An agent-based framework for bio-inspired, value-sensitive decision-making ⋆

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Abstract: We propose a generalizable framework that uses tools of nonlinear dynamics to rigorously connect model-based investigation of the mechanisms of animal group decision-making dynamics to systematic, bio-inspired design of coordinated control of multi-agent systems. We focus on the design of networked multi-agent system dynamics that inherit the remarkable features of value-sensitive decision-making observed in house-hunting honeybees. These features include robustness and adaptability in decision-making, all of which are critical for performance in complex, changing environments.

Keywords: collective decision making, bio-inspired control theory, design, nonlinear dynamics, bifurcation control

1. INTRODUCTION

A fundamental task for many multi-agent system networks is successful collective decision-making among alternatives using information distributed across the network. Groups of individual agents, in applications including transportation, mobile-sensing, power and synthetic biological networks, are often required to make a single choice among alternatives, such as choosing which option is true, which action to take or direction to follow, or when something in the environment or system has changed.

For purposes of designing distributed multi-agent decision-making, we seek to leverage mechanisms used by animal groups whose survival relies on successful collective decisions among alternatives. House-hunting honeybees (Seeley and Buhrman, 2001), schooling fish (Couzin et al., 2011), and migratory birds (Eikenaar et al., 2014) make efficient decisions despite disturbances or significant changes in their environment. They employ decentralized strategies and face limitations on sensing, communication, and computation (Sumpter (2010), Krause and Ruxton (2002)), yet they still perform with speed, accuracy, robustness, and adaptability (Parrish and Edelstein-Keshet, 1999).

Typical mechanisms used to study collective animal behavior depend on the animals' social interactions and on their perceptions of their external environment. A rigorous understanding of these dependencies makes possible the translation of the mechanisms into a systematic bio-inspired design methodology for use in engineered networks. This remains a challenge, however, in part because most studies of collective animal behavior are empirically based or rely on mean-field models.

To address this challenge, we present a generalizable agent-based dynamic model of distributed decision-making between two alternatives. In this type of decision-making, the pitchfork bifurcation is ubiquitous (Leonard, 2014); it appears, for example, in the decision-making dynamics of house-hunting honeybees and schooling golden shiners selecting between food sources. Our approach is to derive the agent-based model so that it too exhibits the pitchfork bifurcation. This allows the animal group dynamics and the multi-agent dynamics to be rigorously connected by mapping to the normal form of the pitchfork bifurcation.

The major contributions of this work are as follows. First, we present a generalizable agent-based model for bio-inspired collective decision-making dynamics and use model reduction and asymptotic expansion to show how the model captures the adaptive and robust features of house-hunting honeybee decision-making dynamics. Remarkably, honeybees reliably select the highest value nest site alternative, and in the case of alternatives of equal value, they quickly make an arbitrary choice if the value is sufficiently high. These honeybee dynamics have been studied in Seeley and Buhrman (2001), Seeley et al. (2012), and Pais et al. (2013), and leveraged in Reina et al. (2015).

Second, we present for this model an investigation of how the value of the alternatives, individual preferences, and interaction topology influence the decision-making dynamics. This is motivated by the problem of designing

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collective decision-making dynamics, as these parameters could serve as control parameters in engineered systems.

In Section 2 the agent-based decision-making dynamic model is proposed. Section 3 describes the dynamics of house-hunting honeybees, highlighting the adaptability through value-sensitive decision-making. In Section 4 a method for reducing the model to a low-dimensional, attractive manifold is presented. The reduced model is used in Section 5 to show how the model recovers the value-sensitivity of the honeybee dynamics, and brief results show the influence of other system parameters.

2. AGENT-BASED DECISION-MAKING MODEL

The proposed model is a specialization of the Hopfield network dynamics (Hopfield (1982), Hopfield (1984)). The model provides a generalizable network decision-making dynamic for a set of N interconnected agents and by design it exhibits a pitchfork bifurcation. To describe decision-making between two alternatives A and B, let $x_i \in \mathbb{R}$ be the state of agent i, representing its opinion, with $i \in \{1, ..., N\}$. Agent i is said to favor alternative A (B) if $x_i > 0$ (< 0), with the strength of agent i's opinion given by $|x_i|$. If $x_i = 0$, agent i is undecided or uncommitted.

The network interconnections define which agents can measure the state of which other agents, and this is encoded using a network adjacency matrix $A \in \mathbb{R}^{N\times N}$. Each $a_{ij} \geq 0$ for $i, j \in \{1, ..., N\}$ and $i \neq j$ gives the weight that agent i puts on its measurement of agent j. Then $a_{ij} > 0$ implies that j is a neighbor of i. We let $a_{ii} = 0$ for all i and $D \in \mathbb{R}^{N\times N}$ be a diagonal matrix with diagonal entries $d_i = \sum_{j=1}^{N} a_{ij}$. $L = D - A$ is the Laplacian matrix of the graph associated with the interaction network.

