A Complexity Trichotomy for k-Regular Asymmetric Spin Systems Using Number Theory

Jin-Yi Cai[1], Zhiguo Fu[2], Kurt Girstmair[3], and Michael Kowalczyk[4]

1 **Computer Sciences Department, University of Wisconsin, Madison, WI, USA**
jyc@cs.wisc.edu
2 **School of Computer Science & Information Technology, Northeast Normal University, Changchun, China**
zfu8@wisc.edu
3 **Institute für Mathematik, Universität Innsbruck, Austria**
Kurt.Girstmair@uibk.ac.at
4 **Math & CS Department, Northern Michigan University, Marquette, MI, USA**
mkowalcz@nmu.edu

Abstract

Suppose φ and ψ are two angles satisfying $\tan(\varphi) = 2\tan(\psi) > 0$. We prove that under this condition φ and ψ *cannot* be both rational multiples of π. We use this number theoretic result to prove a classification of the computational complexity of spin systems on k-regular graphs with general (not necessarily symmetric) real valued edge weights. We establish explicit criteria, according to which the partition functions of all such systems are classified into three classes: (1) Polynomial time computable, (2) #P-hard in general but polynomial time computable on planar graphs, and (3) #P-hard on planar graphs. In particular problems in (2) are precisely those that can be transformed to a form solvable by the Fisher-Kasteleyn-Temperley algorithm by a holographic reduction.

1998 ACM Subject Classification F.1.3 Complexity Measures and Classes

Keywords and phrases Spin Systems, Holant Problems, Number Theory, Characters, Cyclotomic Fields

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

We consider spin systems on finite k-regular graphs $G = (V, E)$. Here every vertex $v \in V$ has degree k, and every edge $(u, v) \in E$ is assigned a constraint function $f : \{0, 1\}^2 \to \mathbb{R}$. The function f is not assumed to be symmetric, and one of u or v is specified as the first input variable of f, and the other one the second. Equivalently one can think of G as a directed graph. Define the *partition function* on G as $Z_f(G) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E(G)} f(\sigma(u), \sigma(v))$. Depending on the nature of the edge function f, we show that the problem $Z_f(\cdot)$ is either computable in polynomial time (denoted as P-time) or #P-hard. Furthermore, for those problems $Z_f(\cdot)$ that are #P-hard in general, if the input is restricted to planar graphs, then some of them become computable in P-time. We prove that for all such problems, it is computable in P-time by a universal algorithm that is a holographic reduction to Kasteleyn's algorithm (this is Valiant's Holographic Algorithm)—all other problems remain #P-hard on planar graphs.[1]

[1] Ladner's theorem [20] states that if P$\neq$ NP, then there are problems in NP that are neither in P nor NP-complete. The same is true for #P. Therefore the assertion that all $Z_f(\cdot)$ can be classified into either

To prove our classification theorem, we will make an unexpected detour into number theory. To state it in general terms, this came about as follows: In our attempt to prove #P-hardness for some particularly tricky cases, we found a pair of constructions. Each is controlled by a pair of eigenvalues of equal norm. If the ratio of the two eigenvalues is a *root of unity* then an iteration of the construction will end up repeating after a fixed number of steps (up to a scalar). This is undesirable because the Vandermonde matrix corresponding to the construction will have bounded rank, making it unable to perform polynomial interpolation for arbitrarily large instance graphs. On the other hand, if the ratio of eigenvalues is not a root of unity then the Vandermonde matrix corresponding to the construction will have full rank, and we can successfully interpolate, and thus prove #P-hardness for those tricky cases.

Unfortunately, it is indeed possible that the ratio of eigenvalues for either of the two constructions is a root of unity, depending on specific f. Having unit norm, being a root of unity is the same as the complex argument being a rational multiple of π. As it turns out, the pair of constructions we found has the following surprising property: If the complex arguments (of the ratio of eigenvalues) of the two constructions are φ and ψ respectively, then the tangent values of φ and ψ satisfy the equation $\tan(\varphi) = 2\tan(\psi) > 0$, in all settings of f. So if we can show, given that $\tan(\varphi) = 2\tan(\psi) > 0$, it is impossible that *both* φ and ψ are rational multiples of π, then we will have proved that in all cases at least one of the two constructions succeeds. This is indeed true and we prove it in Theorem 1.

Proving this rational incommensurability between two tangent values, and at the same time, their angle values divided by π, and then using it to prove the complexity classification is the most surprising aspect of this paper. For any fixed n, questions regarding $\mathbb{Q}$-linear independence among cotangent values of the form $\cot(k\pi/n)$ (for $1 \le k < n/2$ and $\gcd(k, n) = 1$) were first suggested by Chowla and proved by Siegel in 1949 (reported by Chowla [12] in 1964; see also [13]). For any fixed prime p, theorems of this type were found for tangent values $\tan(k\pi/p)$ by Hasse [17], and for cosecant values $\csc(2k\pi/p)$ by Jager and Lenstra [19] ($1 \le k \le (p-1)/2$), although linear dependence for the latter case is possible. For any n, Girstmair gave a representation theoretic treatment to the problem of determining $\mathbb{Q}$-linear relations among numbers of the form, respectively, $\cot(k\pi/n), \tan(k\pi/n), \csc(2k\pi/n)$ or $\sec(2k\pi/n)$, for $\gcd(k, n) = 1$ [15]. While these results do not directly imply what we need (Theorem 1), our proof uses a crucial formula in [15] (Theorem 2, p. 380) regarding Leopoldt's character coordinates of numbers in a number field. (Note that Siegel's theorem [12] does *not*, in view of the requirement $\gcd(k, n) = 1$, imply Theorem 1 because there may not be a common primitive order n for φ and ψ; furthermore, $\cot(\pi/6) = 3\cot(\pi/3)$ provides a counter example to the more general statement of $\mathbb{Q}$-linear independence.)

There have been a number of classification theorems for #CSP and related problems [4, 5, 6, 14, 9, 18, 16, 7, 22, 3]. Spin systems are special cases of #CSP (with a single edge function), and #CSP are special cases of Holant problems in which Equality functions of all arities are assumed to be present. The problem addressed in this paper can be viewed as only allowing Equality function of a fixed arity (regular graphs). Without all Equality functions reduction proofs become more challenging. The immediate predecessors to the present work are the classification for $Z_f(\cdot)$ for k-regular graphs where f is a *symmetric* edge function [10], and the classification for $Z_f(\cdot)$ for 3-regular graphs where f is not necessarily symmetric [11]. There are technical difficulties generalizing the proof in [10, 11] to 4-regular graphs with an asymmetric edge function. On the other hand, aside from its intrinsic interest,

P-time computable or #P-hard is not self-evident. To state our results strictly in Turing machine-based complexity theory, f takes values in algebraic numbers.

spin systems on k-regular graphs for *even* k have another pertinence. Although we do not intend to elaborate it here, the result in this paper fits in a bigger classification program for sum-of-product computations. In particular, to classify all Holant problems, a natural process is arity reduction by taking self loops and some similar operations. This reduces the arity by two, and thus there are two base cases in an inductive proof, arity 3 and arity 4. Often one can holographically transform such a signature to EQUALITY of arity 3 or 4 respectively, which gives rise to a spin system on 3- or 4-regular graphs.

This type of sum-of-product computations is studied in physics, where the term partition function originated. In physics, the 0-1 vertex assignments are called spins, and the edge function values $f(\sigma(u), \sigma(v))$ correspond to local interactions between particles. There is a long history in the statistical physics in the study of "Exactly Solved Models" [2, 26]. A rough correspondence exists between P-time computability and physicists' notion of an "Exactly Solvable" system. A central question is to identify which "systems" can be solved "exactly" and which "systems" are "difficult". While in physics there is no rigorous definition of being "difficult", complexity theory proposes that the right notion is #P-hardness.

This paper is organized as follows: In Section 2 we prove Theorem 1 to establish the incommensurability of (co)tangent values and angle values over π. In Section 3 we state some definitions and needed results. In Section 4 we prove the classification theorem for 4-regular graphs. In Section 5 we prove the classification theorem for k-regular graphs, for all k.

2 A Theorem in Number Theory

Let $0 < \varphi < \psi < \pi/2$ denote two angles. Then $0 < \cot(\psi) < \cot(\varphi) < \infty$. Is it possible that

$$\cot(\varphi) = 2\cot(\psi), \tag{1}$$

and yet φ and ψ are both rational multiples of π? We prove the following theorem. It says that, with exactly one obvious exception, it is *not* possible that *both* the ratio of the cotangent values of φ and ψ is rational, *and* the two angles are rational multiples of π. In particular (1) is not possible when both φ and ψ are rational multiples of π. This incommensurability will be used to prove a key complexity reduction to reach our complexity trichotomy classification.

▶ **Theorem 1.** *Suppose* $0 < \varphi < \psi < \pi/2$, *and* $\cot(\varphi) = r\cot(\psi)$, *for some* $r \in \mathbb{Q}$ *and* $r \neq 3$. *Then* φ *and* ψ *are not both rational multiples of* π.

Proof. We first note that the exception $r = 3$ is witnessed by $\cot(\pi/6) = 3\cot(\pi/3)$.

We write $r = \frac{a}{b}$ for relatively prime integers a and b. We are given $\cot(\varphi) = \frac{a}{b}\cot(\psi)$. For a contradiction, suppose φ and ψ are both rational multiples of π, and we write $\varphi = \frac{k\pi}{n}$ and $\psi = \frac{k'\pi}{n'}$, where $1 \le k < \frac{n}{2}$, $1 \le k' < \frac{n'}{2}$, and $\gcd(k, n) = \gcd(k', n') = 1$.

Let $\zeta_n = \exp(2\pi\mathrm{i}/n)$ be a primitive root of unity. Then it is easy to verify that $\mathrm{i}\cot(\varphi) = \frac{1+\zeta_n^k}{1-\zeta_n^k}$. If we write $t = \mathrm{i}\cot(\varphi)$, then $t \in \Phi_n = \mathbb{Q}(\zeta_n)$, the n-th cyclotomic field (the field extension by adjoining ζ_n to $\mathbb{Q}$). Also $\zeta_n^k = \frac{t-1}{t+1}$. As $\gcd(k, n) = 1$, we have $\Phi_n = \mathbb{Q}(\zeta_n^k) \subseteq \mathbb{Q}(t) \subseteq \Phi_n$, and so $\mathbb{Q}(t) = \Phi_n$.

By $\cot(\varphi) = \frac{a}{b}\cot(\psi)$, we have $\Phi_{n'} = \Phi_n$. It is well known that this implies that either $n = n'$, or n is odd and $n' = 2n$, or n' is odd and $n = 2n'$.

We first consider the case $n = n'$. This case actually follows from Siegel's theorem [12]. For a uniform treatment we give a direct proof here.

For $n = n'$, we have $1 \le k < k' < \frac{n}{2}$, and so $n \ge 5$. A Dirichlet character χ to the modulus n is a function from $\mathbb{Z}$ to $\mathbb{C}$ that is multiplicative, has period n, and $\chi(j) \neq 0$ iff $\gcd(j, n) = 1$. So χ is extended from a group character on $\mathbb{Z}_n^\times$ (i.e., a multiplicative function

taking nonzero values in $\mathbb{C}$ on $\mathbb{Z}_n^\times = \{(j \bmod n) \in \mathbb{Z}_n \mid \gcd(j,n) = 1\})$, and all nonzero values of χ are roots of unity. A Dirichlet character χ is said to be *odd* if $\chi(-1) = -1$.

