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$L(\mathbb{R}, \mu)$ is unique

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Under various appropriate hypotheses it is shown that there is only one determinacy model of the form $L(\mathbb{R}, \mu)$ in which μ is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. In particular, this gives a positive answer to a question asked by W.H. Woodin in 1983.

1. Introduction

This paper deals with several set theories, most of which include ZF + DC. Here, ZF is Zermelo–Fraenkel set theory and DC is the Dependent Choice principle. One such theory is ZFC, which is ZF + AC, where AC is the Axiom of Choice. Other examples are

$$\text{ZFC} + \text{There exists a measurable cardinal}$$

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and

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}\,.$$

We start by reviewing some basic notions.

1.1. Large cardinals and inner model theory

Large cardinal hypotheses, also known as large cardinal axioms, are strong forms of the Axiom of Infinity and provide structures for the analysis of propositions not provable in ZFC alone. Good sources that cover this material are [1] and [2]. We will concentrate primarily on two special kinds of large cardinals that we discuss below.

By definition, an uncountable cardinal κ is *measurable* if and only if there is a non-principal κ-complete and normal ultrafilter on κ. In ZFC, this is equivalent to the existence of a transitive class M and an elementary embedding $j : V \to M$ with critical point κ. The proof of this equivalence uses an ultrapower construction and Łos' Theorem, which in turn uses AC. The existence of a measurable cardinal is an example of a large cardinal axiom. Another example is the existence of a supercompact cardinal. An uncountable cardinal κ is S-*supercompact* if and only if there is a κ-complete ultrafilter on $\mathcal{P}_\kappa(S)$ which is fine and normal.[1] We say κ is *supercompact* if and only if it is S-supercompact for every non-empty set S. In ZFC, this is equivalent to, for every cardinal λ, there exists a transitive class M with ${}^\lambda M \subseteq M$ and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Again we emphasize that AC is used to prove this equivalence. Clearly, in ZFC if κ is supercompact, then κ is measurable and the set of measurable cardinals is unbounded in κ. This can be used to show that the consistency of the theory

$$\mathrm{ZFC} + \text{There is a measurable cardinal}$$

is a theorem of the theory

$$\mathrm{ZFC} + \text{There is a supercompact cardinal.}$$

In other words the second theory has greater consistency strength than the first. It is an empirical fact that large cardinal axioms line up this way.

Another important aspect of this paper is inner model theory, which we describe briefly (more will be discussed in Section 2). Let us assume that V is a model of ZF and that T is any theory of sets. By definition M is an inner model of T if M is a transitive proper class contained in V that satisfies T. The constructible universe, L, is the minimal inner model of ZFC. Gödel proved this fact in ZF. One relativisation of Gödel's universe

[1] For the definitions of fine and normal see [2].

is constructed as follows. For any set S, we construct a transitive proper class $L[S]$ by setting $L_0[S] = \emptyset$ and $L_{\alpha+1}[S]$ to be the family of subsets of $L_\alpha[S]$ that are definable over the structure

$$(L_\alpha[S], \in, S \cap L_\alpha[S])$$

and taking unions at limits. Note that L is the same as $L[\emptyset]$. Let us discuss the connection between these constructions and large cardinals. If $\mathcal{U}$ is a normal measure on $\mathcal{P}(\kappa)$ and

$$\overline{\mathcal{U}} = \mathcal{U} \cap L[\mathcal{U}],$$

then

$$\overline{\mathcal{U}} \in L[\overline{\mathcal{U}}] = L[\mathcal{U}]$$

and

$$L[\overline{\mathcal{U}}] \models \mathrm{ZFC} + \overline{\mathcal{U}} \text{ is a normal measure on } \mathcal{P}(\kappa).$$

This is a theorem of Solovay; see [1]. Extending this, Kunen (cf. [4]) proved the following uniqueness result.

Theorem 1 *(Kunen). Assume* ZFC. *Let κ be an ordinal and assume that for $i < 2$,*

$$L[\mathcal{U}_i] \models \mathrm{ZFC} + \mathcal{U}_i \textit{ is a normal measure on } \kappa.$$

Then $L[\mathcal{U}_0] = L[\mathcal{U}_1]$.

There is more to Kunen's result that we are suppressing for this introduction. For several decades inner model theory has strived to extend such results to more powerful large cardinals. In spite of great progress, supercompact cardinals remain beyond our reach so far in the context of ZFC. Roughly, this paper is on analogs of Kunen's theorem for $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ + There exists an $\mathbb{R}$-supercompact cardinal.

1.2. Determinacy

If S is a set, then AD_S says that, for every game of length ω in which two players alternate choosing members of S, one or the other player has a winning strategy. The instances relevant here are AD_ω, more commonly called AD or the Axiom of Determinacy, and $\mathrm{AD}_\mathbb{R}$. It is an easy well known result that AC implies AD fails. In other words $\mathrm{ZFC} + \mathrm{AD}$ is inconsistent. However, the consistency of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$$

is a theorem of the theory

$$\text{ZFC} + \text{There is a supercompact cardinal.}$$

In fact, combining results of Martin and Steel ([6]) and of Woodin ([25]) one can prove in ZFC that if there is a supercompact cardinal, then $L(\mathbb{R})$ is a model of AD. Here $L(\mathbb{R})$ is the minimal model of ZF + DC containing all ordinals and reals. $L(\mathbb{R})$ is constructed by setting $L_0(\mathbb{R}) = \text{HC}$ (we identify $\mathbb{R}$ with HC), $L_{\alpha+1}(\mathbb{R})$ to be the family of sets definable over $(L_\alpha(\mathbb{R}), \in)$, and taking unions at limits. Woodin reduced the hypothesis of this result to a large cardinal axiom strictly between measurability and supercompactness. In fact, he showed that the existence of a certain countable structure called $\mathcal{M}_\omega^\sharp$ suffices, see [18] for its precise definition. Woodin also showed that the consistency of the theory

$$\text{ZF} + \text{DC} + \text{AD}_\mathbb{R}$$

is a theorem of the theory

$$\text{ZFC} + \text{There is a supercompact cardinal.}$$

Under AD, using fine structure and pointclass theory, there is a complete structural analysis of the Wadge hierarchy (i.e. the pattern of scales) of $L(\mathbb{R})$. In other models, determinacy alone does not yield such a detailed structural analysis. Woodin introduced AD^+ to make up for the difference. Good sources that cover this material are [3] and Chapter 9 of [26]. Before defining AD^+ the following notion gets us started.

Definition 2. Let $A \subseteq \mathbb{R}$. We say A is ∞-Borel if there is a formula $\phi(x, y)$ and a set $S \subset \text{ON}$ such that

$$x \in A \text{ if and only if } L[S, x] \models \phi(x, S).$$

Let us recall that Θ is the least ordinal that is not a surjective image of a function with domain $\mathbb{R}$. In other words,

$$\Theta = \{\alpha \in \text{ON} \mid \text{ there is } f : \mathbb{R} \to \alpha \text{ surjective}\}.$$

Definition 3. AD^+ is the conjunction of the following two sentences:

(1) Every set of reals is ∞-Borel.
(2) Let $\lambda < \Theta$ and $\pi : \lambda^\omega \to \mathbb{R}$ be a continuous function; then $\pi^{-1}[A]$ is determined for every $A \subseteq \mathbb{R}$.

The ordinal Θ has the following approximations. For $A \subseteq \mathbb{R}$, we define

$$\theta(A) = \{\alpha \mid \text{ there is a surjection } f : \mathbb{R} \to \alpha \text{ with } f \in \text{OD}_A\}$$

The Solovay sequence is defined as follows.

Definition 4. Assume AD^+. Define a sequence of ordinals $\leq \Theta$,

$$(\theta_\alpha : \alpha \leq \Omega),$$

as follows.

- $\theta_0 = \theta(\emptyset)$.
- $\theta_\Omega = \Theta$.
- If γ is a limit ordinal, then $\theta_\gamma = \sup\{\theta_\alpha \mid \alpha < \gamma\}$.
- $\theta_{\alpha+1} = \theta(A)$ for A any set of Wadge rank θ_α if $\alpha < \Omega$.

In $L(\mathbb{R})$, the minimal model of AD^+ every set is ordinal definable from a real, hence $\Theta = \theta_0$. Adding conditions on the length of the Solovay sequence yields a hierarchy of strengthenings of AD^+. For example, $\mathrm{AD}^+ + \Theta = \theta_\omega$ implies $\mathrm{AD}_\mathbb{R}$. Another example is $\mathrm{AD}^+ + \Theta = \theta_{\omega_1}$, which implies $\mathrm{DC} + \mathrm{AD}_\mathbb{R}$.

The following ordinal is also an approximation to Θ but in a different sense.

Definition 5. $\boldsymbol{\delta}^2_1$ is the least ordinal that is not the rank of a Δ^2_1 pre-wellorder on $\mathbb{R}$, in other words

$$\boldsymbol{\delta}^2_1 = \{\alpha \in \mathrm{ON} \mid \text{ there exists a } \boldsymbol{\Delta}^2_1 \text{ pre-wellorder on } \mathbb{R} \text{ of rank } \alpha\}$$

Recall that a *bounded quantifier* in a formula is a quantifier that can be rendered as $\forall x \in \mathbb{R}$ or $\exists x \in \mathbb{R}$. A *bounded formula* is a formula all whose quantifiers are bounded. ϕ is a Σ_1-formula if it can be written as $\exists y \psi$ where ψ is a bounded formula. We recall that given a structure M and N a substructure of M we say that N is Σ_1 *elementary in* M and write

$$N \prec_1 M$$

if whenever $\phi(x)$ is a Σ_1 formula and $a \in N$ then $M \models \phi(a)$ implies $N \models \phi(a)$. In $L(\mathbb{R})$, $\boldsymbol{\delta}^2_1$ is the least ordinal α such that $L_\alpha(\mathbb{R}) \prec_1 L(\mathbb{R})$. For this reason we call $\boldsymbol{\delta}^2_1$ the *least stable* ordinal. This is closely related to the fact, that under AD, $\Sigma_1^{L(\mathbb{R})}$ is the largest scaled point-class of $L(\mathbb{R})$ and, for every bounded formula, ϕ, if

$$L(\mathbb{R}) \models \exists A \subset \mathbb{R}\ \phi(A),$$

then $L(\mathbb{R})$ has a Suslin co-Suslin witness for ϕ. Recall that AD^+ was introduced to generalize theorems of $\mathrm{AD} + V = L(\mathbb{R})$. Here is an example we will use later in the paper.

Theorem 6. *(Woodin) Assume* AD^+ *and* $V = L(\mathcal{P}(\mathbb{R}))$*. Suppose that* ϕ *is a bounded formula such that* $\phi(A)$ *holds for some* $A \subseteq \mathbb{R}$*. Then there is a Suslin co-Suslin witness* A *for* ϕ*.*

We end this section by pointing out that it is an open problem whether AD implies AD^+.

1.3. Supercompactness measures under ZF + AD

One important and surprising consequence of determinacy is that ω_1 is a large cardinal. Solovay proved that, under ZF + AD, the club filter on ω_1 is a normal measure and it is the unique such measure (see [3]). He also proved that under $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$, ω_1 is $\mathbb{R}$-supercompact as witnessed by the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (see [10]). We recall that C is a club subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ if there is $\pi : {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that $\sigma \in C$ if and only if σ is closed under π. We define the club filter $\mathcal{C}$ as the collection of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$ that contain a club.

We start to discuss the theory $\mathrm{ZF} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact in further detail. For this we must define another kind of models which is built by a combination of two constructions. Suppose μ is a collection of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. By $L(\mathbb{R}, \mu)$, we mean "throw in $\mathbb{R}$ at the bottom" and "use μ as a predicate". That is, define $L_0(\mathbb{R}, \mu) = \mathrm{HC}$, $L_{\alpha+1}(\mathbb{R}, \mu)$ to be the collection of sets definable over the structure

$$(L_\alpha(\mathbb{R}, \mu), \in, \mu \cap L_\alpha(\mathbb{R}, \mu))$$

and take unions at limits. Notice that μ might not belong to $L(\mathbb{R}, \mu)$ but $\mu \cap L(\mathbb{R}, \mu)$ does and

$$L(\mathbb{R}, \mu) = L(\mathbb{R}, \mu \cap L(\mathbb{R}, \mu))$$

We usually think of $L(\mathbb{R}, \mu)$ as a structure in which the extra symbol $\dot{\mu}$ is interpreted as $\mu \cap L(\mathbb{R}, \mu)$. Perhaps, a more descriptive notation for this structure would be $L(\mathbb{R})[\mu]$, however we will stick to the notation in the literature and refer to this model as $L(\mathbb{R}, \mu)$. It is immediate from Solovay's theorem (cf. [10]) about $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$ and other well-known facts that assuming $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$, $L(\mathbb{R}, \mathcal{C})$ is a model of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact}$$

where the $\mathbb{R}$-supercompactness is witnessed by $\dot{\mu}^{L(\mathbb{R},\mathcal{C})} = \mathcal{C} \cap L(\mathbb{R}, \mathcal{C})$.

Following Solovay, Woodin began the analysis of models of the form $L(\mathbb{R}, \mu)$ and obtained the following uniqueness result, the proof of which can be found in [24].

Theorem 7 *(Woodin). Suppose* $\mathrm{AD}_{\mathbb{R}}$ *holds. Then the club filter is the unique* $\mathbb{R}$*-supercompact measure on* ω_1*.*

Motivated by this result and Kunen's theorem on the uniqueness of $L[\mathcal{U}]$, Woodin asked the following question (cf. [24]).

Question 8 *(Woodin, 1983). Assume* $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}+\mathrm{AD}$*. Is there at most one model of the form* $L(\mathbb{R},\mu)$ *that satisfies* AD + ω_1 *is* $\mathbb{R}$*-supercompact?*

One of our two main results is that the answer is yes. In fact, assuming further that $V = L(\mathcal{P}(\mathbb{R}))$, the unique model is $L(\mathbb{R},\mathcal{C})$, where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$. For future reference:

Theorem 9. *Assume* $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}+\mathrm{AD}$*. Then, there is at most one model of the form* $L(\mathbb{R},\mu)$ *that satisfies* AD + ω_1 *is* $\mathbb{R}$*-supercompact. Moreover if* $V = L(\mathcal{P}(\mathbb{R}))$ *and such a model exists, then* $L(\mathbb{R},\mathcal{C})$ *is the unique one.*

The "moreover" part of the theorem is analogous to another result about measurable cardinals. Namely, if $\kappa = \omega_1$ and $0^\dagger$ exists,[2] then there is exactly one model of the form $L[\mathcal{U}]$ in which $\mathcal{U}\cap L[\mathcal{U}]$ is a normal measure on κ, namely take $\mathcal{U}$ to be the club filter on κ (cf. [4] or [2, Chapter 21]). We note here the analogy of this fact with Theorem 1. The fact implies the uniqueness of $L[\mathcal{U}]$; the extra assumption (that $0^\dagger$ exists) gives us that the club filter $\mathcal{U}$ on κ is indeed a measure on $L[\mathcal{U}]$.

It is also natural to ask Woodin's question with the assumption ZFC instead of $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}+\mathrm{AD}$. We do not know the answer to the modified question, but under ZFC together with a technical large cardinal hypothesis we obtain a positive answer, our second main result.

Theorem 10. *Assume* ZFC *and suppose that* $\mathcal{M}^\sharp_{\omega^2}$ *exists. Then*

(1) $L(\mathbb{R},\mathcal{C}) \models \mathrm{AD}$ + $\mathcal{C}$ *is an* $\mathbb{R}$*-supercompact measure, and*
(2) *if* $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ *is such that*

$$L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1 \textit{ is } \mathbb{R}\textit{-supercompact,}$$

then $L(\mathbb{R},\mathcal{C}) = L(\mathbb{R},\mu)$.

The meaning of $\mathcal{M}^\sharp_{\omega^2}$ and the sense in which it is iterable will be discussed in Section 2. The large cardinal hypothesis that $\mathcal{M}^\sharp_{\omega^2}$ exists is slightly stronger than the consistency strength of the theory $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact. We do not know how to do without this mild "extra" assumption but conjecture that it is possible and have some partial results in this direction that will be mentioned.

[2] $0^\dagger$ is the sharp for a model of the form $L[\mathcal{U}]$ in which $\mathcal{U}\cap L[\mathcal{U}]$ is a measure on a cardinal κ.

Conjecture 11. Assume ZFC. Then there is at most one model of the form $L(\mathbb{R},\mu)$ that satisfies $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact.

Theorems 9 and 10 will come at the end of a series of uniqueness results with varying hypotheses, which we label propositions. One of these, the following, is the main result of Section 2, which Theorem 10 strengthens in that there are no stationarity assumptions for the members of μ.