We define the change in opinion of each agent over time as a function of the agent's current state, the state of their neighbors and a possible external stimulus β_i:

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{N} u a_{ij} S(x_j) + \beta_i. \tag{1}$$

Each $\beta_i \in \{\beta_\mathrm{A}, 0 - \beta_\mathrm{B}\}, \beta_\mathrm{A}, \beta_\mathrm{B} \in \mathbb{R}^+$ describes the external stimulus received by each agent i, and can also be thought of as the agent's preference among alternatives. $\beta_i = \beta_\mathrm{A}$ means agent i has a preference for alternative A, $\beta_i = -\beta_\mathrm{B}$ means agent i has a preference for alternative B, and $\beta_i = 0$ means agent i has no preference. $u \geq 0$ is a non-negative control parameter and $S : \mathbb{R} \to \mathbb{R}$ is a smooth sigmoidal function that satisfies the following conditions: $S'(z) > 0\, \forall z \in \mathbb{R}$ (monotone); $S(z)$ belongs to sector $(0, 1]$; and $\operatorname{sgn} S''(z) = -\operatorname{sgn}(z)$, where $(\cdot)'$ denotes the derivative with respect to the argument of the function.

The term $uS(x_j)$ can be interpreted as the opinion of agent j as perceived by agent i. $S(x)$ is a saturating function, that reduces the influence of larger opinions. The control parameter u determines the function scaling, so u can be thought of as the social effort; larger values of u mean a higher importance is placed on the opinions of others.

Let $\boldsymbol{x} = (x_1, \ldots, x_N)^T$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)^T$ and $S(\boldsymbol{x}) \in \mathbb{R}^N$ be a vector with entries $S(x_i)$. Then (1) can be written in vector form:

$$\dot{\boldsymbol{x}} = -D\boldsymbol{x} + uA\boldsymbol{S}(\boldsymbol{x}) + \boldsymbol{\beta}. \tag{2}$$

To see that these dynamics exhibit a pitchfork bifurcation, let the interconnection graph be fixed and strongly connected. Then $\operatorname{rank}(L) = N - 1$ and $L\mathbf{1}_N = 0$, where $\mathbf{1}_N$ is the N-column vector with unitary entries. L has a zero eigenvalue with corresponding eigenvector $\boldsymbol{x} = \zeta\mathbf{1}_N$, $\zeta \in \mathbb{R}$, and every other eigenvalue has positive real part. Observe that the linearization of (2) at $\boldsymbol{x} = 0$ for $u = 1$ and $\boldsymbol{\beta} = 0$ is the linear consensus dynamics $\dot{\boldsymbol{x}} = -L\boldsymbol{x}$, which converges to the consensus $\boldsymbol{x} = \zeta\mathbf{1}_N$. This implies the possibility of a bifurcation with center manifold tangent to the consensus manifold (Guckenheimer and Holmes, 2002, Theorem 3.2.1). By odd symmetry of (1) for $\boldsymbol{\beta} = \mathbf{0}$, this will generically be a pitchfork (Golubitsky and Schaeffer, 1985, Theorem VI.5.1).

This is illustrated in Theorem 1 and Figure 1 for an all-to-all network and $\boldsymbol{\beta} = 0$:

$$\dot{x}_i = -(N-1)x_i + \sum_{j=1, j\neq i}^{N} uS(x_j). \tag{3}$$

Theorem 1. The following statements hold for the stability of invariant sets of dynamics (3):

(i) The consensus manifold is globally exponentially stable for each $u \in \mathbb{R} \geq 0$;
(ii) $\boldsymbol{x} = 0$ is globally exponentially stable for $u \in [0, 1)$ and globally asymptotically stable for $u = u^* := 1$;
(iii) $\boldsymbol{x} = 0$ is unstable and there exist two stable equilibrium points on the consensus manifold for $u > 1$.

