CORRESPONDENCES WITHOUT A CORE

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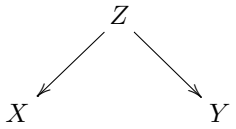
Abstract. We study the formal properties of correspondences of curves without a core, focusing on the case of étale correspondences. The motivating examples come from Hecke correspondences of Shimura curves. Given a correspondence without a core, we construct an infinite graph $\mathcal{G}_{gen}$ together with a large group of "algebraic" automorphisms A. The graph $\mathcal{G}_{gen}$ measures the "generic dynamics" of the correspondence. We construct specialization maps $\mathcal{G}_{gen} \to \mathcal{G}_{phys}$ to the "physical dynamics" of the correspondence. Motivated by the abstract structure of the supersingular locus, we also prove results on the number of bounded étale orbits, in particular generalizing a recent theorem of Hallouin and Perret. We use a variety of techniques: Galois theory, the theory of groups acting on infinite graphs, and finite group schemes.

Contents

1. Introduction

To an étale correspondence of complex hyperbolic curves



one can associated a many-valued function $X \dashrightarrow X$. In [20], Mochizuki proved that if this many-valued function has unbounded dynamics, then X, Y, and Z are all complex Shimura curves. Mochizuki uses a highly non-trivial result of Margulis [17], which characterizes complex Shimura curves via properties of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

The most basic examples of Shimura varieties are the modular curves, parametrizing elliptic curves with level structure. A slightly less familiar example comes from moduli spaces of *fake elliptic curves*; these Shimura varieties are projective algebraic curves. It turns out that the modular curves are the only non-compact Shimura curves. See Deligne [7] for a general introduction to Shimura varieties.

In general, Shimura varieties are quasi-projective algebraic varieties defined over $\overline{\mathbb{Q}}$ [2, 7, 18, 19]. Recent work of Kisin [12] shows that many Shimura varieties of abelian type have natural integral models, which opens up the possibility of studying their reduction modulo p. PEL-type Shimura

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varieties are moduli spaces of abelian varieties with manifestly algebraic conditions (i.e. fixing the data of a polarization, endomorphisms, and level.) Using the moduli interpretation it is straightforward to define PEL-type Shimura varieties directly over finite fields $\mathbb{F}_q$, at least for most q. As far as we know, there is not as-of-yet a direct definition of general non-PEL-type Shimura varieties over $\mathbb{F}_q$.

Jie Xia has recently taken the simplest example of non-PEL-type Shimura curves, what he calls Mumford curves, and given "direct definitions" over $\overline{\mathbb{F}}_p$ [29, 31, 32]. The most basic example of Mumford curves parameterize abelian 4-folds with certain extra Hodge classes, as in Mumford's original paper [21]. Xia proved theorems of the following form: given an abelian scheme $\mathcal{A} \to X$ over a curve $X/\overline{\mathbb{F}}_p$, there are certain conditions that ensure that the pair $(\mathcal{A}, X)$ is the reduction of a Mumford curve together with its universal abelian scheme over $W(\overline{\mathbb{F}}_p)$.

In Chapter 2 of my thesis [15], we posed the question of characterizing Shimura curves over $\mathbb{F}_q$. Unlike Xia, we *did not* assume the existence of an abelian scheme $\mathcal{A}$ over the curves considered. Instead, we took as our starting point Mochizuki's Theorem, which is of group theoretic nature.

Definition. Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k. Then we have an inclusion diagram $k(X) \subset k(Z) \supset k(Y)$ of function fields. We say that the correspondence *has no core* if $k(X) \cap k(Y)$ has transcendence degree 0 over k.

This definition formalizes the phrase "generically unbounded dynamics" (Remark 3.7 and Proposition 5.10.) Shimura curves have many étale correspondences without a core. Inspired by Mochizuki's theorem, we wondered if all étale correspondences of curves without a core are "related to" Shimura curves. Given a smooth projective curve X over $\mathbb{F}_q$, are there other group theoretic conditions on $\pi_1^{\acute{e}t}(X)$ that ensure that X is the reduction modulo p of a classical Shimura curve?

In this article, we explore the formal structure of (étale) correspondences without a core with an aim to understanding the similarities with Hecke correspondences of Shimura curves. We now state the main constructions/results.

Given a correspondence without a core, in Section 5 we construct a pair $(\mathcal{G}_{gen}, A)$ of an infinite graph together with a large topological group of "algebraic" automorphisms. The graph $\mathcal{G}_{gen}$ roughly measures the "generic dynamics." In the case of a symmetric l-adic Hecke correspondence of modular curves, $\mathcal{G}_{gen}$ is a tree and the pair $(\mathcal{G}_{gen}, A)$ is related to the action of $PSL_2(\mathbb{Q}_l)$ on its building. When $\mathcal{G}_{gen}$ is a tree, we prove that the vertices of $\mathcal{G}_{gen}$ are in bijective correspondence with the maximal open compact subgroups of a certain subgroup A^{PQ} of A (Corollary 5.21.) This perhaps suggests that in this case the action of the topological group A^{PQ} on $\mathcal{G}_{gen}$ is similar to the action of the l-adic linear group $PSL_2(\mathbb{Q}_l)$ on its building.

Definition. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over k. A *clump* is a finite set $S \subset Z(\overline{k})$ such that $f^{-1}(f(S)) = g^{-1}(g(S)) = S$. A clump is *étale* if f and g are étale at all points of S.

A clump may be thought of as a "bounded orbit of geometric points." Hecke correspondences of modular curves over $\mathbb{F}_p$ have a natural étale clump: the supersingular locus.

Theorem. *(see Theorem 9.6) Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over a field k without a core. There is at most one étale clump.*

An example: let $l \neq p$. Applying Theorem 9.6 to the Hecke correspondence $Y(1) \leftarrow Y_0(l) \rightarrow Y(1)$ reproves the fact that any two supersingular elliptic curves over $\overline{\mathbb{F}}_p$ are related by an l-primary isogeny. Theorem 9.6 implies a generalization of a theorem of Hallouin and Perret [10], who came upon it in the analysis of optimal towers in the sense of Drinfeld-Vladut. They use spectral graph theory and an analysis of the singularities of a certain recursive tower. In our language, the hypotheses of their "one clump theorem" are

- $k \cong \mathbb{F}_q$
- $X \leftarrow \Gamma \rightarrow X$ is a minimal self-correspondence of type (d, d)
- $\mathcal{H}_{gen}$, a certain directed graph where the in-degree and out-degree of every vertex is d, has no directed cycles.

Our techniques allow one to relax the third condition to "$\mathcal{H}_{gen}$ is infinite"; in particular, $\mathcal{H}_{gen}$ may have directed cycles. Moreover, our proof works over any field k and is purely algebro-geometric. See the lengthy Remark 9.8 for a full translation/derivation.

We remark that Theorem 9.6 is characteristic-independent and hence applies to correspondences without a core over $\mathbb{C}$; in particular, it may be thought of as a result on the dynamics of complex algebraic curves.

Specializing to characteristic 0, we prove the following.

Theorem. *(see Corollary 9.2) Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective curves over a field k without a core. Suppose $char(k) = 0$. Then there are no clumps.*

We unfurl this statement. Think of a symmetric Hecke correspondence $X \leftarrow Z \rightarrow X$ of Shimura curves over $\mathbb{C}$ as a many-valued function from X to X. Then the iterated orbit of any point $x \in X$ under this many-valued function is unbounded. This was likely already known, but we couldn't find it in the literature. We nonetheless believe our approach is new. We now briefly describe the sections.

In Section 3 we state Mochizuki's Theorem (Theorem 3.10). We then reprise the theme: "are all étale correspondences without a core related to Hecke correspondences of Shimura varieties?" in Question 3.16. The phrase "related to" is absolutely vital, and étale correspondences without a core do not always directly deform to characteristic 0. We will see one example in Remark 3.18 via Igusa level structure. More exotic is Example 3.19 of a central leaf in a Hilbert modular variety; according to a general philosophy of Chai-Oort, these leaves should also be considered Shimura varieties in characteristic p. Unlike in characteristic 0, however, these may deform in families purely in characteristic p. There are also examples of étale correspondences of curves over $\overline{\mathbb{F}}_p$ without a core using Drinfeld modular curves. We pose a concrete instantiation of Question 3.16 that doesn't mention Shimura varieties at all (Question 3.21).

In Section 4, starting from a correspondence without a core, we use elementary Galois theory to construct an infinite tower of curves W_∞ with "function field" E_∞. We use this tower to prove that the property of "not having a core" for an étale correspondences specializes in families (Lemma 4.10). As a consequence, there are no global deformations of an étale correspondence of projective hyperbolic curves over $\mathbb{C}$ without a core (Corollary 4.13).

In section 5, given a correspondence without a core, we construct the pair $(\mathcal{G}_{gen}, A)$ of an infinite graph together with a large group of "algebraic" automorphisms. The graph $\mathcal{G}_{gen}$ packages the Galois theory of E_∞ and reflects the generic dynamics of the correspondence. We are especially interested in Question 5.17: given an étale correspondence without a core, is $\mathcal{G}_{gen}$ a tree? Using Serre's theory of groups acting on trees [24], we prove that in this case the action of A^{PQ} on $\mathcal{G}_{gen}$ shares several properties with the action of the l-adic linear group $PSL_2(\mathbb{Q}_l)$ on its building (see Proposition 5.19 and Corollary 5.21).

In Section 6 we develop some basic results for symmetric correspondences. We are interested in the following refinement of Question 5.17: given a symmetric étale correspondence without a core, is the pair $(\mathcal{G}_{gen}, A)$ ∞-transitive (Question 6.11)? In the case of a symmetric, type (3,3) correspondence without a core, we are able to verify this using graph theory due to Tutte (Lemma 6.10); in particular, in this case $\mathcal{G}_{gen}$ is a tree.