Proposition 12. *Assume* ZFC *and suppose that* $\mathcal{M}^{\sharp}_{\omega^2}$ *exists. Then*

(1) $L(\mathbb{R},\mathcal{C})\models \mathrm{AD} + \mathcal{C}$ *is an* $\mathbb{R}$*-supercompact measure, and*
(2) *if* $\mu\subset\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ *is such that for any* $A\in\mu$, A *is stationary and*

$$L(\mathbb{R},\mu)\models \mathrm{AD} + \omega_1 \textit{ is } \mathbb{R}\textit{-supercompact},$$

then $L(\mathbb{R},\mathcal{C})=L(\mathbb{R},\mu)$.

We remark that we just need ZF + DC in the proof of Proposition 12. To read Section 2 the reader should be familiar with the technique of iterating mice to make reals generic, see Theorem 18. We give a brief summary of the results needed at the beginning of subsection 2.1.

In Section 3, we prove Theorem 10. For this, we use the HOD analysis of the models $L(\mathbb{R},\mu)$ satisfying $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact, to show that on a Turing cone of reals x,

$$\mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})}=\mathrm{HOD}_x^{L(\mathbb{R},\mu)}.$$

See [13] for the HOD analysis in $L(\mathbb{R})$; other good sources on this subject are [20] and [22]. The meaning of ordinal definability in $L(\mathbb{R},\mu)$ is different from the usual notion in that the language for the definitions includes the predicate $\dot{\mu}$ which is interpreted as $\mu\cap L(\mathbb{R},\mu)$.

Finally, in Section 4, we use Theorem 10 and its proof to show Theorem 46 after first proving yet another approximation, namely:

Proposition 13. *Assume* $V=L(\mathcal{P}(\mathbb{R}))+\mathrm{AD}^+$*. Then, there is at most one model of the form* $L(\mathbb{R},\mu)$ *that satisfies* $\mathrm{AD}+\omega_1$ *is* $\mathbb{R}$*-supercompact. Moreover if such model exists then* $L(\mathbb{R},\mathcal{C})$ *is the unique such model, where* $\mathcal{C}$ *is the club filter.*

In addition to the prerequisites mentioned earlier, for Section 4, the reader should be familiar with certain concepts of descriptive inner model theory. For example, [8] is a good source.

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2. The ZFC case under the stationarity assumption

In this section, we prove Proposition 12. Recall that we assume the existence of $\mathcal{M}^{\sharp}_{\omega^2}$, we give a brief summary of this special mouse next.

2.1. Mice and genericity iterations

We will assume that the reader has familiarity with the basic concepts of extender models and mice. Sources for this subject are [18] and [7]. However, for the non-expert we will summarize the key parts of the theory of mice we will be using.

We start by recalling that a *premouse* $\mathcal{M}$ is a fine structural model of the form $(\mathcal{J}^{\boldsymbol{E}}_\alpha, \in, \boldsymbol{E} \restriction \alpha, E_\alpha)$, where $\boldsymbol{E}$ is a fine extender sequence in the sense of [18]. Let us fix some notation: supposing $\mathcal{M}$ is as above and $\gamma < \alpha$, we write $\mathcal{M}|\gamma$ for the structure $(\mathcal{J}^{\boldsymbol{E}\restriction\gamma}_\gamma, \in, \boldsymbol{E} \restriction \gamma, E_\gamma)$. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two pre-mice we say that $\mathcal{M}$ *is an initial segment of* N and write $\mathcal{M} \trianglelefteq \mathcal{N}$, if there is $\gamma \leq \mathrm{ON} \cap \mathcal{N}$ such that $\mathcal{M} = \mathcal{N}|\gamma$.

If $\mathcal{M}$ is a k-sound mouse, we say that $\mathcal{M}$ is (k, β, γ) or (k, β)-iterable if player II has a winning strategy for the iteration games $\mathcal{G}_k(\mathcal{M}, \beta, \gamma)$ or $\mathcal{G}_k(\mathcal{M}, \beta)$ respectively, see sections 3 and 4 of [18] for a precise definition. A *mouse* is a k-sound pre-mouse that is $(k, \omega_1 + 1)$-iterable. Whenever $k = \omega$ we will abuse notation and write $\omega_1 + 1$-iterable for $(\omega, \omega_1 + 1)$ etc. $\mathcal{M}^{\sharp}_{\omega^2}$ is the mouse we will be most interested in this work. We warm up for its presentation with the following definition.

Definition 14. A pre-mouse is called ω^2-*small* if whenever κ is the critical point of an extender in the sequence of $\mathcal{M}$, then

$$\mathcal{M}|\kappa \nvDash \text{“There are } \omega^2 \text{ many Woodin cardinals”}.$$

Definition 15. $\mathcal{M}^{\sharp}_{\omega^2}$ is the unique sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable mouse that is not ω^2-small, but all of whose initial segments are ω^2-small.

Notice that $\rho_1(\mathcal{M}^{\sharp}_{\omega^2}) = \omega$ and $p_1(\mathcal{M}^{\sharp}_{\omega^2}) = \emptyset$, hence $\mathcal{M}^{\sharp}_{\omega^2}$ is countable. We will see in Section 3 that $\mathcal{M}^{\sharp}_{\omega^2}$ is related to $L(\mathbb{R}, \mathcal{C})$ very much like $\mathcal{M}^{\sharp}_{\omega}$ is related to $L(\mathbb{R})$ as exposed in [18]. Recall the Solovay sequence defined in subsection 1.2. Its length not only entails stronger versions of determinacy but also the existence of mice with certain large cardinal structure. A fact we will repeatedly use is that under AD^+ if $\Theta > \theta_0$, then there is a *non-tame mouse*, which we introduce now.

Definition 16. Let $\mathcal{M}$ be a pre-mouse. We say $\mathcal{M}$ is *tame* if for any δ such that $\mathcal{M} \models$ “δ is Woodin” and for any E_γ in the sequence of $\mathcal{M}$ with $\mathrm{crit}(E_\gamma) < \delta$, then $\gamma < \delta$.

Note that a non-tame mouse is a mouse that has a cardinal that is strong past a Woodin cardinal. Also, if a non-tame mouse exists, it is easy to see that $\mathcal{M}^\sharp_{\omega^2}$ exists.

Theorem 17 *(Woodin, see [16]). Assume* $\mathrm{AD}^+ + \Theta > \theta_0$*, then there is an* (ω_1+1)*-iterable non-tame mouse.*

One important application of mice is that they help when analyzing certain determinacy models. For example, one of the key ingredients for the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ is that one can iterate $\mathcal{M}^\sharp_\omega$ to make any given real generic. We will also use this technique throughout this paper.

Theorem 18 *(Genericity iterations). Let* Σ *be an* (ω_1+1)*-iteration strategy for a countable mouse* $\mathcal{M}$ *and* $\delta_0 < \delta_1$ *be two ordinals in* $\mathcal{M}$ *such that*

$$\mathcal{M} \models \delta_1 \textit{ is a Woodin cardinal.}$$

Then there is a Boolean algebra $\mathbb{B}^{\mathcal{M}}_{(\delta_0,\delta_1)} \in \mathcal{M}$ *such that* $\mathbb{B}^{\mathcal{M}}_{(\delta_0,\delta_1)} \subset V^{\mathcal{M}}_{\delta_1}$ *and* $\mathcal{M} \models \mathbb{B}_{(\delta_0,\delta_1)}$ *is* δ_1*-c.c. Moreover, for every* $x \in \mathbb{R}$*, there is a countable iteration tree* $\mathcal{T}$ *such that*

- $\mathcal{T}$ *is a play according to* Σ*,*
- $\mathcal{T}$ *has a final model, say* $\mathcal{M}^{\mathcal{T}}_\gamma$*,*
- $\mathcal{T}$ *is nowhere dropping*
- *All extenders used in* $\mathcal{T}$ *have a critical point above* δ_0 *and its images and*
- *there is an* $\mathcal{M}^{\mathcal{T}}_\gamma$*-generic filter* G *for* $\mathbb{B}^{\mathcal{M}^{\mathcal{T}}_\gamma}_{\delta_0, i^{\mathcal{T}}_{0,\gamma}(\delta_1)} = i^{\mathcal{T}}_{0,\gamma}\left(\mathbb{B}^{\mathcal{M}}_{(\delta_0,\delta_1)}\right)$ *such that*

$$\mathcal{M}^{\mathcal{T}}_\gamma[G] = \mathcal{M}^{\mathcal{T}}_\gamma[x]$$

We will abuse notation and call the boolean algebra $\mathbb{B}^{\mathcal{M}}_{(\delta_0,\delta_1)}$ *the extender algebra of* $\mathcal{M}$ *at* δ_1 whenever δ_0 is clear from context. The corresponding iteration is called a *genericity iteration in the interval* (δ_0, δ_1). We will use genericity iterations and $\mathcal{M}^\sharp_{\omega^2}$ to compute the theory of $L(\mathbb{R}, \mathcal{C})$ in Section 3.

We will also use the notion of mice constructed over a set X in Section 3. We define this notion below.

Definition 19. We say that $\mathcal{M} = (\mathcal{J}^{\boldsymbol{E}}_\alpha(tr.cl.(X)), \in, \boldsymbol{E} \restriction \alpha, E_\alpha)$ is an X-premouse if $\boldsymbol{E}$ is a fine extender sequence and all extenders in $\boldsymbol{E}$ have critical points above $o(tr.cl.(X))$.

Definition 20. For a set X we have the following.

- Given an X-premouse $\mathcal{M}$, we say that $\mathcal{M}$ is *countably iterable* if for any $\bar{\mathcal{M}}$ countable and elementarily embeddable into $\mathcal{M}$, we have that $\bar{\mathcal{M}}$ is $\omega_1 + 1$ iterable.

- An X-premouse $\mathcal{M}$ is sound if it is ω-sound (in the sense of [18]).
- An X-premouse $\mathcal{M}$ is said to project to X if there is an $A \subseteq X$ that is definable over $\mathcal{M}$ from $X \cup \{X\}$ and ordinal parameters but A is not in $\mathcal{M}$.
- $Lp(X)$ is the union of all countably iterable and sound X-mice that project to X.

We commonly refer to $Lp(X)$ as the *lower part* of X.

2.2. The proof of Proposition 12

Assume its hypotheses. Recall that $\mathcal{M}^\sharp_{\omega^2}$ is the unique, active, sound mouse projecting to ω, with ω^2-many Woodin cardinals all whose initial segments are ω^2-small. Part of what it means to be a mouse is that $\mathcal{M}^\sharp_{\omega^2}$ has an iteration strategy, which happens to be unique; we call it Σ. For simplicity we make an additional assumption about Σ that we will eliminate when we finish the proof of Proposition 12 at the end of this section. Assume further that Σ extends to an outer model W in which $\mathcal{P}(\mathbb{R}) \cap V$ is countable. Let us fix $\kappa = \left((2^{\mathfrak{c}})^+\right)^V$ and note that $\Sigma \in H^V_\kappa$. We also fix such an outer model W.

If P is a countable Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, then the tail of Σ is an iteration strategy on P. If Q is a Σ-iterate of such a P and there is no dropping on the branch from P to Q, then we write $P \xrightarrow{\Sigma} Q$ for the branch embedding.

We need to review a certain construction that plays a role in the proof of Proposition 12. Consider an arbitrary transitive class model M of ZFC that has ω^2-many Woodin cardinals. Let δ^M_α be the α-th Woodin cardinal of M. Also, let $\lambda^M_\beta = \sup\{\delta^M_\alpha \mid \alpha \in \beta\}$ for β limit; $\lambda^M_0 = 0$. Suppose further that G is an M-generic filter for $\mathrm{col}(\omega, < \lambda^M_{\omega^2})$. Let

$$\sigma_i = \bigcup_{\alpha < \omega i} \mathbb{R}^{M[G \restriction \alpha]} \text{ and } \mathbb{R}^* = \bigcup_{\alpha < \omega^2} \mathbb{R}^{M[G \restriction \alpha]}$$

we say that $\mathbb{R}^*$ is the set of symmetric reals associated to G. In $M[G]$, define *the tail filter*, $\mathcal{F}$, on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$ as follows: for $A \subseteq P_{\omega_1}(\mathbb{R}^*)$

$$A \in \mathcal{F} \text{ if and only if } \exists n \in \omega\, \forall m \geq n\ (\sigma_m \in A)$$

The fact we will use is

Theorem 21 *(Woodin, see [22]). Let M be a transitive proper class model of ZFC with ω^2 many Woodin cardinals whose limit is $\lambda^M_{\omega^2}$. Let G be an M-generic filter for the Levy collapse up to $\lambda^M_{\omega^2}$. Let $\mathcal{F}$ and $\mathbb{R}^*$ be the tail filter and symmetric reals associated to G, then*

$$L(\mathbb{R}^*, \mathcal{F}) \models \mathrm{AD}^+ + \mathcal{F} \textit{ is an } \mathbb{R}\textit{-supercompactness measure.}$$

We will call $L(\mathbb{R}^*, \mathcal{F})$ the *derived model*[3] of M at $\lambda^M_{\omega^2}$. Eventually, we will find a Σ-iterate, M, and an M-generic filter G on $\mathrm{col}(\omega, < \lambda^M_{\omega^2})$ such that the associated $\mathbb{R}^*$ is $\mathbb{R}$ and the corresponding tail filter contains the club filter, $\mathcal{C}$. Towards this, the following gets us started.

Lemma 22. *Suppose that γ is a cardinal of V such that $\gamma \geq \kappa$. Let X_0 and X_1 be countable elementary substructures of H_γ such that $\mathbb{R} \cap X_0 \in X_1$ and $\Sigma \in X_0$. Then there is an iteration tree $\mathcal{T}$ on $\mathcal{M}^\sharp_{\omega^2}$ of successor length $\zeta + 1$ such that $\mathcal{T} \restriction \alpha \in X_0$ for all $\alpha < \zeta$ and $\mathcal{T} \in X_1$, and there exists $G \in X_1$ such that G is $M^{\mathcal{T}}_\zeta$-generic on col$(\omega, < \lambda_\omega^{M^{\mathcal{T}}_\zeta})$ and the associated set of symmetric reals is $\mathbb{R} \cap X_0$.*

Proof. Given the assumptions above note that $\mathbb{R} \cap X_0 \in X_1$, so there is $\langle x_i \,|\, i \in \omega\rangle$ an enumeration of $\mathbb{R} \cap X_0$ in X_1. Now, by Theorem 18, there is an iteration tree $\mathcal{T}_0$ on $\mathcal{M}^\sharp_{\omega^2}$ according to Σ, with last model P_0, such that $i : \mathcal{M}^\sharp_{\omega^2} \to P_0$ exists and x_0 is generic for $\mathbb{B}^{P_0}_{\delta_0}$, the extender algebra at $\delta_0^{P_0}$. Note that since $\Sigma \in X_0$, $\mathcal{T}_0$ belongs to X_0 and is countable there. We continue iterating $P_0 \to P_1$ in the interval $(\delta_0^{P_1}, \delta_1^{P_1})$ (see paragraph after Theorem 18), say via $\mathcal{T}_1$, to make the next real x_1, generic for the extender algebra at $\delta_1^{P_1}$. Note that in this case both x_0 and x_1 are set generic over P_1 for posets in $V^{P_1}_{\lambda^{P_1}_\omega}$. Continuing in this fashion we get Σ-iteration trees $\mathcal{T}_n$ with branch embeddings $P_{n-1} \to P_n$ such that x_n is P_n-generic for the extender algebra at $\delta_n^{P_n}$. Also every x_i for $i < n$ is set generic over P_n.

In X_1, define $\mathcal{T}$ to be the concatenation of the $\mathcal{T}_n$'s. Now $\mathcal{T}$ has a unique cofinal branch b. Let $P = M^{\mathcal{T}}_b$, $\sigma = \mathbb{R} \cap X_0$ and $\lambda = \lambda^P_\omega$. By construction the following hold:

(1) For every $x \in \sigma$ there is a poset $\mathbb{P} \in V^P_\lambda$ such that x is P-generic for $\mathbb{P}$.
(2) $\lambda = \sup\{\omega_1^{P[x]} \,|\, x \in \sigma\}$.
(3) $P \models$ "λ is a strong limit cardinal".

Then essentially by Lemma 3.1.5 of [5] there is a P-generic filter G for $\mathrm{col}(\omega, < \lambda^P_\omega)$ in X_1 such that the associated set of symmetric reals is $\mathbb{R} \cap X_0$. □

Note that in the proof of Lemma 22 all we really need is an iterable mouse $\mathcal{M}$ whose strategy is in X_0 and an interval $(\lambda_0^{\mathcal{M}}, \lambda_\omega^{\mathcal{M}})$ containing ω many Woodin cardinals. In practice (as in Lemma 23) we will use a Σ-iterate $\mathcal{M}$ of $\mathcal{M}^\sharp_{\omega^2}$ with $\mathcal{M} \in X_0$ and some interval $(\lambda^{\mathcal{M}}_{\omega i}, \lambda^{\mathcal{M}}_{\omega(i+1)})$ of ω many Woodins in $\mathcal{M}$.