Proof. Beginning with (i); consider a Lyapunov function $V_{ij}(\boldsymbol{x}) = \frac{(x_i - x_j)^2}{2}$. It follows that

$$\dot{V}_{ij}(\boldsymbol{x}) = -(N-1)(x_i - x_j)(x_i - x_j + u(S(x_i) - S(x_j))) < -(N-1)(x_i - x_j)^2 = -2(N-1)V_{ij},$$

for all $x_i \neq x_j$. Therefore, for $V(\boldsymbol{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}(\boldsymbol{x})$,

$$\dot{V}(\boldsymbol{x}) < -2(N-1)V(\boldsymbol{x}),$$

for all $\boldsymbol{x} \neq \zeta\mathbf{1}_N$, $\zeta \in \mathbb{R}$. $\dot{V}(\boldsymbol{x}) = \mathbf{0}$ for $x_i = x_j = \zeta$, so by LaSalle's invariance principle, the consensus manifold is globally exponentially stable.

Using (i), it suffices to study dynamics (3) on the consensus manifold, where, for $y = \frac{1}{N}\sum_{i=1}^{N} x_i$, they reduce to the scalar dynamics

$$\dot{y} = -(N-1)y + u(N-1)S(y).$$

(ii) and (iii) follow by inspection of these dynamics. □

The pitchfork remains for $\boldsymbol{\beta} \neq \mathbf{0}$ when the symmetry of the system is preserved, i.e., when $\beta_\mathrm{A} = -\beta_\mathrm{B}$ and the number of agents with preference A is equal to the number of agents with preference B. Proof of this, extension to more general cases, and results for the non-symmetric case rely on singularity theory (see Golubitsky and Schaeffer (1985)), and these will be presented in later work. Asymmetry in the system leads to an unfolding of the pitchfork bifurcation, and this unfolding imparts decision-making dynamics organized by the pitchfork and the robustness seen in honeybee decision-making.

3. HOUSE HUNTING HONEYBEES AND VALUE-SENSITIVE DECISION-MAKING

When a honeybee colony grows too large for its hive, part of the colony must leave to find a new nest site.

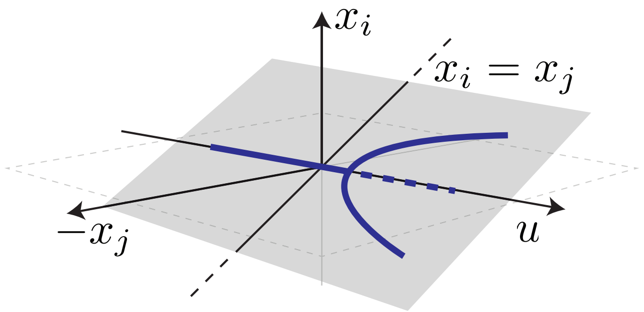

Fig. 1. For $u = 1$, dynamics (3) exhibit a pitchfork bifurcation at $\boldsymbol{x} = 0$. The steady-state branches emerging at the singularity lie on the consensus manifold $\{x_i = x_j | i, j \in \{1, ..., N\}\}$ shown in gray. Branches of stable and unstable solutions are shown as solid and dashed lines, respectively.

The new site must be of high quality, in order for the colony to survive the next winter, and the choice must be made quickly due to limited food supply. The departing bees wait in a swarm while scout bees search out and assess potential nest sites. Each scout returns to the swarm repeatedly to advertise and recruit others to its candidate site. A collective decision by the swarm for one of the alternative sites is made by a quorum.

The value of a site is related to its volume, height above the ground, and the size and location of its entrance cavity. It has been shown that honeybee swarms quickly and accurately choose the highest-value site among alternatives based on these criteria (Seeley and Buhrman, 2001). The adaptability and robustness of this process is one of the features we seek to capture with the proposed agent-based model. Notably, the honeybees efficiently choose one of the sites when they are of equal or near-equal value (Seeley et al., 2012). The decision is sensitive to both the relative and absolute value of the available sites (Pais et al., 2013), and thus the process is referred to as value-sensitive decision-making. Sensitivity to absolute value makes the honeybees adaptive to environmental change: if two equal or near-equal sites have high value, the honeybees will arbitrarily choose one with little effort from the group. However, if they have low value, the honeybees will refrain from choosing either, and will eventually choose one of them after significant group effort.

The mechanisms that explain the value-sensitive decision-making have been extensively studied in (Pais et al., 2013), using the mean-field model presented in (Seeley et al., 2012), which is a well mixed population model. The mean-field model exhibits a pitchfork bifurcation in the case of equal-value alternatives, which is critical to the remarkable decision-making behavior of the honeybees. However, we cannot use the mean-field model to design distributed control strategies or to examine the influence on the dynamics of network topology or distribution of preferences across the group. Our approach instead is to do this with the generalizable agent-based model (2). Because the agent-based model (2) exhibits a pitchfork bifurcation, it can be rigorously connected to the mean-field model and its results. Then, we can examine the value-sensitive decision-making dynamics of the agent-based model in terms of distributed properties of the system, and we can use it for design of adaptive and robust multi-agent network decision-making.