In Section 7 we construct specialization maps $\mathcal{G}_{gen} \rightarrow \mathcal{G}_{phys}$. These roughly specialize the dynamics from the generic point to closed points. When the original correspondence is étale, the maps $\mathcal{G}_{gen} \rightarrow \mathcal{G}_{phys}$ are covering spaces of graphs (Lemma 7.2). Motivated by work of Kohel and Sutherland on *Isogeny Volcanoes* of elliptic curves [13, 25], we speculate on the behavior and asymptotics of these specialization maps in Question 7.5.

The rest of the paper may be read independently. In Section 8 we introduce the notion of an *invariant line bundle* on a correspondence and prove several results about their spaces of sections on (étale) correspondences without a core. In characteristic 0 there are no *invariant pluricanonical differential forms* (Corollary 8.13) on an étale correspondence of projective hyperbolic curves without a core. In characteristic p, however, such forms may exist. The existence of the *Hasse invariant*, a mod-p modular form, is a representative example of the difference. The key to these results is the

introduction of the group scheme $Pic^0(X \leftarrow Z \rightarrow Y)$; when $X \leftarrow Z \rightarrow Y$ does not have a core, we prove that this group scheme is finite (Lemma 8.9). We speculate on the relationship between *invariant differential forms* and $Pic^0(X \leftarrow Z \rightarrow Y)$ in Question 8.18.

In Section 9 we show that an *étale clump* gives rise to an invariant line bundle together with a line of invariant sections. Using the analysis in Section 8, we prove the two sample theorems above and explicate the relationship between our result and that of Hallouin-Perret. In analogy to the supersingular locus, We wonder if every étale correspondence of projective curves without a core in characteristic p has a clump, equivalently an invariant pluricanonical differential form (Question 9.7).

We briefly comment on Chapter 3 of my thesis [15] (see also [14].) Let $(X \xleftarrow{f} Z \xrightarrow{g} X)$ by a symmetric type (3,3) étale correspondence without a core over a finite field $\mathbb{F}_q$. Inspired by the formal similarity between the pair $(\mathcal{G}_{gen}, A^{PQ})$ and $(\mathcal{T}, PSL_2(\mathbb{Q}_2))$,where $\mathcal{T}$ is the building of $PGL_2(\mathbb{Q}_2)$ (i.e. the infinite trivalent tree), we assume that the action of G_P on $\mathcal{G}_{gen}$ is isomorphic to the action of $PSL_2(\mathbb{Z}_2)$ on $\mathcal{T}$, a *purely group-theoretic condition*. Call the associated $\mathbb{Q}_2$ local system $\mathscr{L}$. Then, using 2-adic group theory we prove that $f^*\mathscr{L} \cong g^*\mathscr{L}$. Suppose further that all Frobenius traces of $\mathscr{L}$ are in $\mathbb{Q}$. Using a recent breakthrough in the p-adic Langlands correspondence for curves over a finite field due to Abe [1], we build the following correspondence.

Theorem. *Let C be a smooth, geometrically irreducible, complete curve over $\mathbb{F}_q$. Suppose q is a square. There is a natural bijection between the following two sets.*

$$\left\{\begin{array}{l}\overline{\mathbb{Q}_l}\text{-local systems } \mathscr{L} \text{ on } C \text{ such that}\\ \bullet\ \mathscr{L} \text{ is irreducible of rank 2}\\ \bullet\ \mathscr{L} \text{ has trivial determinant}\\ \bullet\ \text{The Frobenius traces are in } \mathbb{Q}\\ \bullet\ \mathscr{L} \text{ has infinite image,}\\ \text{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}p\text{-divisible groups } \mathscr{G} \text{ on } C \text{ such that}\\ \bullet\ \mathscr{G} \text{ has height 2 and dimension 1}\\ \bullet\ \mathscr{G} \text{ is generically versally deformed}\\ \bullet\ \mathscr{G} \text{ has all Frobenius traces in } \mathbb{Q}\\ \bullet\ \mathscr{G} \text{ has ordinary and supersingular points,}\\ \text{up to isomorphism}\end{array}\right\}$$

such that if $\mathscr{L}$ corresponds to $\mathcal{G}$, then $\mathscr{L} \otimes \mathbb{Q}_l(-1/2)$ is compatible with the F-isocrystal $\mathbb{D}(\mathscr{G}) \otimes \mathbb{Q}$.

If $\mathscr{G}$ is *everywhere versally deformed* on X, Xia's work [30] shows that the pair $(X, \mathscr{G})$ may be canonically lifted to characteristic 0. In this case the whole correspondence is the reduction modulo p of an étale correspondence of Shimura curves. However, examples coming from Shimura curves with Igusa level structure show that $\mathscr{G}$ may be generically versally deformed without being everywhere versally deformed. For more details, see [14] or [15].

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2. Conventions, Notation, and Terminology

We explicitly state conventions and notations. These are in full force unless otherwise stated.

(1) p is a prime number and q is a power of p.
(2) $\mathbb{F}$ is a fixed algebraic closure of $\mathbb{F}_p$.
(3) A *curve* C over a field k is a quasi-projective geometrically integral scheme of dimension 1 over k. Unless otherwise explicitly stated, we assume $C \rightarrow \mathrm{Spec}(k)$ is smooth.
(4) A *morphism of curves* $X \rightarrow Y$ over k is a morphism of k-schemes that is non-constant, finite, and generically separable.

(5) A smooth curve C over a field k is said to be *hyperbolic* if $\mathrm{Aut}_{\overline{k}}(C_{\overline{k}})$ is finite.
(6) A *complex Shimura Curve* is a finite étale cover of a one-dimensional complex Shimura variety. (Other authors call such curves *arithmetic curves* or *Shimura-arithmetic curves*.)
(7) Given a field k, Ω will always be an algebraically closed field of transcendence degree 1 over k.
(8) In general, X, Y, and Z will be a curves over k, with $M = k(Z)$, $L = k(X)$, and $K = k(Y)$ the function fields. We *fix* a k-algebra embedding $PQ : k(Z) \hookrightarrow \Omega$ that identifies Ω as an algebraic closure of $k(Z)$.

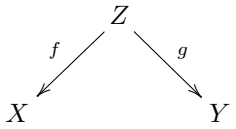
3. Correspondences and Cores

Definition 3.1. A smooth curve X over a field k is said to be hyperbolic if $\mathrm{Aut}_{\overline{k}}(X_{\overline{k}})$ is finite.

This is equivalent to the usual criterion of $2g - 2 + r \geq 1$ where g is the geometric genus of the compactification $\overline{X}$ and r is the number of geometric punctures. Over the complex numbers, this is equivalent to X being uniformized by the upper half plane $\mathbb{H}$.

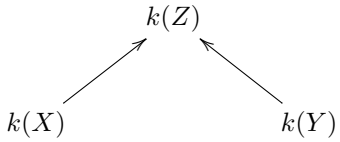
Lemma 3.2. *If $X \to Y$ is a non-constant morphism of curves over k where Y is hyperbolic, then X is hyperbolic.*

Definition 3.3. A *correspondence of curves over k* is a diagram



of smooth curves over a field k where f and g are finite, generically separable morphisms. We call such a correspondence of *type* (d, e) if $\deg f = d$ and $\deg g = e$. We call such a correspondence *étale* if both maps are étale. We call a correspondence *minimal* if the associated map $Z \to X \times Y$ is birational onto its image.

To a correspondence we can associate a containment diagram of function fields:



A correspondence is minimal if and only if there is no proper subfield of $k(Z)$ that contains both $k(X)$ and $k(Y)$.

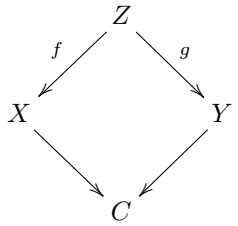
Remark 3.4. Note that we require both f and g to be *finite*; for instance, strict open immersions are not permitted.

Definition 3.5. We say a correspondence $X \leftarrow Z \to Y$ of curves over k has a *core* if the intersection of the two function fields $k(X) \cap k(Y)$ has transcendence degree 1 over k.

Remark 3.6. If a correspondence has a core, then $k(X) \cap k(Y) \subset k(Z)$ is a finite separable field extension. Indeed, suppose it weren't. The morphisms f and g are generically separable. Then at least one of the extensions $k(X) \cap k(Y) \subset k(X)$ or $k(X) \cap k(Y) \subset k(Y)$ is inseparable. Suppose $k(X) \cap k(Y) \subset k(Y)$ is not separable. Then there exists an element $\lambda \in k(X)$ such that $\lambda \notin k(Y)$ but $\lambda^p \in k(Y)$. But $\lambda \in k(Z)$, so g is not separable, contrary to our original assumption

Suppose $X \leftarrow Z \to Y$ is a correspondence of curves over k with a core. If X, Y, and Z are projective, we call the smooth projective curve C associated to the field $k(X) \cap k(Y)$ (considered as a field of transcendence degree 1 over k) the *coarse core* of the correspondence if it exists. One may also define the coarse core if X, Y, and Z are affine, see Remark 4.4.

In particular, a correspondence of curves over k has a core if and only if there exists a curve C over k with finite, generically separable maps from X and Y such that the following diagram commutes.



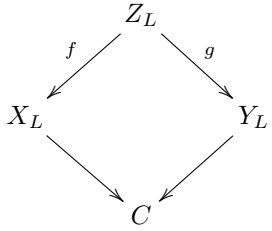
Remark 3.7. Given a correspondence as above, consider the following "many-valued function" $X \dashrightarrow X$ that sends $x \in X$ to the multi-set $f(g^{-1}(g(f^{-1}(x))))$, i.e. start with x, take all pre-images under f, take the image under g, take all pre-images under g and take the image under f. Having a core guarantees that the dynamics of this many-valued function are *uniformly bounded* in the following sense: there exists a positive integer D such that, starting with any point x and iteratively applying the above many-valued function, the size of the image set is no greater than D. In other words, the "orbit" of x under the many-valued function is finite and has size $\leq D$.