Lemma 23. *Let $\mathcal{C}^V$ be the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ as computed in V. Then, in W, there is a Σ-iterate P of $\mathcal{M}^\sharp_{\omega^2}$ and a P-generic filter G for col$(\omega, < \lambda^P_{\omega^2})$ such that if $\mathcal{F}$ is the associated tail filter, then $\mathcal{C}^V$ is contained in $\mathcal{F}$.*

[3] We recall that this is not the standard definition of the *derived model*, see [16] and [25] for the standard definition.

Proof. In W, we let $\langle X_i \mid i \in \omega \rangle$ be a chain of elementary substructures of H^V_κ such that $\bigcup_{i\in\omega} X_i \supseteq \mathcal{P}(\mathbb{R})^V$ and if $\sigma_i = X_i \cap \mathbb{R}$, then $\sigma_i \in X_{i+1}$ and σ_i is countable in V. We may assume that $\mathcal{M}^\sharp_{\omega^2}$, and Σ are in X_0. We construct an iteration of the form

$$\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \to \cdots \to P_i \to P_{i+1} \cdots \to P$$

by recursion using Lemma 22 so that the iteration $P_{i-1} \to P_i$ is done in the interval $(\lambda_{\omega i}, \lambda_{\omega(i+1)})$ and makes σ_i the set of symmetric reals associated to a P_i-generic on $\mathrm{col}(\omega, < \lambda^{P_i}_{\omega i})$. Let P be the direct limit of the P_i, since Σ extends to an iteration strategy in W, we have that P is well-founded. By a variant of Lemma 3.1.5 in [5] there is a P-generic filter G for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{P[G\restriction\alpha]}$.

Note that the set of symmetric reals associated to G and P is $\mathbb{R}^V$. Let $\mathcal{F}$ be the corresponding tail filter. Consider any $A \in \mathcal{C}^V$. Let $\pi \in V$ be such that $\pi : \mathbb{R}^{<\omega} \to \mathbb{R}$ and its closure points belong to A. Then there is an $n \in \omega$ such that $\pi \in X_n$. So for all $m \geq n$, $\pi \in X_m$ and σ_m is closed under π, thus $A \in \mathcal{F}$. □

The two key facts in the proof of Lemma 23 are that if A is an element of $\mathcal{C}^V$, then there is an $i \in \omega$ such that $A \in X_i$, and that every X_i is closed under Σ. This motivates the following definitions.

Definition 24. Suppose N is a set model of ZFC-PowerSet such that $\mathcal{P}(\mathbb{R})^N$ is countable.

(1) We say $\langle X_i \mid i \in \omega \rangle$ is a *good resolution* of N if:
   (a) For all $i \in \omega$, we have $X_i \prec N$,
   (b) $\Sigma \in X_0$,
   (c) $\mathbb{R} \cap X_i \in X_{i+1}$, $\mathbb{R} \cap X_i$ is countable in N and
   (d) $\bigcup_{i\in\omega} X_i \supset \mathcal{P}(\mathbb{R})^N$.

(2) Given $X = \langle X_i \mid i \in \omega \rangle$ a good resolution of N, and $\sigma_i = X_i \cap \mathbb{R}$, we define $\mathcal{F}_X$, the tail filter associated to X by

$$A \in \mathcal{F}_X \text{ if and only if } \exists n \in \omega\, \forall m \geq n\ (\sigma_m \in A)$$

Note that in the proof of Lemma 23 instead of H_κ we could have used any N that is a model of ZFC-PowerSet with a good resolution in W. We give an example of such a situation in the following lemma.

Lemma 25. *Suppose that A is stationary in $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then, in W, there exists a Σ-iterate P of $\mathcal{M}^\sharp_{\omega^2}$, and P-generic filter G for col$(\omega, < \lambda^P_{\omega^2})$ such that A belongs to the tail filter associated to G and P.*

Proof. Consider $\mathbb{P}_A$, the forcing poset whose conditions are countable, closed, increasing sequences from A. In other words, $p = \langle \sigma_\alpha \mid \alpha \leq \beta \rangle$ is a condition in $\mathbb{P}_A$ if

- for all $\alpha \leq \beta$, we have that σ_α belongs to A,
- for every α and α' less or equal β if $\alpha < \alpha'$ then $\sigma_\alpha \subseteq \sigma_{\alpha'}$, and
- if $\alpha \leq \beta$ is a limit ordinal, then $\sigma_\alpha = \bigcup_{i \in \alpha} \sigma_i$.

We say $p \leq q$ if p end-extends q. It is easy to see that this poset shoots a club through A. Also, since A is stationary this forcing is (ω_1, ∞)-distributive, so in particular it does not add any new reals. Note that $\mathbb{P}_A$ has size continuum (in V) and so there is $h \in W$, such that h is V-generic for $\mathbb{P}_A$. Also $\mathbb{R}^{V[h]} = \mathbb{R}$, so if $N = H_\kappa^{V[h]}$ and $\langle X_i | i < \omega \rangle$ is a good resolution for N in W; then the remark after Lemma 23 yields the result. □

Suppose that in W there are two Σ-iterates P and Q of $\mathcal{M}^\sharp_{\omega^2}$ and generic filters G and H for $\text{col}(\omega, < \lambda^P_{\omega^2})$ and $\text{col}(\omega, < \lambda^Q_{\omega^2})$ respectively such that the set of symmetric reals of $P[G]$ and $Q[H]$ is precisely $\mathbb{R}^V$. Let $\mathcal{E}$ and $\mathcal{F}$ be the tail filters associated to P, G and Q, H respectively. We show next that if this is the case then $L(\mathbb{R}, \mathcal{E}) = L(\mathbb{R}, \mathcal{F})$.

Lemma 26. *In W, let N_1 and N_2 be transitive sets models of* ZFC*-PowerSet containing Σ such that $\mathbb{R}^{N_i} = \mathbb{R}^V$ for $i = 1, 2$. Let $\langle X^1_i | i \in \omega \rangle$ and $\langle X^2_i | i \in \omega \rangle$ be good resolutions of N_1 and N_2 respectively and $\mathcal{F}^1$ and $\mathcal{F}^2$ be the associated tail filters. Then $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}^2)$.*

In practice N_1 would be H^V_κ and N_2 would be $H_\kappa^{V[h]}$ for some filter h which is V-generic for a forcing of size $< \kappa$ (that adds no reals).

Proof. Let $\sigma^1_j = X^1_j \cap \mathbb{R}$ and similarly $\sigma^2_j = X^2_j \cap \mathbb{R}$. Iterate $\mathcal{M}^\sharp_{\omega^2}$ inductively as follows. Let $\sigma_0 = \sigma^1_0$, and note that σ_0 can be coded as a single real, so there is i_1 such that $\sigma_0 \in X^2_{i_1}$ and so there is $\mathcal{M}^\sharp_{\omega^2} \to P_0$ an iteration in $X^2_{i_1}$ to make σ_0 generic on the first ω-many Woodins. Define $\sigma_1 = \sigma^2_{i_1}$ and note that there is i_2 such that $\sigma_1 \in X^1_{i_2}$ an hence, in $X^1_{i_2}$, there is an iteration $P_0 \to P_1$ on the second ω-many Woodins to make σ_1 generic. Continue the iteration in this fashion. We get an iteration

$$\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \cdots \to P_i \to P_{i+1} \to \cdots \to P$$

and a P-generic filter G for $\text{col}(\omega, < \lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha < \omega i} \mathbb{R}^{P[G \restriction \alpha]}$. Let $\mathcal{F}$ be the associated tail filter. Note also that for any $i \in \omega$ there are $j > i$ and $k > i$, and natural numbers m and n such that $\sigma^1_j = \sigma_m$ and $\sigma^2_k = \sigma_n$.

Claim. $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}) = L(\mathbb{R}, \mathcal{F}^2)$.

Proof of the claim. By Theorem 21 we have that $\mathcal{F}^1$ is an ultrafilter relative to sets in $L(\mathbb{R}, \mathcal{F}^1)$, and similarly $\mathcal{F}$ is an ultrafilter in $L(\mathbb{R}, \mathcal{F})$. We first show that $\mathcal{F}$ and $\mathcal{F}_1$ agree on

$$L(\mathbb{R}, \mathcal{F}) \cap L(\mathbb{R}, \mathcal{F}_1)$$

To see this let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ be a set in $L(\mathbb{R}, \mathcal{F}) \cap L(\mathbb{R}, \mathcal{F}_1) \cap \mathcal{F}^1$, since $\mathcal{F}$ is an ultrafilter in $L(\mathbb{R}, \mathcal{F})$ we have that either A or its complement is in $\mathcal{F}$. But A contains a tail of the σ_i^1 since it belongs to $\mathcal{F}_1$, hence by construction its complement cannot contain a tail of σ_i, which means $A \in \mathcal{F}$. The other direction is similar.

Now an induction on $\alpha \in \mathrm{ON}$ shows that $L_\alpha(\mathbb{R}, \mathcal{F}^1) = L_\alpha(\mathbb{R}, \mathcal{F})$. Successor steps follow from the observation above, limit steps are clear. We have shown the first equality. The second equality is shown similarly. □

Clearly the claim completes the proof of Lemma 26. □

For simplicity we will refer to the unique model in W coming from constructions a la Lemma 23 as $L(\mathbb{R}, \mathcal{F})$. We refer to $\mathcal{F} \cap V$ as $\mathcal{F}$ when there is no ambiguity. We will use the following useful lemma extensively. We remind the reader that $\dot{\mu}$ is the extra symbol in the language that is interpreted as a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$.

Lemma 27. *Let $L(\mathbb{R}, \mu) \models \mathrm{AD} + \mu$ is an $\mathbb{R}$-supercompactness measure, and suppose that μ contains only stationary sets. Then $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$.*

Proof. We will show again inductively that $L_\alpha(\mathbb{R}, \mathcal{F}) = L_\alpha(\mathbb{R}, \mu)$. For this, as in the proof of the claim of Lemma 26, we only need to see that $\mathcal{F}$ and μ agree on $L(\mathbb{R}, \mathcal{F}) \cap L(\mathbb{R}, \mathcal{F})$. Given $A \in \mathcal{F} \cap L(\mathbb{R}, \mathcal{F}) \cap L(\mathbb{R}, \mu)$ we have that either A or its complement is in μ. For contradiction suppose $A \notin \mu$. Then $A^c \in \mu$, so A^c is stationary, applying Lemma 25 we have that there is a tail filter $\mathcal{E}$ associated to a good resolution such that $A^c \in \mathcal{E}$, now by Lemma 26 we have that $A^c \in \mathcal{F}$, which is a contradiction. □

The careful reader might note that there is a general trend in these kind of inductive arguments. Indeed

Lemma 28. *Let $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ be models of $\mathrm{ZF} + \dot{\mu}$ in an ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$, and suppose further that $\nu \subseteq \mu$. Then $L(\mathbb{R}, \mu) = L(\mathbb{R}, \nu)$.*

Proof. Note that $\mu \cap L(\mathbb{R}, \nu) = \nu \cap L(\mathbb{R}, \nu)$ since $\nu \subseteq \mu$ and ν is an ultrafilter in $L(\mathbb{R}, \nu)$. Now again an induction on the constructive hierarchy shows that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \nu)$ as wanted. □

Lemma 29. $\mathcal{C} \cap L(\mathbb{R}, \mathcal{F}) = \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$.

Proof. Otherwise by Lemma 23, there is $A \in \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$ that does not contain a club, which means that A^c is stationary so by Lemma 25 and 26 we have $A^c \in \mathcal{F}$, which gives a contradiction. □

To summarize we have seen that $L(\mathbb{R}, \mathcal{C})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact under the hypotheses of Proposition 12 and the additional assumption that Σ is has an extension to W, a class outer model in which $P(\mathbb{R})^V$ is countable. Our final step is to eliminate this extra assumption.

Assume that $\mathcal{M}^\sharp_{\omega^2}$ exists and Σ is an iteration strategy. Suppose that μ is as in the statement of the proposition but assume that $L(\mathbb{R}, \mu) \neq L(\mathbb{R}, \mathcal{C})$. Fix γ be such that V_γ reflects this fact, and $\mathcal{P}(\mathbb{R}) \in V_\gamma$. Let $N \prec V_\gamma$ be countable such that Σ and μ are in N. Let H be the transitive collapse of N and $\pi : H \to N$ be the uncollapsing map. Define $\bar{\mu} = \pi^{-1}(\mu)$ and $\overline{\Sigma} = \pi^{-1}(\Sigma)$. On the one hand, by elementarity of π we have that $H \models L(\mathbb{R}, \bar{\mu}) \neq L(\mathbb{R}, \mathcal{C})$. On the other hand $\overline{\Sigma}$ canonically extends to Σ and $\mathcal{P}(\mathbb{R})^H$ is countable in V. The relationship between H and V is similar enough to the relationship between V and W to obtain the following in V. There is a countable iterate P of $\mathcal{M}^\sharp_{\omega^2}$ and a P-generic filter K for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ such that $\mathbb{R}^H$ is the set of symmetric reals of $P[K]$. Moreover, if $\overline{\mathcal{F}}$ is the associated tail filter, then in H, $L(\mathbb{R}^H, \overline{\mathcal{F}}) = L(\mathbb{R}^H, \overline{\mu}) = L(\mathbb{R}^H, \mathcal{C}^H)$, a contradiction. Hence Proposition 12 holds.

We end this section with the remark that the proof of Proposition 12 uses only dependent choice and so it follows from ZF + DC. We will use this fact in the following sections.

3. The general ZFC case

Assume $\mathcal{M}^\sharp_{\omega^2}$ exists. In the last section we saw that if μ consists only of stationary sets and $L(\mathbb{R}, \mu)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact, then $\mu \cap L(\mathbb{R}, \mu) = \mathcal{C} \cap L(\mathbb{R}, \mu)$. Let us fix some notation for the rest of the paper.

Definition 30. Let N be a model of ZFC-PowerSet. We define $\mathcal{S}^N$ to be the (in N) stationary subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})^N$.

Let us give an example that illustrates there is more to do. Consider $\mathcal{S}^V$. By Proposition 12 we have that $L(\mathbb{R}, \mathcal{S}^V)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact. Let $A \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ be a stationary set whose complement is also stationary and let h be a V-generic filter for the poset that shoots a club through A^c (as in the proof of Lemma 25). Applying Proposition 12 in $V[h]$, $L(\mathbb{R}, \mathcal{S}^{V[h]})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact. We would like to conclude that $L(\mathbb{R}, \mathcal{S}^V) = L(\mathbb{R}, \mathcal{S}^{V[h]})$ but it does not follow from Proposition 12 applied in $V[h]$ because $A \in \mathcal{S}^V$ but A is nonstationary. Note, however, that a posteriori since $L(\mathbb{R}, \mathcal{S}^V) = L(\mathbb{R}, \mathcal{S}^{V[h]})$ even though A^c is a club in $V[h]$ we have that $A^c \notin L(\mathbb{R}, \mathcal{S}^{V[h]})$, the point here is that $V[h]$ is unaware of this situation.

Notice that the proof given in the last section relies heavily on the fact that if $A \in \mu$, then one can shoot a club through A without adding reals. Without this available to us we need a different idea. We use Woodin's Analysis of HOD in order to prove Theorem 10. The HOD Analysis for structures of the form $L(\mathbb{R}, \mu)$ was done in [22], however we will use a variant closer to the exposition of [13]. We start by doing the analysis for $L(\mathbb{R}, \mathcal{C})$ and then generalize to $L(\mathbb{R}, \mu)$. We first give some useful definitions and lemmas. We will work, as in the last section, with $\mathcal{M}^{\sharp}_{\omega^2}$ and its strategy Σ, as well as with W an external model in which $P(\mathbb{R})$ is countable and Σ extends to an iteration strategy in W. Ultimately, the extra assumptions will be eliminated at the end of this section using the same ideas from Section 2.

3.1. A HOD *analysis for* $L(\mathbb{R}, \mathcal{C})$

We start the outline of the HOD analysis by adapting the standard notions. This means, we will define, in V, a directed system whose limit agrees with a rank initial segment $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$, and then discuss what the rest of $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$ looks like and define in $L(\mathbb{R}, \mathcal{C})$ a corresponding covering system. Then we will generalize these results to models $L(\mathbb{R}, \mu)$ of $\mathrm{ZF} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. By [22], in $L(\mathbb{R}, \mu)$, AD^+ and Mouse Capturing holds.

