyl module $V(\lambda)$ corresponds to a G-equivariant homomorphism $\delta_b \colon V(\lambda) \to V(\lambda)^*$ via $\delta_b(v)(v') = b(v, v')$. The map δ_b is determined by what it does to a highest weight vector in $V(\lambda)$ and in order that δ_b be nonzero it must be that λ is a weight of $V(\lambda)^*$, and in particular that $\lambda \leq -w_0\lambda$, from which it follows that $\lambda = -w_0\lambda$. As the highest weight spaces in $V(\lambda)$ and $V(\lambda)^*$ are rank 1 $\mathbb{Z}$-modules, we conclude that the space of G-invariant bilinear forms on $V(\lambda)$ is $\mathbb{Z}$ if $\lambda = -w_0\lambda$ and 0 otherwise.

So suppose $\lambda = -w_0\lambda$ and let b be an indivisible G-invariant bilinear form on $V(\lambda)$ — it is determined up to sign. For each field k, the map $\delta_b \colon V(\lambda) \otimes k \to V(\lambda)^* \otimes k$ has kernel the unique maximal proper submodule of $V(\lambda) \otimes k$, see [Jan03, II.2.4, II.2.14], so *$b \otimes k$ is nondegenerate if and only if $V(\lambda) \otimes k$ is irreducible.* Therefore, Theorem 1.1 gives:

Corollary 7.1. *Suppose, in the notation of the previous two paragraphs, that G is simple and split and $\lambda = -w_0\lambda$. Then $b \otimes k$ is nondegenerate for every field k if and only if*

(a) *λ is minuscule; or*
(b) *$G = E_8$ and λ is the highest root $\widetilde{\alpha}$.*

Failure of converse to Theorem 3.1. As another complement to Theorem 1.1, we make precise the settings where the converse to "$V(\lambda)$ irreducible implies $V_J(\lambda)$ irreducible" fails.

Theorem 7.2. *Let λ be a dominant weight for G a split, simple, and simply connected group over $\mathbb{Z}$ with $\operatorname{rank} G > 1$. Then $V_J(\lambda) \otimes k$ is irreducible for every $J \subsetneq \Delta$ if and only if one of the following occurs:*

(a) *λ is minuscule.*
(b) *λ is the highest short root α_0.*
(c) *$\Phi = B_n$ and $\lambda = \omega_1 + \omega_n$.*
(d) *$\Phi = G_2$ and $\lambda = \omega_2$ or $\omega_1 + \omega_2$.*

Proof. First assume that one of conditions (a)–(d) holds. Then one can directly verify that $V_J(\lambda) \otimes k$ is irreducible for every $J \subsetneq \Delta$ (by Theorem 1.1).

On the other hand, suppose that $V_J(\lambda) \otimes k$ is irreducible for every $J \subsetneq \Delta$. Write $\lambda = \sum_i a_i\omega_i$. If some $a_i > 1$ or at least three of the a_i are nonzero, then by Theorem 1.1, we see that $V_J(\lambda)$ is not irreducible with J obtained by removing an end node other than i in the first case or any end node in the second case.

Next consider the case when $\lambda = \omega_i + \omega_j$, $i \neq j$. The result follows unless $\{i, j\}$ correspond to all the end nodes. If there are three end nodes, this is not possible. Thus, we only need consider types A, B, C, F and G. If $\Phi = A_n$, this leads to (b). If $\Phi = B_n$, this leads to (c). Moreover, if $\Phi = C_n, n \geq 3$, then $V_J(\lambda) \otimes k$ is reducible for $J = \Delta - \{\alpha_1\}$ and $\operatorname{char} k = 2$. Similarly, if $\Phi = F_4$, $V_J(\lambda) \otimes k$ is reducible for $J = \Delta - \{\alpha_4\}$ and $\operatorname{char} k = 3$. If $\Phi = G_2$, this leads to one of the cases in (d).

It remains to consider the case that $\lambda = \omega_i$ for some i. If $\Phi = A_n$, then ω_i is minuscule. If G has rank 2, then removing a single node gives a Levi of type A_1 and so we have irreducibility as in (a), (b), and (d). So assume that Φ is not of type A_n and has rank at least 3. It suffices to check that for any J obtained by removing an end node that $V_J(\lambda)$ irreducible implies that λ is either minuscule or $\lambda = \alpha_0$.