We had initially written the following Proposition in the case when L/k was an algebraic field extension. The referee explained that this restriction was unnecessary.

Proposition 3.8. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k. Let L be a field extension of k and $X_L \leftarrow Z_L \rightarrow Y_L$ the base-changed correspondence of curves over L. Then $X \leftarrow Z \rightarrow Y$ has a core if and only if $X_L \leftarrow Z_L \rightarrow Y_L$ has a core.*

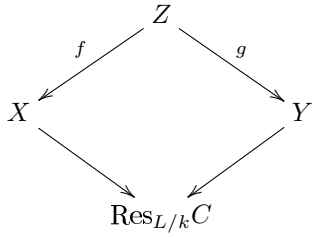
Proof. We may assume X, Y, and Z are projective. If $X \leftarrow Z \rightarrow Y$ has a core, then so does $X_L \leftarrow Z_L \rightarrow Y_L$, so it remains to prove the reverse implication.

First, if $X_L \leftarrow Z_L \rightarrow Y_L$ has a core, then a core exists after a finitely-generated field extension, i.e. we may assume that L/k is finitely generated. Call the coarse core C. Now we may spread out the diagram



to a commutative square of projective curves over a finite-type k-scheme U. There is a finite extension k'/k such that $U(k')$ is non-empty. Specializing to such a point $u \in U(k')$, we see that the correspondence $X_{k'} \leftarrow Z_{k'} \rightarrow Y_{k'}$ of curves over k' has a core. In particular, we reduce to the case of L/k being a finite extension.

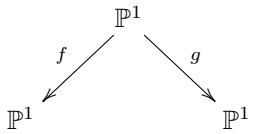
The Weil restriction of scalars $\mathrm{Res}_{L/k}C$ is a smooth, geometrically connected scheme over k by e.g. A.5.9 of [6]. The universal property of $\mathrm{Res}_{L/k}C$ yields the following commutative diagram with non-constant morphisms.



Taking the image of X and Y inside of $\mathrm{Res}_{L/k}C$ allows us to conclude that $X \leftarrow Z \rightarrow Y$ had a core. □

Remark 3.9. In our conventions, a curve C over a field k is geometrically integral. Therefore k is algebraically closed inside of $k(C)$. If $X \leftarrow Z \rightarrow Y$ is a correspondence of curves over k without a core, then the natural map $k \hookrightarrow k(X) \cap k(Y)$ is therefore an isomorphism, as $k(X) \cap k(Y)$ is a finite extension of k that is contained in $k(X)$.

For most of this article, we focus on correspondences without a core. General correspondences of curves will not have cores. Consider, for instance a correspondence of the form



For general f and g, the dynamics of the induced many-valued-function $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ will be unbounded and hence it will not have a core by Remark 3.7. When we restrict to étale correspondences without cores, there is the following remarkable theorem of Mochizuki [20] (due in large part to Margulis [17]), which is the starting point of this article.

Theorem 3.10. [20] *If $X \leftarrow Z \rightarrow Y$ is an étale correspondence of hyperbolic curves without a core over a field k of characteristic 0 , then X, Y, and Z are all Shimura (a.k.a. Shimura-arithemtic) curves (see Definitions 2.2, 2.3 of loc. cit.)*

Remark 3.11. Theorem 3.10 in particular implies that if $X \leftarrow Z \rightarrow Y$ is an étale correspondence of complex hyperbolic curves without a core, then all of the curves and maps can be defined over $\overline{\mathbb{Q}}$: all Shimura curves are defined over $\overline{\mathbb{Q}}$ as in Remark 3.14, and the set of non-constant algebraic morphisms between two complex hyperbolic curves is finite (see also Theorem 4.1 of [20].) This will imply there are *no non-trivial (global) deformations* of étale correspondences of projective hyperbolic curves without a core over $\mathbb{C}$, see Corollary 4.13. This fails in characteristic p; we will see examples, explained by Ching-Li Chai, later in Example 3.19.

The proof of Theorem 3.10 comes down to a reduction to $k \cong \mathbb{C}$ by the Lefschetz principle and an explicit description, due to Margulis [17], of the arithmetic subgroups Γ of $SL(2, \mathbb{R})$. Given a complex hyperbolic curve C, fix a uniformization $\mathbb{H} \rightarrow C$ to obtain an embedding

$$\Gamma := \pi_1(C) \rightarrow SL(2, \mathbb{R})$$

We say $\gamma \in SL(2, \mathbb{R})$ commensurates Γ if the discrete group $\gamma\Gamma\gamma^{-1}$ is commensurable with Γ, i.e. their intersection is of finite index in both groups. Define $Comm(\Gamma)$ to be the subgroup commensurating Γ in $SL(2, \mathbb{R})$ and note that $\Gamma \subset Comm(\Gamma)$. Margulis has proved that Γ is arithmetic if and only if $[Comm(\Gamma) : \Gamma] = \infty$, see e.g. Theorem 2.5 of [20].

Example 3.12. The commensurator of $SL(2, \mathbb{Z})$ in $SL(2, \mathbb{R})$ is $SL(2, \mathbb{Q})$. The modular curve $Y(1) = [\mathbb{H}/SL(2, \mathbb{Z})]$ is arithmetic.

Exercise 3.13. The correspondence of non-projective stacky modular curves $Y(1) \leftarrow Y_0(2) \rightarrow Y(1)$ does not have a core. Here, $Y_0(2)$ is the moduli space of pairs of elliptic curves $(E_1 \overset{2:1}{\rightarrow} E_2)$ with a given degree-2 isogeny between them, and the maps send the isogeny to the source and target elliptic curve respectively. Hint: to prove this over the complex numbers, look at the "orbits" of $\tau \in \mathbb{H}$ as in Remark 3.7.

In our conventions, we declared a *(complex) Shimura curve* to be a finite étale cover of a one-dimensional complex Shimura variety. We briefly comment on this.

Remark 3.14. In Definitions 2.2 and 2.3 of [20], Mochizuki defines two notions of arithmetic hyperbolic Riemann surface: Margulis arithmeticity and Shimura arithmeticity. Margulis arithmeticity is closer in spirit to the classical definition of a Shimura variety, while Shimura arithmeticity is essentially given by the data of a totally real field F and a quaternion algebra D over F that is split at exactly one of the infinite places. Proposition 2.4 then proves these two definitions are equivalent. If X is an arithmetic

curve and $Y \to X$ is a finite étale cover, then Y is manifestly arithmetic by either definition. In particular, the hyperbolic Riemann surfaces associated to *non-congruence subgroups* of $SL_2(\mathbb{Z})$ are arithmetic by definition. What we call a complex Shimura curve is precisely what Mochizuki calls an arithmetic curve.

The algebraic curve $\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}$ is a degree 2 étale cover of the stack $Y(2) = [\mathbb{H}/\Gamma(2)]$ and is hence a complex Shimura curve under our conventions. Therefore by Belyi's theorem, for any curve $X/\overline{\mathbb{Q}}$ and any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, the complex curve $X_{\mathbb{C}}$ is birational to a complex Shimura curve. Conversely, any complex Shimura curve may defined over $\overline{\mathbb{Q}}$: every Shimura variety is defined over $\overline{\mathbb{Q}}$ [2, 7, 18, 19], and a finite étale cover of a $\overline{\mathbb{Q}}$-variety is again defined over $\overline{\mathbb{Q}}$.

Definition 3.15. Let D be an indefinite non-split quaternion algebra over $\mathbb{Q}$ of discriminant d and let $\mathcal{O}_D$ be a fixed maximal order. Let k be a field whose characteristic is prime to d. A *fake elliptic curve with multiplication by* $\mathcal{O}_D$ is a pair (A, i) of an abelian surface A over k and an injective ring homomorphism $i : \mathcal{O}_D \to \mathrm{End}_k(A)$.

Pick $t \in \mathcal{O}_D$ such that $t^2 = -d$ and denote by ι the canonical involution on D. The fake elliptic curve (A, i) is endowed with the unique principal polarization such that the Rosati involution induces the following involution

$$x \to t^{-1}x^{\iota}t$$

on D.

Just as one can construct a modular curve parameterizing elliptic curves, there is a Shimura curve X^D parameterizing fake elliptic curves with multiplication by $\mathcal{O}_D$. Over the complex numbers, these are compact hyperbolic curves. Explicitly, if one chooses an isomorphism $D \otimes \mathbb{R} \cong M_{2\times 2}(\mathbb{R})$, look at the image of $\Gamma = \mathcal{O}_D^1$ of elements of $\mathcal{O}_D^*$ of norm 1 (for the standard norm on $\mathcal{O}_D$) inside of $SL(2, \mathbb{R})$. This is a discrete subgroup and in fact acts properly discontinuously and cocompactly on $\mathbb{H}$. The quotient $[\mathbb{H}/\Gamma]$ is the Shimura curve associated to $\mathcal{O}_D$. There is a notion of isogeny of fake elliptic curves which is required to be compatible with the $\mathcal{O}_D$ structure and the associated "fake degree" of an isogeny. See Buzzard [4] or Boutot-Carayol [3] for more details. These definitions allow us the define Hecke correspondences as in the elliptic modular case. For instance, as long as 2 splits in D, one can define the correspondence

$$\begin{array}{ccc} & X_0^D(2) & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ X^D & & X^D \end{array}$$

where $X_0^D(2)$ parametrizes pairs of fake elliptic curves $(A_1 \to A_2)$ with a given isogeny of fake degree 2 between them and π_1 and π_2 are the projections to the source and target respectively. This is an example of an étale correspondence of (stacky) hyperbolic curves without a core. To get an example without orbifold points, one can add auxiliary level structure by picking an open compact subgroup $K \subset \mathbb{A}^f$ of the finite adeles. This correspondence is in fact defined over $\mathbb{Z}[\frac{1}{2S}]$ for an integer S and so may be reduced modulo p for almost all primes.