Definition 31. We say P is a δ_0-*bounded* Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$ if there is an iteration tree $\mathcal{T}$ on $\mathcal{M}^{\sharp}_{\omega^2}$ built according to Σ, such that,

- P is the last model of $\mathcal{T}$,
- all extenders used in $\mathcal{T}$ have critical point below the image of $\delta_0^{\mathcal{M}^{\sharp}_{\omega^2}}$, and
- there is no drop in model on the branch leading to P so that there is an embedding $i : \mathcal{M}^{\sharp}_{\omega^2} \to P$ given by $\mathcal{T}$.

Let

$$\mathcal{D}^+ = \{P \mid P \text{ is a countable, } \delta_0\text{-bounded iterate of } \mathcal{M}^{\sharp}_{\omega^2}\}.$$

For P and Q in $\mathcal{D}^+$, say $P \preceq^+ Q$ if P iterates to Q via Σ in a δ_0-bounded way, in which case we let $\pi_{P,Q}$ be the corresponding embedding given by Σ. By the Dodd–Jensen property of Σ, $\pi_{P,Q}$ does not depend on any particular Σ-iteration from P to Q. The Dodd–Jensen property also guarantees that $(\mathcal{D}^+, \preceq^+, \pi_{Q,P})$ is a directed system. Take the direct limit of $(\mathcal{D}^+, \preceq^+, \pi_{P,Q})$ and iterate away the sharp (i.e. the top extender of the direct limit) ON-many times to obtain a proper class model M^+_∞. Also, let $\pi_{Q,\infty}$ be the natural map from Q to M^+_∞.

We will eventually prove that, $L[M^+_\infty, \Sigma \restriction X] = \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$, for some set of iteration trees X in $L(\mathbb{R}, \mathcal{C})$. Motivated by the work of Steel and Woodin for $L(\mathbb{R})$ the next step is to adapt the definition of suitability. From now on, we work in $L(\mathbb{R}, \mathcal{C})$ unless otherwise mentioned.

Definition 32. Let $\alpha < \omega^2$. Let P be a premouse. We say P is α-*suitable* if there exist a sequence $\langle \delta_i^P \rangle_{i<\alpha}$ in P such that

(1) For every cut-point η of P, $Lp(P|\eta) \trianglelefteq P$.
(2) If $\eta < o(P)$, but is not a Woodin cardinal of P, then $Lp(P|\eta) \models$ "η is not Woodin".
(3) For all $i < \alpha$, $P \vDash$ "δ_i^P is Woodin", and these are the only Woodin cardinals of P. Furthermore, letting $\lambda = \sup_{i\in\alpha} \delta_i^P$, then $o(P) = \sup_{n\in\omega} (\lambda^{+n})^P$.

We call a premouse *full* if it satisfies (1) of Definition 32. We will say P is *suitable* if there is $\alpha < \omega^2$ such that P is α-suitable and define $\alpha(P) = \alpha$. It is an easy consequence of mouse capturing and the definition of suitability that if P is suitable, ξ a P-cardinal and $A \subseteq \xi$ is such that A is $\mathrm{OD}_P^{L(\mathbb{R},\mathcal{C})}$, then $A \in P$. On the other hand if $A \in P$, then A is $\mathrm{OD}_P^{L(\mathbb{R},\mathcal{C})}$. We will use this fact repeatedly without explicitly mentioning it.

Definition 33. Let $\mathcal{T}$ be normal tree on a suitable mouse P. We say $\mathcal{T}$ is *guided* if and only if for all limit $\eta < \mathrm{lh}(\mathcal{T})$, we have that $Q([0,\eta)_T, \mathcal{T} \restriction \eta)$ exists and is an initial segment of $Lp(\mathcal{M}(\mathcal{T} \restriction \eta))$. We say that $\mathcal{T}$ is *maximal* if $Lp(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin; otherwise we say $\mathcal{T}$ is *short*.

Notice that if $\mathcal{T}$ has successor length, then it is short. Moreover by the way Σ is defined if $\mathcal{T}$ is short and guided then it is according to Σ.[4]

Definition 34 *(Capturing).* Consider an α-suitable premouse P and $A \subseteq \mathbb{R}$. Let η be a cardinal of P. We say that P *captures* A at η if there is a $\mathrm{col}(\omega, \eta)$ name τ, such that whenever g is P-generic on $\mathrm{col}(\omega, \eta)$, we have $\tau[g] \cap \mathbb{R} = A \cap \mathbb{R}$. We say that P captures A if for every $i < \alpha(P)$, P captures A at δ_i^P.

Note that given $A \subset \mathbb{R}$ and a suitable P that captures A at δ_i^P, say via τ, there is a standard term that witnesses the capturing, following [13], we give its definition

$$\tau_{A,i}^P = \{(p, \sigma) \mid \sigma \text{ is a name for a real and } p \Vdash^{\mathrm{col}(\omega,\delta_i^P)} \sigma \in \tau\}.$$

Our next step is to define a notion of iterability that is strong enough so that one can compare suitable mice. Note the connection with [13], where the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ used a system of suitable mice with only finitely many Woodin cardinals. In our situation, however, suitable mice are allowed to have fewer than ω^2 many Woodin cardinals. That is why we need a stronger form of iterability that we describe below.

[4] This is via a standard argument using the uniqueness of Q-structures, cf. [20].

We will define a slight modification of Definition 1.8 from [12]. A suitable P is said to be *weakly* (ω, ω^2)-(quasi-)iterable* if player II has a winning (quasi)-strategy for the game $\mathcal{WG}^*(P, \omega^2)$ in which I and II alternate moves for ω^2 many rounds as follows. The game starts by letting $P_0 = P$. At round α, player I plays a countable normal, guided, putative iteration tree $\mathcal{T}_\alpha$ on P_α. At that point player II has two options. The first option is only available if $\mathcal{T}_\alpha$ has a well-founded final model; then II may accept I's move in which case we set $P_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}_{lh(\mathcal{T}_\alpha)-1}$. The second option is for player II to play a maximal well-founded branch b_α on $\mathcal{T}_\alpha$, such that, if $\mathcal{T}_\alpha$ is short then $Q(b_\alpha, \mathcal{T}_\alpha)$ exists and is an initial segment of $Lp(\mathcal{M}(\mathcal{T}_\alpha))$. The game continues by setting $P_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}_{b_\alpha}$. There are additional requirements for both players at limit rounds. Namely, if I and II have played for all $\beta < \gamma$ and γ is a limit ordinal then:

- If there is $i < \alpha(P)$ such that for infinitely many $\beta < \gamma$, we have that $\mathcal{T}_\beta$ is a tree based on $(\delta^{P_\beta}_{i-1}, \delta^{P_\beta}_i)$ then I loses.[5]
- The direct limit of P_β for $\beta < \gamma$ is well-founded. Otherwise II loses.

After the ω^2 rounds have been played, the only condition for II is that the direct limit along the main branch is well-founded. We illustrate the weak* game, $\mathcal{WG}^*(P, \omega^2)$ game as follows:

Player	0		1	...		ω	...
I	$\mathcal{T}_0$ on P_0		$\mathcal{T}_1$ on $\mathcal{M}^{\mathcal{T}_0}_{b_0}$		$\}P_\omega$	$\mathcal{T}_\omega$	
II		b_0		b_1			b_ω

Note that if P and Q are suitable premice such that II has a winning quasi-strategies τ_P and τ_Q for $\mathcal{WG}^*(P, \omega^2)$ and $\mathcal{WG}^*(Q, \omega^2)$ respectively then one can form guided iteration trees $\mathcal{T}_P$ and $\mathcal{T}_Q$ using the extenders that cause the "least" disagreement, and using τ_P and τ_Q when a maximal tree arises in this comparison. Since each P and Q have $< \omega^2$-many Woodin cardinals, this comparison succeeds. Note also that the end model of this comparison is still weakly* (ω, ω^2)-quasi iterable, since any game on it can be seen as the terminal part of a game on either P or Q.

Recall that a *stack* $\vec{\mathcal{T}}$ on a premouse P is a pair consisting of a sequence of iteration trees $\langle T_i \,|\, i < \gamma \rangle$ and a sequence of premice $\langle P_i \,|\, i \leq \gamma \rangle$ such that

- $P_0 = P$,
- for every $i < \gamma$, $\mathcal{T}_i$ is an iteration tree of successor length on P_i and with last model P_{i+1}, and
- for every limit ordinal $\beta < \gamma$, P_β is the direct limit of $\langle P_i \,|\, i < \beta \rangle$ and the tree embeddings.

[5] Here by convention $\delta^P_{-1} = 0$.

Note that $\langle P_i \,|\, i \leq \gamma \rangle$ is determined by the sequence $\langle \mathcal{T}_i \,|\, i < \gamma \rangle$. Also for a stack $\vec{\mathcal{T}}$ on P, we define $\mathcal{M}^{\vec{\mathcal{T}}}_\infty = P_\gamma$ and $i^{\vec{\mathcal{T}}}_\infty : P \to \mathcal{M}^{\vec{\mathcal{T}}}_\infty$, the natural embedding associated to this stack (if it exists). Notice that in $\mathcal{WG}^*(P, \omega^2)$ players I and II collaborate to form a stack on P.

If P is a suitable mouse capturing some $A \subseteq \mathbb{R}$ it will be desirable that "good" iterations of P maintain the suitability condition and move the terms capturing A correctly.

Definition 35 *(A-iterations).* For a suitable P that captures a set of reals A we define the following.

(1) We say P is A-iterable if II has a winning quasi strategy for the game $\mathcal{WG}^*(P, \omega^2)$ such that whenever $\vec{\mathcal{T}}$ is a stack given by a game according to the quasi strategy[6] and $i^{\vec{\mathcal{T}}}_\infty : P \to \mathcal{M}^{\vec{\mathcal{T}}}_\infty$ exists, then $\mathcal{M}^{\vec{\mathcal{T}}}_\infty$ is suitable and for any $i < \alpha(P)$ we have that $i^{\vec{\mathcal{T}}}_\infty(\tau^P_{A,i}) = \tau^{\mathcal{M}^{\vec{\mathcal{T}}}_\infty}_{A,i}$. We will call such a quasi strategy an A-*quasi strategy* for P.
(2) An A-*iteration* of P is a stack $\vec{\mathcal{T}}$ on P given by a run in $\mathcal{WG}^*(P, \omega^2)$ according to an A-quasi strategy.
(3) We will say Q is an A-*iterate* of P if there is an A-iteration $\vec{\mathcal{T}}$ on P such that $Q = \mathcal{M}^{\vec{\mathcal{T}}}_\infty$ and $i^{\vec{\mathcal{T}}}_\infty$ exists.
(4) For $\mathcal{T}$, a normal guided tree on P of successor length $\eta + 1$, such that $\mathcal{T} \restriction \eta$ is maximal, we let $\mathcal{T}^- = \mathcal{T} \restriction \eta$. In other words $\mathcal{T}^-$ is $\mathcal{T}$ without the last branch.

Also, given $\vec{A}$, a finite sequence of sets of reals, we say P is $\vec{A}$-*iterable* in case there exists a winning quasi strategy in $\mathcal{WG}^*(P, \omega^2)$ that simultaneously witnesses A-iterability for every A in the sequence $\vec{A}$.

Remark. For $A \in L(\mathbb{R}, \mathcal{C})$, being A-iterable is (downward) absolute to $L(\mathbb{R}, \mathcal{C})$. This is because the existence of an A-quasi-strategy is absolute between V and $L(\mathbb{R}, \mathcal{C})$; the latter can figure out Q-structures of a relevant tree $\mathcal{T}$ if $\mathcal{T}$ is short and if $\mathcal{T}$ is maximal, there will be a branch b of $\mathcal{T}$ respecting A if such a b exists in V, using DC in $L(\mathbb{R}, \mathcal{C})$ one gets the desired quasi-strategy. From now on, we will simply write "A-iterable", instead of "A-iterable in $L(\mathbb{R}, \mathcal{C})$".

So far we do not know whether there are A-iterable suitable mice but we prove next that these exist when $\mathcal{M}^\sharp_{\omega^2}$ is present. Lemma 36 and Corollary 37 are adaptations of results in Chapter 3 of [13].

[6] Here, at each normal component $\mathcal{U}$ of $\vec{\mathcal{T}}$, there may be more than one branch choice of $\mathcal{U}$ according to the quasi strategy, if $\mathcal{U}$ is maximal; our convention is when we talk about iteration maps $i^{\vec{\mathcal{T}}}_\infty$, we already make a unique branch choice at each such $\mathcal{U}$.

Lemma 36. *Suppose $A \subseteq \mathbb{R}$ is definable in $L(\mathbb{R}, \mathcal{C})$ from indiscernibles and assume that $\tilde{N}$ is a Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$, such that $i : \mathcal{M}^{\sharp}_{\omega^2} \to \tilde{N}$ (given by Σ) exists. Then any suitable initial segment of $\tilde{N}$ is A-iterable.*

The idea in the proof is the following. Note that if $\tilde{N}$ is a Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$, by Lemma 23, given a Woodin cardinal δ of $\tilde{N}$, we can iterate $\tilde{N} \xrightarrow{\Sigma} K$ above δ to make $L(\mathbb{R}, \mathcal{C})$ realizable as the derived model associated to K and some K-generic filter. Hence, one can define truth in $L(\mathbb{R}, \mathcal{C})$ in K using the homogeneity of the collapse. We show the details below.

Proof. Let $\tilde{N}$ as above and suppose that $A \subseteq \mathbb{R}$ is definable in $L(\mathbb{R}, \mathcal{C})$ from indiscernibles. Let φ be a formula such that for any increasing sequence of indiscernibles $c_0 < c_1 < \cdots < c_n$ for $L(\mathbb{R}, \mathcal{C})$

$$x \in A \Leftrightarrow L(\mathbb{R}, \mathcal{C}) \models \varphi(c_0, \ldots, c_{n-1}, x).$$

Let Q be a suitable initial segment of $\tilde{N}$, and $\alpha = \alpha(Q)$. Let us define N to be the proper class model resulting when iterating the last extender of $\tilde{N}$ ON-many times. Let $\beta < \alpha$ and define τ as $(p, x) \in \tau$ if

$$p \Vdash_N^{\mathrm{col}(\omega, <\delta^N_\beta)} \Vdash_N^{\mathrm{col}(\omega, <\lambda^N_{\omega^2})} L(\dot{\mathbb{R}}, \dot{\mathcal{F}}) \models \varphi(\check{c_0}, \check{c}_1, \ldots, \check{c}_{n-1}, x),$$

where $\dot{\mathbb{R}}$ is the standard name for the symmetric reals under $\mathrm{col}(\omega, <\lambda^N_{\omega^2})$ and $\dot{\mathcal{F}}$ is the name for the tail filter associated to this forcing as defined in Section 2. Now by suitability of Q we have that $\tau \in Q$.

Let us see first that τ captures A at δ^Q_β. For this let G be Q-generic for $\mathrm{col}(\omega, \delta^Q_\beta)$. Note that by suitability G is also N-generic. Now, working in W, we can use Σ to iterate N, above δ^Q_β, in the fashion of Lemma 23 to get an embedding $j : N \to M$ and an $M[G]$-generic filter H such that $G * H$ is M-generic for $\mathrm{col}(\omega, \lambda^M_{\omega^2})$, and $j(\dot{\mathcal{F}})[G * H] = \mathcal{C}$ and $j(\dot{\mathbb{R}})[G * H] = \mathbb{R}$.

Also, we can pick indiscernibles large enough so that they are fixed by j. This implies that if $(p, x) \in \tau$ and $p \in G$, then

$$M[G * H] \models L(\mathbb{R}, \mathcal{C}) \models \varphi(c_0, \ldots, c_{n-1}, x[G])$$

In other words if $x[G] \in \tau[G] \Rightarrow x[G] \in A$. Conversely if $x \in A \cap Q[G]$, then by homogeneity of the second forcing over N we have that

$$\Vdash_N^{\mathrm{col}(\omega, <\lambda^N_{\omega^2})} L(\dot{\mathbb{R}}, \dot{\mathcal{F}}) \models \varphi(\check{c_0}, \check{c}_1, \ldots, \check{c}_{n-1}, (x\check{[G]})),$$

so there is a condition $p \in G$ such that

$$p \Vdash_N^{\mathrm{col}(\omega, <\delta^N_\beta)} \Vdash_N^{\mathrm{col}(\omega, <\lambda^N_{\omega^2})} L(\dot{\mathbb{R}}, \dot{\mathcal{F}}) \models \varphi(\check{c_0}, \check{c}_1, \ldots, \check{c}_{n-1}, x).$$

In other words $A \cap Q[G] \subseteq \tau[G]$, and so $\tau[G] = A \cap Q[G]$.

