Information and Computation 259 (2018) 130–141

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Complexity classification of the six-vertex model

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A R T I C L E I N F O

Article history:
Received 3 November 2016
Received in revised form 27 June 2017
Available online 31 January 2018

Keywords:
Six-vertex model
Spin system
Holant problem
Interpolation

A B S T R A C T

We prove a complexity dichotomy theorem for the six-vertex model. For every setting of the parameters of the model, we prove that computing the partition function is either solvable in polynomial time or #P-hard. The dichotomy criterion is explicit.

1. Introduction

A primary purpose of complexity theory is to provide classifications to computational problems according to their inherent computational difficulty. While computational problems can come from many sources, a class of problems from statistical mechanics has a remarkable affinity to what is naturally studied in complexity theory. These are the *sum-of-product* computations, a.k.a. *partition functions* in physics.

Well-known examples of partition functions from physics that have been investigated intensively in complexity theory include the Ising model and Potts model [10,8,7,12]. Most of these are spin systems. Spin systems as well as the more general counting constraint satisfaction problems (#CSP) are special cases of Holant problems [5] (see Section 2 for definitions). Roughly speaking, Holant problems are tensor networks where edges of a graph are variables while vertices are local constraint functions; by contrast, in spin systems vertices are variables and edges are (binary) constraint functions. Spin systems can be simulated easily as Holant problems, but Freedman, Lovász and Schrijver proved that simulation in the reverse direction is generally not possible [6]. In this paper we study a family of partition functions that fit the Holant problems naturally, but not as a spin system. This is the *six-vertex model*.

The six-vertex model in statistical mechanics concerns crystal lattices with hydrogen bonds. Remarkably it can be expressed perfectly as a family of Holant problems with 6 parameters for the associated signatures, although in physics people are more focused on regular structures such as lattice graphs, and asymptotic limit. In this paper we study the partition functions of six-vertex models purely from a complexity theoretic view, and prove a complete classification of these Holant problems, where the 6 parameters can be arbitrary complex numbers.

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1 Supported by China National 973 program 2014CB340300.

https://doi.org/10.1016/j.ic.2018.01.003
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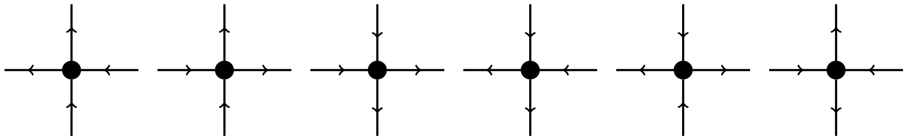


Fig. 1. Valid configurations of the six-vertex model.

The first model in the family of six-vertex models was introduced by Linus Pauling in 1935 to account for the residual entropy of water ice [17]. Suppose we have a large number of oxygen atoms. Each oxygen atom is connected by a bond to four other neighboring oxygen atoms, and each bond is occupied by one hydrogen atom between two oxygen atoms. Physical constraint requires that the hydrogen is closer to either one or the other of the two neighboring oxygens, but never in the middle of the bond. Pauling argued [17] that, furthermore, the allowed configuration of hydrogen atoms is such that at each oxygen site, exactly two hydrogens are closer to it, and the other two are farther away. The placement of oxygen and hydrogen atoms can be naturally represented by vertices and edges of a 4-regular graph. The constraint on the placement of hydrogens can be represented by an orientation of the edges of the graph, such that at every vertex, exactly two edges are oriented toward the vertex, and exactly two edges are oriented away from it. In other words, this is an *Eulerian orientation*. Since there are $\binom{4}{2} = 6$ local valid configurations, this is called the six-vertex model. In addition to water ice, potassium dihydrogen phosphate KH_2PO_4 (KDP) also satisfies this model.

The valid local configurations of the six-vertex model are illustrated in Fig. 1. There are parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_6$ associated with each type of the local configuration. The total energy E is given by $E = n_1\epsilon_1 + n_2\epsilon_2 + \ldots + n_6\epsilon_6$, where n_i is the number of local configurations of type i. Then the partition function is $Z = \sum e^{-E/k_B T}$, where the sum is over all valid configurations, k_B is Boltzmann's constant, and T is the system's temperature. Mathematically, this is a *sum-of-product* computation where the sum is over all Eulerian orientations of the graph, and the product is over all vertices where each vertex contributes a factor $c_i = c^{\epsilon_i}$ if it is in configuration i $(1 \le i \le 6)$ for some constant c.

Some choices of the parameters are well-studied. On the square lattice graph, when modeling ice one takes $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0$. In 1967, Elliott Lieb [14] famously showed that, as the number N of vertices approaches ∞, the value of the "partition function per vertex" $W = Z^{1/N}$ approaches $\left(\frac{4}{3}\right)^{3/2} \approx 1.5396007\ldots$ (Lieb's square ice constant). This matched experimental data 1.540 ± 0.001 so well that it is considered a triumph. The case $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0$ is precisely the problem of counting the number of Eulerian orientations on 4-regular graphs. Mihail and Winkler [16] showed that counting the number of Eulerian orientations (on a general even degree graph, called an Euler graph) is #P-hard, and gave a fully polynomial randomized approximation scheme (fpras) for it. Huang and Lu [9] proved that the problem remains #P-hard for 4-regular graphs, which is exactly the special case for the six-vertex model with $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_6 = 0$. They proved this by a reduction from the #P-hardness of $T_G(3,3)$, the evaluation at $(3,3)$ of the Tutte polynomial T_G, due to Las Vergnas [11]. On (4-regular) *planar* graphs, $T_G(3,3)$ is actually exactly equivalent to a specific six-vertex model; in the notation of Section 2, it is specified by the signature matrix $\begin{bmatrix} 0&0&0&1 \\ 0&1&2&0 \\ 0&2&1&0 \\ 1&0&0&0 \end{bmatrix}$. Welsh has pointed out that $T_G(0,-2)$ is equivalent to counting the number of Eulerian orientations on 4-regular graphs [21]. Luby, Randall and Sinclair [15] gave sampling algorithms for Eulerian orientations on simply connected regions of the grid graph with boundary conditions.

There are other well-known choices in the six-vertex model family. The KDP model of a ferroelectric is to set $\epsilon_1 = \epsilon_2 = 0$, and $\epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 > 0$. The Rys F model of an antiferroelectric is to set $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 > 0$, and $\epsilon_5 = \epsilon_6 = 0$. When there is no ambient electric field, the model chooses the zero field assumption: $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$, and $\epsilon_5 = \epsilon_6$. Historically these are widely considered among the most significant applications ever made of statistical mechanics to real substances. In classical statistical mechanics the parameters are all real numbers while in quantum theory the parameters are complex numbers in general.

In this paper, we give a complete classification of the complexity of calculating the partition function Z on any 4-regular graph defined by an arbitrary choice parameter values $c_1, c_2, \ldots, c_6 \in \mathbb{C}$. (To state our theorem in strict Turing machine model, we take $c_1, c_2, \ldots, c_6$ to be *algebraic* numbers.) Depending on the setting of these values, we show that the partition function Z is either computable in polynomial time, or it is #P-hard, with nothing in between. The dependence of this dichotomy on the values $c_1, c_2, \ldots, c_6$ is explicit.