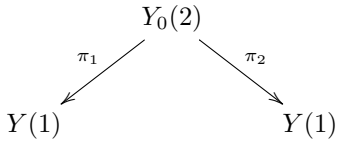
Motivated by these examples, the orienting question of this article is to explore characteristic p analogs of Mochizuki's theorem. More specifically, we wish to understand the abstract structure of étale correspondences of hyperbolic curves without a core.

Question 3.16. *Let k be a field of characteristic p. If $X \leftarrow Z \to Y$ is an étale correspondence of hyperbolic curves over k without a core, then is it related to a Hecke correspondence of Shimura varieties or Drinfeld modular varieties?*

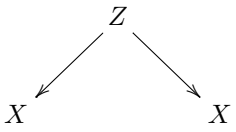
Remark 3.17. In Corollary 4.12 we will in some sense reduce Question 3.16 to the analogous question with $k = \mathbb{F}$.

The clause "is related to" in Question 3.16 is absolutely vital as we will see in the following examples. Nonetheless we take Question 3.16 as a guiding principle.

Remark 3.18. There are examples of étale correspondence of hyperbolic curves without a core that should not deform to characteristic 0. Consider, for instance, the Hecke correspondence



of modular curves over $\mathbb{F}_p$, $p \neq 2$. By definition, there is a universal elliptic curve $\mathcal{E} \to Y(1)$. Let $\mathcal{G} = \mathcal{E}[p^\infty]$ be the associated p-divisible group over $Y(1)$. Note that $\pi_1^*\mathcal{G} \cong \pi_2^*\mathcal{G}$. Let X be the cover of $Y(1)$ that trivializes the finite flat group scheme $\mathcal{G}[p]_{\acute{e}t}$ away from the supersingular locus of $Y(1)$. X is branched exactly at the supersingular points. Let Z be the analogous cover of $Y_0(2)$. Then we have an étale correspondence



which does not have a core (the dynamics of an ordinary point are unbounded) and morally one does not expect this correspondence to lift to characteristic 0. This construction is referred to as adding *Igusa level structure* in the literature: Ulmer's article [27] is a particularly lucid account of this story for modular curves. See Definition 4.8 of Buzzard [4] for the analogous construction for Shimura curves parameterizing fake elliptic curves. We take up the example of Igusa curves once again in Example 8.5, from the perspective of the Hasse invariant and the cyclic cover trick.

Modular curves with Igusa level structure still parametrize elliptic curves with some (purely characteristic p) level structure. Ching-Li Chai has provided the following more exotic example which shows that étale correspondence of hyperbolic curves over a field of characteristic p may deform purely in characteristic p.

Example 3.19. Let F be a totally real cubic field and let p be an inert prime. Consider the Hilbert modular threefold $\mathcal{X}^F$ associated to $\mathcal{O}_F$; $\mathcal{X}^F$ parametrizes abelian threefolds with multiplication by $\mathcal{O}_F$. Let X be the reduction of $\mathcal{X}^F$ modulo p and $\mathscr{A}$ be the universal abelian scheme over X. Oort has constructed a foliation on such Shimura varieties [22]; a leaf of this foliation has the property that the p-divisible group $\mathscr{A}[p^\infty]$ is geometrically constant on the leaf; i.e., if x and y are two geometric points of the leaf, then $\mathscr{A}[p^\infty]_x \cong \mathscr{A}[p^\infty]_y$. We list the possible slopes of a height 6, dimension 3, symmetric p-divisible group.

(1) $(0, 0, 0, 1, 1, 1)$
(2) $(0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1)$
(3) $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$
(4) $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$
(5) $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

The only slope types that could possibly admit multiplication (up to isogeny) by $\mathbb{Q}_{p^3}$ are 1, 4, and 5 by considerations on the endomorphism algebra. By de Jong-Oort purity, the locus where the slope type $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ occurs inside of X is codimension 1. One can prove that a central leaf with this Newton Polygon is a curve. Central leafs of Hilbert modular varieties have the property that they are preserved under l-adic Hecke correspondences and that in fact the l-adic monodromy is as large as possible [5]. In particular, a central leaf has many Hecke correspondences. Moreover, as this Newton polygon stratum has dimension 2, the isogeny foliation is one-dimensional and so this Hecke correspondence deforms in a one-parameter family, purely in characteristic p.

We further remark that, in general, there are central leaves of Shimura varieties that are not defined over $\overline{\mathbb{F}}_p$; in particular there are étale correspondences without a core over fields of characteristic p that *do not descend* to the algebraic closure of a prime field, unlike the case of characteristic 0.

Remark 3.20. Chai and Oort have discussed the possibility that central leaves should be considered Shimura varieties in characteristic p. In particular, one could consider the example of a Hecke correspondence of a central leaf to be a Hecke correspondence of Shimura curves. In any case, both the examples of a Hecke correspondence of modular curves with Igusa level structure and a Hecke correspondence of a central leaf of dimension 1 map finitely onto a Hecke correspondence of Shimura varieties which deform to characteristic 0.

In my thesis, I phrased Question 3.16 only using Shimura varieties. Ambrus Pál has informed us that there are examples of étale correspondences of Drinfeld modular curves (i.e. moduli spaces of $\mathcal{D}$-elliptic modules) over $\mathbb{F}$ without a core. All three of these examples have moduli interpretations. Moreover, they all have *many* Hecke correspondences. This motivates the following variant of Question 3.16, which does not mention Shimura/Drinfeld modular varieties at all.

Question 3.21. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of hyperbolic curves over k without a core. Do there exist infinitely many minimal étale correspondences between X and Y without a core?*

4. A Recursive Tower

In this section we associate an infinite tower of curves to a correspondence without a core. Properties of this tower will allow us to deduce that "being an étale correspondence without a core" specializes in families, Lemma 4.10; Corollary 4.13 then shows that there are no (global) deformations of an étale correspondence of complex projective hyperbolic curves without a core. We also find a simple criterion for an étale correspondence of projective hyperbolic curves over $\mathbb{F}$ to consist of the reductions modulo p of Shimura curves: if the correspondence lifts to $W(\mathbb{F})$ (Corollary 4.15).

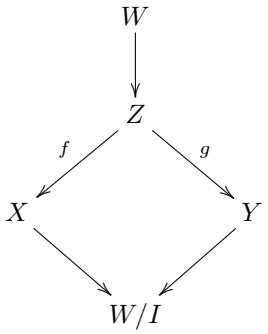
Definition 4.1. Let $f : X \rightarrow Y$ be a finite, non-constant, generically separable map of curves over a field k. We say f is *finite Galois* if $|\mathrm{Aut}_Y(X)| = \deg(f)$. We say it is *geometrically finite Galois* if $f_{\overline{k}} : X_{\overline{k}} \rightarrow Y_{\overline{k}}$ is Galois.

It is well-known that given a finite, generically separable map of curves over a field k, we may take a Galois closure. In the projective case, this is "equivalent" to taking a Galois closure of the associated extension of function fields, and the affine case follows by the operation of "taking integral closure of the coordinate ring in the extension of function fields." However, the output of the "Galois closure" operation will not necessarily be a *curve over* k as in our conventions, i.e. it won't necessarily be a geometrically integral scheme over k, unless k is separably closed. For instance, consider the geometrically Galois morphism $\mathbb{P}^1_{\mathbb{Q}} \rightarrow \mathbb{P}^1_{\mathbb{Q}}$ given by $t \mapsto t^3$. This is not a Galois extension of fields, and a Galois closure is $\mathbb{P}^1_{\mathbb{Q}(\zeta_3)}$, which is not a geometrically irreducible variety over $\mathbb{Q}$. In the language of field theory, the field extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_3)(t)$ is not *regular*. Therefore, when we take a Galois closure, we implicitly extended the field k if necessary to ensure that the output is a *curve over* k.

We begin with a simple Galois-theoretic observation related to the existence of a core.

Lemma 4.2. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence over a field k where Z is hyperbolic. A core exists if and only if there exists a curve W, possibly after replacing k by a finite extension, together with a map $W \rightarrow Z$ such that the composite maps $W \rightarrow X$ and $W \rightarrow Y$ are both finite Galois.*

Proof. Suppose such a curve W existed. Then W is hyperbolic because it maps nontrivially to a hyperbolic curve (Lemma 3.2.) The groups $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$ are both subgroups of $\mathrm{Aut}_k(W)$, which is a finite group because W is hyperbolic. The group I generated by these two Galois groups is therefore finite. Therefore the curve W/I fits into a diagram:



Therefore a core exists. Conversely, if the correspondence has a core, call the coarse core C. Let W be a Galois closure of the map $Z \to C$, finitely extending the ground field if necessary. Then the composite maps $W \to X$ and $W \to Y$ are both finite Galois as desired. □

Lemma 4.3. *Let $X \leftarrow Z \to Y$ be a correspondence of (possibly non-hyperbolic) curves over $\mathbb{F}$. Then a core exists if and only if there exists a curve W together with a map $W \to Z$ such that the composite maps $W \to X$ and $W \to Y$ are both finite Galois.*

Proof. The proof is almost exactly the same as that of Lemma 4.2: the key observation is that everything in sight (including every element of $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$) may be defined over some finite field $\mathbb{F}_q$; therefore the group they generate inside of $\mathrm{Aut}(W)$ consists of automorphisms defined over $\mathbb{F}_q$ and is hence finite. □

Remark 4.4. Let $X \leftarrow Z \to Y$ be a correspondence of affine curves with a core. We prove there exists a curve C together with *finite, generically separable* maps from X and Y making the square commute.