We need to take an odd Dirichlet character χ to the modulus n. An odd Dirichlet character χ (for $n > 2$) exists: The group of Dirichlet characters mod n is isomorphic to $\mathbb{Z}_n^\times$. Since $n > 2$, $\{1, -1\}$ is a subgroup of order two in $\mathbb{Z}_n^\times$. The subgroup of even characters is isomorphic to $\mathbb{Z}_n^\times/\{1,-1\}$. Hence, not every Dirichlet character mod n is even. A more constructive proof is as follows: The character group of $\mathbb{Z}_n^\times$, by Chinese remaindering, is a direct product of the character groups of $\mathbb{Z}_{p_i^{e_i}}^\times$ according to the prime factorization $n = \prod_i p_i^{e_i}$. For each odd prime p_i, the group $\mathbb{Z}_{p_i^{e_i}}^\times$ is cyclic of even order $m = \phi(p_i^{e_i}) = (p_i - 1)p_i^{e_i - 1}$. Let ρ be a generator, then $\rho^{m/2} = -1$. Thus we can define a character χ on $\mathbb{Z}_{p_i^{e_i}}^\times$ by $\chi(\rho^j) = \zeta_m^j$. Then $\chi(-1) = -1$. If $4 \mid n$, then $\mathbb{Z}_n^\times$ has a factor $\mathbb{Z}_{2^e}^\times$ for $e \geq 2$, which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{e-2}}$ as an additive group, with generators $\{-1, 5\}$. Thus every $j \in \mathbb{Z}_{2^e}^\times$ is uniquely expressed as $(-1)^u 5^v \bmod 2^e$, for $u = 0, 1$, and $0 \leq v < 2^{e-2}$. Then an odd character χ on $\mathbb{Z}_{2^e}^\times$ can be defined by $\chi((-1)^u 5^v) = (-1)^u$. If $n \equiv 2 \bmod 4$, then $\mathbb{Z}_n^\times$ has a trivial factor $\mathbb{Z}_2^\times = 1$, and the character group of $\mathbb{Z}_n^\times$ is isomorphic to that of $\mathbb{Z}_{n/2}^\times$, where $n/2 > 1$ is odd. Hence by Chinese remaindering, we can define an odd Dirichlet character χ on $\mathbb{Z}_n^\times$.

An important notion we will use in this proof is that of Leopoldt's character coordinates [21, 15]. In our case, for any odd Dirichlet character χ to the modulus n, the following can be taken as the definition of Leopoldt's character coordinates $y(\chi \mid t) \in \mathbb{C}$, for $t = \mathrm{i}\cot(\frac{k\pi}{n}) \in \Phi_n$,

$$y(\chi \mid t)\ \tau(\overline{\chi_d} \mid 1) = \sum_{j=1}^{n} \overline{\chi(j)}\sigma_j(t), \tag{2}$$

where d is the *conductor* of χ, [2] χ_d is the induced primitive character of χ mod d, overline denotes complex conjugation, the value $\tau(\overline{\chi_d} \mid 1) = \sum_{j=1}^{d} \overline{\chi_d}(j)e^{-2\pi \mathrm{i} j/d}$ is the Gauss sum, and σ_j is the automorphism in the Galois group $G = \mathrm{Gal}(\Phi_n/\mathbb{Q})$ that maps ζ_n to ζ_n^j. The sum $\sum_{j=1}^{n} \overline{\chi(j)}\sigma_j(t)$ is actually over relatively prime integers $j \in \mathbb{Z}_n^\times$, since otherwise the Dirichlet character $\chi(j) = 0$. In the expression $t = \mathrm{i}\cot(\frac{k\pi}{n}) \in \Phi_n$, we have $\gcd(k,n) = 1$, and so $\sigma_k \in G$, and $t = \sigma_k(t_1)$ where $t_1 = \mathrm{i}\cot(\frac{\pi}{n})$. For any fixed $k \in \mathbb{Z}_n^\times$, $\sigma_j \circ \sigma_k = \sigma_{jk}$ runs through all G when j runs through $\mathbb{Z}_n^\times$. Then

$$\sum_{j=1}^{n} \overline{\chi(j)}\sigma_j(t) = \chi(k) \sum_{j \in \mathbb{Z}_n^\times} \chi(k)^{-1}\overline{\chi(j)}\sigma_j(\sigma_k(t_1)) = \chi(k) \sum_{j \in \mathbb{Z}_n^\times} \overline{\chi(kj)}\sigma_{jk}(t_1) = \chi(k) \sum_{j \in \mathbb{Z}_n^\times} \overline{\chi(j)}\sigma_j(t_1).$$

Hence $y(\chi \mid t) = \chi(k)\ y(\chi \mid t_1)$, as the Gauss sum $\tau(\overline{\chi_d} \mid 1) \neq 0$. Similarly for $t' = \mathrm{i}\cot(\frac{k'\pi}{n})$, (recall we have $n' = n$), $y(\chi \mid t') = \chi(k')\ y(\chi \mid t_1)$. Hence the norm of the two Leopoldt's character coordinates are equal, $|y(\chi \mid t)| = |y(\chi \mid t')|$. However if $t' = rt$, where $0 < r \in \mathbb{Q}$, then $y(\chi \mid t') = ry(\chi \mid t)$ by (2), and so $r = 1$ (we will see that $y(\chi \mid t) = \chi(k)\ y(\chi \mid t_1) \neq 0$). This contradicts $r > 1$, which is a consequence of $\cot(\varphi) > \cot(\psi)$ by $0 < \varphi < \psi < \pi/2$.

Now we assume n is odd and $n' = 2n$. We want to take an odd Dirichlet character χ to the modulus $2n$. Since n is odd, the character groups of $\mathbb{Z}_n^\times$ and $\mathbb{Z}_{2n}^\times$ are isomorphic, namely for every $j \in \mathbb{Z}_n^\times$ exactly one of j or $j + n$ is odd and so belongs to $\mathbb{Z}_{2n}^\times$. Since $n > 1$ is odd,

[2] See [1], p. 167–p. 171 for the definitions of induced characters and the conductor of a character. In number theory it is traditional to denote the conductor of a character by f as is written in [15]; we use d here in order not to confuse it with the constraint function $f(u, v)$ in Section 1.

in its prime factorization $n = \prod_i p_i^{e_i}$, every p_i is odd. Then $\mathbb{Z}_n^\times \cong \prod_i \mathbb{Z}_{p_i^{e_i}}^\times$, and every $\mathbb{Z}_{p_i^{e_i}}^\times$ is cyclic of even order $\phi(p_i^{e_i}) = (p_i - 1)p_i^{e_i - 1}$. So we can define an odd Dirichlet character χ on $\mathbb{Z}_n^\times$ by Chinese remaindering, by defining it to be odd on each $\mathbb{Z}_{p_i^{e_i}}^\times$, namely $\chi(-1) = -1$. In particular there is an odd character χ to the modulus $2n$. Since n is an induced modulus, and odd, the conductor d of χ is also odd.

Take any odd Dirichlet character χ mod $2n$. It is proved in [15] (Theorem 2, p. 380) that

$$y(\chi \mid \mathrm{i}\cot(\frac{\pi}{2n})) = \frac{4n}{d} \prod_{p|2n} \left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d}, \tag{3}$$

and

$$y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) = \frac{2n}{d} \prod_{p|n} \left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d}. \tag{4}$$

Here B_{χ_d} is the generalized Bernoulli number. (Eqn. (4) is proved in Theorem 2 of [15] for any non-principal χ mod n without requiring n being odd, and so the proof below that $y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) \neq 0$ is also valid for the previous case $n = n'$.)

By definition the Bernoulli polynomial $B(Z)$ is the first $B^{(1)}(Z)$ defined by

$$\frac{te^{Zt}}{e^t - 1} = \sum_{m=0}^{\infty} B^{(m)}(Z)t^m/m!. \tag{5}$$

And the generalized Bernoulli number B_{χ_d} is defined by

$$\sum_{j=1}^{d} \chi_d(j) \frac{te^{jt}}{e^{dt} - 1} = \sum_{m=0}^{\infty} B_{\chi_d}^{(m)} t^m/m!, \tag{6}$$

with $B_{\chi_d} = B_{\chi_d}^{(1)}$. It follows immediately from (5) (and is also well known) that $B(Z) = Z - \frac{1}{2}$. Substituting t by dt and Z by j/d in (5), we get the following equality from (5) and (6)

$$B_{\chi_d} = \sum_{j=1}^{d} \chi_d(j) B(j/d). \tag{7}$$

It follows easily from the definition that $\sum_{j=1}^{d} \chi_d(j) = 0$. (This uses the fact that χ_d is not principal, namely not identically 1 on $\mathbb{Z}_d^\times = \{j \bmod d \mid \gcd(j,d) = 1\}$; indeed $\chi_d(-1) = -1$, and $\chi_d(j) = 0$ if $\gcd(j,d) > 1$, and so $\sum_j' \chi_d(j) = \sum_j' \chi_d(-j) = -\sum_j' \chi_d(j)$, where each sum $\sum_j'$ is over $\mathbb{Z}_d^\times$.) It follows that $B_{\chi_d} = \sum_{j=1}^{d} \chi_d(j) j/d$.

It is a nontrivial fact that $\sum_{j=1}^{d} \chi_d(j) j \neq 0$ for any odd character χ_d (see [28] Theorem 4.9, p. 37). Hence $B_{\chi_d} \neq 0$, and therefore also $y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) \neq 0$ and $y(\chi \mid \mathrm{i}\cot(\frac{\pi}{2n})) \neq 0$.

For $t' = \mathrm{i}\cot(\frac{k'\pi}{2n})$ and $t = \mathrm{i}\cot(\frac{k\pi}{n})$, it follows from (3) and (4) that

$$\begin{aligned}
y(\chi \mid t') &= \chi(k')\, y(\chi \mid \mathrm{i}\cot(\frac{\pi}{2n})) \\
&= \chi(k') 2 \left(1 - \frac{\overline{\chi_d(2)}}{2}\right) \frac{2n}{d} \prod_{p|n} \left(1 - \frac{\overline{\chi_d(p)}}{p}\right) B_{\chi_d} \\
&= \chi(k') \left(2 - \overline{\chi_d(2)}\right) y(\chi \mid \mathrm{i}\cot(\frac{\pi}{n})) \\
&= \chi(k') \overline{\chi(k)} \left(2 - \overline{\chi_d(2)}\right) y(\chi \mid t)
\end{aligned}$$

On the other hand, since by assumption $t = \frac{a}{b}t'$ for integers a and b, we have

$$y(\chi \mid t) = \frac{a}{b}\, y(\chi \mid t').$$

Hence, by being nonzero, and taking the norm squared, we get

$$b^2 = a^2 \cdot |2 - \overline{\chi_d(2)}|^2. \tag{8}$$

Since χ_d is primitive mod d, and d is odd, we have $\rho = \chi_d(2) \neq 0$. Denote this root of unity by $\rho = \chi_d(2)$. We have

$$b^2 = a^2[5 - 2(\rho + \overline{\rho})].$$

If we started with n' odd and $n = 2n'$, we would have the same equation with a and b exchanged.

$$a^2 = b^2[5 - 2(\rho + \overline{\rho})].$$

If $\rho = 1$ then $a = b$, this is a contradiction to $\varphi \neq \psi$. If $\rho = -1$ then $b^2 = 9a^2$ or $a^2 = 9b^2$. This gives us the unique exceptional case $\varphi = \pi/6$ and $\psi = \pi/3$.

Back to $n' = 2n$ with n odd; the other case being symmetric. Suppose $\rho \neq \pm 1$, then it is a nonreal algebraic integer, and satisfies the equation $2a^2(\rho^2 + 1) = \rho(5a^2 - b^2)$. Its minimal polynomial is monic with integer coefficients. Hence $2a^2 \mid (5a^2 - b^2)$. Hence $a \mid b$. Since $\gcd(a, b) = 1$, we get $a = 1$. Back to (8) we get $b < 3$, since $\rho \neq \pm 1$. And so $b = 2$. But in this case the solution $(1 \pm \sqrt{15}\mathrm{i})/4$ to $2(\rho^2 + 1) = \rho$ is not a root of unity. ◀

We will use Theorem 1 to prove a key complexity reduction, stated in Lemma 18, after we formally define Holant problems and reductions in Section 3.