By the way τ is defined it is easy to see that Σ moves τ correctly and so it is an A-iteration strategy. □

Corollary 37. *Suppose A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals. Then for every $\alpha < \omega^2$ there is an α-suitable P that is A-iterable.*

Proof. By contradiction, suppose that there is a counterexample A. By minimizing the ordinals from which A is defined we may assume that A is actually definable in $L(\mathbb{R}, \mathcal{C})$. Let $\alpha < \omega^2$ and Q is the suitable initial segment of $\mathcal{M}^\sharp_{\omega^2}$ with $\alpha(Q) = \alpha$. By Lemma 36 Q is A-iterable, a contradiction. □

The point in the proof of Corollary 37 is that given a counterexample in $L(\mathbb{R}, \mathcal{C})$ to a statement of the form "for all $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ sets of reals" then one can, by minimizing the counterexample, find a definable one. However it is the case that, usually, the suitable initial segments of $\mathcal{M}^\sharp_{\omega^2}$ witness that there are no definable counterexamples. The same argument essentially gives the following lemma.

Lemma 38 *(Comparison). Suppose that P is a suitable, A-iterable mouse and Q is a suitable, B-iterable mouse. Then there is an $A \oplus B$-iterable suitable mouse R, an A-iteration from P to a suitable initial segment of R and a B iteration from Q to suitable initial segment of R.*

Recall that the notion of A-iteration is definable in $L(\mathbb{R}, \mathcal{C})$. The next step is to define a covering system using pairs (P, A), where A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and P is an A-iterable mouse. However, it could be the case that for such a P, there are two different A-iterations $\pi : P \to Q$ and $\sigma : P \to Q$, and this would be a clear problem in building a directed limit. For this reason we need to work with relevant hulls and a stronger notion of iterability. We define below these concepts.

Definition 39. For an A-iterable mouse P, we let

(1) $P^- = P|(\delta_0^{+\omega})^P$
(2) $\gamma^P_{A,i} = \sup(Hull^P(\tau^P_{A,i}) \cap \delta^P_0)$.
(3) $\gamma^P_A = \sup_{i \in \alpha(P)} \gamma^P_{A,i}$.
(4) $\xi^P_A = \gamma^{P^-}_A$.
(5) $H(P, A) = Hull^P(\xi^P_A \cup \{\tau^P_{A,i} \mid i < \alpha(P)\})$

Note that if P is a suitable A-iterable mouse, then $P^- = P|(\delta_0^{+\omega})^P$ is 1-suitable and A-iterable.

Using the usual "zipper argument" (see [17] or [15]) we get the following lemma.

Lemma 40. *Let $\mathcal{T}$ be a tree of limit length on P, a suitable pre-mouse. Suppose further that there are branches b and c such that $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are A-iterations and $\mathcal{M}_b^\mathcal{T}$ and $\mathcal{M}_c^\mathcal{T}$ are A-iterable. Then $i_b^\mathcal{T} \restriction \gamma_A^P = i_c^\mathcal{T} \restriction \gamma_A^P$ and so $i_b^\mathcal{T} \restriction H(P,A) = i_c^\mathcal{T} \restriction H(P,A)$.*

A delicate point here is that if P is an A-iterable mouse we could potentially have two A-iterations associated to two different trees on P leading to the same end model Q, so Lemma 40 would not apply. Hence we define the notion of strong iterability in the natural way and prove the existence of strongly iterable mice.

Definition 41. For $A \subseteq \mathbb{R}$ and a suitable A-iterable mouse P, we say P is *strongly A-iterable*, if whenever $i : P \to Q$ and $j : P \to Q$ are two A-iterations, then $i \restriction H(P,A) = j \restriction H(P,A)$.

Note again that when proving that for any $A \subseteq \mathbb{R}$ which is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ there is a strongly A-iterable mouse it is sufficient to prove that for any definable set A there is a strongly A-iterable mouse. The following lemma, in contrast to most of what we have discussed so far, is not an "easy" generalization of the HOD Analysis in $L(\mathbb{R})$. The reason for this is the extra complexity in the iteration games considered. We give a detailed proof for the existence of strongly A-iterable mice.

Lemma 42. *Let A be an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and let P be A-iterable. Then there is an A-iterate of P that is strongly A-iterable.*

Proof. By minimizing the ordinals from which a potential counterexample can be defined, we may without loss of generality, assume that A is definable in $L(\mathbb{R},\mathcal{C})$. Given an A-iterable mouse P, by comparison we can A-iterate P to Q, an initial segment of a correct iterate of $\mathcal{M}_{\omega^2}^\sharp$. We claim that Q is as wanted.

Suppose that $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are A-iteration stacks on Q with the same last model R. We want to show that the embeddings given by $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ agree on $H(Q,A)$. We will actually show that both embeddings agree with embeddings given by Σ on $H(Q,A)$. Here we have to be an extra bit more careful than in the analogous situation of $L(\mathbb{R})$, because our iteration games can have more rounds and at limit stages it is not straightforward how to proceed, we will show next the details of how to overcome this difficulty.

We look inductively at the trees in the stack $\vec{\mathcal{T}} = \langle \mathcal{T}_i \mid i \in \alpha \rangle$. Let Q_i (for $i \in \alpha$) be the model starting round i in the weak* game. We will construct trees $\mathcal{S}_i$ inductively such that $\vec{\mathcal{S}} = \langle \mathcal{S} \mid i \in \alpha \rangle$ is according to Σ and has the property that the embedding given by $\vec{\mathcal{S}}$ agrees with $i^{\vec{\mathcal{T}}}$ on ξ_A^Q. Since there are no extenders in Q overlapping a Woodin cardinal, and the trees $\mathcal{T}_i$ are normal we can split each $\mathcal{T}_i$ into a $< \omega^2$-sequence of trees, each of whom is based on a window of the form $(\delta_{k_i}, \delta_{k_i+1})$. Hence, we will assume with no loss of generality that every tree $\mathcal{T}_i$ for $i \in \alpha$ is based on a window of the form $(\delta_{k_i}^{Q_i}, \delta_{k_i+1}^{Q_i})$.

Start with $\mathcal{T}_0$. Let us define $\mathcal{S}_0$ as follows. First suppose that $\mathcal{T}_0$ is based on ${Q_0}^-$. If it is according to Σ we let $\mathcal{S}_0 = \mathcal{T}_0$. Otherwise, if $\mathcal{T}_0$ is not according Σ, then since it is guided it must be a maximal tree with a last branch b (recall that all short and guided trees are according to Σ). Recall that $\mathcal{T}_0^-$ denotes the maximal part of $\mathcal{T}_0$. Let c be the branch given by Σ through $\mathcal{T}_0^-$ and, note that, by the proof of Lemma 36, c respects A. Let $\mathcal{S}_0$ be $\mathcal{T}_0^-{}^\frown c$. Also, by Lemma 40, we have that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ_A^Q. Recall that Q_1 is the last model of $\mathcal{T}_0$, and let $\bar{Q}_1$ be the last model of $\mathcal{S}_0$. By fullness of $\bar{Q}_1$ and Q_1, we have that $Lp(\mathcal{M}(\mathcal{T}_0^-))$ is an initial segment of both Q_1 and $\bar{Q}_1$, so, by and maximality of $\mathcal{T}_0^-$ we have that $\delta(\mathcal{T}_0^-)$ is the first Woodin cardinal of Q_1 and $\bar{Q}_1$, this implies, by suitability, that $Q_1^- = \bar{Q}_1^-$.

If $\mathcal{T}_0$ based on a window above δ_0 then let $\mathcal{S}_0 = \emptyset$, $i^{\mathcal{S}_0} = \mathrm{id}$ and $\bar{Q}_1 = Q_1$. Here we get also, trivially, that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ_A^Q and $Q_1^- = \bar{Q}_1^-$.

Let us consider then $\mathcal{T}_1$. If it is based on ${Q_1}^-$ we can regard it as a tree on $\bar{Q}_1$ and then we can again use Σ to get $\mathcal{S}_1$ on $\bar{Q}_1$ such that $i^{\mathcal{T}_1}$ and $i^{\mathcal{S}_1}$ agree on $\xi_A^{Q_1} = \xi_A^{\bar{Q}_1}$. Again, by fullness we get that if $\bar{Q}_2$ is the last model of $\mathcal{S}_1$, then $Q_2^- = \bar{Q}_2^-$.

Otherwise we just let $\mathcal{S}_1 = \emptyset$ and the desired agreement is maintained so far.

Note that by the rules of the weak* game one has that $\mathcal{T}_i$ can be based on ${Q_i}^-$ only for finitely many $i \in \omega$. Hence $\bar{Q}_\omega$ agrees with Q_ω up to their common 1-suitable initial segment and the embedding on the $\mathcal{T}$-side agrees with the one given by the $\mathcal{S}$-side up to ξ_A^Q.

We proceed inductively in this fashion. At successors simply use Σ if the tree is based below the least Woodin cardinal, and otherwise define the corresponding tree in the $\mathcal{S}$-side as empty.

After α-many steps in this induction we will have that $\bar{Q}_\alpha$ is a Σ-iterate of Q. Let $\bar{\sigma}$ be the branch embedding. Then we have that $\bar{Q}_\alpha$ agrees with $Q_\alpha = R$ up to their common 1-suitable initial segment, and that $\bar{\sigma} \restriction \xi_A^Q = i^{\vec{\mathcal{T}}} \restriction \xi_A^Q$.

Similarly for $\vec{\mathcal{U}}$ one can get the analogous construction. So, we get that $i^{\vec{\mathcal{U}}}$ agrees with $\sigma' : Q \to Q'_\alpha$, an embedding given by Σ, on ξ_a^Q. Furthermore R, $\bar{Q}_\alpha$ and Q'_α agree up to their 1-suitable initial segment, and so since δ_0^R is a cut-point of both $\bar{Q}_\alpha$ and Q'_α by the Dodd Jensen property of Σ we can conclude that $\bar{\sigma}$ and σ' agree up to $\delta_0^{Q_0}$, and so $i^{\vec{\mathcal{T}}}$ and $i^{\vec{\mathcal{U}}}$ agree up to ξ_A^Q. Hence $i^{\vec{\mathcal{T}}} \restriction H(Q, A) = i^{\vec{\mathcal{U}}} \restriction H(Q, A)$ as wanted. □

Our covering system in $L(\mathbb{R}, \mathcal{C})$ will be

$$\mathcal{D} = \{H(P, \vec{A}) \,|\, A \subseteq \mathbb{R},\ P \text{ is strongly } \vec{A}\text{-iterable, and } \vec{A} \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}\}.$$

Also we let $(P, \vec{A}) \preceq (Q, \vec{B})$ if Q is an A-iterate of P and $\vec{A} \subseteq \vec{B}$. We let $\sigma_{(P,\vec{A}),(Q,\vec{B})}$ be the unique embedding from $H(P, \vec{A})$ to $H(Q, \vec{B})$ given by an (any) $\vec{A}$-iteration from P to Q. The following results show that the suitable initial segments of correct iterates of $\mathcal{M}^\sharp_{\omega^2}$ together with the theories of indiscernibles for $L(\mathbb{R}, \mathcal{C})$ are in some sense "dense" in $\mathcal{D}$.

Let

$$M_\infty = \lim(\mathcal{D}, \preceq, \sigma_{(P,A),(Q,B)})$$

and let us define $\sigma_{(P,A),\infty}$ the natural embedding from $H(P,A)$ to this direct limit.[7]

Let $\mathcal{T}_n^{\mathcal{C}}$ be the theory of n-many indiscernibles with real parameters of $L(\mathbb{R},\mathcal{C})$ coded as a subset of $\mathbb{R}$. Lemma 5.6 and Lemma 5.9 in [13] give the following results, we omit their proofs as they are word by word the same, except that we use $\mathcal{M}^\sharp_{\omega^2}$ and $L(\mathbb{R},\mathcal{C})$ instead of $\mathcal{M}^\sharp_\omega$ and $L(\mathbb{R})$ (the key, again, is that one can realize $L(\mathbb{R},\mathcal{C})$ as the derived model of an iterate of $\mathcal{M}^\sharp_{\omega^2}$).

Lemma 43. *Suppose that P is a suitable initial segment of a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, then $\delta_0^P = \sup\{\xi^P_{\mathcal{T}_n^{\mathcal{C}}} \mid n \in \omega\}$.*

Lemma 44. *Assume A is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$, and P is a strongly A-iterable suitable mouse. Then there is R, a suitable initial segment of a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, and natural number n such that $(P,A) \preceq (R, A \oplus \mathcal{T}_n^{\mathcal{C}})$, and moreover*

$$H(R, A \oplus \mathcal{T}_n^{\mathcal{C}}) = H(R, \mathcal{T}_n^{\mathcal{C}}).$$

Let us pause for a moment and discuss the general $L(\mathbb{R},\mu)$ case. The lemma above will also be valid in this context by an application of Σ_1-reflection.

Lemma 45. *Suppose $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, and let A be $\mathrm{OD}^{L(\mathbb{R},\mu)}$. Given P a strongly A-iterable suitable mouse and B an $\mathrm{OD}^{L(\mathbb{R},\mu)}$ set of reals, with $A \leq_W B$, there is R a suitable and $A \oplus B$-iterable mouse, such that $(P,A) \preceq (R, A \oplus B)$ and moreover $H(R, A \oplus B) = H(R,B)$.*

Proof. Otherwise fix A and B a counterexample to the statement. Fix γ large enough such that $L_\gamma(\mathbb{R},\mu) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ and A and B are ordinal definable over $L_\gamma(\mathbb{R},\mu)$, but $L_\gamma(\mathbb{R},\mu)$ has no R and A-iteration of P witnessing the conclusion of the Lemma. This Σ_1 statement about γ can then be reflected below $\boldsymbol{\delta}^2_1$. Hence there is such a $\gamma < \boldsymbol{\delta}^2_1$. But then $L_\gamma(\mathbb{R},\mathcal{C}) = L_\gamma(\mathbb{R},\mu)$ since below $\boldsymbol{\delta}^2_1$ both μ and $\mathcal{C}$ are just the club filter (by results of Woodin, but see [21]). We get then that there are A and B counterexamples of the statement in $L_\gamma(\mathbb{R},\mathcal{C})$ (and moreover OD in this structure). But then we can get the desired R and A-iteration in $L(\mathbb{R},\mathcal{C})$ and, by the closure of γ, an A-iteration of P leading to R can be computed in $L_\gamma(\mathbb{R},\mu)$, so A and B cannot be the counterexample of $L_\gamma(\mathbb{R},\mathcal{C})$, contradiction. □

Lemmas 44 and 45 allow us to compute the direct limit of $\mathcal{D}$ just by looking at suitable initial segments of Σ-iterates of $\mathcal{M}^\sharp_{\omega^2}$ with corresponding theories of indiscernibles. We have the following agreement.

Theorem 46. $M_\infty = M^+_\infty | \lambda^{M^+_\infty}_{\omega^2}$.

[7] Here we identify M_∞ with its transitive collapse.