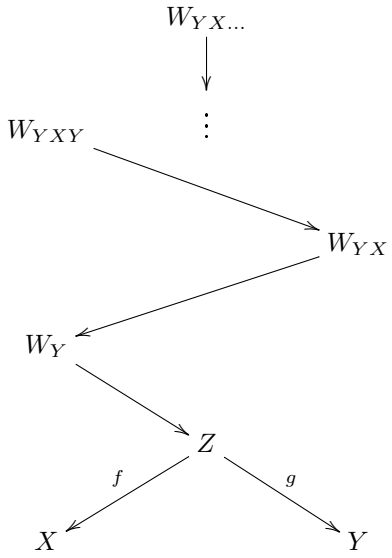
Let $\overline{X} \leftarrow \overline{Z} \to \overline{Y}$ be the compactified correspondence, with coarse core T. Take a Galois closure $\overline{W}$ of $\overline{Z} \to T$. Let W be the affine curve associated to the integral closure of $k[Z]$ in $k(\overline{W})$. Then $W \to X$ and $W \to Y$ are both finite Galois morphisms of affine curves. In fact, $\mathrm{Aut}(W/X) = \mathrm{Aut}(\overline{W}/\overline{X})$ and likewise for Y. The group I generated by $\mathrm{Aut}(W/X)$ and $\mathrm{Aut}(W/Y)$ inside of $\mathrm{Aut}(W)$ is precisely $\mathrm{Aut}(W/T) = \mathrm{Aut}(\overline{W}/T)$, as T was the coarse core of the projective correspondence. Set $C = W/I$, the affine curve with coordinate ring $k[W]^I$. This C is the *coarse core* of the correspondence of affine curves.

Example 4.5. Let us see the relevance both of Z being hyperbolic in Lemma 4.2 and of the base field being $\mathbb{F}$ in Lemma 4.3. Let $Z = \mathbb{P}^1_{\mathbb{F}_p(t)}$ and consider the following finite subgroups of $PGL(2, \mathbb{F}_p(t))$: H_1 is generated by the unipotent element $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and H_2 is generated by the unipotent element $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Quotienting Z gives a correspondence $Z/H_1 \leftarrow Z \to Z/H_2$. Both arrows are Galois, but there is evidently no core because the subgroup of $PGL(2, \mathbb{F}_p(t))$ generated by H_1 and H_2 is infinite. Note that for every specialization of $t \in \mathbb{F}$, the correspondence does in fact have a core, for instance by Corollary 4.3.

Let $X \leftarrow Z \to Y$ be a correspondence of curves without a core where Z is hyperbolic or where $k \cong \mathbb{F}$. We perform the following iterative procedure: take a Galois closure of $Z \to Y$ and call it W_Y. Because we assumed a core does not exist, the associated map $W_Y \to X$ cannot be Galois by Lemma 4.2 (resp. Lemma 4.3). Take a Galois closure of this map and call it W_{YX}. Again, the associated map $W_{YX} \to Y$ cannot be Galois, so we can take a Galois closure to obtain W_{YXY}. Continuing in the

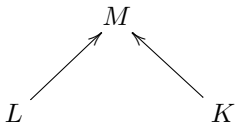
fashion, we get an inverse system of curves $W_{YX\dots}$ over the correspondence.

(4.1)



Note that $W_{YX\dots}$ is Galois over Z. In fact, $W_{YX\dots}$ is Galois over both X and Y because there is a final system of Galois subcovers for each. Note that this procedure may involve algebraic extensions of the field k.

We explicate the based function-field perspective on this construction: let



be the associated diagram of function fields, where $M = k(Z)$, $L = k(X)$, and $K = k(Y)$. Recall that $k \xrightarrow{\sim} L \cap K$; this is exactly the condition that correspondence does not have a core.

Pick an algebraic closure Ω of M, i.e. let Ω be an algebraically closed field of transcendence degree 1 over k and pick *once and for all* an embedding of k-algebras $PQ : M \hookrightarrow \Omega$. (The notation will be justified later, when PQ will correspond to an edge of a graph.) Let E_K be the Galois closure of M/K in Ω. Then E_K/L is no longer Galois by Lemma 4.2 (resp. Lemma 4.3). Let E_{KL} be the Galois closure of E_K/L in Ω. Continuing in this fashion, we get an infinite algebraic field extension $E_{KL\dots}$ of M, Galois over both L and K.

Lemma 4.6. *$W_{YX\dots}$ is isomorphic to $W_{XY\dots}$ as Z-schemes. That is, by reversing the roles of X and Y we get mutually final systems of Galois covers.*

Proof. Equivalently, we must show that $E_{KL\dots} = E_{LK\dots}$ as subfields of Ω. First, note that $E_K \subset E_{LK}$ because E_K is the minimal extension of E in Ω that is Galois over K. Similarly, $E_{KL} \subset E_{LKL}$ because E_{KL} is the minimal extension of E_K in Ω that is Galois over L. Continuing, we see that $E_{KL\dots} \subset E_{LK\dots}$. By symmetry, the reverse inclusion holds as desired. □

Corollary 4.7. *The field extension $E_{KL\dots} = E_{LK\dots}$ of M, thought of as a subfield of Ω, is characterized by the property that it is the minimal field extension of M inside of Ω that is Galois over both L and K.*

For brevity, we denote the inverse system $W_{XYX\dots}$ by W_∞. Let E_∞ be the associated function field, considered as a subfield of Ω. In what follows, unless otherwise specified we consider $E_\infty \subset \Omega$ as inclusions of abstract k-algebras.

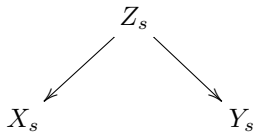
Question 4.8. *Does the "field of constants" of E_∞ have finite degree over k? That is, does $E_\infty \otimes_k \overline{k}$ decompose as an algebra to be the product of finitely many fields? What if the original correspondence is étale?*

If $\mathrm{Gal}(k^{sep}/k)$ is abelian and $\mathrm{Gal}(E_\infty/K)$ has finite abelianization, then Question 4.8 has an affirmative answer. In particular, this applies if $k \cong \mathbb{F}_q$ and $\mathrm{Gal}(E_\infty/K)$ is a semi-simple l-adic group. We will see in Proposition 9.10 that if the correspondence has an *étale clump*, then Question 4.8 has an affirmative answer using the following remark.

Remark 4.9. In Diagram 4.1, the morphism $W_Y \to Z$ is Galois. By precomposing with $\mathrm{Aut}(W_Y/Z)$, we equip W_Y with $\mathrm{Aut}(W_Y/Z)$-many maps to Z. More generally, all curves $W_{YX\dots X}$ will be naturally equipped with $\mathrm{Aut}(W_{YX\dots X}/Z)$-many maps to Z via precompositions by Galois automorphisms. This will be useful in Remark 7.3, when we try to explicitly understand the curves $W_{YX\dots Y}$.

The following lemma allows us to *specialize* étale correspondences without a core.

Lemma 4.10. *Let S be an irreducible Noetherian scheme with generic point η. Let X, Y, and Z be proper, smooth, geometrically integral curves over S. Suppose Z is "hyperbolic" over S; that is, the genus of a fiber is at least 2. Let $X \leftarrow Z \to Y$ be a finite étale correspondence of schemes commuting with the structure maps to S. If s is a geometric point of S such that*



has a core, then $X_\eta \leftarrow Z_\eta \to Y_\eta$ has a core.

Proof. The property of "having a core" does not change under algebraic field extension by Proposition 3.8. By dévissage, we reduce to the case of $S = \mathrm{Spec}(R)$, where R is a discrete valuation ring with algebraically closed residue field κ. Call the fraction field K. We may further replace R by its integral closure in $\overline{K}$ to get a valuation ring having both the residue field and the fraction field algebraically closed. We do this to not worry about the "extension of ground field" question that is always present when taking a Galois closure.

Call the generic point η and the closed point s. First of all $X_\eta \leftarrow Z_\eta \to Y_\eta$ is a correspondence of curves over η. Let us suppose it does not have a core. Then the process of iterated Galois closure, as detailed in Diagram 4.1, continues endlessly to produce a tower of curves over η. On the other hand, any finite étale morphism has a Galois closure. This implies that we can apply the construction of taking iterated Galois closures to the *finite étale correspondence of schemes* $X \leftarrow Z \to Y$ to build a tower $W_{YX\dots}$ over S.

As R has algebraically closed residue field and fraction field, $W_{YX\dots X}$ is a smooth proper curve over S; in particular the geometric fibers of the morphism $W_{YX\dots X} \to S$ are irreducible. Moreover, all of the maps $W_{YX\dots Y} \to Z$ are finite étale. In fact, the fiber of $W_{XY\dots X}$ over the generic point η of $W_{XY\dots X}$ is isomorphic, as a scheme over Z_η, to the corresponding curve in the tower associated to the correspondence $X_\eta \leftarrow Z_\eta \to Y_\eta$ of curves over η. For instance, $(W_Y)_\eta \to Y_\eta$ is a Galois closure of the finite étale morphism $Z_\eta \to Y_\eta$. Therefore, if we could prove $(W_{YX\dots X})_\eta$ were disconnected, we would get a contradiction with the original assumption that $X_\eta \leftarrow Z_\eta \to Y_\eta$ had no core.

The fact that the maps $Z \to X$, $Z \to Y$, and $W_{YX\dots Y} \to Z$ are finite étale implies that taking a Galois closure and then restricting to s yields a finite Galois étale cover of Z_s. For example, $(W_Y)_s$ is a (possibly disconnected) finite Galois cover of Y_s that maps surjectively to $(W_s)_{Y_s}$, a Galois closure of the map $Z_s \to Y_s$.