3 Definitions and Known Results

A constraint function f of arity k is a map $\{0,1\}^k \to \mathbb{C}$. Let $\mathcal{F}$ denote a set of constraint functions. A signature grid $\Omega = (G, \pi)$ is a tuple, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$ of arity $\deg(v)$, and the incident edges $E(v)$ at v with input variables of f_v. We consider all 0-1 edge assignments σ, each gives an evaluation $\prod_{v \in V} f_v(\sigma|_{E(v)})$, where $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\mathrm{Holant}_\Omega(\mathcal{F}) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

The Holant problem parameterized by the set $\mathcal{F}$ is denoted by Holant($\mathcal{F}$). If the underlying graph is a planar graph, then we denote it by Pl-Holant($\mathcal{F}$). Replacing f by $c \cdot f$ for any $c \neq 0$ only changes the value $\mathrm{Holant}_\Omega(\mathcal{F})$ by c^n where n is the number of times f appears in Ω. Thus it does not change its complexity, therefore we can ignore such constant factors. We also write Holant($\mathcal{F}, f$) for Holant($\mathcal{F} \cup \{f\}$). We use Holant($\mathcal{F}|\mathcal{G}$) to denote the Holant problem over signature grids with a bipartite graph $G = (U, V, E)$, where each vertex in U or V is assigned a signature in $\mathcal{F}$ or $\mathcal{G}$ respectively.

A constraint function is also called a signature. A signature f of arity k can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^k}$, or as a tensor in $(\mathbb{C}^2)^{\otimes k}$. A binary signature $f(x_1, x_2) = (f_{00}, f_{01}, f_{10}, f_{11})$ can be represented as a

matrix $M(f) = \left[\begin{smallmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{smallmatrix}\right]$. A function is symmetric if its value depends only on the Hamming weight of its input. A symmetric function f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w. For example, $(=_k)$ is the EQUALITY signature $[1, 0, \ldots, 0, 1]$ (with $k-1$ 0's) of arity k.

In this paper, we consider the complexity of spin systems on k-regular graphs with real-valued edge functions. This can be defined as Holant problems of the form Holant$(=_k \mid f)$, where $f(x_1, x_2) = (f_{00}, f_{01}, f_{10}, f_{11}) \in \mathbb{R}^4$ is a binary signature. If $k = 1$, the spin system is a union of disjoint edges (the bipartite vertex-edge incidence graph form for Holant$(=_k \mid f)$ is a union of disjoint 2-paths). If $k = 2$, the spin system is a union of disjoint cycles. Thus, for $k \leq 2$, the Holant is trivially computable in polynomial time. We assume $k \geq 3$.

For $T \in \mathrm{GL}_2(\mathbb{C})$ and a signature f of arity n, written as a column vector $f \in \mathbb{C}^{2^n}$, we denote by $T^{-1}f = (T^{-1})^{\otimes n} f$ the transformed signature. For a signature set $\mathcal{F}$, define $T^{-1}\mathcal{F} = \{T^{-1}f \mid f \in \mathcal{F}\}$. For signatures written as row vectors we define $\mathcal{F}T$ similarly. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H, \pi)$ of Holant $(\mathcal{F} \mid \mathcal{G})$, for the same bipartite graph H, we get a new signature grid $\Omega' = (H, \pi')$ of Holant $\left(\mathcal{F}T \mid T^{-1}\mathcal{G}\right)$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T$ or $T^{-1}\mathcal{G}$.

▶ **Theorem 2** (Valiant's Holant Theorem [27]). *For any* $T \in \mathrm{GL}_2(\mathbb{C})$,

$$\mathrm{Holant}_{\Omega}(\mathcal{F} \mid \mathcal{G}) = \mathrm{Holant}_{\Omega'}(\mathcal{F}T \mid T^{-1}\mathcal{G}).$$

Therefore, a holographic transformation does not change the value, and so it does not change the complexity of the Holant problem in the bipartite setting.

3.1 Gadget Construction

One basic notion used throughout the paper is realization. If f is realizable from a set $\mathcal{F}$, then we can freely add f into $\mathcal{F}$ while preserving the complexity. This notion is defined by an $\mathcal{F}$-gate. An $\mathcal{F}$-gate (G, π) is similar with a signature grid for Holant$(\mathcal{F})$ except that $G = (V, E, D)$ is a graph with some dangling edges D. The dangling edges define external variables for the $\mathcal{F}$-gate. We name the regular edges in E by $1, 2, \ldots, m$ and the dangling edges in D by $m+1, \ldots, m+n$. Then we can define a function f for this $\mathcal{F}$-gate as

$$f(y_1, \ldots, y_n) = \sum_{x_1, \ldots, x_m \in \{0,1\}} H(x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where $(y_1, \ldots, y_n) \in \{0,1\}^n$ is an assignment on the dangling edges and $H(x_1, \ldots, y_n)$ is the value of the signature grid on an assignment of all edges in G, which is the product of evaluations at all vertices in V. We also call this function f the signature of the $\mathcal{F}$-gate.

If f is a binary signature, and g has arity $n > 2$, we may connect f to two consecutive variables of g. We call this operation "adding a loop to g using f". This produces a signature of arity $n-2$. Note that this $\{f, g\}$-gate (a gadget construction) is planar.

In an instance of Holant$(\mathcal{F} \mid \mathcal{G})$, if we have $(=_2)$ on both sides, then we can move any signature f on one side to another side by connecting one copy of $(=_2)$ to each variable of f. So in this case, we can ignore the bipartite restriction when constructing gadgets.

3.2 Tractable Signature Sets

We define some sets of signatures that are known to define polynomial time computable problems (we call them tractable).

3.2.0.1 Affine Signatures $\mathcal{A}$

▶ **Definition 3.** Let f be a signature of arity n. We say f has affine support of dimension k if the support of f is an affine subspace of dimension k over $\mathbb{Z}_2$.

▶ **Definition 4.** A signature $f(x_1, \ldots, x_n)$ of arity n is *affine* if it has the form

$$\lambda \cdot \chi_{AX=0} \cdot \mathrm{i}^{Q(X)},$$

where $\lambda \in \mathbb{C}$, $X = (x_1, x_2, \ldots, x_n, 1)$, A is a matrix over $\mathbb{Z}_2$, $Q(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_4[x_1, \ldots, x_n]$ is a quadratic (total degree at most 2) multilinear polynomial with the additional requirement that the coefficients of all cross terms are even, i.e., Q has the form

$$Q(x_1, x_2, \ldots, x_n) = a_0 + \sum_{k=1}^{n} a_k x_k + \sum_{1 \le i < j \le n} 2b_{ij} x_i x_j,$$

and χ is a 0-1 indicator function such that $\chi_{AX=0}$ is 1 iff $AX = 0$ over $\mathbb{Z}_2$. We use $\mathcal{A}$ to denote the set of all affine signatures.

The following lemma is an easy criterion for binary signatures in $\mathcal{A}$.

▶ **Lemma 5.** *Let $f = \lambda(\mathrm{i}^{r_1}, \mathrm{i}^{r_2}, \mathrm{i}^{r_3}, \mathrm{i}^{r_4})$ be a binary signature, where λ is a nonzero constant and $r_i \in \{0, 1, 2, 3\}$, then $f \in \mathcal{A}$ iff $r_1 + r_2 + r_3 + r_4 \equiv 0 \pmod 2$.*

Proof. If we normalize f, by dividing the constant i^{r_1}, whether $f \in \mathcal{A}$ is unchanged, nor is the stated criterion. So we may assume $r_1 = 0$. Then it is easy to check that $f(x_1, x_2) = \lambda \mathrm{i}^{Q(x_1, x_2)}$, where $Q(x_1, x_2) = r_3 x_1 + r_2 x_2 + (r_4 - r_2 - r_3) x_1 x_2$. The lemma follows. ◀

3.2.0.2 Product-Type Signatures $\mathcal{P}$

▶ **Definition 6.** A signature on a set of variables X is of *product type* if it can be expressed as a product of unary functions, binary equality functions ($[1, 0, 1]$), and binary disequality functions ($[0, 1, 0]$), each on not necessarily disjoint subsets of variables of X. We use $\mathcal{P}$ to denote the set of product-type functions.

For example, the binary signatures $(w, 0, 0, z)$ and $(0, x, y, 0)$ are in $\mathcal{P}$ for any $w, x, y, z \in \mathbb{C}$. If $\det \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right] = 0$, then $f = (w, x, y, z) \in \mathcal{P}$ and we say that f is degenerate.

3.2.0.3 Matchgate Signatures $\mathcal{M}$

Matchgates were introduced by Valiant [27] to give polynomial-time algorithms by the FKT algorithm for a collection of counting problems over planar graphs. We use $\mathcal{M}$ to denote the set of all matchgate signatures and Pl-Holant($\mathcal{M}$) is tractable. In this paper, we only need the following facts about $\mathcal{M}$ (see [8]):

1. A binary signature $f \in \mathcal{M}$ iff $f = (w, 0, 0, z)$ or $f = (0, x, y, 0)$ for any $w, x, y, z \in \mathbb{C}$;
2. The symmetric signature $[1, 1]^{\otimes n} + [1, -1]^{\otimes n} = (=_n) H^{\otimes n} \in \mathcal{M}$ for any positive integer n, where $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.
3. The symmetric signature $[1, 1]^{\otimes n} + \mathrm{i}^n [1, -1]^{\otimes n} = (=_n) Z^{\otimes n}$ is in $\mathcal{M}$ iff n is even, where $Z = \left[\begin{smallmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{smallmatrix}\right]$.

Moreover, we have the following lemma.

▶ **Lemma 7.** *Let $f = (w, x, y, z)$ be a binary signature, where $w, x, y, z \in \mathbb{C}$.*

- *If $z = \epsilon w$ and $y = \epsilon x$, where $\epsilon = \pm 1$, then $(H^{-1})^{\otimes 2} f \in \mathcal{M}$.*
- *If $z = -\epsilon w$ and $y = \epsilon x$, where $\epsilon = \pm 1$, then $(Z^{-1})^{\otimes 2} f \in \mathcal{M}$.*

Proof. Note that $H^{-1} = \frac{1}{2}H$. Ignoring a constant factor,

$$H^{\otimes 2} f = (w + x + y + z, w - x + y - z, w + x - y - z, w - x - y + z).$$

Then $H^{\otimes 2} f$ satisfies the parity constraint (item **1.** above for the properties of $\mathcal{M}$) and is therefore in $\mathcal{M}$, if either $w = z, x = y$ or $w = -z, x = -y$.

For $(Z^{-1})^{\otimes 2} f$, note that up to a constant factor, $Z^{-1} = H \left[\begin{smallmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{smallmatrix}\right]$. Thus

$$(Z^{-1})^{\otimes 2} f = H^{\otimes 2} \left[\begin{smallmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{smallmatrix}\right]^{\otimes 2} f = H^{\otimes 2} f'$$

is in $\mathcal{M}$, where $f' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{smallmatrix}\right]^{\otimes 2} f = (w, -x\mathrm{i}, -\epsilon x\mathrm{i}, \epsilon w)$ transforming it to the first case. ◀

3.2.0.4 Transformable

▶ **Definition 8.** We say a pair of signature sets $(\mathcal{G}|\mathcal{F})$ is $\mathcal{C}$-transformable for Holant $(\mathcal{G} \mid \mathcal{F})$ if there exists $T \in \mathbf{GL}_2(\mathbb{C})$ such that $\mathcal{G}T \subseteq \mathcal{C}$ and $T^{-1}\mathcal{F} \subseteq \mathcal{C}$.

If Holant$(\mathcal{C})$ is tractable and $(\mathcal{G}|\mathcal{F})$ is $\mathcal{C}$-transformable, then Holant$(\mathcal{G}|\mathcal{F})$ is tractable by a holographic transformation.