Proof. We define a map $i : M_\infty \to M_\infty^+$ that is surjective below $\lambda_{\omega^2}^{M_\infty^+}$ and respects the membership relation as follows. For $x \in M_\infty$ there is a natural n and a suitable initial segment of a correct iterate of $\mathcal{M}_{\omega^2}^\sharp$, say P, such that x is in the range of $\sigma_{(P,A\oplus\mathcal{T}_n^{\mathcal{C}}),\infty}$, and there is $z \in H(P, \mathcal{T}_n^{\mathcal{C}})$ such that $\sigma_{(P,A\oplus\mathcal{T}_n^{\mathcal{C}}),\infty}(z) = x$. Now we have an iteration $\mathcal{M}_{\omega^2}^\sharp \xrightarrow{\Sigma} \mathcal{N}$ such that P is a suitable initial segment of $\mathcal{N}$. Note that $\mathcal{N}$ might not be a δ_0-bounded iterate of $\mathcal{M}_{\omega^2}^\sharp$. We can however split the iteration from $\mathcal{M}_{\omega^2}^\sharp$ to $\mathcal{N}$ in a δ_0-bounded part and the rest. Namely, there is $\mathcal{N}^*$ such that $\mathcal{M}_{\omega^2}^\sharp \xrightarrow{\Sigma} \mathcal{N}^*$ into a δ_0-bounded way, and $\mathcal{N}^* \xrightarrow{\Sigma} \mathcal{N}$. Note that this second iteration does not move $\delta_0^{\mathcal{N}^*}$ (because all its extenders have critical point above the first Woodin cardinal). This implies that z is in the range of $\pi_{\mathcal{N}^*,\mathcal{N}}$, let $\bar{z}$ be its pre-image. Then we define $i(x) = \pi_{\mathcal{N}^*,\infty}(\bar{z})$, it is routine to show that i is well defined (see Theorem 5.10 of [13]). Now Lemma 44 gives us the surjectivity as follows: Let $x \in M_\infty^+|\lambda_{\omega^2}^{M_\infty^+}$ so there is $z \in \mathcal{N}$ a correct iterate of $\mathcal{M}_{\omega^2}^\sharp$ such that $\pi_{\mathcal{N},\infty}(z) = x$. Let P be a suitable initial segment of $\mathcal{N}$ such that $z \in P$. Because $\mathcal{N}$ is a δ_0 bounded iterate of $\mathcal{M}_{\omega^2}^\sharp$ we have that z is definable from ordinals less than $\delta_0^{\mathcal{N}}$ and indiscernibles, but this is easily computable from $\mathcal{T}_n^{\mathcal{C}}$ for a suitable n (again this follows essentially by Corollary 5.7 of [13]). Because $\xi_{\mathcal{T}_n^{\mathcal{C}}}^P$ is unbounded in $\delta_0^{\mathcal{N}}$ we conclude that $z \in H(P, \mathcal{T}_n^{\mathcal{C}})$ for a sufficiently large n. This readily implies x is in the range of i as wanted. □

Recall that W is the outer model in which Σ has an extension and $\mathcal{P}(\mathbb{R})$ is countable. Hence, it is forced (over V) that in $V^{\mathrm{col}(\omega,\mathbb{R})}$ Σ has an extension. Let us work for a moment in $V^{\mathrm{col}(\omega,\mathbb{R})}$. Here we have that M_∞^+ is a countable Σ-iterate of $\mathcal{M}_{\omega^2}$. Also if G is M_∞^+-generic for $\mathrm{col}(\omega, < \lambda_{\omega^2}^{M_\infty})$, and $\mathbb{R}^*$ and $\mathcal{F}$ are the symmetric reals and associated tail filter, then, by Theorem 21, $L(\mathbb{R}^*, \mathcal{F})$ is model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Following the notation and the content of Chapter 6 from [13], if A is in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ we can define, $A^* \subseteq \mathbb{R}^*$, by pieces as follows. For (P, A), an element of $\mathcal{D}$, and for $i < o(P)$ let

$$\tau_{A,i}^* = \sigma_{(P,A),\infty}(\tau_{A,i}^P)$$

and

$$A^* = \bigcup_{i\in\omega^2} \tau_{A,i}^*[G \restriction \delta_i^{M_\infty}].$$

Recall that $\mathcal{T}_n^{\mathcal{C}}$ is the theory of n-many indiscernibles with real parameters. We will be in particular interested in $\mathcal{T}_n^{\mathcal{C}*}$. Note that any suitable initial segment of M_∞ is strongly $\mathcal{T}_n^{\mathcal{C}*}$-iterable (in $V^{\mathrm{col}(\omega,\mathbb{R})}$ as witnessed by Σ and in $L(\mathbb{R}^*, \mathcal{F})$ by absoluteness). Recall that M_∞^- is the 1-suitable initial segment of M_∞. We summarize the relevant facts of these sets in Lemmas 47 and 48.

Lemma 47. *For any set of reals A which is OD in $L(\mathbb{R}, \mathcal{C})$ we have that A^* is OD in $L(\mathbb{R}^*, \mathcal{F})$. Moreover for any such A,*

$$L(\mathbb{R}^*, \mathcal{F}) \models M_\infty^- \text{ is strongly } A^*\text{-iterable.}$$

Proof. This follows exactly as in the case of $L(\mathbb{R})$ so we omit details. These proofs can be found essentially in Chapter 6: Claims 1, 2 and 3 of [13]. □

Let X be the set of finite full stacks on M_∞^- in $M_\infty|(\lambda_{\omega^2}^{M_\infty})$. As in [13], when computing the branches that Σ would have picked through $\vec{\mathcal{T}} \in X$ it is enough to choose the unique branch that moves terms for A^* (such that $A \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$) correctly. That is to say

Lemma 48. *Suppose $\mathcal{T} \in L(\mathbb{R}^*, \mathcal{F})$ is a guided maximal tree on M_∞^- in $L(\mathbb{R}^*, \mathcal{F})$. Then $\Sigma(\mathcal{T}) = b$ if and only if $\mathcal{T}^\frown b$ is an A^*-iteration for all A in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$.*

Proof. This is claim 4 of Chapter 6 in [13]. Here we use Lemma 44 instead of Lemma 5.8 of [13], everything else follows word by word. □

From this and the homogeneity of the collapse it follows that $L[M_\infty, \Sigma \restriction X] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$. Also, note that if M_∞^* is the direct limit defined in $L(\mathbb{R}^*, \mathcal{F})$ of the system $\{(P, \vec{A}^*) \,|\, P \text{ is strongly } \vec{A}^*\text{-iterable and } \vec{A} \in OD^{L(\mathbb{R},\mathcal{C})}\}$, then there is an embedding $\sigma : M_\infty^- \to M_\infty^*$, where $\sigma = \bigcup_{A \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}} \sigma_{(M_\infty^-, A^*), \infty}$.

Lemma 49. *Suppose that N_0 and N_1 are countable Σ-iterates of $\mathcal{M}_{\omega^2}^\sharp$, G_i is N_i-generic for $col(\omega, \lambda_{\omega^2}^{N_i})$ and $L(\mathbb{R}_i, \mathcal{F}_i)$ are the associated derived models.*

Then given $x \in \mathbb{R}_0 \cap \mathbb{R}_1$ we have

$$\langle L(\mathbb{R}_0, \mathcal{F}_0), x, \mathcal{T}_n^0 \rangle \equiv \langle L(\mathbb{R}_1, \mathcal{F}_1), x, \mathcal{T}_n^1 \rangle,$$

where $\mathcal{T}_n^i$ is the theory of n indiscernibles for $L(\mathbb{R}_i, \mathcal{F}_i)$.

Proof. Fix x as in the hypotheses. Then there exist $k < \omega^2$ such that for $i = 0, 1$ we have $x \in \mathbb{R}^{N_i[G_i \restriction \delta_k^{N_i}]}$. Fix $c_0 < c_1 < \cdots < c_{n-1}$ indiscernibles for $L(\mathbb{R}_0, \mathcal{F}_0)$, $L(\mathbb{R}_1, \mathcal{F}_1)$ and $L(\mathbb{R}, \mathcal{C})$. Let φ be a formula and assume $L(\mathbb{R}_0, \mathcal{F}_0) \models \varphi(x, c_0, \ldots, c_{n-1})$. We will see that $L(\mathbb{R}, \mathcal{C})$ satisfies the same formula. By homogeneity of the collapse we have that

$$\Vdash_{N_0[G \restriction \delta_k^{N_0}]}^{\mathrm{col}(\omega, <\lambda_{\omega^2}^{N_0})} L(\dot{\mathbb{R}}, \dot{\mathcal{F}}) \models \varphi(\check{x}, \check{c}_0, \ldots, \check{c}_{n-1})$$

Now in W we can iterate N_0 above $\delta_k^{N_0}$ to realize $L(\mathbb{R}, \mathcal{C})$ as a derived model (see Lemma 36). Picking $c_0, \ldots, c_n$ large enough we get that

$$L(\mathbb{R}, \mathcal{C}) \models \varphi(x, c_0, \ldots, c_{n-1})$$

By symmetry of the argument, we cannot have $L(\mathbb{R}_1, \mathcal{F}_1) \models \neg\varphi(x, c_0, \ldots, c_{n-1})$, which completes the proof. □

Theorem 50. *Suppose $\mathcal{M}^\sharp_{\omega^2}$ exists and its iteration strategy extends to an outer model W in which $P(\mathbb{R})^V$ is countable. Then the following are the same model.*

(1) $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$
(2) $L[M_\infty, \Sigma \restriction X]$
(3) $L[M_\infty, \sigma]$

Proof. This proof follows exactly as Claims 6 and 7 in chapter 6 of [13]. Here instead of Lemma 6.2 of [13] we use Lemma 49 and that by Theorem 3.1 in [22] $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})} = L[B]$ for some $B \subset \Theta^{L(\mathbb{R},\mathcal{C})}$. □

Also for an arbitrary $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact we can define the corresponding internal direct limit system

$$\mathcal{D}_\mu = \{H(P,\vec{A}) \mid A \subseteq \mathbb{R},\ P \text{ is strongly } \vec{A}\text{-iterable, and } \vec{A} \in \mathrm{OD}^{L(\mathbb{R},\mu)}\}.$$

Let $M_{\infty,\mu}$ be its direct limit (see for example Theorem 3.13, and subsequent discussion in [22]). Furthermore by [22] we have the following result.

Theorem 51. *Suppose $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then*

$$\mathrm{HOD}^{L(\mathbb{R},\mu)} = L[M_{\infty,\mu}, \Sigma_\mu]$$

where Σ_μ is defined in $L(\mathbb{R},\mu)$ using the corresponding definition given in Lemma 48.

Note that the construction recovering HOD can be relativized to any particular real y as follows. The existence of $\mathcal{M}^\sharp_{\omega^2}$ implies the existence of $\mathcal{M}^\sharp_{\omega^2}(y)$ and so one has $\mathrm{HOD}_y^{L(\mathbb{R},\mathcal{C})} = L[M_{\infty,\mu}(y), \Sigma_\mu(y)]$, where $M_{\infty,\mu}(y)$ is the direct limit of

$$D_\mu(y) = \{H(P,A) \mid P \text{ is a strongly } A\text{-iterable } y\text{-mouse and } A \in \mathrm{OD}_y^{L(\mathbb{R},\mathcal{C})}\}.$$

And $\Sigma_\mu(y)$ is the strategy whose domain consists of finite full stacks of trees on $M^-_{\infty,\mu}(y)$ that are in

$$M_{\infty,\mu}(y)|(\lambda_{\omega^2}^{M_{\infty,\mu}(y)})$$

and $\Sigma_\mu(y)$ picks branches b such that respect every A^* for $A \in \mathrm{OD}_y^{L(\mathbb{R},\mu)}$. Here we define A^* in $L(\mathbb{R}^*,\mathcal{F})$, the derived model given by a generic filter over $M_{\infty,\mu}$ for the collapse up to the sup of its Woodins. For the Record

Lemma 52. *Suppose $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact and let y be a real. Let $M_{\infty,\mu}(y)$ and $\Sigma_\mu(y)$ be as above. Then*

$$\mathrm{HOD}_y^{L(\mathbb{R},\mu)} = L[M_{\infty,\mu}(y), \Sigma_\mu(y)].$$

3.2. $\mathcal{P}(\mathbb{R})$ in models of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact

We start this subsection by analyzing $\mathcal{P}(\mathbb{R})$ in "minimal" AD models of ω_1 is $\mathbb{R}$-supercompact. The following terminology will get us started.

Definition 53. Given μ, a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$, we use the following notation:

- $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$
- $\boldsymbol{\delta}^2_1(\mu) = \boldsymbol{\delta}^{2\,L(\mathbb{R},\mu)}_1$
- $\Theta(\mu) = \Theta^{L(\mathbb{R},\mu)}$

The following lemma says that the power sets of the reals of such models line up with that of $L(\mathbb{R}, \mathcal{C})$.

Lemma 54. *Suppose that $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ is such that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then, either $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\mathcal{C}(\mathbb{R})$ or $\mathcal{P}_\mathcal{C}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$.*

Proof. Suppose neither $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\mathcal{C}(\mathbb{R})$ nor $\mathcal{P}_\mathcal{C}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$. Let $\Gamma = \mathcal{P}_\mathcal{C}(\mathbb{R}) \cap \mathcal{P}_\mu(\mathbb{R})$. By Theorem 3.7.1 of [23] $L(\mathbb{R}, \Gamma) \models \mathrm{AD}_\mathbb{R}$. Hence by the theorem of Solovay mentioned in the first paragraph of Section 1.3, if ν is the club filter defined in $L(\mathbb{R}, \Gamma)$, then $L(\mathbb{R}, \nu) \models \mathrm{AD} + \nu$ is an $\mathbb{R}$-supercompactness measure. Moreover, we have that ν is a subset of $\mathcal{C}$, so by Lemma 28 we have $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mathcal{C})$, which readily gives a contradiction. □

We will need the notion of the *envelope* of a point-class. For a complete exposition of this subject the reader may consult Chapter 3 of [23]. We will mostly be interested in envelopes of point-classes of the form $\Sigma_1^{Lp(\mathbb{R})|\gamma}$. We recall the definitions below. It should be noted that the definition below is not the original definition of Envelope by D.A. Martin but it is due to Steel. In the context of AD, it is equivalent to the original definition.

Definition 55. Suppose that γ is an admissible ordinal of $Lp(\mathbb{R})$. Let $\Gamma = \Sigma_1^{Lp(\mathbb{R})|\gamma}$. For $A \subseteq \mathbb{R}$

- We say $A \in \mathrm{OD}^{<\gamma}$ if there is $\alpha < \gamma$ such that A is $\mathrm{OD}^{Lp(\mathbb{R})|\alpha}$.
- We say $A \in Env(\Gamma)$ if for every $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ there is $A' \in \mathrm{OD}^{<\gamma}$ such that $A \cap \sigma = A' \cap \sigma$.

We also note that the definition of the envelope can be relativized to any real x. Recall that $\mathbf{Env}(\Gamma)$, the boldface envelope, is $\bigcup_{x \in \mathbb{R}} Env(\Gamma(x))$. The notion of the envelope is particularly useful when analyzing the Σ_1-gaps and the pattern of scales in the structure $Lp(\mathbb{R})$ (see [11], [14] and [9]).

We turn now to prove that for any μ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, we have that $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\mathcal{C}(\mathbb{R})$.

Lemma 56. *Suppose that μ is a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu)$ satisfies $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $L(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\mu)$ have the same sets of reals.*

Proof. For contradiction, suppose that this is not the case. Without loss of generality we may assume that μ and $\mathcal{C}$ measure some subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ differently, as otherwise the lemma would follow trivially. By Lemma 54 we have the following two cases.

Case 1: $\mathcal{P}_\mu(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mathcal{C}(\mathbb{R})$.

In this case without loss of generality, we will assume that μ is such that $\mathcal{P}_\mu(\mathbb{R})$ is minimal. In other words, given any other $\nu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$.

By $(\mathbb{R},\mu)^\sharp$, we mean the theory of the reals and indiscernibles of $L(\mathbb{R},\mu)$ in a language with predicates for membership and μ and constant symbols $\dot{x}$ for each real x. Let B belong to $\mathcal{P}_\mathcal{C}(\mathbb{R})$ but not to $\mathcal{P}_\mu(\mathbb{R})$. Then $(\mathbb{R},\mu)^\sharp = \oplus_{n\in\omega}\mathcal{T}^\mu_n$, where each $\mathcal{T}^\mu_n$ is Wadge reducible to B; here, $\mathcal{T}^\mu_n$ is the theory of the first n indiscernibles with real parameters in $L(\mathbb{R},\mu)$. Since there is a real x that codes all these reductions, $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},\mathcal{C})$. Also, recall that $L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C}) \prec_1 L(\mathbb{R},\mathcal{C})$ (see [21]), hence there is such a sharp in $L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Let $\bar{\mu}$ be such that $(\mathbb{R},\bar{\mu})^\sharp \in L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\bar{\mu}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Claim. *In $L(\mathbb{R},\mathcal{C})$, $\mathcal{M}^\sharp_{\omega^2}$ exists and is $\omega_1 + 1$-iterable.*

Proof of the claim. Let us work in $L(\mathbb{R},\mathcal{C})$. First, by results of [22] we have that $\mathcal{P}_C(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ and $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$. Note that if M is an $\mathbb{R}$-mouse in $L(\mathbb{R},\bar{\mu})$ projecting to $\mathbb{R}$, there is a set of reals in $\mathcal{P}_{\bar{\mu}}(\mathbb{R})$ coding it. Thus $M \in L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Also, if M is countably iterable in $L(\mathbb{R},\bar{\mu})$, by definition, if $\bar{M}$ is a countable hull of M it is (ω_1+1)-iterable in $L(\mathbb{R},\bar{\mu})$. As $\mathbb{R} \subset L(\mathbb{R},\mathcal{C})$ any such $\bar{M}$ is ω_1-iterable in $L(\mathbb{R},\mathcal{C})$. But ω_1 is measurable in $L(\mathbb{R},\mathcal{C})$ hence $\bar{M}$ is (ω_1+1)-iterable in $L(\mathbb{R},\mathcal{C})$. So by definition of $Lp(\mathbb{R})$, M is an initial segment of $Lp(\mathbb{R})$ in $L(\mathbb{R},\mathcal{C})$. This gives us that:

$$Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})} \lhd (Lp(\mathbb{R})|\boldsymbol{\delta}^2_1(\mathcal{C}))^{L(\mathbb{R},\mathcal{C})}.$$

This implies that $\boldsymbol{\delta}^2_1(\bar{\mu})$ starts a Σ_1-gap in $Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ (and not the last gap).[8] This is because "starting a Σ_1-gap" is downward absolute and also $\boldsymbol{\delta}^2_1(\mathcal{C})$ starts the last gap of $(Lp(\mathbb{R})|\boldsymbol{\delta}^2_1(\mathcal{C}))^{L(\mathbb{R},\mathcal{C})}$. Let

$$\Gamma = \Sigma_1^{Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}}.$$

We claim that $\mathbf{Env}(\Gamma) = P_{\bar{\mu}}(\mathbb{R})$, where the envelope is as defined in $L(\mathbb{R},\mathcal{C})$.