As the correspondence specialized to s has a core and $Z \to S$ is assumed to be hyperbolic, Lemma 4.2 implies that there exists a curve $W_{YX\dots Y}$ of our tower over S such that the fiber $(W_{YX\dots Y})_s$ is disconnected. We therefore have a smooth proper curve $W_{XY\dots Y} \to S$ such that the fiber over s is disconnected. Zariski's connectedness principle (e.g. Corollary 8.3.6 of [16] for the case of curves) implies

$(W_{YX\ldots Y})_\eta$ is disconnected (this is where we use properness), contradicting our original assumption that $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ had no core. □

Remark 4.11. Example 4.5 shows that the argument of Lemma 4.10 *does not work* if the correspondence is not assumed to be étale.

The following Corollary shows we may "reduce" the study of Question 3.16 to where $k = \mathbb{F}$.

Corollary 4.12. *Given an étale correspondence of hyperbolic curves $X \leftarrow Z \rightarrow Y$ without a core over a field k of characteristic p, we can specialize to an étale correspondence without a core over $\mathbb{F}$.*

Proof. By spreading out, we may ensure that we are in the situation of Lemma 4.10. Then the nonexistence of a core implies the same for all of the geometric fibers by Lemma 4.10. □

Corollary 4.13. *Let S be an integral scheme of finite type over $\mathbb{C}$ with generic point η. Let X, Y, and Z be proper, smooth, geometrically integral, hyperbolic curves over S. Let $X \leftarrow Z \rightarrow Y$ be a finite étale correspondence of schemes commuting with the structure maps to S. Suppose that $X_\eta \leftarrow Z_\eta \rightarrow Y_\eta$ has no core. Then this family of correspondences is étale locally constant.*

Proof. Let $\mathcal{M}$ denote the moduli space of finite étale correspondences of projective curves with the same genera as X, Y, and Z. This is a finite-type moduli "space" (i.e. DM algebraic stack) that may be defined over $\overline{\mathbb{Q}}$ and has quasi-projective coarse space by the analogous assertions for $\mathcal{M}_g$ for $g \geq 2$. In particular, we obtain an algebraic map

$$u : S \rightarrow \mathcal{M}_{\mathbb{C}}$$

classifying the family $X \leftarrow Z \rightarrow Y$ over S. By Lemma 4.10, for every point $s \in S(\mathbb{C})$, the correspondence $X_s \leftarrow Z_s \rightarrow Y_s$ has no core; therefore Remark 3.11 implies that the image $u(s)$ is a $\overline{\mathbb{Q}}$-valued point of $\mathcal{M}(\mathbb{C})$. On the other hand, $u(S)$, being the image of a morphism between an irreducible finite-type $\mathbb{C}$-scheme and a finite-type $\mathbb{C}$ DM-stack, either has image uncountable or a single point by Chevalley's theorem: a constructible subset of a finite-type DM stack over $\mathbb{C}$ cannot be countably infinite.

We therefore see that the image of u is a single point; as S is reduced, there exists an étale cover $S' \rightarrow S$ such that when we pull $X \leftarrow Z \rightarrow Y$ to S', the family is locally constant. □

In other words, Corollary 4.13 says that if $X \leftarrow Z \rightarrow Y$ is an étale correspondence of compact hyperbolic Riemann surfaces without a core, then inside of the moduli space $\mathcal{M}$ of finite étale correspondences of projective curves with the appropriate genera, the moduli point $[X \leftarrow Z \rightarrow Y]$ is isolated.

Lemma 4.10 says that for an étale correspondence of projective hyperbolic curves, the property of "not having a core" specializes. We now show that the property of "not having a core" generalizes, even without the assumption of étale-ness. This is rather useful: it implies that one way to answer Question 3.16 is to directly lift the correspondence to characteristic 0.

Lemma 4.14. *Let $S = Spec(R)$ be the spectrum of a discrete valuation ring with closed point s and generic point η. Let X, Y, and Z be smooth, projective, geometrically irreducible curves over S and let $X \leftarrow Z \rightarrow Y$ be a correspondence of schemes, commuting with the structure maps to S, that is a correspondence of curves when restricted to s and to η. If over η the correspondence has a core, then over s the correspondence has a core.*

Proof. Let π be a uniformizer of R. Denote by κ residue field of R and by K the fraction field of R. Pick a non-constant rational function f in the intersection $K(X) \cap K(Y)$ (the intersection takes place in $K(Z)$.) By multiplying by an appropriate power of π, we can guarantee that f extends to rational functions on the special fiber and in fact that f has nonzero reduction in $0 \neq \overline{f} \in \kappa(X_s) \cap \kappa(Y_s)$. Suppose f is constant modulo π, or equivalently that $f \equiv c (\mathrm{mod} \pi)$ for some $c \in R$. Then $\frac{f-c}{\pi}$ may again be reduced modulo π. If $\frac{f-c}{\pi}$ is non-constant on the special fiber, we are done, so suppose not and repeat the procedure. This procedure terminates because our original choice of $f \in K(X)$ was non-constant and the result will be a non-constant function in $\kappa(X_s) \cap \kappa(Y_s)$. □

Corollary 4.15. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of smooth projective hyperbolic curves over $\mathbb{F}$ without a core. If correspondence lifts to a correspondence of curves $\tilde{X} \leftarrow \tilde{Z} \rightarrow \tilde{Y}$ over $W(\mathbb{F})$, then X, Y, and Z are the reductions modulo p of Shimura curves.*

Proof. The lifted correspondence is automatically étale as the étale locus is open. Lemma 4.14 implies that the generic fiber does not have a core. Mochizuki's Theorem 3.10 then implies that $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ are all Shimura curves as desired. □

5. The Generic Graph of a Correspondence

Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k. and let Ω be an algebraically closed field of transcendence degree 1 over k, thought of as a k-algebra. We construct an infinite 2-colored graph $\mathcal{G}_{gen}^{full}$, which we call the *full generic graph* of the correspondence. The blue vertices of $\mathcal{G}_{gen}^{full}$ are the Ω-valued points of X; more precisely, a blue vertex is given by a k-algebra homomorphism $k(X) \rightarrow \Omega$. Similarly, the red vertices are the Ω-valued points of Y and the edges are the Ω-valued points of Z. A blue vertex $p : k(X) \hookrightarrow \Omega$ and red vertex $q : k(Y) \hookrightarrow \Omega$ are joined by an edge if there exists an embedding $k(Z) \hookrightarrow \Omega$ that restricts to p and to q on the subfields $k(X)$ and $k(Y)$ respectively. Note that $\mathrm{Aut}_k(\Omega)$ naturally acts on the graph $\mathcal{G}_{gen}^{full}$ by post-composition.

Remark 5.1. The original correspondence is minimal if and only if there are no multiple edges in $\mathcal{G}_{gen}^{full}$. (Recall that the morphisms of curves were generically separable by definition.)

Condition 5.2. For the rest of the sections involving graph theory, we suppose that the correspondence $X \leftarrow Z \rightarrow Y$ is minimal in order that we get a graph and not a multigraph.

Definition 5.3. Given any subgraph $H \subset \mathcal{G}_{gen}^{full}$, we define the subfield $E_H \subset \Omega$ by taking the compositum of the subfields $e(k(Z)) \subset \Omega$, $p(k(X)) \subset \Omega$, and $q(k(Y)) \subset \Omega$ corresponding to all of the edges and vertices e, p, and q in H.

There is no reason to believe that $\mathcal{G}_{gen}^{full}$ is connected. We give $\mathcal{G}_{gen}^{full}$ a distinguished blue vertex P, red vertex Q, and edge PQ between them by picking the k-embedding

$$PQ : k(Z) \hookrightarrow \Omega$$

and we set the graph $\mathcal{G}_{gen}$ (the *generic graph*) to be the connected component of $\mathcal{G}_{gen}^{full}$ containing this distinguished edge. All connected components of $\mathcal{G}_{gen}^{full}$ arise in this way and all connected components of $\mathcal{G}_{gen}^{full}$ are isomorphic. We denote by $P(k(X))$ the image of the distinguished blue point P as a subfield of Ω and similarly for $Q(k(Y))$.

Lemma 5.4. *Let $H \subset \mathcal{G}_{gen}$ be the full subgraph consisting of all vertices of distance at most n from a fixed vertex v; that is, H is the closed ball $H = B(v, n)$. Then E_H is Galois over E_v.*

Proof. First of all, E_v is the field corresponding to v as in Definition 5.3. We may suppose WLOG that v is a blue vertex, so $E_v = v(k(X))$ as v is by definition a k-embedding $k(X)$ to Ω. In other words, v gives Ω the structure of a $k(X)$-algebra. Now, E_H is the compositum of all of the fields associated to all of the edges and vertices in H in Ω. In particular, if $\mathcal{P} = \{P\}$ is the collection of all paths of length n starting at v, then E_H is the compositum of $(E_P)_{P \in \mathcal{P}}$ inside of Ω. Here each E_P and E_H has a $k(X)$-algebra structure via v and our goal is to prove that E_H is Galois over $k(X)$ with respect to this algebra structure $v : k(X) \hookrightarrow E_H$.