For example, we consider Pl-Holant$(=_k |f)$, where $f = (w, x, y, z)$ with $w, x, y, z \in \mathbb{C}$. If $z = \epsilon w, y = \epsilon x$, where $\epsilon = \pm 1$, then $(=_k |f)$ is $\mathcal{M}$-transformable using the holographic transformation H by Lemma 7 and Pl-Holant$(=_k |f)$ can be computed in polynomial time by the FKT algorithm. Similarly, if k is even and $z = -\epsilon w, y = \epsilon x$, where $\epsilon = \pm 1$, then Pl-Holant$(=_k |f)$ can be computed in polynomial time by the FKT algorithm.

3.3 Some results

In [10], the following trichotomy theorem for k-regular *symmetric* spin systems is given.

▶ **Theorem 9.** *Let $k \geq 3$.* Holant$(=_k |f)$*, where $f = [w, x, z]$ is a symmetric binary signature $(w, x, z \in \mathbb{C})$, is* #P*-hard for all $w, x, z \in \mathbb{C}$, except in the following cases, for which the problem is in* P*:*

- *$f \in \mathcal{P}$: $wz = x^2$, or $w = z = 0$, or $x = 0$,*
- *f is $\mathcal{A}$-transformable: $wz = -x^2$ and $w^{4k} = x^{4k} = z^{4k}$.*

If the input is restricted to planar graphs, then another class becomes tractable but everything else remains #P*-hard.*

- *$(=_k |f)$ is $\mathcal{M}$-transformable: $w^k = z^k$.*

By Theorem 9, we have the following corollary.

▶ **Corollary 10.** *Let $f = [w, x, z]$ be a symmetric binary signature, where $w, x, z \in \mathbb{C}$, and $f \notin \mathcal{P}$, i.e., $wz \neq x^2$, $x \neq 0$ and there is at most one zero in $\{w, z\}$. If $|w| \neq |z|$, then* Pl-Holant$(=_k |f)$*, where $k \geq 3$, is* #P*-hard.*

In [25], a trichotomy theorem for 3-regular asymmetric spin systems is given.

▶ **Theorem 11.** *Suppose $w, x, y, z \in \mathbb{C}$. Then* Holant$(=_3 |(w, x, y, z))$ *is* #P*-hard except in the following classes, for which the problem is in* P*.*

- *$f \in \mathcal{P}$: $wz = xy$, or $w = z = 0$ or $x = y = 0$;*
- *f is $\mathcal{A}$-transformable: $wz = -xy, w^6 = \epsilon z^6, x^2 = \epsilon y^2$, where $\epsilon = \pm 1$.*

If the input is restricted to planar graphs, then another class becomes tractable but everything else remains #P*-hard.*

- $(=_3 \mid f)$ *is* $\mathcal{M}$*-transformable:* $w^3 = \epsilon z^3, x = \epsilon y$*, where* $\epsilon = \pm 1$*.*

By Theorem 11, we have the following corollary.

▶ **Corollary 12.** *Let* $f = (w, x, y, z)$ *be a binary signature, where* $w, x, y, z \in \mathbb{C}$ *and* $f \notin \mathcal{P}$*, i.e.,* $wz \neq xy$ *and there is at most one zero in* $\{w, x, y, z\}$*. If* $|w| \neq |z|$ *or* $|x| \neq |y|$*, then* Pl-Holant$(=_3 \mid f)$ *is* #P*-hard.*

▶ **Lemma 13.** *Let* f *be a binary signature with the signature matrix* $N = P \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right] P^{-1}$*, where* P *is an invertible* 2×2 *matrix. Suppose* $\lambda\mu \neq 0$ *and* $\frac{\lambda}{\mu}$ *is not a root of unity, then for any* $\mathcal{F}$ *and any* $a, b \in \mathbb{C}$*, if* g *has signature matrix* $P \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] P^{-1}$*, then*

$$\text{Holant}(\mathcal{F}, =_2 \mid f, g) \leq_T^p \text{Holant}(\mathcal{F}, =_2 \mid f).$$

Proof. Let l be any positive integer. In Pl-Holant$(\mathcal{F}, =_2 \mid f)$, by connecting l copies of f on the RHS via $=_2$ on the LHS, we can implement f_l with the signature matrix $N^l = P \begin{bmatrix} \lambda^l & 0 \\ 0 & \mu^l \end{bmatrix} P^{-1}$. Since $\frac{\lambda}{\mu}$ is not a root of unity, for any positive integer l, $(\frac{\lambda}{\mu})^l \neq 1$.

Consider an instance Ω of Pl-Holant$(\mathcal{F}, =_2 \mid f, g)$. Suppose that g appears t times. We obtain Ω_l from Ω by replacing each occurrence of g with f_l. Since f_l has the signature matrix N^l, we can view our construction of Ω_l as replacing f_l by 3 signatures, with matrix P, $\begin{bmatrix} \lambda^l & 0 \\ 0 & \mu^l \end{bmatrix}$, and P^{-1}, respectively. We stratify the assignments in Ω_l with nonzero evaluations based on the assignments to the t occurrences of the signature with the signature matrix $\begin{bmatrix} \lambda^l & 0 \\ 0 & \mu^l \end{bmatrix}$. Suppose there are i times it was assigned 00 with function value λ^l, and j times 11 with function value μ^l. To have a nonzero evaluation clearly $i + j = t$. Let c_{ij} be the sum over all such assignments of the products of evaluations of all signatures (including the signatures corresponding to matrices P and P^{-1}) in Ω_l except for $\begin{bmatrix} \lambda^l & 0 \\ 0 & \mu^l \end{bmatrix}$. Then

$$\begin{aligned} \text{Holant}_{\Omega_l} &= \sum_{i+j=t} \left(\lambda^l\right)^i \left(\mu^l\right)^j c_{ij} \\ &= \mu^{lt} \sum_{0 \leq i \leq t} \left(\left(\frac{\lambda}{\mu}\right)^l\right)^i c_{i,t-i}. \end{aligned}$$

By oracle calls to Pl-Holant$(\mathcal{F}, =_2 \mid f)$, we can get Holant_{Ω_l} for any $1 \leq l \leq t+1$. Since $(\frac{\lambda}{\mu})^l \neq 1$ for $l \geq 1$, we have $(\frac{\lambda}{\mu})^u \neq (\frac{\lambda}{\mu})^v$, for any two distinct integers $u, v \geq 0$. Therefore we get a non-singular Vandermonde system. We can solve all c_{ij} for $i + j = t$ given Holant_{Ω_l} for all $1 \leq l \leq t+1$. Then we can compute $\sum_{i+j=t} c_{ij} a^i b^j$, the desired Holant value. Hence,

$$\text{Pl-Holant}(\mathcal{F}, =_2 \mid f, g) \leq_T^p \text{Pl-Holant}(\mathcal{F}, =_2 \mid f).$$ ◀

▶ **Lemma 14.** *Let* f *be a non-degenerate binary signature, then for any* $\mathcal{F}$*,*

$$\text{Pl-Holant}(\mathcal{F}, =_2 \mid f, =_2) \leq_T^p \text{Pl-Holant}(\mathcal{F}, =_2 \mid f).$$

Proof. Since f is non-degenerate, by the Jordan normal form, there exists a non-singular matrix P such that the signature matrix of f takes the form $\begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} = P \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right] P^{-1}$ with $\lambda\mu \neq 0$ or, up to a nonzero constant multiple, $\begin{bmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{bmatrix} = P \left[\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right] P^{-1}$ with $\lambda \neq 0$.

In the first case, if there is a positive integer j such that $\lambda^j = \mu^j$, then we may directly implement $=_2$ on the RHS by connecting j copies of f via $=_2$ on the LHS. Otherwise, $\frac{\lambda}{\mu}$ is not a root of unity and we get $=_2$ on the RHS by Lemma 13.

In the second case $P\left[\begin{smallmatrix}1 & \lambda\\ 0 & 1\end{smallmatrix}\right]P^{-1}$, by connecting l copies of f on the RHS via $=_2$ on the LHS, where l is a positive integer, we can implement f_l with the signature matrix $P\left[\begin{smallmatrix}1 & l\lambda\\ 0 & 1\end{smallmatrix}\right]P^{-1}$.

The following proof is similar to Lemma 13. Consider an instance Ω of Pl-Holant$(\mathcal{F}, =_2 \mid f, =_2)$. Suppose the signature $=_2$ on the RHS appears t times. We obtain a planar signature grid Ω_l, a problem in Pl-Holant$(\mathcal{F}, =_2 \mid f)$, by replacing each occurrence of $=_2$ on the RHS with f_l. We can view our construction of Ω_l as replacing f_l by 3 signatures, with matrix P, $\left[\begin{smallmatrix}1 & l\lambda\\ 0 & 1\end{smallmatrix}\right]$, and P^{-1}, respectively. We stratify the assignments in Ω_l with nonzero evaluations based on the assignments to the t occurrences of the signature with the signature matrix $\left[\begin{smallmatrix}1 & l\lambda\\ 0 & 1\end{smallmatrix}\right]$. Suppose there are i times it was assigned $00, 11$ with function value 1, and j times 01 with function value $l\lambda$. Then $i + j = t$. Let c_{ij} be the sum over all such assignments of the products of evaluations of all signatures (including the signatures corresponding to matrices P and P^{-1}) in Ω_l except for $\left[\begin{smallmatrix}1 & l\lambda\\ 0 & 1\end{smallmatrix}\right]$. Then

$$\text{Holant}_{\Omega_l} = \sum_{i+j=t} (l\lambda)^j c_{ij}.$$

By oracle calls to Pl-Holant$(\mathcal{F}, =_2 \mid f)$, we can get Holant_{Ω_l} for any $1 \le l \le t+1$. For any two distinct integers $l, l' \ge 0$, $l\lambda \ne l'\lambda$ since $\lambda \ne 0$. Therefore we get a non-singular Vandermonde system. We can solve for all c_{ij} $(i + j = t)$ given Holant_{Ω_l} for all $1 \le l \le t+1$. Then notice that c_{t0} is the desired Holant value. Therefore,

$$\text{Pl-Holant}(\mathcal{F}, =_2 \mid f, =_2) \le_T^p \text{Pl-Holant}(\mathcal{F}, =_2 \mid f).$$ ◀

Using a similar proof idea as in Lemma 13 we can prove

▶ **Corollary 15.** *Let $\mathcal{F}$ and $\mathcal{G}$ be any two signature sets, then we have*

$$\text{Pl-Holant}(=_4, \mathcal{G} \mid \mathcal{F}, =_2) \le_T^p \text{Pl-Holant}([1, 0, 0, 0, x], \mathcal{G} \mid \mathcal{F}, =_2),$$

for any $x \ne 0$.

Lin and Wang proved the following lemma (Lemma 3.1 in [22]).

▶ **Lemma 16** (Lin-Wang). *Let $\mathcal{F}$ be a set of signatures, and f be a signature. Then*

$$\text{Holant}(\mathcal{F}, f) \le_T^p \text{Holant}(\mathcal{F}, f^{\otimes k}),$$

for any $k \ge 1$.

The proof of Lemma 16 is non-planar. Thus it cannot be applied directly to planar Holant problems. We give the following lemma for planar graphs.

▶ **Lemma 17.** *Let $\mathcal{F}$ be a set consisting of signatures of even arities and let f be a non-degenerate binary signature, then*

$$\text{Pl-Holant}(=_4 \mid \mathcal{F}, f, [1, 1]) \le_T^p \text{Pl-Holant}(=_4 \mid \mathcal{F}, f, [1, 1]^{\otimes 2}).$$

Proof. In the setting of Pl-Holant$(=_4 \mid \mathcal{F}, f, [1, 1]^{\otimes 2})$, by adding a loop to $=_4$ using $[1, 1]^{\otimes 2}$, we have $=_2$ on the LHS. Then by Lemma 14, we have $=_2$ on the RHS. Now that we have $=_2$ on both sides, we can ignore the bipartite restriction. Thus we just need to prove that

$$\text{Pl-Holant}(=_4 \mid \mathcal{F}, f, [1, 1]) \le_T^p \text{Pl-Holant}(=_4, =_2, \mathcal{F}, f, [1, 1]^{\otimes 2}).$$

Given any instance Ω of Pl-Holant$(=_4 \mid \mathcal{F}, f, [1, 1])$, we may assume the plane graph of Ω is connected, since the Holant value on Ω is the product over its connected components.