Suppose that $\Phi = D_n$, $n \geq 4$. If λ is not minuscule and $\lambda \neq \alpha_0 = \omega_2$, then we can remove the first node and see that $V_J(\lambda) \otimes k$ is reducible for $\operatorname{char} k = 2$.

It remains to consider types B, C, E and F. If i does not correspond to an end node, then we can choose J in such a way that the Levi factor L_J of the reduced system does not have type A_n and $V_J(\lambda)$ does not correspond to an end node, whence by Theorem 1.1, $V_J(\lambda)$ is not irreducible.

If G has type B_n or C_n, then ω_1 and ω_n either correspond to the short root or are minuscule for L_J. In the case when G has type E_6, then ω_i corresponding to an end node is either α_0 or minuscule for L_J. If G has type F_4 or $E_n, n \geq 7$, then one checks the only end node satisfying the hypotheses is α_0. □

Connection with B-cohomology. Let B be the Borel of G corresponding to the negative roots. For 2ρ the sum of the positive roots and N the number of positive roots, one can use Serre duality to show that

$$\mathrm{Hom}_G(k, V(-w_0\lambda)) \cong \mathrm{Ext}^N_B(k, \lambda - 2\rho) \cong \mathrm{H}^N(B, \lambda - 2\rho), \tag{5}$$

see [HN06] and [GN16, Theorem 5.5].

For $\lambda = \widetilde{\alpha}$ the highest root, $\widetilde{\alpha} = -w_0\widetilde{\alpha}$ and $V(\widetilde{\alpha})$ is the Lie algebra of the simply connected cover of G. The adjoint representation for E_8 is simple for all $p > 0$, so

$$\mathrm{H}^{120}(B, \widetilde{\alpha} - 2\rho) = 0.$$

On the other hand, if G is of type A_n then

$$\mathrm{H}^{n(n+1)/2}(B, \widetilde{\alpha} - 2\rho) \cong \begin{cases} k & \text{if } p \mid n+1 \\ 0 & \text{if } p \nmid n+1. \end{cases}$$

Similar statements can be formulated for other types.

The calculation of the B-cohomology with coefficients in a one-dimensional representation is an open problem in general. Complete answers are known for degrees 0, 1, and 2 and for most primes in degree 3. See [BNP14] for a survey.

Quantum Groups. For quantum groups (Lusztig $\mathcal{A}$-form) at roots of unity, one can ask when the quantum Weyl modules are globally irreducible. The Weyl modules with minuscule highest weights will yield globally irreducible representations. One can prove an analog of Theorem 3.1 to use Levi factors to reduce to considering fundamental weights or weights of the form $\omega_1 + \omega_n$.

For type A_n, if the root of unity has order l and $l \mid n+1$ then $V(\omega_1 + \omega_n)$ is not simple (see [Fay05]). This uses representation theory of the Hecke algebra of type A_n. From this one can prove the analog of our main theorem (Theorem 1.1) for quantum groups in the A_n case.

In order to handle root systems other than A_n, more detailed information needs to be worked out such as the the tables given in [Jan91] and analogs of results for Weyl modules in type C_n as given in [PS83].

Further Directions. Suppose now that G is a split simply connected algebraic group over $\mathbb{Z}$ and λ is a dominant weight. In a preliminary version of this manuscript, we asked to what extent is the following statement true: *If μ is a dominant weight that is maximal among the dominant weights $< \lambda$, then there is a field k such that $V(\lambda) \otimes k$ has $L(\mu)$ as a composition factor.* Certainly, it is false for $G = E_8$, λ the highest root, and $\mu = 0$. Jantzen has recently shown in [Jan16] that, apart from this one counterexample, the statement holds when G is simple. Note that, in contrast to Theorem 1.1, this result does not include an upper bound on $\mathrm{char}\, k$ that only depends on the rank of G. For example, take $G = \mathrm{SL}_2$ and pick a prime p and a $d > p$ not divisible by p. Then $d - 2$ is a weight of the irreducible representation $L(d)$ with highest weight d over $\mathbb{F}_p$, so $L(d-2)$ is not in the composition series for $V(d) \otimes \mathbb{F}_p$.

