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On acyclic anyon models

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Abstract
Acyclic anyon models are non-abelian anyon models for which thermal anyon errors can be corrected. In this note, we characterize acyclic anyon models and raise the question whether the restriction to acyclic anyon models is a deficiency of the current protocol or could it be intrinsically related to the computational power of non-abelian anyons. We also obtain general results on acyclic anyon models and find new acyclic anyon models such as $SO(8)_2$ and the representation theory of Drinfeld doubles of nilpotent finite groups.

Keywords Nilpotent modular category · Braiding · Anyon · Error correction

Mathematics Subject Classification 16W30 · 18D10 · 19D23

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1 Introduction

In topological quantum computing (TQC), information is encoded in the ground-state manifolds of topological phases of matter which are error correction codes. Therefore, TQC is intrinsically fault tolerant against local errors. But at any finite temperature $T > 0$, thermal anyon pairs created from the vacuum due to thermal fluctuations can diffuse and braid with computational anyons to cause errors, the so-called thermal anyon errors. In practice, thermal anyon creations are suppressed by the energy gap Δ and low temperature T as $\alpha e^{-\frac{\Delta}{T}}$ for some positive constant α, so it might not pose a serious challenge. But if the suppression by gap and temperature is not enough, then thermal anyon errors could become a serious issue for long quantum computation. In [8], the authors found an error correction scheme for acyclic anyon models (called non-cyclic in [8]). In this paper, we characterize acyclic anyon models as anyon models with nilpotent fusion rules. We obtain several general results on acyclic anyon models and find many more acyclic anyon models such as $SO(8)_2$, which has Property F.

Our characterization of acyclic anyon models raises the question whether the restriction to acyclic anyon models is a deficiency in the current protocol or could it be intrinsically related to the computational power of non-abelian anyons. A triality exists for the computational power of non-abelian anyons as illustrated by the anyon models $SU(2)_k$, $k = 2, 3, 4$. The type of anyons in $SU(2)_k$ is labeled by the truncated angular momenta in $\{0, 1/2, \ldots, k/2\}$ and let s be the spin$=1/2$ anyon. When $k = 2$, s is essentially the Ising anyon σ, not only it is not braiding universal, but also all braiding circuits can be efficiently simulated by a Turing machine. Moreover, it is believed that all measurements of total charges can also be efficiently simulated classically. When $k = 3$, s is the Fibonacci anyon, which is braiding universal [12]. When $k = 4$, s is a metaplectic anyon which is not braiding universal. But supplemented by a total charge measurement, a universal quantum computing model can be designed based on the metaplectic anyon s [6]. While $SU(2)_2$ is acyclic, neither $SU(2)_3$ nor $SU(2)_4$ is. Since acyclic anyon models are weakly integral (proved below), they should not be braiding universal as the property F conjecture suggests [16]. Therefore, it would be interesting to know whether any acyclic model can be made universal when supplemented with total charge measurements. If not, then whether or not the protocols in [8] can be generalized to go beyond acyclic anyon models.

2 Preliminaries

An anyon model is mathematically a unitary modular tensor category—a very difficult and complicated structure [17]. But the fusion rule of an anyon model is completely elementary. Our main result is a theorem about fusion rules, so we start with the basics of fusion rules to make the characterization of acyclic anyon model self-contained.

2.1 Fusion rules

A *fusion rule (A,N)* based on the finite set A is a collection of nonnegative integers $\{N_{ij}^k\}$ as below, where the elements of A will be called anyon types or particle types or topological charges. The elements in A will be denoted by $x_1, x_2, x_3, \ldots$, and the number of elements of A will be called the *rank* of (A, N). A fusion rule is really the pair (A, N), but in the following we sometimes simply refer to the set A or the set of integers $\{N_{ij}^k\}$ as the fusion rule when no confusion would arise.

For every particle type x_i, there exists a unique dual or anti-particle type that we denote by $\overline{x}_i = x_{\bar{i}}$. There is a trivial or "vacuum" particle type denoted by 1.