In the decision-making process honeybees are known to use two communication mechanisms: the "waggle dance" for recruitment and the "stop-signal" for cross-inhibition. Seeley et al. (2012) showed that the stop-signal is used to inhibit the dancing and recruitment of bees for competing sites, allowing the bees to break a deadlock between near-equal alternatives. They derived a model of the mean-field population-level dynamics, assuming a large total bee population N. The model describes the evolution of three population fractions: $y_{\mathrm{A}}(t) = \frac{N_{\mathrm{A}}(t)}{N}$, $y_{\mathrm{B}}(t) = \frac{N_{\mathrm{B}}(t)}{N}$, and $y_U(t) = \frac{N_U(t)}{N}$, where N_{A}, N_{B}, and N_U are the sub-populations of bees committed to sites A, B and uncommitted bees respectively. Because $N_A + N_B + N_U = N$ and therefore $y_{\mathrm{A}} + y_{\mathrm{B}} + y_U = 1$, it suffices to study the evolution of the two committed populations only:

$$\begin{aligned}\frac{dy_{\mathrm{A}}}{dt} &= \gamma_{\mathrm{A}} y_U - y_{\mathrm{A}}(\alpha_{\mathrm{A}} - \rho_{\mathrm{A}} y_U + \sigma_{\mathrm{B}} y_{\mathrm{B}}) \\ \frac{dy_{\mathrm{B}}}{dt} &= \gamma_{\mathrm{B}} y_U - y_{\mathrm{B}}(\alpha_{\mathrm{B}} - \rho_{\mathrm{B}} y_U + \sigma_{\mathrm{A}} y_{\mathrm{A}}).\end{aligned} \tag{4}$$

Here γ_i is the rate of scouting discovery and commitment, α_i is the rate of spontaneous abandonment, ρ_i is the rate of recruitment and σ_i is the rate of stop-signalling. It is assumed that $\gamma_i = \rho_i = \nu_i$ and $\alpha_i = \frac{1}{\nu_i}$ where ν_i is the assessed value of nest site i. Also, $\sigma_i = \sigma$, and so is equal for all sub-populations. A quorum decision is reached when y_A or y_B crosses some threshold $\omega \in (0.5, 1]$.

In the equal-alternative case ($\nu_{\mathrm{A}} = \nu_{\mathrm{B}} = \nu$), there exists a critical value of stop signalling strength $\sigma^* = \frac{4\nu^3}{(\nu^2-1)^2}$. If $\sigma < \sigma^*$ the system has one globally stable equilibrium at $y_{\mathrm{A}} = y_{\mathrm{B}}$, i.e. deadlock or no decision. For $\sigma > \sigma^*$ the deadlock solution is unstable and there are two stable equilibria, each corresponding to a decision for one of the two alternatives. This is a pitchfork bifurcation with bifurcation parameter σ and bifurcation value $\sigma = \sigma^*$. The critical value of σ is inversely dependent on the value ν of the two alternatives, which allows the bees to adapt to their environment. Suppose that the bees use a fixed rate of stop-signalling σ. Then, when choosing between two low-valued alternatives they will remain in deadlock, presumably waiting for another nest site candidate to be discovered. But if the two equal alternatives are of high value, they will quickly choose one arbitrarily. There is also the possibility that the bees could increase the rate of stop-signalling over time, when it becomes apparent that no better alternatives will appear.

Figure 2 shows the range of values ν and stop-signalling rate σ for which one solution (deadlock) and three solutions (two stable decisions and one unstable deadlock) exist, as well as the two-dimensional simplexes on which the dynamics evolve. The curve between regions describes the inverse relationship between the bifurcation point σ^* and the value ν.

To explore these dynamics for equal alternatives in the agent-based model, we define a mapping of (2) using a time scale change $\tau = \nu t$, which gives

$$\frac{d\boldsymbol{x}}{d\tau} = -\frac{1}{\nu} D\boldsymbol{x} + \mu A \boldsymbol{S}(\boldsymbol{x}) + \boldsymbol{\nu} \tag{5}$$