[8] $[\alpha,\beta]$ is a Σ_1-gap in $Lp(\mathbb{R})$ if $Lp(\mathbb{R})|\alpha \prec_1 Lp(\mathbb{R})|\beta$, $\forall\gamma < \alpha\ \neg(Lp(\mathbb{R})|\gamma \prec_1 Lp(\mathbb{R})|\alpha)$, and $\forall\gamma > \beta\ \neg(Lp(\mathbb{R})|\beta \prec_1 Lp(\mathbb{R})|\gamma)$. See [17] for more detailed discussions and precise definition of $\prec_1$.

For this note that by results of [14], we have $\mathbf{Env}(\Gamma) = P(\mathbb{R})^{Lp(\mathbb{R})|\gamma}$, where γ is the largest ordinal such that $Lp(\mathbb{R})|\boldsymbol{\delta}^2_1(\bar{\mu}) \prec_1 Lp(\mathbb{R})|\gamma$. Note that $\gamma \geq \Theta^{L(\mathbb{R},\bar{\mu})}$, and for all we know this inequality could be strict. However, since $[\boldsymbol{\delta}^2_1(\bar{\mu}), \gamma]$ is a Σ_1-gap and $\gamma < \boldsymbol{\delta}^2_1(\mathcal{C})$, we have that $Lp(\mathbb{R})|(\gamma+1)$ is the first initial segment of $L(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ that has a subset of the reals not in $Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$, in fact $(\mathbb{R}, \bar{\mu})^\sharp \in Lp(\mathbb{R})|(\gamma+1)$. Thus $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})|\gamma$ and so $\mathbf{Env}(\Gamma) = \mathcal{P}_{\bar{\mu}}(\mathbb{R})$, as wanted.

Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$.[9] Since $\vec{B}$ is countable, there exists a real x such that each element in $\vec{B}$ is $\mathrm{OD}_x^{L(\mathbb{R},\bar{\mu})}$. Let $\mathcal{M}_{\mu,\infty}(x)$ as defined in Lemma 52. For ease of notation put $\mathcal{M} = \mathcal{M}_{\mu,\infty}(x)$. Then $\mathcal{M}$ has ω^2-many Woodin cardinals and terms capturing every B in $\vec{B}$ at every Woodin cardinal δ of $\mathcal{M}$. Let $\tau_{B,\delta}$ be the standard term[10] witnessing this. Let $\kappa >> \Theta^{L(\mathbb{R},\mu)}$ be an inaccessible cardinal in $\mathcal{M}$ and a V-cardinal and let us define $\mathcal{N} = Hull^{\mathcal{M}|\kappa}(\{\tau_{B,\delta} \mid B \in \vec{B} \text{ and } \delta \text{ Woodin in } \mathcal{M}\})$; note that $\mathcal{N}$ is a model of ZFC. Hence $\mathcal{N}$ is an x-mouse that captures all the elements of a self-justifying system. Thus, by a theorem of Woodin, the strategy that picks branches that are realizable into $\mathcal{M}$[11] and moves these term relations correctly is an iteration strategy for $\mathcal{N}$ (see [8]). In other words, $\mathcal{N}$ is ω_1-iterable in $L(\mathbb{R}, \mathcal{C})$, and $\mathcal{N}^\sharp$ exists and is ω_1-iterable (hence ω_1 + 1-iterable) in $L(\mathbb{R}, \mathcal{C})$ (sketch: since ω_1 is measurable, and $\mathcal{N}$ is countable, $\mathcal{N}^\sharp$ exists. Furthermore, the Woodin cardinals of $\mathcal{N}$ remain Woodin in $\mathcal{N}^\sharp$ because we have put $\tau_{B,\delta}$'s in the hull.). Therefore $\mathcal{M}^\sharp_{\omega^2}$ exists and it is ω_1 (and hence ω_1+1) iterable. □

We claim that if ν is the club filter in $L(\mathbb{R}, \mathcal{C})$, then $L(\mathbb{R}, \nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We cannot apply Proposition 12 directly but we can work our way into a situation where the proof can be adapted. For this, let α be large such that $L_\alpha(\mathbb{R}, \mathcal{C}) \vDash$ ZF + DC. Therefore, there is a countable set N, such that $N \prec L_\alpha(\mathbb{R}, \mathcal{C})$; furthermore, we get that $\mathcal{M}^\sharp_{\omega^2}$ and its unique strategy are in N. Let $\bar{N}$ be the transitive collapse of N. Then Proposition 12 and the remark after it imply that $\bar{N}$ believes that "if $\bar{\nu}$ is the club filter, then $L(\mathbb{R}, \bar{\nu}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact". By elementarity and the choice of α, we get that $L(\mathbb{R}, \mathcal{C})$ believes this as well. Also, $\nu \subseteq \mathcal{C}$, and so by Lemma 28 we get $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mathcal{C})$, implying $L(\mathbb{R}, \mathcal{C}) = L(\mathcal{P}_{\mathcal{C}}(\mathbb{R}))$. Now, Theorem 9.100 of [26] implies $L(\mathbb{R}, \mathcal{C}) \models \mathrm{AD}_\mathbb{R}$ but this is impossible since $L(\mathbb{R}, \mathcal{C}) \models \Theta = \theta_0$.

Case 2: $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mu(\mathbb{R})$.

By Σ_1-reflection we have that in $L(\mathbb{R}, \mu)$ there is $\bar{\mathcal{C}}$ in $L_{\boldsymbol{\delta}^2_1(\mu)}(\mathbb{R}, \mu)$ such that $L(\mathbb{R}, \bar{\mathcal{C}})$ is a model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. By Case 1, $\mathcal{P}_{\mathcal{C}}(\mathbb{R}) \subseteq \mathcal{P}_{\bar{\mathcal{C}}}(\mathbb{R})$ and so $\mathcal{P}_{\mathcal{C}}(\mathbb{R}) \in L_{\boldsymbol{\delta}^2_1(\mu)}(\mathbb{R}, \mu)$. By [21] we have that $\mu \cap L_{\boldsymbol{\delta}^2_1(\mu)}(\mathbb{R}, \mu)$ is a subset of the club filter of $L(\mathbb{R}, \mu)$. So if $A \in \mathcal{P}_{\mathcal{C}}(\mathbb{R}) \cap \mu$ then A contains a club in V. Hence $\mathcal{C} \cap L(\mathbb{R}, \mathcal{C}) \subseteq \mu$ and so Lemma 28 implies $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$. □

[9] See e.g. [23] for a detailed discussion of self-justifying systems and sealing. Roughly, $\vec{B}$ is a countable sequence of sets of reals telling us where the gap ends and where the next scaled pointclass begins.

[10] This was defined after Definition 34.

[11] Suppose $\pi : \mathcal{N} \to \mathcal{M}$ is the uncollapse map and let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$. b is a branch of $\mathcal{T}$ that is realizable into $\mathcal{M}$ if whenever $i_b^{\mathcal{T}}$ exists, there is a map $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{M}$ such that $\pi = \sigma \circ i_b^{\mathcal{T}}$.

Note that the proof of Lemma 56 implies that any two models, $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ that satisfy $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact would have the same Θ and so they would also share the same $\boldsymbol{\delta}^2_1$. This justifies referring to $\Theta(\mu)$ and $\boldsymbol{\delta}^2_1(\mu)$ simply as Θ and $\boldsymbol{\delta}^2_1$ respectively.

3.3. The proof of Theorem 10

Let us fix a μ such that $L(\mathbb{R},\mu)$ is a model of $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact.

The crux of the main theorem of this section is the following observation.

Note that by Lemma 56 we have that $\mathcal{P}_{\mathcal{C}}(\mathbb{R})=\mathcal{P}_\mu(\mathbb{R})$. This implies that the notion of suitability is the same in $L(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\mu)$. The notion of ordinal definability might however be different. Recall that $\mathcal{T}^\mu_n$ is the theory of n many indiscernibles with real parameters of $L(\mathbb{R},\mu)$.

We have that for any n there is k such that $\mathcal{T}^{\mathcal{C}}_n\leq_W\mathcal{T}^\mu_k$, and vice versa, for any n there is a k such that $\mathcal{T}^\mu_n\leq_W\mathcal{T}^{\mathcal{C}}_k$. From now on let us fix a real number x that codes all of these reductions in a natural way.[12] We also choose x so that $\mathcal{T}^{\mathcal{C}}_k$ is $\mathrm{OD}_x^{L(\mathbb{R},\mu)}$ for all k. If P is a suitable initial segment of a Σ_x-iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$ then by Lemmas 36 and 42 we have that P is strongly $\mathcal{T}^{\mathcal{C}}_n$-iterable. Furthermore as $x\in P$ by Lemma 5.9 of [13] we have that P captures $\mathcal{T}^\mu_n$ for every natural n. Moreover we have that if x codes a reduction $\mathcal{T}^\mu_n\leq_W\mathcal{T}^{\mathcal{C}}_k$ then for any $i<o(P)$, $\tau^P_{\mathcal{T}^\mu_n,i}\in H(P,\mathcal{T}^{\mathcal{C}}_k)$ and moreover every $\mathcal{T}^{\mathcal{C}}_k$-iteration of P is also a $\mathcal{T}^\mu_n$ iteration. The following lemma will show that as in the case of $L(\mathbb{R},\mathcal{C})$ the pairs of the form $(P,\mathcal{T}^{\mathcal{C}}_n)$ are dense in $\mathcal{D}_\mu$ in the sense of Lemma 44. In other words.

Lemma 57. *Suppose $L(\mathbb{R},\mu)\models\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact. Let A be $\mathrm{OD}_x^{L(\mathbb{R},\mu)}$ and P is an x-mouse that is A-iterable. Then there is a natural number n and a suitable initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$, say Q, that is $A\oplus\mathcal{T}^{\mathcal{C}}_n$-iterable, $\tau^Q_A\in H(Q,\mathcal{T}^{\mathcal{C}}_n)$ and $(P,A)\preceq(Q,\mathcal{T}^{\mathcal{C}}_n)$.*

Proof. Here just note that $\{\mathcal{T}^{\mathcal{C}}_n\mid n\in\omega\}$ is Wadge cofinal in the Wadge hierarchy of $L(\mathbb{R},\mu)$. Also for every n we have that $\mathcal{T}^{\mathcal{C}}_n$ is $\mathrm{OD}_x^{L(\mathbb{R},\mu)}$. We can then apply Lemma 45 and comparison to get the desired Q. □

Theorem 58. *Suppose that $L(\mathbb{R},\mu)\models\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact. Then for a Turing cone of $y\in\mathbb{R}$ we have that $\mathrm{HOD}_y^{L(\mathbb{R},\mu)}=\mathrm{HOD}_y^{L(\mathbb{R},\mathcal{C})}$.*

Proof. Using Lemma 57, the proof of Theorem 46 can be adapted to yield

$$M_{\infty,\mu}(x)=M^+_\infty(x)|\lambda^{M^+_\infty(x)}_{\omega^2}$$

[12] Fix $z\mapsto\langle(z)_i\rangle_{i\in\omega}$ a recursive bijection between $\mathbb{R}$ and $\mathbb{R}^\omega$ and fix x such that given $n\in\omega$ there exists i and j naturals such that $(x)_i$ codes a continuous reduction witnessing $\mathcal{T}^\mu_n\leq_W\mathcal{T}^{\mathcal{C}}_k$ (for some k) and the similarly $(x)_j$ codes a reduction $\mathcal{T}^{\mathcal{C}}_n\leq_W\mathcal{T}^\mu_k$ (fore some other k).

Also, essentially, by Theorem 46 $M_\infty(x) = M_\infty^+(x)|\lambda_{\omega^2}^{M_\infty^+(x)}$. Hence we have that

$$M_{\infty,\mu}(x) = M_\infty(x).$$

Note that by Theorem 51 we only need to show the following claim.

Claim. $\Sigma_{\mu,x} = \Sigma_x$ *when restricted to countable stacks of trees based on* $M_\infty(x)^-$.[13]

Proof of the claim. We will prove inductively that if $\vec{\mathcal{T}}$ is a stack of n trees, and is according to both Σ_x and $\Sigma_{\mu,x}$ then these strategies pick the next branch the same way. Note that by the definitions of Σ_x and $\Sigma_{\mu,x}$ we have that $\Sigma_x(\vec{\mathcal{T}}) = b$ if and only if $\vec{\mathcal{T}}^\frown b$ is an $\mathcal{T}_n^{\mathcal{C}*}$-iteration on M_x^- for all $n \in \omega$ (here again the key fact is that the $\xi_{\mathcal{T}_n^{\mathcal{C}}}^{M_x^-}$ and the $\xi_{\mathcal{T}_n^\mu}^{M_x^-}$ are cofinal in $\delta_o^{M_x^-}$). As discussed in the discussion at the beginning of this section this implies that $\vec{\mathcal{T}}^\frown b$ is a $\mathcal{T}_n^{\mu *}$-iteration for all $n \in \omega$, in other words $\Sigma_{\mu,x}(\vec{\mathcal{T}}) = b$, which finishes the proof of the claim. □

But then this implies $\mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})} = \mathrm{HOD}_x^{L(\mathbb{R},\mu)}$ by Theorem 51. Note also that if $y \geq_T x$ then same proof relativized to y is still valid. This completes the proof of the theorem. □

Proof of Theorem 10. First let us suppose that Σ extends to W, so all the previous results of this section hold. By Theorem 58 we can fix a real x such that $\mathrm{HOD}_x^{L(\mathbb{R},\mu)} = \mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})}$. Also, by [21] we have that

$$L(\mathbb{R},\mu) = \mathrm{HOD}_x^{L(\mathbb{R},\mu)}(\mathbb{R}) \text{ and } \mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})}(\mathbb{R}) = L(\mathbb{R},\mathcal{C}),$$

which clearly implies $L(\mathbb{R},\mathcal{C}) = L(\mathbb{R},\mu)$.

Now, if Σ is just an ω_1+1-iteration strategy, by contradiction suppose that there is μ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact but $L(\mathbb{R},\mathcal{C}) \neq L(\mathbb{R},\mu)$. Pick γ such that V_γ is a model of ZF-PowerSet, and reflects the existence of such μ. Let $N \prec V_\gamma$ be countable and H its transitive collapse. Then we are in the same situation as when proving Proposition 12. Hence the result follows word by word from the proof of Proposition 12. □

4. The AD case

We give in this section a proof of Theorem 9. We will first assume AD^+ and for contradiction suppose that the theorem does not hold and then we reflect this statement to a Suslin co-Suslin set. Then we can use [19] and [8] to construct models with Woodin

[13] We refer as Σ_x the strategy given by Lemma 48 and $\Sigma_{\mu,x}$ the one defined in the paragraph before Theorem 51.

cardinals and run a version of the last chapter's arguments. Lastly we show how to reduce the hypotheses to $\mathrm{AD}+\mathrm{DC}_{\mathbb{R}}$. We start by noting some preliminary facts.

Lemma 59. *Suppose* $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$ *and let* μ *be a filter such that* $L(\mathbb{R},\mu)$ *satisfies* $\mathrm{AD} + \omega_1$ *is* $\mathbb{R}$*-supercompact. Then* $\mathcal{P}_\mu(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.

Proof. Otherwise we have that $V = L(\mathcal{P}(\mathbb{R}))$ believes there is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Also $V = L(\mathbb{R},\mu)$, so by [26, Theorem 9.100] $L(\mathbb{R},\mu) \models \mathrm{AD}_{\mathbb{R}}$ but this is impossible since we have $L(\mathbb{R},\mu) \models \Theta = \theta_0$ by [22]. □

From now on we will also assume that $V = L(\mathcal{P}(\mathbb{R})) \models \Theta = \theta_0$, as otherwise by Theorem 17 there exists a non-tame mouse and hence $\mathcal{M}^\sharp_{\omega^2}$ exists and it is iterable so the results of last section would hold. Since $\Theta = \theta_0$ we have that, in particular, DC holds in V. We now prove the first approximation to our main result.