Moreover, since all signatures in $\mathcal{F}$ have even arities, the number of occurrences of $[1,1]$ must be even.

Let T be a spanning tree of the dual graph of Ω, and pick any node as the root of T. For definiteness we pick the node of T that corresponds to the external face of Ω as root. Let $\mathfrak{F}$ be a leaf node of T, corresponding to a face F of Ω. Suppose there are an even number of $[1,1]$ inside F, then we can connect them in pairs within the face by copies of $[1,1]^{\otimes 2}$, maintaining planarity. Suppose there are an odd number of $[1,1]$ in $\mathfrak{F}$ and suppose $\mathfrak{F}$ is not the root of T. Let the parent node of $\mathfrak{F}$ correspond to the face F' of Ω, and F and F' share the edge e in Ω. Then we replace e by a path of length 2, put $=_4$ on the new node, and connect two input variables of $=_4$ each to a copy of $[1,1]$, one inside F and one inside F'. This operation effectively changes the new $=_4$ to $=_2$, thus not changing the Holant value, while at the same time changing the parity of the numbers of $[1,1]$'s inside F *and* F'. This is illustrated in Figure 1.

Then we can replace those $[1,1]$'s inside F in pairs by $[1,1]^{\otimes 2}$. We delete the leaf node from T, and complete the proof by induction. Note that finally at the root of T, there must be an even number of $[1,1]$, because the parity of the total number of $[1,1]$ is unchanged during this process. Thus we can simulate Pl-Holant$(=_4 \mid \mathcal{F}, f, [1,1])$ by Pl-Holant$(=_4 \mid \mathcal{F}, f, [1,1]^{\otimes 2})$. ◀

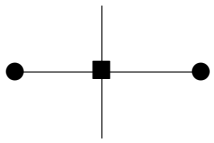


Figure 1: The operation replacing the edge between F and F', drawn vertically. The circle vertices are labeled by $[1,1]$ and the square is labeled by $=_4$. Effectively the new node has signature $=_2$, thus keeping the Holant value unchanged.

4 Trichotomy for Spin Systems on 4-regular Graphs

In this section, we prove Theorem 24 for the special case $k = 4$.

We say a non-singular M has infinite projective order if M^n is not a scalar multiple of I for any $n \geq 1$. Let $y \in \mathbb{R}$. The matrix $M = \left[\begin{smallmatrix} 1 & y \\ -y & 1 \end{smallmatrix}\right]$ is diagonalizable, $M = Z \left[\begin{smallmatrix} 1+y\mathrm{i} & 0 \\ 0 & 1-y\mathrm{i} \end{smallmatrix}\right] Z^{-1}$, where $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{smallmatrix}\right]$. The ratio of the two eigenvalues is $\frac{1+y\mathrm{i}}{1-y\mathrm{i}}$. Therefore this M has infinite projective order iff $\frac{1+y\mathrm{i}}{1-y\mathrm{i}}$ is not a root of unity.

The following lemma is a reduction that follows from Theorem 1.

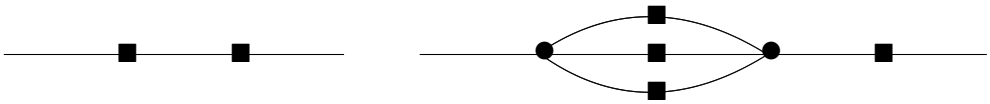
▶ **Lemma 18.** *Let $\mathcal{F}$ be any signature set containing a binary signature $(1, x, -x, 1)$, where $x \in \mathbb{R}$ and $x \neq 0, \pm 1$. Then for some $y \in \mathbb{R}$,*

$$\text{Holant}(=_4 \mid \mathcal{F}, (1, y, -y, 1)) \leq_T^p \text{Holant}(=_4 \mid \mathcal{F}),$$

where the signature matrix $\left[\begin{smallmatrix} 1 & y \\ -y & 1 \end{smallmatrix}\right]$ has eigenvalues $1 \pm y\mathrm{i}$, with ratio $\frac{1+y\mathrm{i}}{1-y\mathrm{i}}$ not a root of unity, and thus the matrix has infinite projective order.

Proof. In Holant$(=_4 \mid \mathcal{F})$, by adding a loop to $=_4$ using $(1, x, -x, 1) \in \mathcal{F}$, we have $=_2$ on the LHS. Since $(1, x, -x, 1)$ is non-degenerate, by Lemma 14 we obtain $=_2$ on the RHS. Once we have $=_2$ on both sides we can freely move signatures from either side, and so we can ignore the bipartite restriction.

By the construction in Figure 2, using $f = \left[\begin{smallmatrix} 1 & x \\ -x & 1 \end{smallmatrix}\right]$, we can realize binary functions g_1 with $M(g_1) = \left[\begin{smallmatrix} 1-x^2 & 2x \\ -2x & 1-x^2 \end{smallmatrix}\right]$, and g_2 with $M(g_2) = \left[\begin{smallmatrix} 1-x^4 & x+x^3 \\ -x-x^3 & 1-x^4 \end{smallmatrix}\right] = (x^2+1) \left[\begin{smallmatrix} 1-x^2 & x \\ -x & 1-x^2 \end{smallmatrix}\right]$.



g_1 g_2

Figure 2: The gadgets realizing g_1 and g_2. The circle vertices are labeled by $=_4$ and squares are labeled by f. For the squares, the edge on the left side corresponds to the variable x_1 of f and the edge on the right side corresponds to x_2.

This means that, if we assign $(b_1, b_2) \in \{0,1\}^2$ to the two external edges, and form the sum of product over 0-1 assignments on internal edges we get the value in the matrix in row b_1 and column b_2.

The matrix $\left[\begin{smallmatrix} 1-x^2 & x \\ -x & 1-x^2 \end{smallmatrix}\right]$ has two nonzero eigenvalues $1 - x^2 \pm x\mathrm{i}$, with ratio $\frac{a+b\mathrm{i}}{a-b\mathrm{i}}$, where $a = 1 - x^2$ and $b = x$. This ratio is a root of unity iff the complex argument φ of $a + b\mathrm{i} = |a + b\mathrm{i}|e^{\mathrm{i}\varphi}$ is a rational multiple of π, where $\cot(\varphi) = \frac{a}{b}$.

Similarly, $\left[\begin{smallmatrix} 1-x^2 & 2x \\ -2x & 1-x^2 \end{smallmatrix}\right]$ has two nonzero eigenvalues $1 - x^2 \pm 2x\mathrm{i}$, with ratio $\frac{a+2b\mathrm{i}}{a-2b\mathrm{i}}$. This ratio is a root of unity iff the complex argument ψ of $a + 2b\mathrm{i} = |a + 2b\mathrm{i}|e^{\mathrm{i}\psi}$ is a rational multiple of π, where $\cot(\psi) = \frac{a}{2b}$.

By Theorem 1 these cannot both happen. Therefore at least one of the two constructions defines a matrix that has infinite projective order. ◀

Let $f = (w, x, y, z)$ be a binary signature where $w, x, y, z \in \mathbb{R}$. If $wz = xy$ or there are two or more zeros in $\{w, x, y, z\}$, then $f \in \mathcal{P}$ and Holant$(=_4 \,|f)$ can be computed in polynomial time. Moreover, if $x = y$, then f is symmetric and Theorem 24 follows Theorem 9. Thus we now assume the following:

Assumption: The binary signature $f = (w, x, y, z)$ satisfies $wz \neq xy, x \neq y$ and there is at most one zero in $\{w, x, y, z\}$.

First, we consider the case that there is exactly one zero in $\{w, z\}$.

▶ **Lemma 19.** *Let $f = (w, x, y, z)$, where $w, x, y, z \in \mathbb{R}$. If there is exactly one zero in $\{w, z\}$ and $xy \neq 0$, then* Pl-Holant$(=_4 \,|f)$ *is* #P*-hard.*

Proof. By flipping 0 and 1, we may assume that $w \neq 0, z = 0$. By normalizing $w = 1$ we can assume that $f = (1, x, y, 0)$.

In the setting of Pl-Holant$(=_4 \,|f)$, by adding a loop using f to $=_4$, we have $[1, 0]^{\otimes 2}$ on the LHS. Then taking two copies of f and connecting $[1, 0]$ to the variable x_1 of each copy we get $[1, x]^{\otimes 2}$ on the RHS. This operation used $[1, 0]^{\otimes 2}$ on the LHS. By adding a loop using $[1, x]^{\otimes 2}$ to $=_4$, we have $[1, 0, x^2]$ on the LHS. So we have

$$\text{Pl-Holant}(=_4, [1, 0]^{\otimes 2}, [1, 0, x^2]|f) \leq_T^p \text{Pl-Holant}(=_4 |f). \tag{9}$$

By a holographic transformation using $\left[\begin{smallmatrix} 1 & 0 \\ 0 & x^{-1} \end{smallmatrix}\right]$, we have

$$\text{Pl-Holant}([1, 0, 0, 0, x^{-4}], [1, 0]^{\otimes 2}, =_2 |(1, x^2, xy, 0)) \equiv_T^p \text{Pl-Holant}(=_4, [1, 0]^{\otimes 2}, [1, 0, x^2]|f). \tag{10}$$

Since $(1, x^2, xy, 0)$ is non-degenerate, we can get $(=_2)$ on the RHS by Lemma 14,

$$\begin{aligned} & \text{Pl-Holant}([1,0,0,0,x^{-4}], [1,0]^{\otimes 2}, =_2 \mid (1, x^2, xy, 0), =_2) && (11)\\ \leq_T^p \ & \text{Pl-Holant}([1,0,0,0,x^{-4}], [1,0]^{\otimes 2}, =_2 \mid (1, x^2, xy, 0)) && (12)\end{aligned}$$

Now that we have $=_2$ on both sides of (11), we will ignore the bipartiteness restriction. We construct $[1,1]^{\otimes 2}$ in (11) as follows.

- If $|x| = 1$, as $x \in \mathbb{R}$, we have $(1, x^2, xy, 0) = (1, 1, xy, 0)$. Then by taking two copies of $(1, 1, xy, 0)$ and connecting $[1, 0]$ to the variable x_1 for each copy (using $[1, 0]^{\otimes 2}$), we get $[1, 1]^{\otimes 2}$.
- If $|x| \neq 1$, by adding a loop using $=_2$ to $[1, 0, 0, 0, x^{-4}]$ we have $[1, 0, x^{-4}]$. Since $|x^{-4}| \neq 1$, we can get $[1, 0, x^{-2}]$ by Lemma 13. Connecting one variable of $[1, 0, x^{-2}]$ to the variable x_2 of $(1, x^2, xy, 0)$, we get the signature $(1, 1, xy, 0)$ and proceed as above.