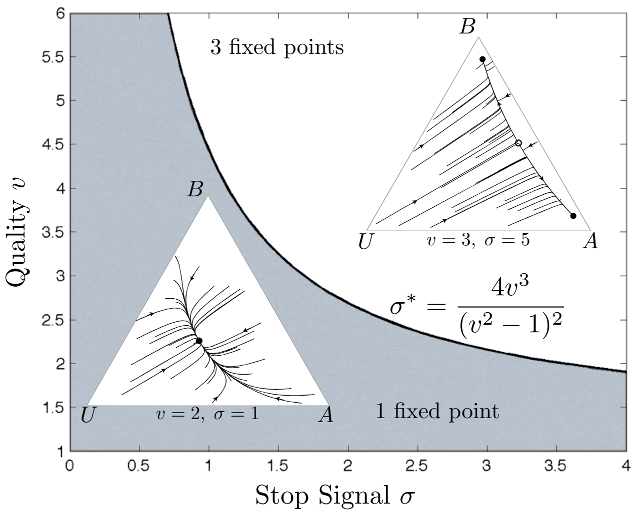

Fig. 2. From Pais et al. (2013). Value-sensitive decision-making for alternatives with equal value ν in mean-field model (4). The simplex on the left, representative of the grey area, shows convergence to the single stable equilibrium (black circle) at deadlock. The simplex on the right, representative of the white area, shows convergence to two stable equilibria at a decision for A or B. The curve that separates the region describes the inverse relationship of σ^* to ν.

where $\boldsymbol{\nu} = (\nu_1, ..., \nu_N)^T$, $\nu_i \in \{\nu, -\nu, 0\}$ such that $\nu = \sqrt{\beta}$, and $\mu = \frac{u}{\nu}$. This mapping will be used in Section 5 to connect results of the agent-based model to the honeybee mean-field model dynamics.

4. MODEL REDUCTION TO LOW-DIMENSIONAL, ATTRACTIVE MANIFOLD.

Returning now to the agent-based model, for certain classes of network graph it is possible to identify a globally-attractive, low-dimensional manifold on which to reduce the dynamics (1), and to perform analysis on the reduced model. The dimensionality N of the system is treated as a discrete parameter, allowing for the study of the sensitivity of the dynamics to the sizes of the committed and uncommitted populations.

Let n_1 and n_2 be the number of agents with preference $\beta_i = \beta_A = \bar{\beta}_1$ and $\beta_i = -\beta_B = \bar{\beta}_2$ respectively, and let $n_3 = N - n_1 - n_2$ be the number of agents with no preference ($\beta_i = 0 = \bar{\beta}_3$). From Mesbahi and Egerstedt (2010), a partition of vertices into cells $C_1, ..., C_r$ is said to be equitable if each node in C_i has the same number of neighbors in C_j for all i, j. Let $\mathscr{I}_k \subset \{1, ..., N\}$, $k \in \{1, 2, 3\}$ be the index set associated with each of the three groups n_1, n_2, n_3, such that $\mathscr{I}_k \subset \{1, ..., N\}$, $k \in \{1, 2, 3\}$ defines an equitable partition. We can then define the opinion dynamics of each agent $i \in \mathscr{I}_k$, $k \in \{1, 2, 3\}$ as

$$\dot{x}_i = -\bar{d}_k x_i + u \sum_{j \in \mathscr{I}_k} S(x_j) + u \sum_{\substack{m \in \{1,2,3\} \\ m \neq k}} \sum_{j \in \mathscr{I}_m} \bar{a}_{km} S(x_j) + \bar{\beta}_k, \tag{6}$$

for $i \in \mathscr{I}_k$, where $\bar{d}_k$ is the in-degree of each agent in group k, and $\bar{a}_{km}$ is the number of neighbors that each node in group k has in group m (including $m = k$).

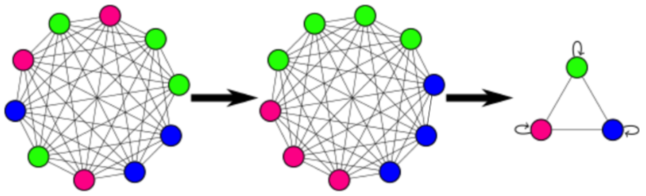

Fig. 3. Model reduction for (2) defined on an all-to-all graph with $N = 10$ nodes (left) reduces to a 3-dimensional system described by the graph on the right. Agents are grouped by the information value (middle), and opinion within each group converges to consensus according to dynamics (7). Blue agents have $\beta_i = 1$, pink agents $\beta_i = -1$ and green agents $\beta_i = 0$. In the reduced model, the agents have self loops, which represent the influence of the others in the same group.

The following theorem allows the analysis of (6) to be restricted to the subspace where each agent in the same group has the same opinion. The central idea of the theorem is summarized in Figure 3.