A number of complexity dichotomy theorems for counting problems have been proved previously. These are mostly on spin systems, or on #CSPs (counting Constraint Satisfaction Problems), or on Holant problems with *symmetric* local constraint functions. #CSP is the special case of Holant problems where Equalities of all arities are auxiliary functions assumed to be present. Spin systems are a further specialization of #CSP, where there is a single binary constraint function (see Section 2). The six-vertex model cannot be expressed as a #CSP problem. It is a Holant problem where the constraint functions are *not symmetric*. Thus previous dichotomy theorems do not apply. This is the first complexity dichotomy theorem proved for a class of Holant problems on *non-symmetric* constraint functions and without auxiliary functions assumed to be present.

However, one important technical ingredient of our proof is to discover a direct connection between some subset of the six-vertex models with spin systems. Another technical highlight is a new interpolation technique that carves out subsums of a partition function by assembling a suitable sublattice, and partitions the sum over an exponential range according to an enumeration of the intersections of cosets of the sublattice with this range.

2. Preliminaries and notations

A constraint function f of arity k is a map $\{0,1\}^k \to \mathbb{C}$. Fix a set of constraint functions $\mathcal{F}$. A signature grid $\Omega = (G, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$ of arity $\deg(v)$, and the incident edges $E(v)$ at v with input variables of f_v. We consider all 0–1 edge assignments σ, each gives an evaluation $\prod_{v\in V} f_v(\sigma|_{E(v)})$, where $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute $\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma|_{E(v)})$. The Holant problem parameterized by the set $\mathcal{F}$ is denoted by Holant($\mathcal{F}$). We denote by Holant($\mathcal{F} \mid \mathcal{G}$) the Holant problem on bipartite graphs where signatures from $\mathcal{F}$ and $\mathcal{G}$ are assigned to vertices from the Left and Right.

A spin system on $G = (V, E)$ has a variable for every $v \in V$ and a binary function g for every edge $e \in E$. The partition function is $\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} g(\sigma(u), \sigma(v))$. Spin systems are special cases of #CSP($\mathcal{F}$) (counting CSP) where $\mathcal{F}$ consists of a single binary function. In turn, #CSP($\mathcal{F}$) is the special case of Holant where $\mathcal{F}$ contains Equality of all arities.

A constraint function is also called a signature. A function f of arity k can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^k}$, or as a tensor in $(\mathbb{C}^2)^{\otimes k}$, or as a matrix in $\mathbb{C}^{2^{k_1}} \times \mathbb{C}^{2^{k_2}}$ if we partition the k variables to two parts, where $k_1 + k_2 = k$. A function is symmetric if its value depends only on the Hamming weight of its input. A symmetric function f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w. For example, $(=_k)$ is the Equality signature $[1, 0, \ldots, 0, 1]$ (with $k-1$ 0's) of arity k. We use $\neq_2$ to denote binary Disequality function $[0, 1, 0]$. The support of a function f is the set of inputs on which f is nonzero.

Given an instance $\Omega = (G, \pi)$ of Holant($\mathcal{F}$), we add a middle point on each edge as a new vertex to G, then each edge becomes a path of length two through the new vertex. Extend π to label a function g to each new vertex. This gives a bipartite Holant problem Holant($g \mid \mathcal{F}$). It is obvious that Holant($=_2 \mid \mathcal{F}$) is equal to Holant($\mathcal{F}$).

For $T \in \text{GL}_2(\mathbb{C})$ and a signature f of arity n, written as a column vector $f \in \mathbb{C}^{2^n}$, we denote by $T^{-1}f = (T^{-1})^{\otimes n} f$ the transformed signature. For a signature set $\mathcal{F}$, define $T^{-1}\mathcal{F} = \{T^{-1}f \mid f \in \mathcal{F}\}$. For signatures written as row vectors we define $\mathcal{F}T$ similarly. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H, \pi)$ of Holant ($\mathcal{F} \mid \mathcal{G}$), for the same bipartite graph H, we get a new signature grid $\Omega' = (H, \pi')$ of Holant $\left(\mathcal{F}T \mid T^{-1}\mathcal{G}\right)$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T$ or $T^{-1}\mathcal{G}$.

In this paper we focus on Holant($\neq_2 \mid f$) when f has support among strings of Hamming weight 2. They are the six-vertex models on general graphs. We note that as an orientation problem the six-vertex model is naturally represented as a Holant problem on the edge-vertex incidence graph $\Gamma(G)$ of a given 4-regular graph G, where the edge node in $\Gamma(G)$ are given the binary Disequality function $(\neq_2)$, and the degree 4 vertices are given the constraint function f. We can also transform this to a set of (non-bipartite) Holant problems by a holographic reduction [18–20]. Let $Z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. The matrix form of $(\neq_2)$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = Z^{\mathrm{T}}Z$. Under a holographic transformation with bases Z, Holant($\neq_2 \mid f$) becomes Holant($=_2 \mid Z^{\otimes 4}f$), where $Z^{\otimes 4}f$ is a constraint function represented by a column vector, which is the matrix tensor power $Z^{\otimes 4}$ multiplied by the column vector form of f. The bipartite Holant problems of the form Holant($\neq_2 \mid f$) naturally correspond to the non-bipartite Holant problems Holant($Z^{\otimes 4}f$). In general f and $Z^{\otimes 4}f$ are non-symmetric functions.

A signature f of arity 4 has the signature matrix $M = M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} f_{0000} & f_{0010} & f_{0001} & f_{0011} \\ f_{0100} & f_{0110} & f_{0101} & f_{0111} \\ f_{1000} & f_{1010} & f_{1001} & f_{1011} \\ f_{1100} & f_{1110} & f_{1101} & f_{1111} \end{bmatrix}$. If $\{i, j, k, \ell\}$ is a permutation of $\{1, 2, 3, 4\}$, then the 4×4 matrix $M_{x_ix_j,x_kx_\ell}(f)$ lists the 16 values with row index $x_ix_j \in \{0,1\}^2$ and column index $x_kx_\ell \in \{0,1\}^2$ in lexicographic order.

Let $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Note that N is the double Disequality, which is the function of connecting two pairs of edges by $(\neq_2)$.

If f and g have signature matrices $M(f) = M_{x_ix_j,x_kx_\ell}(f)$ and $M(g) = M_{x_sx_t,x_ux_v}(g)$, by connecting x_k to x_s, x_ℓ to x_t, both with Disequality $(\neq_2)$, we get a signature of arity 4 with the signature matrix $M(f)NM(g)$ by matrix product with row index x_ix_j and column index x_ux_v.

The six-vertex model is Holant($\neq_2 \mid f$), where $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$. We also write this matrix by $M(a, x, b, y, c, z)$. When $a = x, b = y$ and $c = z$, we abridge it as $M(a, b, c)$. Note that all nonzero entries of f are on Hamming weight 2. Denote the 3 pairs of ordered complementary strings by $\lambda = 0011, \overline{\lambda} = 1100$, $\mu = 0110, \overline{\mu} = 1001$, and $\nu = 0101, \overline{\nu} = 1010$. The support of f is the union $\{\lambda, \overline{\lambda}, \mu, \overline{\mu}, \nu, \overline{\nu}\}$ of the pairs $(\lambda, \overline{\lambda})$, $(\mu, \overline{\mu})$ and $(\nu, \overline{\nu})$, on which f has values (a, x), (b, y) and (c, z). If f has the same value in a pair, say $a = x$ on λ and $\overline{\lambda}$, we say it is a twin.