The *fusion rules* can be conveniently organized into formal fusion product and sum of particle types (mathematically such formal product and sum can be made into operations of a fusion algebra where particle types are bases elements of the fusion algebra):

$$x_i x_j = \sum_k N_{i,j}^k x_k,$$

where $N_{i,j}^k \in \mathbb{Z}^{\geq 0}$. The fusion rules obey the following relations

(a) Associativity: $(x_i x_j)x_k = x_i(x_j x_k)$,
(b) The vacuum is the identity for the fusion product,

$$x_i 1 = x_i = 1 x_i,$$

(c) The anti-particle type $x_i \mapsto \overline{x_i} = x_{\bar{i}}$ defines an involution, that is,

$$\overline{1} = 1, \quad \overline{x_{\bar{i}}} = x_i, \quad x_{\bar{j}} x_{\bar{i}} = \overline{x_i x_j},$$

where

$$\overline{x_i x_j} := \sum_k N_{i,j}^k x_{\bar{k}},$$

(d) The fusion of x_i with its anti-particle $x_{\bar{i}}$ contains the vacuum with multiplicity one, that is,

$$N_{i,\bar{i}}^1 = 1.$$

A fusion rule is called *abelian* (or pointed) if

$$\sum_k N_{i,j}^k = 1$$

for every x_i and x_j. If A is an abelian fusion rule, then the fusion product defines a group structure on A and conversely every group defines an abelian fusion rule.

2.2 Nilpotent fusion rules

Let (A, N) be a fusion rule on the set A. A *sub-fusion rule* of (A, N) is a subset $B \subset A$ such that

(a) $1 \in B$,
(b) $x_i \in B$ if and only if $x_{\bar{i}} \in B$,
(c) if $x_i, x_j \in B$, then $N_{i,j}^k > 0$ implies $x_k \in B$.

The rank of the fusion rule A is $|A|$, the cardinality of the set A.

Definition 2.1 [13] Let A_{ad} be the minimal sub-fusion rule of A with the property that $x_i x_{\bar{i}}$ belongs to A_{ad} for all $x_i \in A$; that is, A_{ad} is generated by all particle types $x_k \in A$ such that $N_{i,\bar{i}}^k > 0$ for some $x_k \in A$.

Definition 2.2 [13, Definition 4.2] The *descending central series* of A is the sequence of sub-fusion rules

$$\cdots A^{(n+1)} \subseteq A^{(n)} \subseteq \cdots \subseteq A^{(1)} \subseteq A^{(0)} = A,$$

defined recursively as $A^{(n+1)} = A_{\text{ad}}^{(n)}$, for all $n \geq 0$.

Definition 2.3 [13, Definition 4.4] A fusion rule is called *nilpotent*, if there exists an $n \in \mathbb{N}$ such that $A^{(n)}$ has rank one. The smallest number n for which this happens is called the *nilpotency class* of A.

3 Acyclic fusion rules are nilpotent

In this section, we prove our main result.

3.1 Acyclic fusion rules

Definition 3.1 [8] A fusion rule A is called *acyclic* if for any value of $n \in \mathbb{N}$ and for any sequence

$$(x_{i_1} = x_{i_{n+1}}, x_{i_n}, \ldots, x_{i_3}, x_{i_2}, x_{i_1})$$

with $x_{i_1} \neq \mathbf{1}$, we have that

$$\prod_{k=1}^{n} N_{i_{k+1}, \overline{i_{k+1}}}^{i_k} = 0.$$

To any fusion rule, we may associate its *adjoint graph* defined as follows [8]: The vertices are pairs $X_i := (x_i, x_{\bar{i}})$ and a directed edge is drawn from $X_i \neq (\mathbf{1}, \mathbf{1})$ to X_j if $N_{i,\bar{i}}^j \neq 0$. Notice that this is unambiguous since $N_{i,\bar{i}}^j = N_{i,\bar{i}}^{\bar{j}}$.