Consider E_H together with the subfields E_P, $P \in \mathcal{P}$, as abstract $k(X)$-algebras. Let ϕ_0 be the original embedding $E_H \hookrightarrow \Omega$. To prove E_H is Galois over $k(X)$, we must show that for every

$$\phi \in \mathrm{Hom}_{k(X)}(E_H, \Omega)$$

the image of ϕ is contained in $\phi_0(E_H)$. Note that ϕ is determined by where all of the E_P are sent. Any ϕ can be obtained from ϕ_0 via an element of $\mathrm{Aut}(\Omega/k(X))$, as Ω is algebraically closed, and so a path P of length n originating at v is sent to another such path P'. In other words, $\phi(E_P) = \phi_0(E_{P'})$ for another path P' of length n originating at v. As E_H was the compositum of all such E_P, it follows that the extension $E_H/k(X)$ is Galois as desired. □

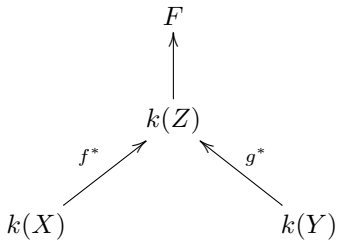
The graph $\mathcal{G}_{gen}$ is a full subgraph of $\mathcal{G}_{gen}^{full}$ so, as in Definition 5.3, we can take the associated field $E_{\mathcal{G}_{gen}} \subset \Omega$ given by the compositum of the subfields of Ω associated to the edges. Let $E \subset \Omega$ be the minimal field extension of $k(Z)$ (with respect to the embedding $PQ : k(Z) \hookrightarrow \Omega$) that is Galois over both $k(X)$ and $k(Y)$. We prove that $E = E_{\mathcal{G}_{gen}}$ with the next series of results.

Corollary 5.5. *The subfield $E_{\mathcal{G}_{gen}} \subset \Omega$ is Galois over both $P(k(X))$ and $Q(k(Y))$. Therefore $E \subset E_{\mathcal{G}_{gen}}$*

Proof. The connected graph $\mathcal{G}_{gen}$ is the union of the subgraphs $\cup_n B(P, n)$ of closed balls of radius n around P, so by Lemma 5.4 the field $E_{\mathcal{G}_{gen}}$ is Galois over $P(k(X))$. Similarly, $E_{\mathcal{G}_{gen}}$ is Galois over $Q(k(Y))$. Therefore $E \subset E_{\mathcal{G}_{gen}}$ as desired. □

Lemma 5.6. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k and embed the function fields into Ω via $PQ : k(Z) \hookrightarrow \Omega$. If there is a subfield $F \subset \Omega$ that is Galois over both $k(X)$ and $k(Y)$, then $E_{\mathcal{G}_{gen}} \subset F$.*

Proof. We have the following diagram of fields



where F is Galois over *both* $k(X)$ and $k(Y)$. The field F is naturally equipped with the structure of a $k(Z)$ algebra. Extend $PQ : k(Z) \hookrightarrow \Omega$ any which way to a $k(Z)$-algebra embedding $\phi : F \rightarrow \Omega$. Then the image of any edge adjacent to P in $\mathcal{G}_{gen}$ lands inside of the image $\phi(F)$ because F is Galois over $k(X)$. Similarly, the image of any edge adjacent to Q in $\mathcal{G}_{gen}$ lives inside the image of $\phi(F)$.

Let $q \neq Q$ be a vertex adjacent to P. There exists an automorphism $\alpha \in \mathrm{Gal}(\phi(F)/P(k(X)))$ that sends $Q(k(Y))$ to E_q because F is Galois over $k(X)$. Conjugating by α, we deduce that $\phi(F)$ is Galois over E_q and hence the image of all edges emanating from q lie in $\phi(F)$. By propagating, we get that $E_{\mathcal{G}_{gen}} \subset F$ as desired. □

Corollary 5.7. *We have an equality of fields $E = E_{\mathcal{G}_{gen}}$, considered as subfields of Ω. Equivalently, $E_{\mathcal{G}_{gen}}$ is the minimal field extension of $PQ(k(Z))$ inside of Ω that is Galois over the fields $P(k(X))$ and $Q(k(Y))$.*

Proof. Combine Lemma 5.6 and Corollary 5.5. □

Corollary 5.8. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k without a core with Z hyperbolic or with $k \cong \mathbb{F}$. Then $E_\infty \cong E_{\mathcal{G}_{gen}}$.*

Proof. The field E_∞ is also the minimal field extension of $PQ(k(Z))$ inside of Ω that is Galois over $P(k(X))$ and $Q(k(Y))$ by Corollary 4.7. □

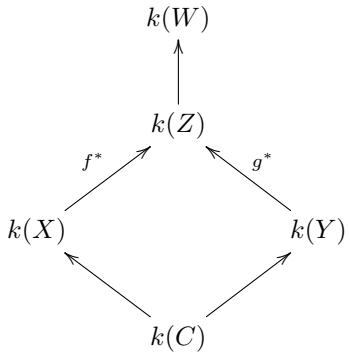
Remark 5.9. One is tempted to make a converse definition to Definition 5.3: given any subfield $E \subset \Omega$ (respectively $E \subset E_\infty$), define $\mathcal{G}_E^{full}$ (respectively $\mathcal{G}_E$) to be the subgraph of $\mathcal{G}_{gen}^{full}$ (respectively $\mathcal{G}_{gen}$) whose points and edges are have image inside of E. This definition is rather poorly behaved; for instance if one starts out with a finite connected subgraph $H \subset \mathcal{G}_{gen}$, takes $E_H \subset E_\infty$, and then looks at the associated graph G_{E_H}, there is no reason to believe that this graph is connected.

The graph $\mathcal{G}_{gen}$ informally reflects the "generic dynamics" of the correspondence. We will see one way of making this precise in Section 7: via a specialization map. Nevertheless, we have the following proposition, which says that a core exists if and only if $\mathcal{G}_{gen}$ is finite (i.e. the "generic dynamics" are bounded), in line with Remark 3.7.

Proposition 5.10. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves over k where Z is hyperbolic or where $k \cong \mathbb{F}$. This correspondence has no core if and only if $\mathcal{G}_{gen}$ is an infinite graph.*

Proof. If $\mathcal{G}_{gen}$ is finite, then $E_{\mathcal{G}_{gen}}$ is a finite Galois extension of both $k(X)$ and $k(Y)$, so the correspondence has a core by Lemma 4.2 (resp. Lemma 4.3.)

Conversely, if the correspondence had a core, then let C be the coarse core. Let W be a Galois closure of $Z \rightarrow C$. We have the following diagram of fields, where we again fix $PQ : k(Z) \hookrightarrow \Omega$ and any extension $\phi : k(W) \hookrightarrow \Omega$.



Call P, Q, and R the restriction of the algebra embedding PQ to $k(X)$, $k(Y)$, and $k(C)$ respectively. Let v be blue vertex in $\mathcal{G}_{gen}$ adjacent to Q, given by a k-algebra embedding $v : k(X) \hookrightarrow \phi(k(W)) \subset \Omega$ by Lemma 5.6. As $k(W)/k(Y)$ is Galois, there exists an automorphism

$$\alpha \in \mathrm{Gal}(\phi(k(W))/Q(k(Y))) \cong \mathrm{Gal}(k(W)/k(Y))$$

that sends P to v. As $k(C) \subset k(Y)$, this implies that $v|_{k(C)} = R$. By propagating, we see that for every vertex v of $\mathcal{G}_{gen}$, $v|_{k(C)} = R$. Therefore, for every edge $e \in \mathcal{G}_{gen}$, thought of as a k-algebra embedding $e : k(Z) \hookrightarrow \Omega$, we have that $e|_{k(C)} = R$. On the other hand, $k(Z)$ is a finite extension of $k(C)$, so there are only finitely many ways to extend R to a k-algebra homomorphism $k(Z) \hookrightarrow \Omega$. Therefore the number of edges is finite, as desired. □

We record the following easy proposition for later use in proving the surjectivity of the specialization morphism in the case of an étale correspondence without a core.

Proposition 5.11. *For any finite subgraph $H \in \mathcal{G}_{gen}$, the field E_H is contained in a finite extension F of $PQ(k(Z))$.*

Proof. We have the the following two facts.

- E_H lands inside of E_∞, which is exhausted by fields of the form $E_{KL\ldots K}$, by Lemma 5.6
- E_H is finitely generated over k.

Therefore E_H lands inside of some $E_{KL\ldots K}$, a finite extension of $PQ(k(Z))$, as desired. □

We now analyze the action of various subgroups of $\mathrm{Aut}(E_\infty)$ on $\mathcal{G}_{gen}$.

Remark 5.12. We take a brief digression into the structure of automorphism groups of fields. Let Ω be any field. We endow the group $\mathrm{Aut}(\Omega)$ with the compact-open topology, considering Ω to be a discrete set. Given any finite subset $S \subset \Omega$, the subgroup $\mathrm{Stab}(S) \subset \mathrm{Aut}(\Omega)$ is an open subgroup and as S ranges these form a neighborhood base of the identity in $\mathrm{Aut}(\Omega)$. If $K \subset \Omega$ is a separable Galois extension with K finitely generated over its prime field, the natural map $\mathrm{Gal}(\Omega/K) \subset \mathrm{Aut}(\Omega)$ is an open embedding of topological groups; in other words, the topology just defined is compatible with the usual profinite topology on Galois groups.

Note that this procedure generalizes: if $k \subset \Omega$ is a field extension, we may give the group $\mathrm{Aut}_k(\Omega)$ the structure of a topological group, where a neighborhood base of the identity is given by $\mathrm{Stab}(S)$ for finite subsets $S \subset \Omega \backslash k$. However, $\mathrm{Aut}_k(\Omega)$ is not an *open subgroup* of $\mathrm{Aut}(\Omega)$ unless k is finitely generated over its prime field.