Theorem 2. Every trajectory of the opinion dynamics (6) converges exponentially to the three-dimensional manifold

$$\mathscr{E} = \{\boldsymbol{x} \in \mathbb{R}^N | x_i = x_j,\ \forall i, j \in \mathscr{I}_k, k = 1, 2, 3\}.$$

The dynamics on $\mathscr{E}$ are

$$\begin{aligned}
\dot{y}_1 &= -\bar{d}_1 y_1 + u(n_1 - 1)S(y_1) + u\big(n_2 \bar{a}_{12} S(y_2) \\
&\qquad + n_3 \bar{a}_{13} S(y_3)\big) - \beta_A \\
\dot{y}_2 &= -\bar{d}_2 y_2 + u(n_2 - 1)S(y_2) + u\big(n_1 \bar{a}_{21} S(y_1) \\
&\qquad + n_3 \bar{a}_{23} S(y_3)\big) + \beta_B \\
\dot{y}_3 &= -\bar{d}_3 y_3 + u(n_3 - 1)S(y_3) + u\big(n_1 \bar{a}_{31} S(y_1) \\
&\qquad + n_2 \bar{a}_{32} S(y_2)\big).
\end{aligned} \tag{7}$$

Proof. Consider a Lyapunov function $V(\boldsymbol{x}) = \sum_{k=1}^{3} V_k(\boldsymbol{x})$, where

$$V_k(\boldsymbol{x}) = \frac{1}{2} \sum_{i \in \mathscr{I}_k} \sum_{j \in \mathscr{I}_k} (x_i - x_j)^2, \text{ for } k \in \{1, 2, 3\}.$$

If follows that

$$\begin{aligned}
\dot{V}_k(\boldsymbol{x}) &= \sum_{i \in \mathscr{I}_k} \sum_{j \in \mathscr{I}_k} (x_i - x_j)(\dot{x}_i - \dot{x}_j) \\
&= \sum_{i \in \mathscr{I}_k} \sum_{j \in \mathscr{I}_k} \big(-\bar{d}_k (x_i - x_j)^2 - u(x_i - x_j)(S(x_i) - S(x_j))\big) \\
&\leq -\bar{d}_k V_k(\boldsymbol{x}),
\end{aligned}$$

so $\dot{V}(\boldsymbol{x}) \leq -\bar{d}_k V(\boldsymbol{x})$. By LaSalle's invariance principle, every trajectory of (6) converges exponentially to the largest invariant set in $\dot{V}(\boldsymbol{x}) = 0$, which is the manifold $\mathscr{E}$. Let $y_k = x_i$, for any $i \in \mathscr{I}_k, k \in \{1, 2, 3\}$. Then dynamics (6) reduce immediately to (7). $\square$

5. RECOVERING VALUE-SENSITIVITY AND THE INFLUENCE OF SYSTEM PARAMETERS

A vector field $F : \mathbb{R}^N \to \mathbb{R}^N$ is Z_2-symmetric when it commutes with the linear transformation

$$\gamma = \begin{bmatrix} 0_{n \times n} & -I_n & 0_{n \times (N-2n)} \\ -I_n & 0_{n \times n} & 0_{n \times (N-2n)} \\ 0_{(N-2n) \times n} & 0_{(N-2n) \times n} & -I_{N-2n} \end{bmatrix},$$

for some even $2n < N$. Thus, the opinion dynamics (2) are Z_2-symmetric if $\beta_A = -\beta_B = \beta$ and $n_1 = n_2 = n$. That is, reversing the sign of β_A and β_B is equivalent to applying the transformation $\boldsymbol{x} \mapsto -\boldsymbol{x}$. Considering (7) under Z_2 symmetry, an approximation $\hat{u}^*$ to the bifurcation point u^* can be found.

5.1 Approximating the bifurcation point for a Z_2 symmetric, all-to-all network.

Under the above assumptions, and with $n_3 = N - 2n$ uncommitted agents, dynamics (7) reduce to

$$\begin{aligned}\dot{y}_1 &= -(N-1)y_1 + u\big((n-1)S(y_1)\\ &\qquad + nS(y_2) + n_3 S(y_3)\big) + \beta\\ \dot{y}_2 &= -(N-1)y_2 + u\big(nS(y_1)\\ &\qquad + (n-1)S(y_2) + n_3 S(y_3)\big) - \beta\\ \dot{y}_3 &= -(N-1)y_3 + u\big(nS(y_1)\\ &\qquad + nS(y_2) + (n_3-1)S(y_3)\big).\end{aligned} \tag{8}$$

Then $\boldsymbol{y}^* = (y^*, -y^*, 0)$ is always an equilibrium, where y^* is the solution to

$$(N-1)y^* + uS(y^*) - \beta = 0. \tag{9}$$

When $\beta = 0$, the deadlock state $\boldsymbol{y} = \boldsymbol{0}$ is an equilibrium of (8) for all $u \in \mathbb{R}$. For $\beta \neq 0$, under Z_2-symmetry, the implicit function theorem implies that the deadlock manifold perturbs smoothly and that the equilibrium point $\boldsymbol{y}^* = (y^*, -y^*, 0)$ where $y^* = y^*(u, \beta)$, such that $y^*(u, 0) \equiv 0$, depends smoothly on u and β. Then, an approximation to $\boldsymbol{y}^*$ can be found.