We can place the constructed $[1, 1]^{\otimes 2}$ on the RHS of (11). Moreover, note that we have $[1, 0, 0, 0, x^{-4}]$ on the LHS and $=_2$ on the RHS in (11). Thus we have $=_4$ on the LHS by Corollary 15. This implies that

$$\begin{aligned} & \text{Pl-Holant}(=_4 \mid [1,1]^{\otimes 2}, (1, x^2, xy, 0)) \\ & \leq_T^p \text{Pl-Holant}([1,0,0,0,x^{-4}], [1,0]^{\otimes 2}, =_2 \mid (1, x^2, xy, 0), =_2). \end{aligned} \tag{13}$$

Then by Lemma 17 we have

$$\text{Pl-Holant}(=_4 \mid [1,1], (1, x^2, xy, 0)) \leq_T^p \text{Pl-Holant}(=_4 \mid [1,1]^{\otimes 2}, (1, x^2, xy, 0)). \tag{14}$$

In Pl-Holant$(=_4 \mid [1, 1], (1, x^2, xy, 0))$, by connecting $[1, 1]$ to $=_4$ we have $=_3$ on the LHS. This implies that

$$\text{Pl-Holant}(=_3 \mid (1, x^2, xy, 0)) \leq_T^p \text{Pl-Holant}(=_4 \mid [1,1], (1, x^2, xy, 0)). \tag{15}$$

By Theorem 11, Pl-Holant$(=_3 \mid (1, x^2, xy, 0))$ is #P-hard. Then by (9), (10), (11), (13), (14) and (15), Pl-Holant$(=_4 \mid f)$ is #P-hard. ◀

Now we can assume that $wz \neq 0$ and $f = (1, x, y, z)$ by normalizing $w = 1$.

▶ **Lemma 20.** *Let $f = (1, x, y, z)$, where $x, y, z \in \mathbb{R}$ and $z \neq 0$. If $|z| \neq 1$, then* Pl-Holant$(=_4 \mid f)$ *is* #P*-hard.*

Proof. In Pl-Holant$(=_4 \mid f)$, by adding a loop using f to $=_4$, we have $[1, 0, z]$ on the LHS, i.e. we have

$$\text{Pl-Holant}(=_4, [1,0,z] | f) \leq_T^p \text{Pl-Holant}(=_4 \mid f). \tag{16}$$

For Pl-Holant$(=_4, [1, 0, z]|f)$, by the holographic transformation using $\begin{bmatrix} 1 & 0 \\ 0 & z^{-\frac{1}{2}} \end{bmatrix}$, we have

$$\text{Pl-Holant}([1,0,0,0,z^{-2}], =_2 \mid (1, xz^{\frac{1}{2}}, yz^{\frac{1}{2}}, z^2)) \equiv_T^p \text{Pl-Holant}(=_4, [1,0,z] | f). \tag{17}$$

Note that $z^{\frac{1}{2}}$ can be a complex number.

Now we consider the LHS problem in (17). Firstly, since $(1, xz^{\frac{1}{2}}, yz^{\frac{1}{2}}, z^2)$ is non-degenerate, by Lemma 14, we have $=_2$ on the RHS. Then by Corollary 15, we have $=_4$ on the LHS. Moreover, by adding a loop using $=_2$ to $[1, 0, 0, 0, z^{-2}]$, we have $[1, 0, z^{-2}]$ on the LHS. This implies that

$$\begin{aligned} & \text{Pl-Holant}([1,0,z^{-2}], =_2, =_4 \mid =_2, (1, xz^{\frac{1}{2}}, yz^{\frac{1}{2}}, z^2)) && (18)\\ \leq_T^p \ & \text{Pl-Holant}([1,0,0,0,z^{-2}], =_2 \mid (1, xz^{\frac{1}{2}}, yz^{\frac{1}{2}}, z^2)). && (19)\end{aligned}$$

In problem (18), we have $=_2$ on both sides, and so we can ignore the bipartiteness restriction. Since $|z| \neq 1$, by Lemma 13, we have $[1,0]^{\otimes 2}$ and $[1,0,x^{-1}z^{-\frac{1}{2}}]$. Connecting $[1,0,x^{-1}z^{-\frac{1}{2}}]$ to the variable x_2 of $(1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)$, we have $(1,1,yz^{\frac{1}{2}},x^{-1}z^{\frac{3}{2}})$. By taking two copies of $(1,1,yz^{\frac{1}{2}},x^{-1}z^{\frac{3}{2}})$ and connecting $[1,0]$ to the variable x_1 for each copy, we have $[1,1]^{\otimes 2}$. This makes use of $[1,0]^{\otimes 2}$. This implies that

$$\begin{aligned}&\text{Pl-Holant}(=_4 \mid [1,1]^{\otimes 2}, (1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2))\\ &\leq_T^p \text{Pl-Holant}([1,0,z^{-2}],=_2,=_4 \mid =_2,(1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)).\end{aligned} \tag{20}$$

Then by Lemma 17, we have

$$\text{Pl-Holant}(=_4 \mid [1,1],(1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)) \leq_T^p \text{Pl-Holant}(=_4 \mid [1,1]^{\otimes 2},(1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)). \tag{21}$$

By connecting $[1,1]$ to $=_4$, we have $=_3$ on the LHS. This implies that

$$\text{Pl-Holant}(=_3 \mid (1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)) \leq_T^p \text{Pl-Holant}(=_4 \mid [1,1],(1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2)). \tag{22}$$

By Corollary 12, Pl-Holant$(=_3 \mid (1,xz^{\frac{1}{2}},yz^{\frac{1}{2}},z^2))$ is #P-hard since $|z^2| \neq 1$ and there is at most one zero in $\{xz^{\frac{1}{2}},yz^{\frac{1}{2}}\}$. Then by (16), (17), (18), (20), (21) and (22), Pl-Holant$(=_4 \mid f)$ is #P-hard. ◀

In addition to $w = 1$, $wz \neq xy$, $x \neq y$, we may now assume $|z| = 1$. Being real, $z = \pm 1$. We next consider the case that $x \neq \pm y$.

▶ **Lemma 21.** *Let $f = (1,x,y,z)$, where $x,y,z \in \mathbb{R}$ and $|z| = 1, x \neq \pm y$, then* Pl-Holant$(=_4 \mid f)$ *is* #P*-hard.*

Proof. For Pl-Holant$(=_4 \mid f)$, by adding a loop using f to $=_4$ we have $[1,0,z]$ on the LHS. Take two copies of f and one copy of $[1,0,z]$, and connect them in a symmetric path, i.e., connect x_2 of both copies of f to $[1,0,z]$, leaving x_1 of both copies of f as external edges, we get the following symmetric signature g on the RHS with the signature matrix (using $z^2 = 1$),

$$\left[\begin{smallmatrix}1 & x\\ y & z\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & z\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & y\\ x & z\end{smallmatrix}\right] = \begin{bmatrix}1+x^2z & x+y\\ x+y & z+y^2\end{bmatrix}.$$

For $z = 1$, $g = [1+x^2, x+y, 1+y^2]$. We have $x+y \neq 0$ and $1+x^2 \neq 1+y^2$ by $x \neq \pm y$. Thus Pl-Holant$(=_4 \mid g)$ is #P-hard by Corollary 10. So Pl-Holant$(=_4 \mid f)$ is #P-hard.

For $z = -1$, $g = [1-x^2, x+y, y^2-1]$. Still $x+y \neq 0$.

- If $|1-x^2| \neq |y^2-1|$, then Pl-Holant$(=_4 \mid g)$ is #P-hard by Corollary 10. So Pl-Holant$(=_4 \mid f)$ is #P-hard.
- If $1-x^2 = 1-y^2$, then $x^2 = y^2$. This is a contradiction.
- If $1-x^2 = y^2-1$, we have $x^2 \neq 1$. Otherwise $y^2 = 1$ and again a contradiction. Thus we have $1-x^2 = y^2-1 \neq 0$. So we can assume that $g = [1, \frac{x+y}{1-x^2}, 1]$ after the nonzero scalar $1-x^2$. By adding a loop using g to $=_4$, we have $=_2$ on the LHS. By connecting two copies of f using $=_2$ on the LHS, we get the signature h with the signature matrix

$$\left[\begin{smallmatrix}1 & x\\ y & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & y\\ x & -1\end{smallmatrix}\right] = \begin{bmatrix}1+x^2 & y-x\\ y-x & 1+y^2\end{bmatrix}.$$

Note that h is symmetric and $y-x \neq 0$, $1+x^2 \neq 1+y^2$ by $x \neq \pm y$. Thus Pl-Holant$(=_4 \mid h)$ is #P-hard by Corollary 10. So Pl-Holant$(=_4 \mid f)$ is #P-hard.

◀

Now we can assume that $w = 1$, $z = \pm 1$, and $x = -y \neq 0$, i.e., $f = (1, x, -x, \pm 1)$. By Lemma 7 and the statements before Lemma 7, the pairs $(=_4 |(1, x, -x, -1))$ and $(=_4 |(1, x, -x, 1))$ are $\mathcal{M}$-transformable under the holographic transformation H and Z respectively. Thus Pl-Holant$(=_4 |(1, x, -x, \pm 1))$ can be computed in polynomial time. Moreover, if $x = \pm 1$ then $f \in \mathcal{A}$ by Lemma 5 and Holant$(=_4 |f)$ can be computed in polynomial time on general graphs. In the following, we consider Holant$(=_4 |f)$ on general graphs with $x \neq \pm 1$. These are cases where Holant$(=_4 |f)$ is #P-hard, but Pl-Holant$(=_4 |f)$ is in P. It is for the proof of these cases that we ultimately use Theorem 1 from number theory.

▶ **Lemma 22.** *Let $f = (1, x, -x, z)$, where $x \in \mathbb{R}$, $z = \pm 1$ and $x \neq 0, \pm 1$. Then* Holant$(=_4 |f)$ *is* #P*-hard.*

Proof. For $z = 1$, by Lemma 18, there exists $y \in \mathbb{R}$ such that

$$\text{Holant}(=_4 |f, (1, y, -y, 1)) \leq_T^p \text{Holant}(=_4 |f), \tag{23}$$

where $\left[\begin{smallmatrix} 1 & y \\ -y & 1 \end{smallmatrix}\right]$ has infinite projective order, i.e., the ratio of eigenvalues $\frac{1+y\mathrm{i}}{1-y\mathrm{i}}$ is not a root of unity. Recall that

$$\left[\begin{smallmatrix} 1 & y \\ -y & 1 \end{smallmatrix}\right] = Z \left[\begin{smallmatrix} 1+y\mathrm{i} & 0 \\ 0 & 1-y\mathrm{i} \end{smallmatrix}\right] Z^{-1},$$

where $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{smallmatrix}\right]$. Then by Lemma 13, we can choose g having the signature matrix

$$Z \left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right] Z^{-1} = \left[\begin{smallmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{smallmatrix}\right],$$

such that

$$\text{Holant}(=_4 |f, g) \leq_T^p \text{Holant}(=_4 |f, (1, y, -y, 1)). \tag{24}$$

Note that g is degenerate, being the tensor product of two unary signatures, $g = [1, \mathrm{i}] \otimes [1, -\mathrm{i}]$. If we take 4 copies of g and connect each of their first variable corresponding to $[1, -\mathrm{i}]$ to $=_4$, we obtain $2[1, \mathrm{i}]^{\otimes 4}$ on the RHS. The proof of Lemma 16 can be easily adapted to the bipartite case with $=_4$ on the LHS, and we get

$$\text{Holant}(=_4 |f, [1, \mathrm{i}]) \leq_T^p \text{Holant}(=_4 |f, [1, \mathrm{i}]^{\otimes 4}). \tag{25}$$

In Holant$(=_4 |f, [1, \mathrm{i}])$, by connecting $[1, \mathrm{i}]$ to $=_4$, we have $[1, 0, 0, \mathrm{i}]$ on the LHS, i.e.,

$$\text{Holant}([1, 0, 0, \mathrm{i}]|f) \leq_T^p \text{Holant}(=_4 |f, [1, \mathrm{i}]). \tag{26}$$

Then by the holographic transformation using $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{i}^{-\frac{1}{3}} \end{smallmatrix}\right]$, we have

$$\text{Holant}(=_3 |(1, x\mathrm{i}^{\frac{1}{3}}, -x\mathrm{i}^{\frac{1}{3}}, z\mathrm{i}^{\frac{2}{3}})) \equiv \text{Holant}([1, 0, 0, \mathrm{i}]|f). \tag{27}$$

Since $|x\mathrm{i}^{\frac{1}{3}}| \neq 0, 1$, Holant$(=_3 |(1, x\mathrm{i}^{\frac{1}{3}}, -x\mathrm{i}^{\frac{1}{3}}, z\mathrm{i}^{\frac{2}{3}}))$ is #P-hard by Theorem 11. Then by (23), (24), (25), (26) and (27), Holant$(=_4 |f)$ is #P-hard.