The permutation group S_4 on $\{x_1, x_2, x_3, x_4\}$ induces a group action on $\{s \in \{0,1\}^4 \mid \mathrm{wt}(s) = 2\}$ of size 6. This is a faithful representation of S_4 in S_6. Since the action of S_4 preserves complementary pairs, this group action has non-trivial blocks of imprimitivity, namely $\{A, B, C\} = \{\{\lambda, \overline{\lambda}\}, \{\mu, \overline{\mu}\}, \{\nu, \overline{\nu}\}\}$. The action on the blocks is a homomorphism of S_4 onto S_3, with kernel $K = \{1, (12)(34), (13)(24), (14)(23)\}$. In particular one can calculate that the subgroup $S_{\{2,3,4\}} = \{1, (23), (34), (24), (243), (234)\}$ maps to $\{1, (AC), (BC), (AB), (ABC), (ACB)\}$. By a permutation from S_4, we may permute the matrix $M(a, x, b, y, c, z)$ by any permutation on the values $\{a, b, c\}$ with the corresponding permutation on $\{x, y, z\}$, and moreover we can further flip an even number of pairs (a, x), (b, y) and (c, z). In particular, we can arbitrarily reorder the three rows in $\begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}$, and we can also reverse the order of arbitrary two rows together. In the proof, after one construction, we may use this property to get a similar construction and conclusion, by quoting this symmetry of three pairs or six values.

Definition 2.1. A 4-ary signature is redundant iff in its 4 by 4 signature matrix the middle two rows are identical and the middle two columns are identical.

Theorem 2.2 *([2]). If f is a redundant signature and the determinant*

$$\det \begin{bmatrix} f_{0000} & f_{0010} & f_{0011} \\ f_{0100} & f_{0110} & f_{0111} \\ f_{1100} & f_{1110} & f_{1111} \end{bmatrix} \neq 0,$$

then $\mathrm{Holant}(\neq_2 \mid f)$ *is #P-hard.*

Now we define the tractable function classes $\mathscr{A}$ and $\mathscr{P}$.

Affine signatures $\mathscr{A}$

Definition 2.3. A signature $f(x_1, \ldots, x_n)$ of arity n is *affine* if it has the form

$$\lambda \cdot \chi_{AX=0} \cdot \mathfrak{i}^{Q(X)},$$

where $\lambda \in \mathbb{C}$, $X = (x_1, x_2, \ldots, x_n, 1)$, A is a matrix over $\mathbb{Z}_2$, $Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, x_2, \ldots, x_n]$ is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, i.e., Q has the form

$$Q(x_1, x_2, \ldots, x_n) = a_0 + \sum_{k=1}^{n} a_k x_k + \sum_{1 \le i < j \le n} 2 b_{ij} x_i x_j,$$

and χ is a 0-1 indicator function such that $\chi_{AX=0}$ is 1 iff $AX = 0$. We use $\mathscr{A}$ to denote the set of all affine signatures.

Product-type signatures $\mathscr{P}$

Definition 2.4. A signature on a set of variables X is of *product type* if it can be expressed as a product of unary functions, binary equality functions ($[1, 0, 1]$), and binary disequality functions ($[0, 1, 0]$), each on (not necessarily disjoint subsets of) one or two variables of X. We use $\mathscr{P}$ to denote the set of product-type functions.

The classes $\mathscr{A}$ and $\mathscr{P}$ are identified as tractable for #CSP [3]. Problems defined by $\mathscr{A}$ are tractable essentially by Gauss Sums (See Theorem 6.30 of [13]). The signatures in $\mathscr{P}$ are tensor products of signatures whose supports are among two complementary bit vectors. Problems defined by them are tractable by a propagation algorithm. The full version [1] contains complete definitions and characterizations of these classes.

Theorem 2.5 *([3]). Let $\mathcal{F}$ be any set of complex-valued signatures in Boolean variables. Then #CSP($\mathcal{F}$) is #P-hard unless $\mathcal{F} \subseteq \mathscr{A}$ or $\mathcal{F} \subseteq \mathscr{P}$, in which case the problem is computable in polynomial time.*

By Theorem 2.5, we have the following corollary, which is easier to apply for binary signatures.

Corollary 2.6. *Let $f = \begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be a binary signature. Then #CSP(f) is #P-hard unless*

- *$f \in \mathscr{P}$: $\alpha\delta - \beta\gamma = 0$, or $\alpha = \delta = 0$, or $\beta = \gamma = 0$;*
- *$f \in \mathscr{A}$: $(\alpha, \beta, \gamma, \delta) = \lambda(\mathfrak{i}^{r_1}, \mathfrak{i}^{r_1}, \mathfrak{i}^{r_3}, \mathfrak{i}^{r_4})$, where $r_1 + r_2 + r_3 + r_4 \equiv 0 \pmod 2$ and $\lambda \in \mathbb{C}$.*

If $f \in \mathscr{P} \cup \mathscr{A}$, then #CSP($f$) is computable in polynomial time.

Definition 2.7. $\mathscr{M}$ is the set of functions, whose support is composed of strings of Hamming weight at most one. $\mathscr{M}' = \{g \mid \exists f \in \mathscr{M},\ g(\mathbf{x}) = f(\overline{\mathbf{x}})\}$, where $\overline{\mathbf{x}}$ is the complement of $\mathbf{x}$.

Note that all unary functions are in $\mathscr{M} \cap \mathscr{M}'$. Theorem 2.8 is a consequence of Theorem 2.2 in [4].

Theorem 2.8. Holant($\neq_2 \mid \mathscr{M}$) *and* Holant($\neq_2 \mid \mathscr{M}'$) *are polynomial time computable.*

3. Main theorem

Theorem 3.1. *Let f be a 4-ary signature with the signature matrix $M_{x_1x_2,x_4x_3}(a, x, b, y, c, z)$, then Holant($\neq_2 \mid f$) is #P-hard except for the following cases:*

- *$f \in \mathscr{P}$;*
- *$f \in \mathscr{A}$;*
- *there is a zero in each pair (a, x), (b, y), (c, z);*

in which cases Holant($\neq_2 \mid f$) is computable in polynomial time.

We prove the complexity classification by categorizing the six values a, b, c, x, y, z in the following way.

1. There is a zero pair. If $f \in \mathscr{A} \cup \mathscr{P}$, then it is tractable. Otherwise it is #P-hard.
2. All values in $\{a, x, b, y, c, z\}$ are nonzero. We prove these are #P-hard.
   (a) Three twins. We prove this case mainly by an interpolation reduction from redundant signatures, then apply Theorem 2.2.
   (b) There is one pair that is not twin. We prove this by a reduction from Case 2a.
3. There is exactly one zero in $\{a, x, b, y, c, z\}$. All are #P-hard by reducing from Case 2.
4. There are exactly two zeros which are from different pairs. All are #P-hard by reducing from Case 2.
5. There is one zero in each pair. These are tractable according to Theorem 2.8.

By definition, in Case 1 and Case 5, f may have more zero values than the stated ones.