For illustration, let $S(\cdot) = \tanh(\cdot)$ and begin with the Taylor series expansion of $y^*(u, \beta)$ with respect to β:

$$y^*(u,\beta) = \beta y_{\mathrm{I}} + \beta^2 y_{\mathrm{II}} + \beta^3 y_{\mathrm{III}} + \beta^4 y_{\mathrm{IV}} + \mathscr{O}(\beta^5).$$

Then substitute $y^*(u, \beta)$ into (9) and differentiate with respect to β to obtain

$$(N-1)y^{*\prime}(u,\beta) + u\,\mathrm{sech}^2\big(y^*(u,\beta)\big)y^{*\prime}(u,\beta) - 1 = 0.$$

Substituting in $\beta = 0$ yields $y_{\mathrm{I}} = \frac{1}{N-1+u}$. Proceeding similarly for higher orders gives $y_{\mathrm{II}} = y_{\mathrm{IV}} = 0$ and

$$y_{\mathrm{III}} = \frac{u}{3(N-1+u)^4}.$$

Therefore

$$y^* = \frac{1}{N-1+u}\beta + \frac{u}{3(N-1+u)^4}\beta^3 + \mathscr{O}(\beta^5).$$

At a bifurcation point u^*, the Jacobian of (8) computed at y^* drops rank. The expression for the Jacobian, too lengthy to reproduce here, is a function of N, n_3 and y^*.

Recall that y^* is also a function of u^*, so by setting the determinant of the Jacobian equal to zero we get a transcendental equation in u^* that can be solved numerically for u^*. To approximate u^* begin with the Taylor series expansion $u^*(\beta) = 1 + \beta u_1^* + \beta^2 u_2^* + \beta^3 u_3^* + \mathscr{O}(\beta^4)$. Proceeding as for y^* above, leads to the following $\mathscr{O}(\beta^4)$ approximation to u^*:

$$\hat{u}^* = 1 + \frac{(1+3N^3)^2(N-n_3)}{9N^9}\beta^2. \tag{10}$$

The approximation (10) of bifurcation point u^* is a function of value β, total group size N and the size of the uncommitted group n_3; it explicitly describes the sensitivity of the bifurcation to group size and preference strength.

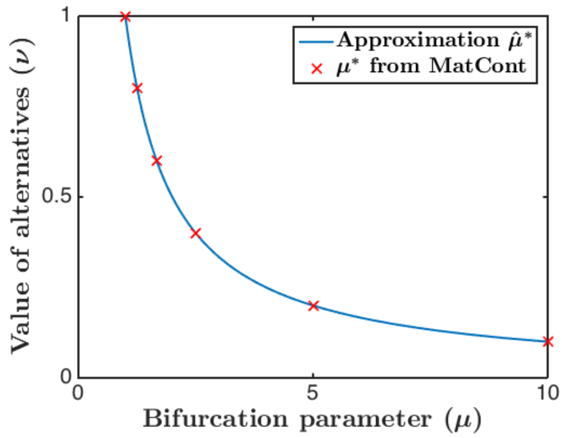

Fig. 4. Value-sensitive decision-making for alternatives with equal value ν in agent-based model (5). The curve shows how μ^* depends inversely on ν, recovering the value sensitivity of the honeybee mean-field model (4); compare with Figure 2. The blue line shows the approximation μ^* (11) while the red crosses show μ^* computed numerically using continuation software. The group sizes are $n = n_3 = 30$.

5.2 Recovering the value-sensitivity of honeybee dynamics

To recover the value-sensitivity of the honeybee mean-field model in the agent-based model dynamics, we consider the dynamics in the form (5) for equal alternatives, where the bifurcation parameter is $\mu = \frac{u}{\nu}$ with $\nu = \sqrt{\beta}$. Applying the approximation (10) for u^* gives the approximation to the bifurcation point μ^* for dynamics (5) as

$$\hat{\mu}^* = \frac{1}{\nu} + \frac{(1+3N^3)^2(N-n_3)}{9N^9}\nu^3 + \mathscr{O}(\nu^7). \tag{11}$$

Figure 4 shows how well $\hat{\mu}^*$ approximates μ^* computed using MatCont continuation software (Govaerts and Kuznetsov, 2015). As in the case of the honeybee mean-field model (see Figure 2), the bifurcation point in the agent-based model depends inversely on the value of the alternatives ν (see (11) and Figure 4). Thus, our agent-based decision-making model recovers the value-sensitive decision-making of the honeybee mean-field model. This value-sensitivity allows for efficient and adaptable decision-making dynamics.