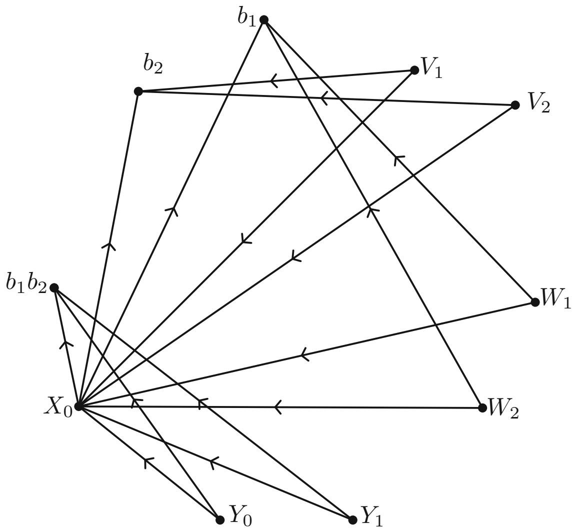

Fig. 1 $SO(8)_2$ adjoint graph, with **1** removed: b_1, b_2, b_1b_2 are bosons and the remaining objects have dimension 2. All objects are self-dual, so we may abbreviate the vertex labels

Now we can say that a fusion rule is acyclic if its adjoint graph contains no directed cycles, as in [8]. Moreover, since (by definition) no directed edges emanate from **1**, we may ignore the vertex (**1**, **1**) and all incident edges when searching for cycles: No cycle can have (**1**, **1**) as a vertex.

We illustrate this with an example of an acyclic fusion rule: The adjoint graph found in Fig. 1 corresponds to $SO(8)_2$, an integral modular category of dimension 32 and rank 11: The explicit fusion rules are found in [2]. Notice that there are no directed cycles in the adjoint graph of $SO(8)_2$ so its fusion rule is acyclic.

The direct product of two acyclic fusion rules is acyclic as well. Here, the direct product of two fusion rules (A, N) and (B, M) is the fusion rule on $A \times B$ with $(x_i, y_a)(x_j, y_b) = \sum_{k,c} N_{ij}^k M_{ab}^c (x_k, y_c)$. Since $N_{i,\bar{i}}^j M_{a,\bar{a}}^b \neq 0$ iff both $N_{i,\bar{i}}^j \neq 0$ and $M_{a,\bar{a}}^b \neq 0$, the adjoint graph of $(A, N) \times (B, M)$ is the (tensor) product of the adjoint graphs of (A, N) and (B, M). If $(A, N) \times (B, M)$ has a cycle, then either (A, N) or (B, M) has a cycle by projecting, proving that direct products preserve acyclicity.

Lemma 3.2 *Let A be finite acyclic fusion rule with $|A| > 1$. Then, the rank of A_{ad} is strictly smaller than the rank of A.*

Proof Assume that A is acyclic and $A_{\mathrm{ad}} = A$.

For each $n \in \mathbb{N}$, we will define inductively a sequence of bases elements

$$(x_{i_n}, \ldots, x_{i_2}, x_{i_1}) \tag{3.1}$$

such that

(a) $N_{i_{k+1},\overline{i_{k+1}}}^{i_k} > 0$ and $N_{i_k,\overline{i_k}}^{i_{k+1}} = 0$ for all $k < n$.
(b) $x_{i_k} \neq 1$ for all k.
(c) The elements in the sequence are pairwise distinct.

Since A has rank bigger than one, there is an $x_{i_1} \neq 1$. Using that $A_{\mathrm{ad}} = A$, we have that there is x_{i_2} such that $N_{i_2,\overline{i_2}}^{i_1} > 0$. Now, since A is acyclic, using the sequence $(x_{i_1}, x_{i_2}, x_{i_1})$, we have that $N_{i_1,\overline{i_1}}^{i_2} = 0$. In particular, $N_{i_1,\overline{i_1}}^{i_2} = 0$ implies $x_{i_2} \neq 1$. Using the same argument, we can construct for each $n \in \mathbb{N}$ a sequence $(x_{i_n}, \dots, x_{i_1})$ that satisfies (a) and (b).