These cases above cover all possibilities: After Case 1 we may assume that there is no zero pair. Then after Case 2 we may assume there is at least one zero and there is no zero pair. Similarly after Case 3 we may assume there are at least two zeros and there is no zero pair. So Case 4 finishes off the case when there are exactly two zeros. After Case 4 we may assume there are at least three zeros, but there is no zero pair. Therefore we may assume the only case remaining is where there are exactly three zeros in three distinct pairs, and Case 5 finishes the proof.

In the following we prove the 5 cases to prove the main theorem.

4. Case 1: one zero pair

In this section we prove Case 1. Note that by renaming the variables x_1, x_2, x_3, x_4 we may assume the signature f of arity 4 with one zero pair has the form in (4.1).

Lemma 4.1. *Let f be a 4-ary signature with the signature matrix*

$$M_{x_sx_t,x_ux_v}(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.1}$$

where $\{s, t, u, v\}$ is a permutation of $\{1, 2, 3, 4\}$. Then Holant($\neq_2 \mid f$) is #P-hard unless $f \in \mathscr{A}$ or $f \in \mathscr{P}$, in which case the problem is computable in polynomial time.

Proof. By the S_4 group symmetry, we only need to prove the lemma for $(s, t, u, v) = (1, 2, 4, 3)$. Tractability follows from Corollary 2.6.

Let $g(x, y)$ be the binary signature $g = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ in matrix form. We prove that #CSP(g) $\leq_T$ Holant($\neq_2 \mid f$) in two steps. In each step, we begin with a signature grid and end with a new signature grid such that the Holant values of both signature grids are the same.

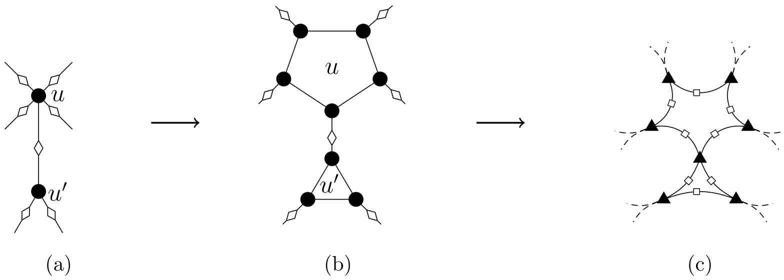


Fig. 2. The reduction from #CSP(g) to Holant($\neq_2 | f$). The circle vertices are assigned $=_d$, where d is the degree of the corresponding vertex, the diamond vertices are assigned g, the triangle vertices are assigned f, and the square vertices are assigned $\neq_2$. In the first step, we replace a vertex by a cycle, where the length of the cycle is the degree of the vertex. The vertices on the cycle are assigned $=_3$. In the second step, we merge two vertices that are connected to the diamond with g and assign f to the new vertex.

For step one, consider an instance of #CSP(g). Equivalently, we will view the instance as a bipartite graph $G = (U, V, E)$, where each $u \in U$ is a variable, and each $v \in V$ has degree two and is labeled g. We define a cyclic order of the edges incident to each vertex $u \in U$, and decompose u into $k = \deg(u)$ vertices. Then we connect the k edges originally incident to u to these k new vertices so that each vertex is incident to exactly one edge. We also connect these k new vertices in a cycle according to the cyclic order (see Fig. 2b). Thus, in effect we have replaced u by a cycle of length $k = \deg(u)$. (If $k = 1$ there is a self-loop. If $k = 2$ there is a cycle of length 2, i.e., a pair of parallel edges.) Each of the k vertices has degree 3, and we assign them $(=_3)$. Clearly this does not change the value of the partition function. The resulting graph has the following properties: (1) every vertex has either degree 2 or degree 3; (2) each degree 2 vertex is connected to degree 3 vertices; (3) each degree 3 vertex is connected to exactly one degree 2 vertex.

Now step two. For every $v \in V$, v has degree 2 and is labeled by g. We contract the two edges incident to v. The resulting graph $G' = (V', E')$ is 4-regular. We put a node on every edge of G' (which are all the edges on the cycles introduced in step 1 for each $u \in U$) and assign $(\neq_2)$ to the new node (see Fig. 2c). Next we assign a copy of f to every $v' \in V'$ after this contraction. The input variables x_1, x_2, x_3, x_4 are carefully assigned at each copy of f as illustrated in Fig. 3. More specifically, suppose originally the binary constraint g is applied to the ordered pair of variables u and u', then after the contraction we have a degree 4 vertex common to the cycles corresponding to u and u' respectively. In this case, variables x_1 and x_2 will be on the incident edges from the cycle for u, variables x_3 and x_4 will be on the incident edges from the cycle for u', so that clockwise cyclically the four variables are ordered x_1, x_2, x_4, x_3. (Note the flipped order of x_4 and x_3 in the cyclic order.) This careful assignment of variables is to ensure that in any nonzero term of the Holant sum there are only two possible configurations to each original cycle corresponding to a variable $u \in U$. Indeed, notice that the support of f is contained in $(x_1 \neq x_2) \wedge (x_3 \neq x_4)$. Hence to have a nonzero term in the Holant sum, any cycle corresponding to a variable $u \in U$ has only two configurations corresponding to two cyclic orientations, by the support of f and the $(\neq_2)$ on the cycle. These correspond to the 0-1 assignment values at the original variable $u \in U$. Moreover in each case, the value of the function g is perfectly mirrored by the value of the function f under the orientations. So we have #CSP(g) $\leq_T$ Holant($\neq_2 | f$).

We have $f(x_1, x_2, x_3, x_4) = g(x_1, x_4) \cdot \chi_{x_1 \neq x_2} \cdot \chi_{x_3 \neq x_4}$. By Definition 2.3, $g \in \mathscr{A} \cup \mathscr{P}$ implies $f \in \mathscr{A} \cup \mathscr{P}$, because the factor $\chi_{x_1 \neq x_2} \cdot \chi_{x_3 \neq x_4}$ simply gets absorbed into the 0-1 characteristic function of the affine support of g. Therefore if $f \notin \mathscr{A} \cup \mathscr{P}$, then $g \notin \mathscr{A} \cup \mathscr{P}$. Then #CSP($g$) is #P-hard by Corollary 2.6. It follows that Holant($\neq_2 | f$) is #P-hard. This finishes the proof. □

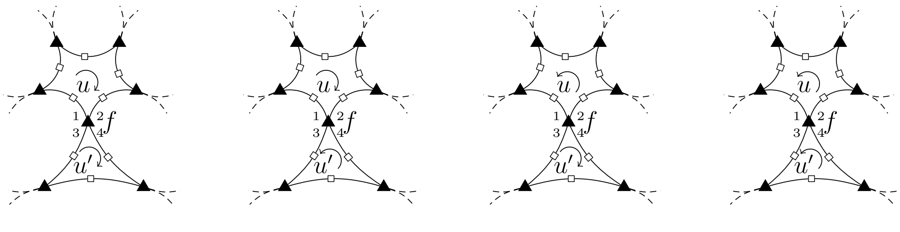
5. Case 2: all six values are nonzero

In this section, we handle the case $axbycz \neq 0$, by proving all problems in this case are #P-hard. Firstly, we give a technical lemma for interpolation reduction. Then we prove the 3-twins case. Finally, we prove the other cases by realizing a 3-twins problem.