8. Appendix: the Killing form over $\mathbb{Z}$

The reduced Killing form. Let G be a split simple algebraic group over $\mathbb{Z}$. There is a canonical indivisible G-invariant bilinear form on $\mathrm{Lie}(G)$, the *reduced Killing form*, which we denote b_G. In this appendix, we prove the following result, which is similar in flavor to Theorem 1.1 and answers a question raised by George Lusztig.

Theorem 8.1. *The reduced Killing form on* $\mathrm{Lie}(G) \otimes k$ *is nondegenerate for every field* k *if and only if* G *is one of the following groups:*

(a) E_8;
(b) SO_{2n} *for some* $n \geq 4$;
(c) HSpin_{2n} *for* n *divisible by* 4; *or*
(d) SL_{m^2}/μ_m *for some* $m > 1$.

We actually prove something more. The Lie algebra of G is a free $\mathbb{Z}$-module, so it has a basis $v_1, \ldots, v_n$ for some n. The determinant of b_G (denoted $\det b_G$) is the determinant of the matrix with (i, j)-entry $b_G(v_i, v_j)$; note that $\det b_G$, as an element of $\mathbb{Z}$, does not depend on the choice of basis. Furthermore, $b_G \otimes k$ is degenerate if and only if $\det b_G = 0$ in k. The point of this appendix is to calculate $\det b_G$ for every simple G over $\mathbb{Z}$, from which Theorem 8.1 quickly follows.

The case where G is simply connected. In case G is simply connected, the Killing form on $\mathrm{Lie}(G)$ — a bilinear form over $\mathbb{Z}$ — is divisible by $2h^\vee$ for $h^\vee$ the dual Coxeter number of G; we define b_G to be the quotient. It is even and indivisible [GN04, Prop. 4]. It is natural to call b_G the reduced Killing form because it is obtained from the Killing form by dividing by the greatest common divisor of its values. (Note that b_G has the advantage that $b_G \otimes k$ is nonzero for every k, a property not satisfied by the Killing form.)

For simply connected G, $\det b_G$ was calculated in [SS70, I.4.8(a)]. Specifically, let N denote the number of positive roots, N_s denote the number of short roots, and N_{ss} the number of short simple roots. Put c for the ratio of the square-lengths of the long to short roots (so $c \in \{1, 2, 3\}$) and f for the determinant of the Cartan matrix. Let T be a split maximal torus in G over $\mathbb{Z}$. Then $\mathrm{Lie}(G)$ is an orthogonal sum of $\mathfrak{t} := \mathrm{Lie}(T)$ and a subspace $\mathfrak{n}$ spanned by the root subalgebras of G with respect to T. One checks that $\det b_G|_{\mathfrak{n}} = (-1)^N c^{N_{ss}}$. The Lie algebra $\mathfrak{t}$ is naturally identified with the coroot lattice $Q^\vee$, and this identifies the restriction of b_G to $\mathfrak{t}$ with the Weyl-group invariant bilinear form such that $b_G(\alpha^\vee, \alpha^\vee) = 2$ for every short coroot $\alpha^\vee$. In summary, one finds that

$$\det b_G = (-1)^N c^{N_s + N_{ss}} f \quad \text{for } G \text{ simply connected.} \tag{6}$$

The value of $\det b_G$ can be found in Table 2. The conflicting values given in the table on p. 634 of [GN04] are typos.

Definition of reduced Killing form for G not simply connected. Suppose that G is split simple over $\mathbb{Z}$ and let $f \colon \widetilde{G} \to G$ denote the simply connected cover in the sense of [DG11, XXI.6.2.6, XXII.4.3.3]. Note that $\widetilde{G}$ and G may have distinct Lie algebras; the natural map $\mathrm{Lie}(\widetilde{G}) \otimes k \to \mathrm{Lie}(G) \otimes k$ has kernel $\mathrm{Lie}(\ker f \times k)$, which may be nonzero. The differential $\mathrm{d}f : \mathrm{Lie}(\widetilde{G}) \to \mathrm{Lie}(G)$ gives an isomorphism $\mathrm{d}f_{\mathbb{Q}} : \mathrm{Lie}(\widetilde{G}) \otimes \mathbb{Q} \to \mathrm{Lie}(G) \otimes \mathbb{Q}$; pushing forward $b_{\widetilde{G}}$ gives a G-invariant bilinear form $\hat{b}_G$ on $\mathrm{Lie}(G) \otimes \mathbb{Q}$ and we define the reduced Killing form on $\mathrm{Lie}(G)$ to be $b_G := e \cdot \hat{b}_G$, where e is the smallest positive rational number such that the resulting b_G has integer values. It follows from the indivisibility of $b_{\widetilde{G}}$ that e is an integer.