Theorem 60. *Suppose* $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$*. Then there is at most one model of the form* $L(\mathbb{R},\mu)$ *satisfying* $\mathrm{AD}+\omega_1$ *is* $\mathbb{R}$*-supercompact. Moreover if such model exists then the unique such model is* $L(\mathbb{R},\mathcal{C})$ *where* $\mathcal{C}$ *is the club filter on* $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Proof. Suppose that there is $\mu \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu) \models \mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact. Let μ be chosen such that $\mathcal{P}_\mu(\mathbb{R})$ is minimal in that given any ν such that $L(\mathbb{R},\nu) \models \mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact then we have that $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$. Note that by Lemma 59 we have that there is a set of reals B of Wadge rank bigger than the Wadge rank of any sets of reals in $L(\mathbb{R},\mu)$. For $x \in \mathbb{R}$, let E_x be the Wadge reduction of B coded by x. Let

$$\bar{A} = \{x \in \mathbb{R} \mid E_x \in \mu\}.$$

Let A be a set of reals from which $\bar{A}$ and B are definable. Then, in $L(\mathbb{R},A)$ μ is definable from parameters and moreover by AD^+ we have $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},A)$. Now by Theorem 6 and minimality of μ we may assume that A is Suslin and co-Suslin.

Let us work from now on in $L(\mathbb{R},A)$. By minimality of μ we get that $(\mathbb{R},\mu)^\sharp$ is Suslin and co-Suslin in $L(\mathbb{R},A)$. The presence of $(\mathbb{R},\mu)^\sharp$ and the proof of Lemma 56 imply the existence of $\mathcal{N} = (M_{\infty,\mu})^\sharp$. Here we identify $\mathcal{N}$ with the least active mouse extending $M_{\infty,\mu}$. Let Γ be $\Sigma_1^{L(\mathbb{R},\mu)}$ and $\vec{B}$ a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Let us fix ζ to be the largest Suslin cardinal in $L(\mathbb{R},A)$.

Claim 1. $\mathbf{Env}(\Gamma) \subset Lp(\mathbb{R})$.

Proof of Claim 1. Let $C \in Env(\Gamma)$. First, note that C is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$; this is because by the choice of A, B is Wadge reducible to A and the Wadge rank of A is at most ζ. By the definition of $\mathbf{Env}$, for any $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, we have that $C \cap \sigma \in \mathrm{OD}^{L(\mathbb{R},\mu)}_{\{\sigma,A\}\cup\sigma}$; so by mouse capturing in $L(\mathbb{R},\mu)$ we have that $C \cap \sigma \in Lp(\sigma)$. We then have that $C \cap \sigma \in M_\sigma$

where $M_\sigma = \mathrm{HOD}^{L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))}_{\{C,\sigma\}\cup\sigma}$; this is because $Lp(\sigma)^{L(\mathbb{R},\mu)} \in M_\sigma$. Define $\mathcal{M}_\sigma \lhd Lp(\sigma)$ to be the least initial segment of $Lp(\sigma)$ having $C \cap \sigma$ as an element. Note that $\mathcal{M}_\sigma \in M_\sigma$ and $M_\sigma \models$ "$\mathcal{M}_\sigma$ is countably iterable" because the unique iteration strategy for $\mathcal{M}_\sigma$ is definable from σ, C.

Also the club filter $\mathcal{C}$ is an ultrafilter on $\mathcal{P}_\zeta(\mathbb{R})$ (basically by Moschovaki's third periodicity theorem, see [22]). So, we can define $M = \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma/\mathcal{C}$, where the functions of this ultraproduct are $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma$ and $f \in L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$. Note that by [21], $\mathcal{C}$ is normal and countably complete. Then we have that Σ_1-Łos holds, since $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ satisfies Σ_1-replacement. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mathcal{C}}$; we claim that M believes "$\mathcal{M}$ is countably iterable". To see this let $\bar{\mathcal{M}}$ be a countable transitive hull of $\mathcal{M}$, then we have that $\bar{\mathcal{M}} \in \sigma$ for club-many σ. Also $[\sigma \mapsto \bar{\mathcal{M}}]_{\mathcal{C}} = \bar{\mathcal{M}}$ (by countable completeness of $\mathcal{C}$). Now by Σ_1 -Łos we have that for club-many σ, $\bar{\mathcal{M}}$ is a countable hull of $\mathcal{M}_\sigma$ and so $M_\sigma \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable". Let Σ_σ be the unique iteration strategy of $\bar{\mathcal{M}}$, then the function $\sigma \mapsto \Sigma_\sigma$ is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ and is such that $M_\sigma \models (HC, \Sigma_\sigma)$" $\models \Sigma_\sigma$ is an ω_1 strategy for $\bar{\mathcal{M}}$". By Łos, again, we get that $M \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable".

Also, $C = [\sigma \mapsto C \cap \sigma]_{\mathcal{C}}$ hence $C \in \mathcal{M}$. Note that in $L(\mathbb{R}, A)$, $\mathcal{M}$ is actually countably iterable, so we have $\mathcal{M} \lhd Lp(\mathbb{R})$ and so $C \in Lp(\mathbb{R})$. □

Arguing as in the proof of the claim in Lemma 56 we then get that $\mathbf{Env}(\Gamma) = \mathcal{P}_\mu(\mathbb{R})$. Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Recall that $\mathcal{N}$ captures every B in $\vec{B}$, say via τ_B. Define then

$$\mathcal{M} = Hull^{\mathcal{N}}(\{\tau_B^{\mathcal{N}} \mid B \in \vec{B}\}).$$

Here we think of $\mathcal{M}$ as the transitive collapse of this hull. Then as in the proof of the claim in Lemma 56, we have that $\mathcal{M}$ is $\omega_1 + 1$ iterable and so $\mathcal{M}^\sharp_{\omega^2}$ exists and is $\omega_1 + 1$-iterable.

Claim 2. *$L(\mathbb{R}, \mathcal{C})$ is a model of "AD $+ \omega_1$ is $\mathbb{R}$-supercompact" and the only such model.*

Proof of Claim 2. Here we use the results of Section 2. The key point is that the iteration strategy for $\mathcal{M}^\sharp_{\omega^2}$ might not extend to big generic collapses. For this though we use instead a countable elementary substructure of $L_\alpha(\mathbb{R}, A)$, where $\alpha >> \Theta^{L(\mathbb{R},A)}$ large and is such that $L_\alpha(\mathbb{R}, A) \models$ ZF + DC-PowerSet and contains all relevant objects. Let $N \prec L_\alpha(\mathbb{R}, A)$ be countable and elementary such that $\mathcal{M}^\sharp_{\omega^2} \in N$ (here we use that DC holds in V). Let $\bar{H}$ be the transitive collapse of N. Then as in the proof of Proposition 12 the results of Section 2 give that $\bar{H}$ models "$L(\mathbb{R}, \mathcal{C})$ satisfies AD $+\ \omega_1$ is $\mathbb{R}$-supercompact", but then N does and so does V.

The same argument combined with the results of Section 3 will show that since $\mathcal{M}^\sharp_{\omega^2}$ exists, $L(\mathbb{R}, \mathcal{C})$ is the unique model of AD $+\ \omega_1$ is $\mathbb{R}$-supercompact. This concludes the proof. □

Let us mention that the key fact about AD^+ we used in the proof of Theorem 60 is that given μ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, then one can reflect the existence of such a μ to the Suslin co-Suslin part of a model of the form $L(\mathbb{R}, A)$, where A is a set of reals. This is particularly useful as then one can take ultraproducts using the club filter. In the absence of AD^+ this can be a little bit more tricky as we may not be able to reflect, but we show how to overcome this difficulty and get the proof of the result under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$.

Proof of Theorem 9. First let us assume AD^+ holds, and then we will use this proof to get a proof under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$. Suppose that there are μ and ν such that $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are models of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We may assume with no loss that $V = L(\mathbb{R},\mu,\nu)$ and $\Theta = \theta_0$, as otherwise there is a non-tame mouse and we would finish the proof as before.[14]

Note that the proof of Lemma 56 holds in this case too, so $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$.

Claim. *$\mathcal{P}(\mathbb{R})$ is strictly larger that $\mathcal{P}_\mu(\mathbb{R})$.*

Proof of the claim. Otherwise we have that $\mathcal{P}(\mathbb{R}) = \mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$. We can fix then an $\mathrm{OD}^{L(\mathcal{P}(\mathbb{R}))}$ tree T that projects to a universal Σ^2_1. Following [22] we let $\mathbb{D} = \{\langle d_i \mid i \in \omega\rangle \mid \forall i \in \omega\ d_i$ is a Σ^2_1 degree and $d_i <_{\Sigma^2_1} d_{i+1}\}$.[15] We recall in the following lines the definition of the auxiliary measures $\bar{\mu}$ and $\bar{\nu}$ on $\mathbb{D}$ from [22].

For $A \subseteq \mathbb{D}$, let $S \subset \mathrm{ON}$ be an ∞-Borel code for A, then

$$A \in \bar{\mu} \text{ iff } \forall^*_\mu \sigma L[T,S](\sigma) \models \text{“}\mathrm{AD}^+ + \sigma = \mathbb{R} \text{ and } \exists(\emptyset,U) \in \bar{\mathbb{P}}\ (\emptyset,U) \Vdash \dot{G} \in \mathcal{A}_S\text{”}$$

where $\bar{\mathbb{P}}$ is the usual Prikry forcing using Σ^2_1-degrees in $L[S,T](\sigma)$ and the Martin measure (see section 6.3 of [3]), also $\dot{G}$ is the name of the corresponding Prikry sequence and $\mathcal{A}_S$ is the interpretation of the set of reals coded by S.

By results of [22] we have:

- For any $S \subset \mathrm{ON}$ we have that $\forall^*_\mu \sigma L[T,S](\sigma) \models$ “$\mathrm{AD}^+ + \sigma = \mathbb{R}$”.
- Whether $A \in \bar{\mu}$ does not depend on the code S.
- Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and for $d \in \mathbb{D}$ let

$$\sigma_d = \{y \mid \text{there are } i \text{ and } x \text{ such that } y \leq_{\Sigma^2_1} d(i)\}.$$

  Then we have that if $\bar{A} = \{d \in \mathbb{D} \mid \sigma_d \in A\}$

[14] Here $L(\mathbb{R},\mu,\nu)$ is constructed by induction as follows. $L_0(\mathbb{R},\mu,\nu) = \mathbb{R}$, for $\alpha \in \mathrm{ON}$ we let $L_{\alpha+1}(\mathbb{R},\mu,\nu)$ be the collection definable sets over $(L_\alpha(\mathbb{R},\mu,\nu), \in, \nu \cap L_\alpha(\mathbb{R},\mu,\nu), \mu \cap L_\alpha(\mathbb{R},\mu,\nu))$ and taking unions at limit stages.

[15] Define $x \leq_{\Sigma^2_1} y$ if and only if $x \in L[T,y]$, $x \equiv_{\Sigma^2_1} y$ if and only if $x \leq_{\Sigma^2_1} y$ and $y \leq_{\Sigma^2_1} x$, and $x <_{\Sigma^2_1} y$ if and only if $x \leq_{\Sigma^2_1} y$ and $\neg(x \equiv_{\Sigma^2_1} y)$. A Σ^2_1 degree is a $\equiv_{\Sigma^2_1}$-equivalence class. It can be shown that the above definition of $\leq_{\Sigma^2_1}$ does not depend on the choice of T; in fact, $x \leq_{\Sigma^2_1} y$ if and only if $x \in \mathrm{HOD}_y$.

$$A \in \mu \text{ if and only if } \bar{A} \in \bar{\mu}.$$

Let us recall the construction of the Prikry Forcing done in Section 2 of [22]; we however, will alternate using μ and ν when choosing measure one sets. More precisely, for $n \in \omega$, given $X \subseteq \mathbb{D}^{n+1}$ we say $X \in \mathcal{U}_n$ if

$$\forall^*_{\bar{\mu}} \vec{z}(0) \forall^*_{\bar{\nu}} \vec{z}(1) \cdots \forall^*_{\mathcal{G}} \vec{z}(n) \left(\langle \vec{z}(i) \mid i < n+1 \rangle \in X \right).$$

In the definition above, $\mathcal{G} = \bar{\mu}$ if n is even and $\mathcal{G} = \bar{\nu}$ otherwise. We also define $\mathbb{P}$ as follows. Conditions will be pairs $(p, \vec{U})$, with $\vec{U}(n) \in \mathcal{U}_n$ for all $n \in \omega$ and such that $p = \langle \vec{d}_i \mid i < n \rangle$ is a sequence of elements in $\mathbb{D}$, such that $\vec{d}_i$ is in $L[x, T]$ for any (all) $x \in \vec{d}_{i+1}(0)$ and it is countable there. We say $(q, \vec{W}) \leq_{\mathbb{P}} (p, \vec{U})$ if $\exists n \exists r \in \mathbb{D}^n \; q = p^\frown r$ and $r^\frown s \in \vec{U}(n+k)$ for all k and all $s \in \vec{W}(k)$. As in Section 6 of [3] we will have that $\mathbb{P}$ has the Prikry property, which is to say that given a forcing statement Φ and a condition $(p, \vec{U}) \in \mathbb{P}$, there is $\vec{W}$ such that $(p, \vec{W})$ decides Φ. We summarize the facts of this forcing that we will use (see [22]).

- For a given set a that admits a well order rudimentary in a, there is a cone of reals x such that $\mathrm{HOD}^{L[T,x]}_{T,a} \models \omega_2^{L[T,x]}$ is Woodin. For a real x we let $\delta(x) = \omega_2^{L[T,x]}$. And for a Σ^2_1-degree d, we let $\delta(d) = \delta(x)$ for any (all) $x \in d$.
- Given $\langle \vec{d}_i \mid i < n \rangle \in \mathbb{D}^n$, we let

  $$Q_0(\vec{d}) = \mathrm{HOD}^{L[\vec{d}_0, T]}_{\vec{d}_0, T} \mid \sup\{\delta(d_0(n)) \mid n \in \omega\},$$

  and

  $$Q_{i+1}(\vec{d}) = \mathrm{HOD}^{L[T, \vec{d}_{i+1}]}_{Q_i, \vec{d}_{i+1}, T} \,|(\sup\{\delta(d_{i+1}(n)) \mid n \in \omega\}).$$

- Given G generic for $\mathbb{P}$ define $g = \bigcup\{p \mid (p, \vec{U}) \in G \text{ for some } \vec{U}\}$. Let $Q_i(g)$ be $Q_i(g \restriction i)$. Then $L[\cup_{i\in\omega} Q_i(g), T]$ has ω^2 many Woodin cardinals.
- If $\sigma_i = \{x \mid \exists n (x \in \vec{d}_i(n))\}$ then the tail filter $\mathcal{F}$ generated by $(\sigma_i : i \in \omega)$ is such that $L(\mathbb{R}, \mathcal{F}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Let us fix G a V-generic filter for $\mathbb{P}$ and let $\mathcal{F}$ be its associated tail filter. We claim that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F}) = L(\mathbb{R}, \nu)$. For this, suppose that $A \in \mathcal{F} \cap V$, we will show $A \in \mu$. Otherwise we have $A \notin \mu$. Let $(p, \vec{U}) \Vdash A \in \mathcal{F}$. Let $\vec{W}$ be defined as $\vec{W}(2n) = \vec{U}(2n) \cap \mathbb{D} \setminus \bar{A}$, and $\vec{W}(2n+1) = \vec{U}(2n+1)$ for $n \in \omega$ (here $\bar{A}$ is the translation of A to $\mathbb{D}$ as defined before). But then it is clear that $(p, \vec{W}) \Vdash A \notin \mathcal{F}$, a contradiction. Hence Lemma 28 implies that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$, similarly $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mathcal{F})$. So $V = L(\mathbb{R}, \mu)$ which is impossible. □

Hence $\mathcal{P}(\mathbb{R})$ is strictly larger than $\mathcal{P}_\mu(\mathbb{R})$, and we can choose $A \subseteq \mathbb{R}$ such that $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are definable (from parameters) in $L(\mathbb{R}, A)$ and hence the result follows from Theorem 60.

Now, assume AD^+ does not hold, then we have that $\mathcal{P}_\mu(\mathbb{R})$ is strictly smaller than $\mathcal{P}(\mathbb{R})$ (because AD^+ holds in $L(\mathbb{R}, \mu)$). Let $\Gamma = \{A \subset \mathbb{R} \mid L(\mathbb{R}, A) \models \mathrm{AD}^+\}$. By [26, Theorem 9.14], we have that $L(\mathbb{R}, \Gamma) \models \mathrm{AD}^+$. We have two cases. If Γ strictly contains $\mathcal{P}_\mu(\mathbb{R})$, then we have that $L(\mathbb{R}, \mu)$ is definable from parameters in $L(\mathbb{R}, \Gamma)$ and hence one can work in $L(\mathbb{R}, \Gamma)$ and the theorem follows from Theorem 60.

If $\Gamma = \mathcal{P}_\mu(\mathbb{R})$, then $\Gamma \neq \mathcal{P}(\mathbb{R})$ and, by Theorem 9.14 of [26] again, we get $L(\mathbb{R}, \Gamma) \models \mathrm{AD}_\mathbb{R}$, and so $L(\mathbb{R}, \mu) \models \mathrm{AD}_\mathbb{R}$, which is a contradiction. □

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