Lemma 5.1. *Suppose $\alpha, \beta \in \mathbb{C} - \{0\}$, and the lattice $L = \{(j,k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\}$ has the form $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s,t) \neq (0,0)$. Let ϕ and ψ be any numbers satisfying $\phi^s \psi^t = 1$. If we are given the values $N_\ell = \sum_{j,k \geq 0,\ j+k \leq m} (\alpha^j \beta^k)^\ell x_{j,k}$ for $\ell = 1, 2, \ldots \binom{m+2}{2}$, then we can compute $\sum_{j,k \geq 0,\ j+k \leq m} \phi^j \psi^k x_{j,k}$ in polynomial time.*

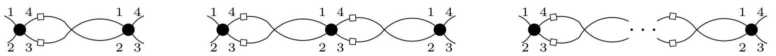
Proof. We treat $\sum_{j,k \geq 0,\ j+k \leq m} (\alpha^j \beta^k)^\ell x_{j,k} = N_\ell$ (where $1 \leq \ell \leq \binom{m+2}{2}$) as a system of linear equations with unknowns $x_{j,k}$. The coefficient vector of the first equation is $(\alpha^j \beta^k)$, indexed by the pair (j,k), where $0 \leq j,k \leq m$ and $j + k \leq m$. The coefficient matrix of the linear system is a Vandermonde matrix, with row index ℓ and column index (j,k). However, this Vandermonde matrix is rank deficient. If $(j,k) - (j',k') \in L$, then columns (j,k) and (j',k') have the same value.

We can combine the identical columns (j,k) and (j',k') if $(j,k) - (j',k') \in L$, since for each coset T of L, the value $\alpha^j \beta^k$ is constant. Thus, the sum $\sum_{j,k \geq 0,\ j+k \leq m} (\alpha^j \beta^k)^\ell x_{j,k}$ can be written as $\sum_T (\alpha^j \beta^k)^\ell \left(\sum_{j,k \geq 0,\ j+k \leq m,\ (j,k) \in T} x_{j,k} \right)$, where the



(a) $g_{00} \leftrightarrow f_{0110}$ (b) $g_{01} \leftrightarrow f_{0101}$ (c) $g_{10} \leftrightarrow f_{1010}$ (d) $g_{11} \leftrightarrow f_{1001}$

Fig. 3. Assigning input variables at one copy of f: Suppose the binary function g is applied to (the ordered pair) (u, u'). The variables u and u' have been replaced by cycles of length $\deg(u)$ and $\deg(u')$ respectively. (In the figure, they have $\deg(u) = 5$ and $\deg(u') = 3$.) For the cycle C_u representing a variable u, we associate the value $u = 0$ with a clockwise orientation, and $u = 1$ with a counter-clockwise orientation. We assign x_i to the edge labeled by i for $i = 1, 2, 3, 4$. Then by the support of f, $x_1 = 0, 1$ forces $x_2 = 1, 0$ respectively, and similarly $x_4 = 0, 1$ forces $x_3 = 1, 0$ respectively. Thus there is a natural 1–1 correspondence between $u = 0$ (respectively, $u = 1$) with clockwise (respectively, counter-clockwise) orientation of the cycle C_u, and similarly for $C_{u'}$. Under this 1–1 correspondence, the value of the function g is perfectly mirrored by the value of the function f.



(a) MNM' (b) $M(NM')^2$ (c) $M(NM')^{s-1}$

Fig. 4. Recursive construction of the interpolation in Lemma 5.2. The circles are assigned f and the squares are assigned $\neq_2$.

sum over T is for all cosets T of L having a non-empty intersection with the cone $C = \{(j,k) \mid 0 \le j, k \le m,\ j + k \le m\}$. Now the coefficient matrix, indexed by $1 \le \ell \le \binom{m+2}{2}$ for the rows and the cosets T with $T \cap C \neq \emptyset$ for the columns, has full rank. And so we can solve $\left(\sum_{j,k \ge 0,\ j+k \le m,\ (j,k) \in T} x_{j,k}\right)$ for each coset T with $T \cap C \neq \emptyset$. Notice that for the sum $\sum_{j+k \le m} \phi^j \psi^k x_{j,k}$, we also have the expression $\sum_T \phi^j \psi^k \left(\sum_{j,k \ge 0,\ j+k \le m,\ (j,k) \in T} x_{j,k}\right)$, since $\phi^j \psi^k$ on each coset T of L is also constant. The lemma follows. □

Now we prove the #P-hardness for the 3-twins case. In this case $a = x$, $b = y$ and $c = z$. We denote by $M(a, b, c)$ the problem defined by the signature matrix $M_{x_1x_2,x_4x_3}(a, a, b, b, c, c)$.

Lemma 5.2. *Let f be a 4-ary signature with the signature matrix* $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$ *with $abc \neq 0$. Then* Holant($\neq_2 \mid f$) *is #P-hard.*

Proof. We construct a series of gadgets by a chain of one leading copy of f and a sequence of twisted copies of f linked by two ($\neq_2$)'s in between. It has the signature matrix $D_s = M(NM')^{s-1}$, for $s \ge 1$, where $M = M_{x_1x_2,x_4x_3}(f)$, $M' = M_{x_2x_1,x_4x_3}(f)$ is a permuted copy of M, and N is the double Disequality. See Fig. 4. This is in the right side of Holant($\neq_2 \mid f$).

The signature matrix of this gadget is given as a product of matrices. Each matrix is a function of arity 4. Notice that the two row indices in $M_{x_2x_1,x_4x_3}(f)$ exchange their positions compared with the standard one $M_{x_1x_2,x_4x_3}(f)$. Thus the rows of M under go the permutation $(00, 01, 10, 11) \to (00, 10, 01, 11)$ to get M'. In other words, M' is obtained from M by exchanging the middle two rows. Also NM' reverses all 4 rows of M'. So we have

$$NM' = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, \quad \text{and} \quad D_s = \begin{bmatrix} 0 & \mathbf{0} & a^s \\ \mathbf{0} & \begin{bmatrix} b & c \\ c & b \end{bmatrix}^s & \mathbf{0} \\ a^s & \mathbf{0} & 0 \end{bmatrix}.$$

We diagonalize the 2 by 2 matrix in the middle using $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (note that $H^{-1} = H$), and get $D_s = P \Lambda_s P$, where

$$P = \begin{bmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & H & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix}, \quad \text{and} \quad \Lambda_s = \begin{bmatrix} 0 & 0 & 0 & a^s \\ 0 & (b+c)^s & 0 & 0 \\ 0 & 0 & (b-c)^s & 0 \\ a^s & 0 & 0 & 0 \end{bmatrix}.$$

The matrix Λ_s has a good form for polynomial interpolation. Suppose we have a problem Holant($\neq_2 \mid F$) to be reduced to Holant($\neq_2 \mid M$). Let F appear m times in an instance Ω. We replace each appearance of F by a copy of the gadget D_s, to get an instance Ω_s of Holant($\neq_2 \mid M$). We can treat each of the m appearances of D_s as a new gadget composed of three functions in sequence P, Λ_s and P, and denote this new instance by Ω'_s. We divide Ω'_s into two parts. One part is composed of m functions Λ_s. The second part is the rest of the functions, including $2m$ occurrences of P, and its signature is represented by X (which is a tensor expressed as a row vector). The Holant value of Ω'_s is the dot product $\langle X, \Lambda_s^{\otimes m}\rangle$, which is a summation over $4m$ bits, that is, the values of the $4m$ edges connecting the two parts. We can stratify all 0–1 assignments of these $4m$ bits having a nonzero evaluation of $\text{Holant}_{\Omega'_s}$ into the following categories. For each tuple of nonnegative integers (i, j, k) such that $i + j + k = m$, the category parameterized by (i, j, k) is:

- there are i many copies of Λ_s receiving inputs 0011 or 1100;
- there are j many copies of Λ_s receiving inputs 0110; and
- there are k many copies of Λ_s receiving inputs 1001.