We will use induction on the length of the sequence (3.1) to see that the elements in the sequence (3.1) are pairwise distinct. For $n = 2$, we have that $N_{i_2,\overline{i_2}}^{i_1} \neq N_{i_1,\overline{i_1}}^{i_2}$, then $x_{i_1} \neq x_{i_2}$. Assume that any sequence of $n-1$ elements satisfying (a) and (b) has pairwise distinct elements. Then, $(x_{i_n}, \dots, x_{i_2})$ and $(x_{i_{n-1}}, \dots, x_{i_1})$ are pairwise distinct. Since A is acyclic, if $x_{i_1} = x_{i_n}$, using the sequence $(x_{i_1} = x_{i_n}, x_{i_{n-1}}, \dots, x_{i_1})$ we have that

$$\prod_{k=1}^{n} N_{i_{k+1},\overline{i_{k+1}}}^{i_k} = 0.$$

But by construction, $N_{i_{k+1},\overline{i_{k+1}}}^{i_k} > 0$; hence, we have a contradiction. In conclusion, the elements in the sequence $(x_{i_n}, \dots, x_{i_1})$ are pairwise distinct.

Finally, since the rank of A is a finite number, and we can construct an arbitrary large sequence of pairwise distinct basic elements, we obtain a contradiction. Thus, if A is a non-trivial acyclic fusion rule, the rank of A_{ad} is strictly smaller than the rank of A. □

Theorem 3.3 *Let A be a fusion rule. Then, A is acyclic if and only if A is nilpotent.*

Proof Clearly, any sub-fusion rule of an acyclic fusion rule is acyclic.

Assume that A is acyclic. Using Lemma 3.2, we obtain that in the sequence

$$\cdots A^{(n+1)} \subseteq A^{(n)} \subseteq \cdots \subseteq A^{(1)} \subseteq A^{(0)} = A,$$

the rank of $A^{(n+1)}$ is strictly smaller than the rank of $A^{(n)}$ if the rank of $A^{(n)}$ is bigger than one. Since the rank of A is finite, there is $m \in \mathbb{N}$ such that the rank of $A^{(m)}$ is one, that is, A is nilpotent.

Assume that A is nilpotent, in particular $A_{\mathrm{ad}} \neq A$. We will use induction on the nilpotency class of A. If A has nilpotency class one, then A is abelian (pointed in mathematical terminology) and thus acyclic. If A has nilpotency class $n > 1$, the nilpotency class of A_{ad} is $n-1$; thus, by induction hypothesis, A_{ad} is acyclic. Let

$$\left(x_{i_1} = x_{i_{n+1}}, x_{i_n}, \dots, x_{i_2}, x_{i_1}\right)$$

be a sequence of basic elements with $x_{i_1} \neq 1$. If $N_{i_{k+1},\overline{i_{k+1}}}^{i_k} > 0$ for all k, then $x_{i_k} \in A_{\mathrm{ad}}$ for all k and

$$\prod_{k=1}^{n-1} N_{i_{k+1},\overline{i_{k+1}}}^{i_k} > 0,$$

a contradiction since A_{ad} is acyclic. □

Now, we will present some consequences of Theorem (3.3) to modular categories. We refer the reader to [3,9] for a general theory of modular categories.

Let $\mathcal{B}$ be a modular category, the set $\mathrm{Irr}(\mathcal{B}) = \{x_0 = [1], x_1, \dots, x_n\}$, of the isomorphism classes of the distinct simple objects of $\mathcal{B}$ with $N_{i,j}^k = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathcal{B}}(x_k, x_i \otimes x_j))$ are *the fusion rules of* $\mathcal{B}$. A modular category is called acyclic (respectively nilpotent) if its fusion rules are acyclic (respectively nilpotent). It follows from Theorem 3.3 that nilpotent and acyclic modular categories are equivalent definitions. The simplest example of a gauging corresponds to the trivial action of a finite group G on the trivial modular category Vec_G; in this case, the associated modular category is the representation theory of the Drinfeld double of G, denoted by $\mathcal{Z}(\mathrm{Rep}(G))$, see [15, Section 3].

When a finite group G acts on a modular category $\mathcal{B}$ by braided autoequivalences, then gauging this symmetry, when possible, leads to a new modular tensor category denoted by $\mathcal{B}_G^{\times, G}$. A physical and mathematical theory of gauging based on the notion of G-crossed braided fusion category was developed in [1,5], respectively.