For $z = -1$, by adding a loop to $=_4$ using f, we have $[1, 0, -1]$ on the LHS. This implies that

$$\text{Holant}(=_4, [1, 0, -1]|f) \leq_T^p \text{Holant}(=_4 |f). \tag{28}$$

Then by the holographic transformation using $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{smallmatrix}\right]$, we have

$$\text{Holant}(=_4, =_2 |(1, x\mathrm{i}, -x\mathrm{i}, 1)) \equiv \text{Holant}(=_4, [1, 0, -1]|f). \tag{29}$$

Note that $(1, x\mathrm{i}, -x\mathrm{i}, 1)$ has the signature matrix

$$\left[\begin{smallmatrix} 1 & x\mathrm{i} \\ -x\mathrm{i} & 1 \end{smallmatrix}\right] = Z' \left[\begin{smallmatrix} 1+x & 0 \\ 0 & 1-x \end{smallmatrix}\right] Z'^{-1},$$

where $Z' = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{smallmatrix}\right]$. Note that the ratio $\frac{1+x}{1-x}$ is not a root of unity since $x \in \mathbb{R}$ and $x \neq 0, \pm 1$. By Lemma 13, we have the signature with the signature matrix

$$Z' \left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right] Z'^{-1} = \left[\begin{smallmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{smallmatrix}\right]$$

on the RHS, i.e., we have $[1, -\mathrm{i}] \otimes [1, \mathrm{i}]$ on the RHS. The remaining proof is the same as the previous case that $z = 1$ and we omit it here. ◀

Now we give the main theorem of this section.

▶ **Theorem 23.** *Let $f = (w, x, y, z)$ with $w, x, y, z \in \mathbb{R}$. Then* Holant$(=_4 \mid f)$ *is* #P*-hard except in the following cases, where the problem is computable in polynomial time.*

- *$f \in \mathcal{P}$: $wz = xy$, or $w = z = 0$, or $x = y = 0$;*
- *$f \in \mathcal{A}$: $w^2 = x^2 = y^2 = z^2$.*

If the input is restricted to planar graphs, then another case becomes polynomial time computable but everything else remains #P*-hard.*

- *The pair $(=_4 \mid f)$ is $\mathcal{M}$-transformable: $w^2 = z^2$ and $x^2 = y^2$.*

Proof. If $wz = xy$, or $w = z = 0$, or $x = y = 0$, then $f \in \mathcal{P}$ and Holant$(=_4 \mid f)$ can be computed in polynomial time. In the following, we assume that $wz \neq xy$ and there is at most one zero in $\{w, x, y, z\}$.

- If $wz = 0$, then Pl-Holant$(=_4 \mid f)$ is #P-hard by Lemma 19;
- if $wz \neq 0, |w| \neq |z|$, then Pl-Holant$(=_4 \mid f)$ is #P-hard by Lemma 20.

Now we can assume that $|w| = |z| \neq 0$, i.e., $f = (w, x, y, \pm w)$, with $w \neq 0$.

- If $x \neq \pm y$, then Pl-Holant$(=_4 \mid f)$ is #P-hard by Lemma 21;
- if $x = y$, then f is symmetric and the theorem has been proved as Theorem 9.

Now we can assume that $|w| = |z| \neq 0$ and $x = -y$, i.e., $f = (w, x, -x, \pm w)$, with $w \neq 0$. Since there is at most one zero among $\{w, x, y, z\}$, with the given form there is actually no zero entry. Since $(=_4 \mid (w, x, -x, -w))$ and $(=_4 \mid (w, x, -x, w))$ are $\mathcal{M}$-transformable under the holographic transformation H and Z respectively by Lemma 7, Pl-Holant$(=_4 \mid (w, x, -x, \pm w))$ can be computed in polynomial time. For general graphs,

- if $w^2 = x^2$, then $f \in \mathcal{A}$ by Lemma 5 and Holant$(=_4 \mid (w, x, -x, \pm w))$ can be computed in polynomial time;
- if $w^2 \neq x^2$, and since there are no zero entries, then Holant$(=_4 \mid f)$ is #P-hard by Lemma 22.

◀

5 Trichotomy for k-regular Graphs

In this section, we prove our main theorem, Theorem 24, a complexity trichotomy for spin systems with not necessarily symmetric real edge weights over k-regular graphs, for any $k \geq 3$.

▶ **Theorem 24.** *Let $f = (w, x, y, z)$ with $w, x, y, z \in \mathbb{R}$. Then* Holant$(=_k \mid f)$*, where $k \geq 3$, is* #P*-hard except in the following cases, where the problem is computable in polynomial time.*

- *$f \in \mathcal{P}$: $wz = xy$, or $w = z = 0$, or $x = y = 0$;*

- $f \in \mathcal{A} : w^2 = x^2 = y^2 = z^2$.

If the input is restricted to planar graphs, then another case becomes polynomial time computable but everything else remains #P-hard.

- *The pair $(=_k \mid f)$ is $\mathcal{M}$-transformable: $w = \epsilon z, x = \epsilon y$, or k is even and $w = \epsilon z, x = -\epsilon y$, where $\epsilon = \pm 1$.*

Proof. If $wz = xy$ or there are two or more zeros in $\{w, x, y, z\}$, then $f \in \mathcal{P}$ and Holant$(=_k \mid f)$ can be computed in polynomial time. If $x = y$, then f is symmetric and the theorem follows Theorem 9. In the following, we assume that $wz \neq xy$, $x \neq y$ and there is at most one zero in $\{w, x, y, z\}$.

For $k = 3$ or 4, the theorem has been proved in Theorem 11 and Theorem 23 respectively. So we can assume that $k \geq 5$.

Firstly, we consider the case that $wz = 0$. By assumption, we have $xy \neq 0$ and there is exact one zero in $\{w, z\}$. Without loss of generality, we assume that $w \neq 0, z = 0$. Then we may assume that $f = (1, x, y, 0)$ by normalizing $w = 1$.

- If k is odd, by adding $\frac{k-1}{2}$ loops using f to $=_k$, we have $[1, 0]$ on the LHS of Holant$(=_k \mid f)$. By connecting $[1, 0]$ to the variable x_1 of f, we get $[1, x]$ on the RHS. By connecting $k-3$ copies of $[1, x]$ to $=_k$, we have $[1, 0, 0, x^{k-3}]$ of arity 3 on the LHS, i.e., we have

$$\text{Pl-Holant}([1, 0, 0, x^{k-3}] \mid f) \leq_T^p \text{Pl-Holant}(=_k \mid f). \tag{30}$$

Then by the holographic transformation using $\begin{bmatrix} 1 & 0 \\ 0 & x^{-\frac{k-3}{3}} \end{bmatrix}$, we have

$$\text{Pl-Holant}(=_3 \mid (1, x^{\frac{k}{3}}, yx^{\frac{k-3}{3}}, 0)) \equiv_T^p \text{Pl-Holant}([1, 0, 0, x^{k-3}] \mid f). \tag{31}$$

By Theorem 11, Pl-Holant$(=_3 \mid (1, x^{\frac{k}{3}}, yx^{\frac{k-3}{3}}, 0))$ is #P-hard. Thus Pl-Holant$(=_k \mid f)$ is #P-hard by (30) and (31).

- If k is even, by adding $\frac{k-2}{2}$ loops using f to $=_k$, we have $[1, 0]^{\otimes 2}$ on the LHS. Then we take two copies of f and connect $[1, 0]$ to the variable x_1 for each copy to get $[1, x]^{\otimes 2}$ on the RHS. This can be realized by $[1, 0]^{\otimes 2}$ on the LHS. By adding $\frac{k-4}{2}$ loops using $[1, x]^{\otimes 2}$ to $=_k$, we have $[1, 0, 0, 0, x^{k-4}]$ of arity 4 on the LHS, i.e., we have

$$\text{Pl-Holant}([1, 0, 0, 0, x^{k-4}] \mid f) \leq_T^p \text{Pl-Holant}(=_k \mid f). \tag{32}$$

Since k is even and at least 5 by assumption, we have $k \geq 6$ and $k - 4 \geq 2$ is even. Hence $x^{k-4} > 0$. Thus we may choose a 4th root $x^{\frac{k-4}{4}} \in \mathbb{R}$. (Any statement in a holographic transformation involving a quantity such as $z^{1/n}$ is valid for any choice as long as a consistent choice is made.)

Then by the holographic transformation using $\begin{bmatrix} 1 & 0 \\ 0 & x^{-\frac{k-4}{4}} \end{bmatrix}$, we have

$$\text{Pl-Holant}(=_4 \mid (1, x^{\frac{k}{4}}, yx^{\frac{k-4}{4}}, 0)) \equiv_T^p \text{Pl-Holant}([1, 0, 0, 0, x^{k-4}] \mid f). \tag{33}$$

Note that all the entries of $(1, x^{\frac{k}{4}}, yx^{\frac{k-4}{4}}, 0)$ are real numbers. Therefore we may apply Theorem 23, and conclude that Pl-Holant$(=_4 \mid (1, x^{\frac{k}{4}}, yx^{\frac{k-4}{4}}, 0))$ is #P-hard. It follows that Pl-Holant$(=_k \mid f)$ is #P-hard by (32) and (33).

Now we consider the case that $wz \neq 0$. So we may assume that $f = (1, x, y, z)$ by normalizing $w = 1$.

Firstly, we consider the case that k is odd. By adding $\frac{k-3}{2}$ loops using f to $=_k$, we have $[1, 0, 0, z^{\frac{k-3}{2}}]$ on the LHS, i.e., we have

$$\text{Pl-Holant}([1, 0, 0, z^{\frac{k-3}{2}}] \mid f) \leq_T^p \text{Pl-Holant}(=_k \mid f). \tag{34}$$

Then by the holographic transformation using $\begin{bmatrix} 1 & 0 \\ 0 & z^{-\frac{k-3}{6}} \end{bmatrix}$, we have

$$\text{Pl-Holant}(=_3 |(1, xz^{\frac{k-3}{6}}, yz^{\frac{k-3}{6}}, z^{\frac{k}{3}})) \equiv_T^p \text{Pl-Holant}([1, 0, 0, z^{\frac{k-3}{2}}]|f). \tag{35}$$

- If $z \neq \pm 1$, then since $z \in \mathbb{R}$, we have $|z^{\frac{k}{3}}| \neq 1$ and Pl-Holant$(=_3 |(1, xz^{\frac{k-3}{6}}, yz^{\frac{k-3}{6}}, z^{\frac{k}{3}}))$ is #P-hard by Corollary 12. Thus Pl-Holant$(=_k |f)$ is #P-hard by (34) and (35).
- If $x \neq \pm y$, then since $x, y \in \mathbb{R}$, we have $|x| \neq |y|$, and thus $|xz^{\frac{k-3}{6}}| \neq |yz^{\frac{k-3}{6}}|$. Then Pl-Holant$(=_3 |(1, xz^{\frac{k-3}{6}}, yz^{\frac{k-3}{6}}, z^{\frac{k}{3}}))$ is #P-hard by Corollary 12. So Pl-Holant$(=_k |f)$ is #P-hard by (34) and (35).
- The remaining case is that $z = \pm 1$ and $x = -y$ since $x \neq y$, i.e., $f = (1, x, -x, \pm 1)$. (Note that in this case there are no zero entries, since there could have been at most one zero entry; in particular $x \neq 0$.)
  - If $x = \pm 1$, then $f \in \mathcal{A}$ by Lemma 5 and Holant$(=_k |f)$ can be computed in polynomial time.
  - Suppose $x \neq \pm 1$. For $z = -1$, since $(=_k |(1, x, -x, -1))$ is $\mathcal{M}$-transformable under the holographic transformation H by Lemma 7, Pl-Holant$(=_k |(1, x, -x, -1))$ is computable in polynomial time. But for $z = 1$, i.e., $f = (1, x, -x, 1)$, for the problem in the left-hand side of (35), the signature $(1, xz^{\frac{k-3}{6}}, yz^{\frac{k-3}{6}}, z^{\frac{k}{3}})$ is just $(1, x, -x, 1)$. By $|x| \neq 0, 1$, Pl-Holant$(=_3 |(1, x, -x, 1))$ is #P-hard by Theorem 11. Thus Pl-Holant$(=_k |(1, x, -x, 1))$ is #P-hard by (34) and (35).
    Moreover, for general graphs, for either case of $z = +1$ and $= -1$, note that $|xz^{\frac{k-3}{6}}| = |yz^{\frac{k-3}{6}}| \neq 1$. Thus Holant$(=_3 |(1, xz^{\frac{k-3}{6}}, yz^{\frac{k-3}{6}}, z^{\frac{k}{3}}))$ is #P-hard by Theorem 11. So Holant$(=_k |f)$ is #P-hard by (34) and (35).