For any assignment in the category with parameter (i, j, k), the evaluation of $\Lambda_s^{\otimes m}$ is clearly $a^{si}(b+c)^{sj}(b-c)^{sk}$. We can rewrite the dot product summation and get

$$\text{Holant}_{\Omega_s} = \text{Holant}_{\Omega'_s} = \langle X, \Lambda_s^{\otimes m}\rangle = \sum_{i+j+k=m} a^{si}(b+c)^{sj}(b-c)^{sk} x_{i,j,k}, \tag{5.2}$$

where $x_{i,j,k}$ is the summation of values of the second part X over all assignments in the category (i, j, k). Because $i + j + k = m$, we also use $x_{i,j}$ to denote the value $x_{i,j,k}$. Similarly we use $x_{j,k}$ or $x_{i,k}$ to denote the same value $x_{i,j,k}$ when there is no confusion.

Generally, in an interpolation reduction, we pick polynomially many values of s, and get a system of linear equations in $x_{i,j,k}$. When all $a^i(b+c)^j(b-c)^k$ are distinct, for $i + j + k = m$, we get a full rank Vandermonde coefficient matrix, and then we can solve for each $x_{i,j,k}$. Once we have $x_{i,j,k}$ we can compute any function in $x_{i,j,k}$.

When $a^i(b+c)^j(b-c)^k$ are not distinct, say $a^i(b+c)^j(b-c)^k = a^{i'}(b+c)^{j'}(b-c)^{k'}$, we may define a new variable $y = x_{i,j,k} + x_{i',j',k'}$. We can combine all $x_{i,j,k}$ with the same $a^i(b+c)^j(b-c)^k$. Then we have a full rank Vandermonde system of linear equations in these new unknowns. We can solve all new unknowns and then sum them up to get $\sum_{i+j+k=m} x_{i,j,k}$. This is one special function in $x_{i,j,k}$.

The above are two typical application methods in this kind of interpolation. Unfortunately in our case, we may have a rank deficient Vandermonde system, and the sum $\sum_{i+j+k=m} x_{i,j,k}$ does not give us anything useful. This is because if we replace $a^{si}(b+c)^{sj}(b-c)^{sk}$ by the constant value 1 in equation (5.2), we get $\sum_{i+j+k=m} x_{i,j,k}$. Thus, $\sum_{i+j+k=m} x_{i,j,k}$ corresponds to Ω'_s with all nonzero values in Λ_s replaced by the constant 1, i.e., we get a reduction from the problem $M(1, 1, 0)$. But $M(1, 1, 0)$ is a tractable problem, and so we do not get any hardness result by such a reduction. Instead, we consider weighted sums of the form $\sum_{i,j,k} \phi^j \psi^k x_{i,j,k}$ for appropriate values of ϕ and ψ, so that we end up in a #P-hard problem.

To prove this lemma, there are three cases when there are 3 twins.

1. Two elements in $\{a, b, c\}$ are equal. By the symmetry of the group action of S_4, without loss of generality, we may assume $b = c$. We have

$$\Lambda_s = \begin{bmatrix} 0 & 0 & 0 & a^s \\ 0 & (2b)^s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^s & 0 & 0 & 0 \end{bmatrix},$$

and equation (5.2) becomes $\text{Holant}_{\Omega_s} = \sum_{i+j=m} x_{i,j} a^{si}(2b)^{sj}$. Note that all terms $x_{i,j,k}$ with $k \neq 0$ have disappeared. We can interpolate to get $\sum_{i+j=m} x_{i,j}$. (This can be argued as follows: If $\frac{a}{2b}$ is not a root of unity, then we can get a system of linear equations about $x_{i,j}$ with a full-ranked coefficient matrix. By solving the system of linear equations we have all $x_{i,j}$. If $\frac{a}{2b}$ is a root of unity, we pick the minimum positive $u \in \mathbb{Z}$ such that $(\frac{a}{2b})^u = 1$. Let $X_k = \sum_{0 \le \ell \le \lfloor \frac{m}{u} \rfloor} x_{k+u\ell, m-k-u\ell}$ for $0 \le k \le u-1$. Then we can get a system of linear equations about X_k with a full-ranked coefficient matrix. By solving this system of linear equations we have all X_k and then we can get $\sum_{i+j=m} x_{i,j}$.) This sum corresponds to the problem defined by the matrix $\Lambda' = \begin{bmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix}$, which in turn corresponds to the signature $M(1, \frac{1}{2}, \frac{1}{2}) = P\Lambda' P = \begin{bmatrix} 0&0&0&1 \\ 0&\frac{1}{2}&\frac{1}{2}&0 \\ 0&\frac{1}{2}&\frac{1}{2}&0 \\ 1&0&0&0 \end{bmatrix}$. Since $\text{Holant}(\neq_2 | M(1, \frac{1}{2}, \frac{1}{2}))$ is #P-hard by the determinant criterion, we obtain that $\text{Holant}(\neq_2 | M(a, b, c)$ is #P-hard.
2. Two elements in $\{a, b, c\}$ have the opposite value. By the symmetry of the group action of S_4, without loss of generality, we may assume $b = -c$. We have

$$\Lambda_s = \begin{bmatrix} 0 & 0 & 0 & a^s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (2b)^s & 0 \\ a^s & 0 & 0 & 0 \end{bmatrix},$$

and equation (5.2) becomes $\text{Holant}_{\Omega_s} = \sum_{i+k=m} x_{i,k} a^{si} (2b)^{sk}$. Similarly note that all terms $x_{i,j,k}$ with $j \neq 0$ have disappeared. We can interpolate to get $\sum_{i+k=m} x_{i,k}$. Similarly, this sum corresponds to the problem defined by the matrix $\Lambda'' = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}$, which in turn corresponds to the signature $M(2,1,-1) = P\Lambda'' P = \begin{bmatrix} 0&0&0&1 \\ 0&\frac{1}{2}&-\frac{1}{2}&0 \\ 0&-\frac{1}{2}&\frac{1}{2}&0 \\ 1&0&0&0 \end{bmatrix}$. By the group action we also have $M(-1,2,1)$. If we link two copies of $M(-1,2,1)$ by N, we get $M(1,5,4)$, because $\begin{bmatrix} 2&1 \\ 1&2 \end{bmatrix}^2 = \begin{bmatrix} 5&4 \\ 4&5 \end{bmatrix}$. Then $M(1,5,4) = P\Lambda P$, where $\Lambda = \begin{bmatrix} 0&0&0&1 \\ 0&9&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}$. There are only two nonzero values 9 and 1 in Λ. For $M(1,5,4)$, we have $\text{Holant}_{\Omega_s} = \sum_{0 \le i \le m} x_i 9^{si}$, from which we can solve all x_i $(i = 0, 1, \ldots, m)$, we can compute $\sum_{0 \le i \le m} x_i 3^{si}$. This realizes the following problem $\begin{bmatrix} 0&0&0&1 \\ 0&3&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}$, which gives us $\text{Holant}(\neq_2 \mid M(1,2,1))$. By the symmetry of group action we also have $\text{Holant}(\neq_2 \mid M(2,1,1))$, which is #P-hard by Theorem 2.2.