Killing-Cartan type of G	G simply connected $\det b_G$	G adjoint e	$\det b_G$
A_n	$(-1)^{(n^2+n)/2}(n+1)$	$n+1$	$(-1)^{(n^2+n)/2}(n+1)^{n^2+2n-1}$
B_n	$(-1)^n 2^{2n+2}$	1	$(-1)^n 2^{2n}$
C_n	$(-1)^n 2^{2n^2-n}$	$\rho(n)$	$(-1)^n 2^{2n^2-n-2}\rho(n)^{2n^2+n}$
D_n $(n \geq 4)$	2^2	$2\rho(n)$	$2^{2n^2-n-2}\rho(n)^{2n^2-n}$
G_2	3^7		$\cdots$
F_4	2^{26}		$\cdots$
E_6	3	3	3^{77}
E_7	-2	2	-2^{132}
E_8	1		$\cdots$

TABLE 2. Determinant of the reduced Killing form for G simply connected or adjoint. The value of e is taken from [Gar09, §3]. The notation $\rho(n)$ means 1 if n is even and 2 if n is odd.

Pick a pinning of G with respect to a split maximal torus T, and fix a corresponding pinning of $\widetilde{G}$ with respect to the maximal torus $\widetilde{T} := f^{-1}(T)^\circ$. The two groups have the same root system and $\mathrm{d}f$ restricts to give an isomorphism for each of the 1-dimensional root subalgebras of $\mathrm{Lie}(G)$ with the corresponding root subalgebra of $\mathrm{Lie}(\widetilde{G})$. The map $\mathrm{d}f$ embeds $\mathrm{Lie}(\widetilde{T}) = Q^\vee$ in $\mathrm{Lie}(T)$.

In case G is adjoint, [SS70, I.4.8(b)] says that $\det \hat{b}_G = (\det b_G)/f^2$ and therefore $\det b_G = e^{\dim G}(\det b_{\widetilde{G}})/f^2$. This gives the values in the last column of Table 2. Note that $\mathrm{Lie}(T)$ is naturally identified with the lattice $P^\vee$ of weights for the dual root system.

Groups that are neither simply connected nor adjoint. It remains to treat the case where G is neither simply connected nor adjoint. Recall that for even $n > 4$, the simply connected group Spin_{2n} has two non-isomorphic quotients by a central μ_2: SO_{2n} and one more called a *half-spin group*; we denote it by HSpin_{2n}. (For $n = 4$, HSpin_8 is defined, but it is isomorphic to SO_8.) So suppose G is SL_n/μ_m for some $m \mid n$, SO_{2n} for some $n \geq 4$, or HSpin_{2n} for some even $n \geq 4$. For each of these three possibilities, we determine $\hat{b}_G|_{\mathrm{Lie}(T)}$ and e.

As all roots have the same length, we use the canonical identification of the root system with its dual and calculate $\mathrm{Lie}(T)$ as a sublattice of the weight lattice P. We may identify $\mathrm{Lie}(T)$ by noting that for each possibility for G, $\mathrm{Lie}(T)/Q$ is cyclic, so it suffices to find a fundamental weight ω such that $\mathrm{Lie}(T)$ is generated by ω and Q, and to find a simple root α and $c \in \mathbb{N}$ so that $c\omega$ is in Q and is a sum of α and a linear combination of the other simple roots. Then $\{\omega\} \cup \Delta \setminus \{\alpha\}$ is a basis for $\mathrm{Lie}(T)$. We take, with roots numbered as in Table 1:

- for $G = \mathrm{SL}_n/\mu_m$: $\omega = (n/m)\omega_{n-1}$, $\alpha = \alpha_1$, $c = m$;
- for $G = \mathrm{SO}_{2n}$: $\omega = \omega_1$, $\alpha = \alpha_n$, $c = 2$; or
- for $G = \mathrm{HSpin}_{2n}$: $\omega = \omega_n$, $\alpha = \alpha_1$, $c = 2$,

see [Gar09, 3.6, 3.7, 5.2], although a slightly different choice of basis was taken for HSpin_{2n} there. (We can now see e: as $\hat{b}_G(\omega, \omega) = n(n-1)/m^2$ (as in [Gar09, Lemma

5.2]), 1, and $n/4$ respectively, and $\hat{b}_G(\omega, Q) \subseteq \mathbb{Z}$, we find that $e = m/\gcd(m, n/m)$, 1, and $\rho(n/2)$ respectively.)

We can write down the Gram matrix for $\det \hat{b}_G|_{\mathrm{Lie}(T)}$ with respect to the basis ordered by taking the simple roots in the Bourbaki ordering, deleting α, and appending ω. This leads to $\det b_G$ via the formula

$$\det b_G = (-1)^N e^{\dim G} \det \hat{b}_G|_{\mathrm{Lie}(T)}, \tag{7}$$

which follows from the definition of e and the fact that $\mathrm{d}f$ restricts to an isomorphism on the span of the root subalgebras.

For $G = \mathrm{SL}_n/\mu_m$, the Gram matrix is $(n-1)$-by-$(n-1)$, with a Cartan matrix of type A_{n-2} in the upper left corner and the right column and bottom row are both $(0, \ldots, 0, n/m, n(n-1)/m^2)$, so its determinant is n/m^2. Equation (7) gives that

$$\det b_{\mathrm{SL}_n/\mu_m} = (-1)^{\binom{n}{2}} \frac{n}{m^2} \left(\frac{m}{\gcd(m, n/m)} \right)^{n^2-1}. \tag{8}$$

For $G = \mathrm{SO}_{2n}$, the Gram matrix is n-by-n, has a Cartan matrix of type A_{n-1} in the upper left corner, and has right column and bottom row both $(1, 0, \ldots, 0, 1)$. So its determinant is 1, and $\det b_G = 1$.

For $G = \mathrm{HSpin}_{2n}$, the Gram matrix is n-by-n, has a Cartan matrix of type D_{n-1} in the upper left corner, and has right column and bottom row both $(0, \ldots, 0, 1, n/4)$. So its determinant is 1, and $\det b_G = \rho(n/2)$.

Proof of Theorem 8.1. Given the calculation of $\det b_G$ above, it suffices to consider the case $G = \mathrm{SL}_n/\mu_m$ where $n \geq 4$. If $n = m^2$, then $\det b_G = \pm 1$ and b_G is nondegenerate. Conversely, assume b_G is nondegenerate, so the restriction to the root subalgebras must be nondegenerate, i.e., $e = 1$, equivalently, $\gcd(m, n/m) = m$, equivalently, m^2 divides n. Then $\det b_G = \pm n/m^2$. □

Remark. The subdivision of groups of type D_n for n even into the cases $n \equiv 0, 2 \bmod 4$, as seen in Theorem 8.1(c), can also be seen in the representation theory of these groups: the half-spin representation of Spin_{2n} over $\mathbb{C}$ is orthogonal for n divisible by 4 and symplectic for $n \equiv 2 \bmod 4$, see [Bou05, Ch. VIII, Table 1] or [KMRT98, 8.4].

Example 8.2. Let $G := \mathrm{SL}_{p^d}/\mu_p$ for some prime p and some $d \geq 3$. Then the reduced Killing form $b_G \otimes k$ on $\mathrm{Lie}(G) \otimes k$ is degenerate when $\mathrm{char}\, k = p$, where the element ω in $\mathrm{Lie}(T) \otimes k$ is in the radical. Yet $\mathrm{Lie}(G) \otimes k$ is self-dual because $\mathrm{Lie}(G) \otimes k$ is a direct sum of the self-dual representations $L(\widetilde{\alpha})$ and k by [Hum67, 9.4]. Furthermore, by (8), $b_G \otimes k'$ is non-degenerate for every field k' of characteristic different from p.

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