Now we consider the case that k is even. To ensure that all the signatures we discuss are real-valued, we need to consider the cases $k \equiv 0 \pmod 4$ and $k \equiv 2 \pmod 4$ separately.

- If $k \equiv 0 \pmod 4$, by adding $\frac{k-4}{2}$ loops using f to $=_k$, we have $[1, 0, 0, 0, z^{\frac{k-4}{2}}]$ on the LHS, i.e., we have

$$\text{Pl-Holant}([1, 0, 0, 0, z^{\frac{k-4}{2}}]|f) \leq_T^p \text{Pl-Holant}(=_k |f). \tag{36}$$

  As $\frac{k-4}{2}$ is even, we have $z^{\frac{k-4}{2}} > 0$. Thus we can choose $z^{-\frac{k-4}{8}} \in \mathbb{R}$. It also follows that $z^{\frac{k}{4}} \in \mathbb{R}$. Then by the holographic transformation using $\begin{bmatrix} 1 & 0 \\ 0 & z^{-\frac{k-4}{8}} \end{bmatrix}$, we have

$$\text{Pl-Holant}(=_4 |(1, xz^{\frac{k-4}{8}}, yz^{\frac{k-4}{8}}, z^{\frac{k}{4}})) \equiv_T^p \text{Pl-Holant}([1, 0, 0, 0, z^{\frac{k-4}{2}}]|f). \tag{37}$$

  - If $z \neq \pm 1$ or $x \neq \pm y$, then Pl-Holant$(=_4 |(1, xz^{\frac{k-4}{8}}, yz^{\frac{k-4}{8}}, z^{\frac{k}{4}}))$ is #P-hard by Theorem 23, and Pl-Holant$(=_k |f)$ is #P-hard by (36) and (37).
  - For $z = \pm 1$ and $x = -y$, i.e., $f = (1, x, -x, \pm 1)$, if $x = \pm 1$, then $f \in \mathcal{A}$ by Lemma 5 and Holant$(=_k |f)$ can be computed in polynomial time.
    Suppose $x \neq \pm 1$. since $(=_k |(1, x, -x, -1))$ and $(=_k |(1, x, -x, 1))$ are $\mathcal{M}$-transformable under the holographic transformations H and Z respectively by Lemma 7, the planar Holant problem Pl-Holant$(=_4 |(1, x, -x, \pm 1))$ is computable in polynomial time. But for general graphs, Holant$(=_4 |(1, xz^{\frac{k-4}{8}}, -xz^{\frac{k-4}{8}}, z^{\frac{k}{4}}))$ is #P-hard by Theorem 23 since $|xz^{\frac{k-4}{8}}| \neq 0, 1$. Thus Holant$(=_k |f)$ is #P-hard by (36) and (37) (These reductions also hold for non-planar Holant problems respectively).
- If $k \equiv 2 \pmod 4$, by adding $\frac{k-2}{2}$ loops using f to $=_k$, we have $[1, 0, z^{\frac{k-2}{2}}]$, i.e.,

$$\text{Pl-Holant}(=_k, [1, 0, z^{\frac{k-2}{2}}]|f) \leq_T^p \text{Pl-Holant}(=_k |f). \tag{38}$$

Note that $\frac{k-2}{2}$ is even, and thus $z^{\frac{k-2}{2}} > 0$. Then we can choose $z^{-\frac{k-2}{4}} \in \mathbb{R}$. By the holographic transformation $\begin{bmatrix} 1 & 0 \\ 0 & z^{-\frac{k-2}{4}} \end{bmatrix}$, we have

$$\begin{aligned}&\text{Pl-Holant}([1,0,\cdots,0,z^{-\frac{k(k-2)}{4}}],=_2 \mid (1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}}))\\ &\leq_T^p \text{Pl-Holant}(=_k,[1,0,z^{\frac{k-2}{2}}]|f).\end{aligned} \tag{39}$$

Then by Lemma 14, we have

$$\begin{aligned}&\text{Pl-Holant}([1,0,\cdots,0,z^{-\frac{k(k-2)}{4}}],=_2 \mid =_2,(1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}}))\\ &\leq_T^p \text{Pl-Holant}([1,0,\cdots,0,z^{-\frac{k(k-2)}{4}}],=_2 \mid (1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}})).\end{aligned} \tag{40}$$

In the left-hand side of (40) by adding $\frac{k-4}{2}$ loops to $[1,0,\cdots,0,z^{-\frac{k(k-2)}{4}}]$ using $=_2$, we have a signature of arity 4, $[1,0,0,0,z^{-\frac{k(k-2)}{4}}]$ on the LHS, i.e,

$$\begin{aligned}&\text{Pl-Holant}([1,0,0,0,z^{-\frac{k(k-2)}{4}}]|(1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}}))\\ &\leq_T^p \text{Pl-Holant}([1,0,\cdots,0,z^{-\frac{k(k-2)}{4}}],=_2 \mid =_2,(1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}})).\end{aligned} \tag{41}$$

Note that $z^{\frac{k(k-2)}{4}} > 0$. Thus we can choose $z^{\frac{k(k-2)}{16}} \in \mathbb{R}$. By the holographic transformation $\begin{bmatrix} 1 & 0 \\ 0 & z^{\frac{k(k-2)}{16}} \end{bmatrix}$, we have

$$\begin{aligned}&\text{Pl-Holant}(=_4 \mid (1,xz^{-\frac{(k-2)(k-4)}{16}},yz^{-\frac{(k-2)(k-4)}{16}},z^{\frac{k(k+2)}{8}}))\\ &\leq_T^p \text{Pl-Holant}([1,0,0,0,z^{-\frac{k(k-2)}{4}}]|(1,xz^{\frac{k-2}{4}},yz^{\frac{k-2}{4}},z^{\frac{k}{2}})).\end{aligned} \tag{42}$$

The remaining proof is similar with the previous case that $k \equiv 0 \pmod 4$ and we omit it here.

◀

Acknowledgement

We thank Kurt Kilpela for his contributions, especially for a conversation which triggered the discovery that incommensurability between tangent values and angles over π could be exploited. We thank Tonghai Yang for discussions on number theory. We also thank Yijia Chen, Pinyan Lu, Xiaoming Sun and Tyson Williams for insightful discussions.

References

1 Tom M. Apostol. Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer, 1976.
2 Rodney J. Baxter. Exactly solved models in statistical mechanics. Academic Press London, 1982.
3 Miriam Backens: A New Holant Dichotomy Inspired by Quantum Computation. ICALP 2017: 16:1-16:14.
4 Andrei A. Bulatov: The complexity of the counting constraint satisfaction problem. J. ACM 60(5): 34:1-34:41 (2013).
5 Andrei A. Bulatov, Martin E. Dyer, Leslie Ann Goldberg, Mark Jerrum, Colin McQuillan: The expressibility of functions on the boolean domain, with applications to counting CSPs. J. ACM 60(5): 32:1-32:36 (2013).

6 Jin-Yi Cai, Xi Chen: Complexity of Counting CSP with Complex Weights. J. ACM 64(3): 19:1-19:39 (2017).

7 Jin-Yi Cai, Zhiguo Fu: Holographic algorithm with matchgates is universal for planar #CSP over boolean domain. STOC 2017: 842-855. The full version is available at https://arxiv.org/pdf/1603.07046.pdf.

8 Jin-Yi Cai, Aaron Gorenstein: Matchgates Revisited. Theory of Computing 10: 167-197 (2014).

9 Jin-Yi Cai, Pinyan Lu, Mingji Xia: The complexity of complex weighted Boolean #CSP. J. Comput. Syst. Sci. 80(1): 217-236 (2014).

10 Jin-Yi Cai, Michael Kowalczyk: Spin systems on k-regular graphs with complex edge functions. Theor. Comput. Sci. 461: 2-16 (2012).

11 Jin-Yi Cai, Michael Kowalczyk, Tyson Williams: Gadgets and anti-gadgets leading to a complexity dichotomy. ITCS 2012: 452-467. The full version is available at https://arxiv.org/pdf/1108.3383.pdf

12 S. Chowla. A special infinite series, Kong. Norsk. Vidensk. Selsk. Forhandl. 37, 85-87, (1964).

13 S. Chowla. The nonexistence of nontrivial relations between the roots of a certain irreducible equation, J. Number Th., 2, 120-123, (1970).

14 Martin E. Dyer, Leslie Ann Goldberg, Mark Jerrum: The Complexity of Weighted Boolean CSP. SIAM J. Comput. 38(5): 1970-1986 (2009).

15 Kurt Girstmair: Character coordinates and annihilators of cyclotomic numbers. Manuscripta Math. 59 (1987), no. 3, 375-389.

16 Heng Guo and Tyson Williams: The Complexity of Planar Boolean #CSP with Complex Weights. ICALP (1) 2013: 516-527. The full version is available at https://arxiv.org/pdf/1212.2284.pdf

17 Helmut Hasse: On a question of S. Chowla. Acta Arith. 18(1971), 275-280.

18 Sangxia Huang, Pinyan Lu: A Dichotomy for Real Weighted Holant Problems. Computational Complexity 25(1): 255-304 (2016).

19 H. Jager, H. W., Lenstra: Linear independence of cosecant values, Nieuw Archief Wisk. (3) 23, 131-144 (1975).

20 Richard E. Ladner: On the Structure of Polynomial Time Reducibility. J. ACM 22(1): 155-171 (1975).

21 H. W. Leopoldt, Über die Hauptordnung der ganzen Elemente eines abelschen Zahlkörpers, J. reine angew. Math. 201, 119-149 (1959).

22 Jiabao Lin, Hanpin Wang: The Complexity of Holant Problems over Boolean Domain with Non-Negative Weights. ICALP 2017: 29:1-29:14. The full version is available at https://arxiv.org/pdf/1611.00975.pdf.

23 P. W. Kasteleyn, The Statistics of Dimers on a Lattice, Physica 27 (1961) 1209-1225.

24 P. W. Kasteleyn, Graph Theory and Crystal Physics. In Graph Theory and Theoretical Physics, (F. Harary, ed.), Academic Press, London, 43-110 (1967).

25 Michael Kowalczyk, Jin-Yi Cai: Holant Problems for 3-Regular Graphs with Complex Edge Functions. Theory Comput. Syst. 59(1): 133-158 (2016).

26 H. N. V. Temperley and M. E. Fisher. Dimer Problem in Statistical Mechanics - an Exact Result. Philosophical Magazine 6: 1061- 1063 (1961).

27 Leslie G. Valiant: Holographic Algorithms. SIAM J. Comput. 37(5): 1565-1594 (2008).

28 L. C. Washington, Introduction to Cyclotomic Fields, Graduate Text in Mathematics, 83, Springer, 1980.