Corollary 3.4 *(1) If a gauging* $\mathcal{B}_G^{\times, G}$ *of a modular category* $\mathcal{B}$ *by a finite group* G *has acyclic fusion rules, then* $\mathcal{B}$ *has acyclic fusion rules and* G *is nilpotent.*
(2) The category of representation of the Drinfeld double of a finite group G *has acyclic fusion rules if and only if* G *is a nilpotent group.*

Proof (1) If $\mathcal{B}_G^{\times, G}$ is nilpotent, any fusion subcategory is also nilpotent [13, Proposition 4.6]. Since $\mathrm{Rep}(G) \subset \mathcal{B}^G \subset \mathcal{B}_G^{\times, G}$, we have that $\mathrm{Rep}(G)$ and $\mathcal{B}^G$ are nilpotent, and by Gelaki and Nikshych [13, Remark 4.7], we have that G is a nilpotent group. It follows from [4, Corollary 4.25] that the forgetful functor $\mathcal{B}^G \to \mathcal{B}$ is surjective, and then by Gelaki and Nikshych [13, Proposition 4.6], $\mathcal{B}$ is nilpotent.

(2) The category of representation of the Drinfeld double of a finite group G is $\mathcal{Z}(\mathrm{Rep}(G))$ the Drinfeld center of the category of finite-dimensional representation of G, see [14, Theorem XIII.5.1]. Thus, by Gelaki and Nikshych [13, Theorem 6.11] and Theorem 3.3, the modular category $\mathcal{Z}(\mathrm{Rep}(G))$ is acyclic if and only if G is a nilpotent group. □

The Perron–Frobenius dimension of a fusion rules (A, N) is the *unique* function $\mathrm{FPdim} : A \to \mathbb{R}^{>0}$, that satisfies $\mathrm{FPdim}(1) = 1$ and $\mathrm{FPdim}(x_i x_j) = \mathrm{FPdim}(x_i)\mathrm{FPdim}(x_j)$ for all $x_i, x_j \in A$, (see [9, Proposition 3.3.6]). A modular category $\mathcal{B}$ is called weakly integral if $\mathrm{FPdim}(\mathcal{B}) := \sum_{x_i \in \mathrm{Irr}(\mathcal{B})} \mathrm{FPdim}(x_i)^2 \in \mathbb{Z}$.

A braided fusion category $\mathcal{C}$ is said to have property F [16] if, for every simple object X, the braid group representation associated with X has finite image. Conjecturally, the class of braided fusion categories with property F coincides with the class of braided weakly integral fusion categories. It follows from [16] that the acyclic braided fusion category $SO(8)_2$ mentioned above has property F. A large class of braided fusion categories that have property F are the (braided) group-theoretical categories, i.e., categories Morita equivalent to a pointed category, see [11] for a proof of this fact. A larger class of weakly integral categories for which property F is not currently known are the weakly group-theoretical categories, i.e., categories Morita equivalent to a nilpotent fusion category. Although we do not know whether all acyclic braided fusion categories have property F, some partial results in this direction are as follows, which strongly suggest that anyon models with acyclic fusion rules are never braiding universal.

Corollary 3.5 *(1) A fusion category with acyclic fusion rules is weakly group theoretical. In particular, it is weakly integral.*
(2) An integral braided fusion category with acyclic fusion rules is group theoretical and hence has property F.
(3) A braided fusion category $\mathcal{B}$ has acyclic fusion rules if and only if $\mathcal{B}$ is the Deligne product of braided fusion categories of prime powers.

Proof In light of Theorem 3.3, the statements follow from the results of [7,10,11] dealing with nilpotent categories. In more detail, a nilpotent category is weakly group theoretical by Etingof et al. [10, Definition 1.1], from which the first statement follows. By Drinfeld et al. [7, Corollary 6.2], any integral nilpotent fusion category is group theoretical and hence has property F by Etingof et al. [11, Corollary 4.4], proving (2) while [7, Theorem 1.1] implies (3). □

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