The dependence of μ^* on ν is demonstrated further in Fig. 5a, where bifurcation diagrams for the agent-based model are given for a range of value ν. We observe that the bifurcation point decreases as ν is increased. There is also an increase in the sharpness of the bifurcation branches as ν is increased; this corresponds to a faster increase in average opinion.

5.3 Influence of system parameters

An advantage of the agent-based framework is that it makes it possible to study sensitivity of the dynamics to system parameters that vary over the group. This will be explored in more detail in later work, but some brief results are presented here. Fig. 5b - Fig. 5c show the same relationship for $\hat{\mu}^*$ in terms of ν as shown in Fig. 4, but with different numbers of agents and different subgroup arrangements. The effects in this case are small, but there are still some clear trends.

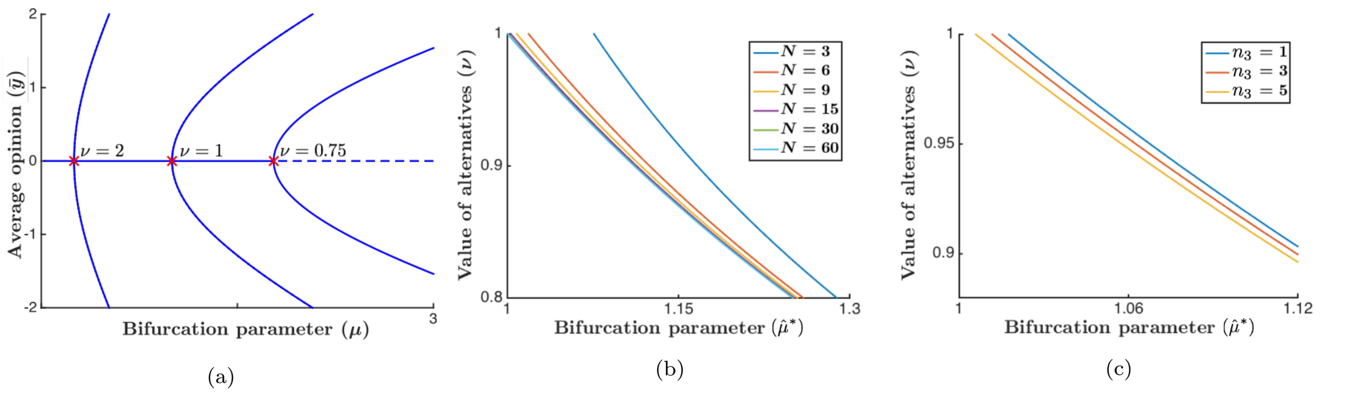

Fig. 5. (a) Bifurcation diagrams for agent-based model (5) with $n_3 = 20$, $N = 60$ and a range of ν, showing the dependence of the bifurcation point and diagram sharpness on ν. (b)-(c) The approximation to the bifurcation point $\hat{\mu}^*$ as a function of ν for different group sizes. (b) shows increasing N with fixed $n_3 = \frac{N}{3}$, and (c) shows increasing n_3 for fixed $N = 7$.

Fig. 5b shows how $\hat{\mu}^*$ decreases with increasing total group size N, implying that less social effort is required to make a decision with a larger group. Also, there is a limiting value of $\mu^* = \frac{1}{\nu}$ that is approached quickly for $N > 30$. From a design perspective, this shows that if the aim is to minimze the required social effort for a group of agents, there is a diminishing return on increasing group size.

Fig. 5c shows that the amount of social effort also decreases when the number of uncommitted agents n_3 increases. Couzin et al. (2011) have shown that the number of uninformed agents plays an important role in the decision dynamics of schooling fish, so this result suggests that the agent-based model could also be mapped these dynamics.

6. CONCLUSION

The agent-based decision-making model introduced in this paper connects animal group and multi-agent network dynamics. It provides a generalizable framework to enforce the salient properties of animal collective decision-making (robustness, adaptability) in engineered network systems. In the simplest case of all-to-all communication, it captures the value-sensitivity of decision making in house-hunting honeybees. Preliminary results suggest that it may capture democratic consensus dynamics in schooling fish. Future work will rely on singularity theory to develop a principled sensitivity analysis of the proposed model, both in the all-to-all and balanced interconnection cases. Other extensions include the design of suitable evolutionary dynamics for the different control parameters.

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