3. After the previous two cases, we can assume that there are no two elements in $\{a,b,c\}$ that are equal or opposite. If we consider a, b and c as three nonzero complex numbers on the plane, there must be two elements in $\{a,b,c\}$ that are not orthogonal as vectors. This is easy to see, since if a and b are orthogonal, and also b and c are orthogonal, then $a/c = (a/b) \cdot (b/c)$ has complex argument a multiple of π. By the symmetry of the group action of S_4, we may assume b and c are not orthogonal. We have already considered the cases $b + c = 0$ or $b - c = 0$, So we may assume $b \neq \pm c$. Let $\alpha = \frac{b+c}{a}$ and $\beta = \frac{b-c}{a}$. Then they have different norms $|\alpha| \neq |\beta|$. Indeed, if $|\alpha| = |\beta|$ then $|1 + c/b| = |1 - c/b|$ which means that $c/b \in i\mathbb{R}$ is purely imaginary, i.e., b and c are orthogonal.
Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. By the interpolation method, we have a system of linear equations in $x_{i,j,k}$, whose coefficient matrix $((a^i (b+c)^j (b-c)^k)^s)$ has row index s and column index from $\{(i,j,k) \mid i,j,k \in \mathbb{N}, i+j+k = m\}$.
The matrix $((a^i (b+c)^j (b-c)^k)^s)$, after dividing the sth row by a^{sm}, has the form $((\alpha^j \beta^k)^s)$, which is a Vandermonde matrix with row index s and column index from $\{(j,k) \mid j,k \in \mathbb{N}, j+k \le m\}$. Define $L = \{(j,k) \in \mathbb{Z}^2 \mid \alpha^j \beta^k = 1\}$. This is a sublattice of $\mathbb{Z}^2$. Every lattice has a basis. There are 3 cases depending on the rank of L.
   (a) $L = \{(0,0)\}$. All $\alpha^j \beta^k$ are distinct. Thus the coefficient matrix has full rank. By solving a system of linear equations, we can realize $\begin{bmatrix} 0&0&0&1 \\ 0&3&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}$, which in turn corresponds to the signature $M(2,1,1)$. Since $\text{Holant}(\neq_2 \mid M(2,1,1))$ is #P-hard by Theorem 2.2, we obtain that $\text{Holant}(\neq_2 \mid M(a,b,c))$ is #P-hard.
   (b) L contains two vectors (j_1,k_1) and (j_2,k_2) independent over $\mathbb{Q}$. Then the nonzero vectors $j_2(j_1,k_1) - j_1(j_2,k_2) = (0, j_2 k_1 - j_1 k_2)$ and $k_2(j_1,k_1) - k_1(j_2,k_2) = (k_2 j_1 - k_1 j_2, 0)$ are in L. Hence, both α and β are roots of unity, but this contradicts with $|\alpha| \neq |\beta|$.
   (c) $L = \{(ns, nt) \mid n \in \mathbb{Z}\}$, where $s, t \in \mathbb{Z}$ and $(s,t) \neq (0,0)$. We know that $s + t \neq 0$, otherwise we get $|\alpha| \neq |\beta|$. By Lemma 5.1, for any numbers ϕ and ψ satisfying $\phi^s \psi^t = 1$, we can compute $\sum_{j+k \le m} \phi^j \psi^k x_{j,k}$ efficiently.
   Define $A = \begin{bmatrix} 0&0&0&1 \\ 0&\phi&0&0 \\ 0&0&\psi&0 \\ 1&0&0&0 \end{bmatrix}$, and we have $2PAP = \begin{bmatrix} 0&0&0&2 \\ 0&\phi+\psi&\phi-\psi&0 \\ 0&\phi-\psi&\phi+\psi&0 \\ 2&0&0&0 \end{bmatrix}$. We get $\text{Holant}(\neq_2 \mid M(2, \phi+\psi, \phi-\psi))$.
      i. $t = 0$. Without loss of generality $s > 0$. Let $\phi = 1$ and $\psi = 1/2$. We get $M(4,3,1)$, from which we can get $M(1,4,3)$ by the S_4 group symmetry. This is #P-hard by the same proof method as we prove $M(1,5,4)$ is #P-hard in Case 2.
      ii. $t > 0$ and $s \ge 0$. Let $\phi = \psi + 2$. We need $f(\psi) = (\psi+2)^s \psi^t = 1$. Because $f(0) = 0 < 1$ and $f(1) \ge 1$, there is a root $\psi_0 \in (0,1]$. We get $M(2, 2\psi_0 + 2, 2)$, which is #P-hard by Case 1.
      iii. $t > 0$, $s < 0$ and $|t| > |s|$. Let $\phi = \psi + 2$. $\psi^{|t|} = (\psi+2)^{|s|}$ has a solution ψ_0 in $(1, \infty)$. We get $M(2, 2\psi_0 + 2, 2)$, which is #P-hard by Case 1.
      iv. $t > 0$, $s < 0$ and $|t| < |s|$. Let $\psi = \phi + 2$. $\phi^{|s|} = (\phi+2)^{|t|}$ has a solution ϕ_0 in $(1, \infty)$. We get $M(2, 2\phi_0 + 2, -2)$, which is #P-hard by Case 2. □

We finish this section by proving the other no zero cases can realize 3-twins.

Lemma 5.3. *Let f be a 4-ary signature with the signature matrix $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0&0&0&a \\ 0&b&c&0 \\ 0&z&y&0 \\ x&0&0&0 \end{bmatrix}$ with $abcxyz \neq 0$. Then* $\text{Holant}(\neq_2 \mid f)$ *is #P-hard.*

Proof. Note that $M_{x_4x_3,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & b & z & 0 \\ 0 & c & y & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$. Connecting two copies of f back to back by double Disequality N, we get the gadget whose signature has the signature matrix

$$M_{x_1x_2,x_4x_3}(f) N M_{x_4x_3,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & 2bc & by+cz & 0 \\ 0 & by+cz & 2yz & 0 \\ ax & 0 & 0 & 0 \end{bmatrix}.$$

If $by+cz \neq 0$, we have realized a function $M(ax, ax, 2bc, 2yz, by+cz, by+cz)$ of two twins, with all nonzero values. We can use $M(2bc, 2yz, ax, ax, by+cz, by+cz)$ to construct the following function by the same gadget

$$M(4bcyz, 4bcyz, 2ax(by+cz), 2ax(by+cz), a^2x^2+(by+cz)^2, a^2x^2+(by+cz)^2).$$

If furthermore $a^2x^2+(by+cz)^2 \neq 0$, we get a nonzero 3-twins function and we can finish the proof by Lemma 5.2. If this process fails, we get a condition that either $by+cz=0$ or $\mathfrak{i}ax+by+cz=0$ or $-\mathfrak{i}ax+by+cz=0$. Recall the symmetry among the 3 pairs (a,x), (b,y), (c,z). If we apply this process with a permuted form of M, we will get either $ax+cz=0$ or $ax+\mathfrak{i}by+cz=0$ or $ax-\mathfrak{i}by+cz=0$. There is one more permutation of M which gives us either $ax+by=0$ or $ax+by+\mathfrak{i}cz=0$ or $ax+by-\mathfrak{i}cz=0$.

We claim that, when $axbycz \neq 0$, the 3 Boolean disjunction conditions can not hold simultaneously. Hence, one of three constructions will succeed and give us #P-hardness.

To prove the claim, we assume that all 3 disjunction conditions hold. Then we get 3 conjunctions, each a disjunction of 3 linear equations. Each equation is a homogeneous linear equation on (ax, by, cz). The 3 equations in the first conjunction all have the form $\alpha \cdot ax + 1 \cdot by + 1 \cdot cz = 0$ where $\alpha \in \{0, \mathfrak{i}, -\mathfrak{i}\}$. Similarly the 3 equations in the second and third conjunction all have the form $1 \cdot ax + \beta \cdot by + 1 \cdot cz = 0$ and $1 \cdot ax + 1 \cdot by + \gamma \cdot cz = 0$ respectively. If at least one equation holds in each of the 3 sets of linear equations with nonzero solution (ax, by, cz), the following determinant

$$\det \begin{bmatrix} \alpha & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \gamma \end{bmatrix} = 0, \tag{5.3}$$

for some $\alpha, \beta, \gamma \in \{0, \mathfrak{i}, -\mathfrak{i}\}$. However, there are no choices of $\alpha, \beta, \gamma \in \{0, \mathfrak{i}, -\mathfrak{i}\}$ such that Equation (5.3) holds: The determinant is $\alpha\beta\gamma + 2 - \alpha - \beta - \gamma$. For $\alpha, \beta, \gamma \in \{0, \mathfrak{i}, -\mathfrak{i}\}$, the norm $|2-\alpha-\beta-\gamma| \geq 2$, but $|\alpha\beta\gamma| = 0$ or 1. □

6. Case 3: exactly one zero

Lemma 6.1. *Let f be a 4-ary signature with the signature matrix*

$$M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix},$$

where there is exactly one of $\{a, b, c, x, y, z\}$ that is zero, then Holant$(\neq_2 | f)$ *is #P-hard.*

Proof. Without loss of generality, we can assume that $b=0$. Note that $M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & c & y & 0 \\ 0 & 0 & z & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$. Connecting a copy of f with this via N, we get a signature g with signature matrix

$$M_{x_1x_2,x_4x_3}(f) N M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & c^2 & cy & 0 \\ 0 & cy & y^2+z^2 & 0 \\ ax & 0 & 0 & 0 \end{bmatrix}.$$

If $y^2+z^2 \neq 0$, by Lemma 5.3, Holant$(\neq_2 | g)$ is #P-hard. Thus Holant$(\neq_2 | f)$ is #P-hard. Otherwise, we have

$$y^2+z^2=0.$$

Similarly, $M_{x_3x_4,x_1x_2}(f) N M_{x_3x_4,x_2x_1}(f)$ gives us

$$y^2+cz=0.$$

$M_{x_4x_3,x_1x_2}(f) N M_{x_2x_1,x_4x_3}(f)$ gives us

$$y^2 + c^2 = 0.$$

From these equations, we get $c^2 = z^2 = cz = -y^2$. This gives us $z = c$ and $y = \pm ic$, and $M = M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & c & 0 \\ 0 & c & \pm ic & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$.

For this matrix M, we may construct $MNM^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & 0 & c^2 & 0 \\ 0 & c^2 & \pm 2ic^2 & 0 \\ ax & 0 & 0 & 0 \end{bmatrix}$. Now we may repeat the construction from the beginning using MNM^{T} instead of M. Because $(c^2)^2 + (\pm 2ic^2)^2 \neq 0$, we get a function of 6 nonzero values. By Lemma 5.3, $\operatorname{Holant}(\neq_2 |\, f)$ is #P-hard. □

7. Case 4: exactly two zeros from distinct pairs

Lemma 7.1. *Let f be a 4-ary signature with the signature matrix*

$$M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & z & y & 0 \\ x & 0 & 0 & 0 \end{bmatrix},$$

where there are exactly two zero entries in $\{a, b, c, x, y, z\}$ and they are from distinct pairs, then $\operatorname{Holant}(\neq_2 |\, f)$ *is #P-hard.*

Proof. Recall from Section 2 that we can arbitrarily reorder the three rows in $\begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}$, and we can also reverse arbitrary two rows. Thus, we can assume that $ax \neq 0, bz \neq 0$ and $c = y = 0$. Note that $M_{x_1x_2,x_4x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & z & 0 & 0 \\ x & 0 & 0 & 0 \end{bmatrix}$ and $M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & b & z & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$. Take two copies of f. If we connect the variables x_4, x_3 of the first function with the variables x_3, x_4 of the second function using $(\neq_2)$, we get a signature g with the signature matrix

$$M_{x_1x_2,x_4x_3}(f) N M_{x_3x_4,x_1x_2}(f) = \begin{bmatrix} 0 & 0 & 0 & ax \\ 0 & b^2 & bz & 0 \\ 0 & bz & z^2 & 0 \\ ax & 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 5.3, $\operatorname{Holant}(\neq_2 |\, g)$ is #P-hard. Thus $\operatorname{Holant}(\neq_2 |\, f)$ is #P-hard. □

8. Case 5: one zero in each pair

Lemma 8.1. *If there is one zero in each pair of (a, x), (b, y), (c, z), then* $\operatorname{Holant}(\neq_2 |\, f)$ *is computable in polynomial time.*

Proof. We will list the three strings of weight 2 where f may be nonzero, by the symmetry of the group action of S_4. We may assume the first string is $\xi = 0011$. The second string η, being not complementary to ξ and of weight two, we may assume it is 0101.

The third string ζ, being not complementary of either ξ or η, and of weight two, must be either 0110 or 1001. Hence,

$$\begin{matrix} \xi & = & 0 & 0 & 1 & 1 \\ \eta & = & 0 & 1 & 0 & 1 \\ \zeta & = & 0 & 1 & 1 & 0 \end{matrix} \quad \text{or} \quad \begin{matrix} \xi & = & 0 & 0 & 1 & 1 \\ \eta & = & 0 & 1 & 0 & 1 \\ \zeta & = & 1 & 0 & 0 & 1 \end{matrix}.$$

Then $f(x_1, x_2, x_3, x_4) = \textsc{Is-Zero}(x_1) \cdot g(x_2, x_3, x_4)$ or $\textsc{Is-One}(x_4) \cdot h(x_1, x_2, x_3)$, where $h \in \mathscr{M}$ and $g \in \mathscr{M}'$. Note that the $\textsc{Is-Zero}$ and $\textsc{Is-One}$ are both unary functions and both belong to $\mathscr{M} \cap \mathscr{M}'$. By Theorem 2.8, $\operatorname{Holant}(\neq_2 |\, f)$ is computable in polynomial time. □

Acknowledgment

We benefited greatly from the comments and suggestions of the anonymous referees, to whom we are grateful. We also sincerely thank Xi Chen and Pinyan Lu for their interest and comments.

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