Any element $g \in \mathrm{Aut}_k(E_\infty)$ gives a map of graphs $\mathcal{G}_{gen} \to \mathcal{G}_{gen}^{full}$ by post-composition: for instance, an edge $e : k(Z) \to E_\infty \subset \Omega$ is sent to the edge $g \circ e : k(Z) \to E_\infty \subset \Omega$. In fact, the Galois groups $G_P := \mathrm{Gal}(E_\infty / P(k(X)))$ and $G_Q := \mathrm{Gal}(E_\infty / Q(k(Y)))$ actually act on the connected graph: $g \in G_P$ sends an edge $e : k(Z) \to E_\infty \subset \Omega$ to $g \circ e : k(Z) \to E_\infty \subset \Omega$, and $g \circ e$ is an edge of the connected graph $\mathcal{G}_{gen}$ because g fixes P.

Definition 5.13. Let $A \subset \mathrm{Aut}_k(E_\infty)$ the subgroup of $\mathrm{Aut}_k(E_\infty)$ sends $\mathcal{G}_{gen}$ to itself with the induced topology, as in Remark 5.12. Let $A^{PQ} \subset A$ be the subgroup of A generated by G_P and G_Q with the induced topology from A.

Question 5.14. *Is $A \hookrightarrow Aut_k(E_\infty)$ an isomorphism?*

Remark 5.15. The topology on A^{PQ} is uniquely determined by declaring the compact subgroups G_P and G_Q to be open.

By definition, A acts faithfully on $\mathcal{G}_{gen}$: if $g \in A$ acts trivially on $\mathcal{G}_{gen}$, then it acts trivially on the field generated by all of the vertices and the edges of $\mathcal{G}_{gen}$, i.e. it is the trivial automorphism of E_∞. If we give $\mathcal{G}_{gen}$ the discrete topology, A^{PQ} acts continuously on $\mathcal{G}_{gen}$; that is, the stabilizer of a vertex is an open subgroup. Let $d = \deg(Z \to X)$ and $e = \deg(Z \to Y)$. Then the degree of a blue vertex is d and the degree of a red vertex is e. Moreover, G_P acts transitively on the edges coming out of P by Galois theory and similarly G_Q acts transitively on the edges coming out of Q. By conjugating we see that $A^{PQ} \subset \mathrm{Aut}(\mathcal{G}_{gen})$ acts transitively on the edges coming out of any vertex. Therefore the group A^{PQ} acts transitively on the edges of $\mathcal{G}_{gen}$, subject to the constraint that colors of the vertices are preserved. This is recorded in the following corollary.

Corollary 5.16. *In the notation above, A^{PQ} and hence also A act transitively on the edges of $\mathcal{G}_{gen}$, subject to the constraint that the colors of the vertices are preserved. We say the pair $(\mathcal{G}_{gen}, A^{PQ})$ is colored-edge-symmetric.*

Question 5.17. *If $X \leftarrow Z \to Y$ is a minimal correspondence with no core, does $\mathcal{G}_{gen}$ have any cycles? What if it is étale?*

The graph $\mathcal{G}_{gen}$ being a tree has consequences for the structure of the group A^{PQ}. To state these, we need a theorem of Serre.

Theorem 5.18. *(Serre) Let G be a group acting on a graph X, and let e be an edge of X connecting vertices p and q. Suppose that e is a fundamental domain for the action. Let G_p, G_q, and G_e be the stabilizers in G of p, q, and e respectively. Then the following are equivalent.*

1. *X is a tree*
2. *The homomorphism $G_p *_{G_e} G_q \to G$ induced by the inclusions $G_p \to G$ and $G_q \to G$ is an isomorphism*

Proof. This is a direct translation of Théorème 6 on Page 48 of [24]. □

Proposition 5.19. *Suppose $\mathcal{G}_{gen}$ is a tree. Then the natural map $G_P *_{G_{PQ}} G_Q \to A^{PQ}$ is an isomorphism of topological groups.*

Proof. There is no element $a \in A^{PQ}$ that flips any edge e of $\mathcal{G}_{gen}$ because A^{PQ} preserves the coloring. By Corollary 5.16, the segment PQ is a fundamental domain for the action of A^{PQ} on $\mathcal{G}_{gen}$. Therefore, by Serre's Theorem, the fact that $\mathcal{G}_{gen}$ is a tree implies the induced map $G_P *_{G_{PQ}} G_Q \to A^{PQ}$ is an isomorphism of abstract groups. The group $G_P *_{G_{PQ}} G_Q$ has a natural topology generated by the topologies of G_P and G_Q (because G_{PQ} is an *open subgroup* of both G_P and G_Q), and endowed with this topology the above map is an isomorphism of topological groups. □

When $\mathcal{G}_{gen}$ is a tree, we may describe the pair $(\mathcal{G}_{gen}, A^{PQ})$ in a different way. Given any compact open subgroup $G \subset A^{PQ}$ and any vertex $v \in \mathcal{G}_{gen}$, the orbit $G.v$ is compact and discrete (as we gave $\mathcal{G}_{gen}$ the discrete topology) and is hence finite. Therefore G acts on a finite subtree T of $\mathcal{G}_{gen}$ and hence the action factors through a finite quotient H of G.

Lemma 5.20. *A finite group H acting on a finite tree T always has a fixed point (though not necessarily a fixed vertex.)*

Proof. This is well known and Aaron Bernstein explained the following simple proof to us.

Let the height $h(v)$ of a vertex v be the maximal distance of v to any leaf. Any automorphism of T preserves heights. If there is a unique vertex v of minimal height, we are done, so suppose there is another vertex w of minimal height. Then v and w must be connected by an edge: if the unique path between them contained an intermediate vertex u, then some thought shows that $h(u) < h(v)$. As T is a tree, there can be *at most two* vertices of minimal height. If there are two, then their midpoint is a fixed point for *any* automorphism of T. □

Therefore there must be a point $p \in T$ that is fixed by H; here T is thought of as a topological space. If p were not a vertex T, H would fix the two neighboring vertices of the edge p is on because H respects the coloring of the graph. Therefore H fixes at least one vertex v. On the other hand, given any vertex v, the subgroup G_v fixing v is a compact open subgroup. Therefore, the vertices of $\mathcal{G}_{gen}$ are in natural bijective correspondence with the maximal open compact subgroups G of A^{PQ}.

Corollary 5.21. *If $\mathcal{G}_{gen}$ is a tree, any maximal compact open subgroup G of A^{PQ} is conjugate to either G_P or G_Q.*

Proof. The discussion above shows that every maximal compact open subgroup G of A^{PQ} is G_v for some vertex v of $\mathcal{G}_{gen}$. The group G_v is conjugate in A^{PQ} to G_P or G_Q by Corollary 5.16. Finally, G_P is not conjugate to G_Q in A^{PQ} because the action of A^{PQ} on $\mathcal{G}_{gen}$ preserves the coloring. □

Remark 5.22. If $\mathcal{G}_{gen}$ is a tree, then the action of A^{PQ} on $\mathcal{G}_{gen}$ is the conjugation action on the maximal compact subgroups.

We may similarly describe the adjacency relation in $\mathcal{G}_{gen}$ from the group A^{PQ} when $\mathcal{G}_{gen}$ is a tree. Recall our standing assumption that the original correspondence $X \leftarrow Z \rightarrow Y$ is minimal (in order for $\mathcal{G}_{gen}$ to not have multiple edges.) As above, we suppose the correspondence is of type (d, e). Then a blue vertex G_v and a red vertex G_w are joined by an edge if the intersection $G_v \cap G_w$ (inside of A^{PQ}) has index d inside of G_v and index e inside of G_w.

6. Symmetric Correspondences

Definition 6.1. A *symmetric correspondence* of curves over k is a self-correspondence $X \xleftarrow{f} Z \xrightarrow{g} X$ over curves over k such that there is an involution $w \in \mathrm{Aut}(Z)$ with $f \circ w = g$, i.e. w swaps f and g. We denote by w^* the induced involution on $k(Z)$.

Note that if the correspondence is minimal, w is unique if it exists. Therefore being symmetric is a property and not a structure of a minimal correspondence.

Lemma 6.2. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be a symmetric correspondence of curves over k without a core. Suppose Z is hyperbolic or $k \cong \mathbb{F}$. Any $w \in Aut(Z)$ that swaps f and g lifts to an automorphism $\tilde{w}$ of W_∞. We denote by $\tilde{w}^*$ the associated automorphism of $E_\infty = k(W_\infty)$.*

Proof. We proceed exactly as in the discussion at the beginning of Section 4: let W_f (resp. W_g) denote a Galois closure of arrow f (resp. g). The automorphism w of Z swaps f and g and hence we can choose an isomorphism $w_1 : W_g \rightarrow W_f$ living over w on Z:

$$\begin{array}{ccc} W_g & \xrightarrow{w_1} & W_f \\ \downarrow & & \downarrow \\ W & \xrightarrow{w} & W \end{array}$$

Similarly, we can chose an isomorphism $w_2 : W_{gf} \rightarrow W_{fg}$ living over w on Z, again because w swaps the roles of f and g. Continuing in this fashion, we get an isomorphism of towers

$$\tilde{w} : W_{gf\ldots} \rightarrow W_{fg\ldots}$$

By Lemma 4.6, $W_{fg\ldots}$ is isomorphic to W_{gf} as a pro-curve over W and we may think of $\tilde{w}$ as an automorphism of W_∞ living over $w \in \mathrm{Aut}(Z)$. □

Remark 6.3. Another way of phrasing Lemma 6.2 is as follows. If $X \xleftarrow{f} Z \xrightarrow{g} X$ is a symmetric correspondence without a core with Z hyperbolic, then for any choice of symmetry w, the following map is (infinite) Galois.

$$W_\infty \to Z/ < w >$$

From this perspective, it is clear that the lift $\tilde{w}$ is not unique.

Definition 6.4. Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be a symmetric correspondence of curves over k without a core where Z is hyperbolic or $k \cong \mathbb{F}$. Pick a symmetry w and a lift $\tilde{w}$ to W_∞, which exists by Lemma 6.2. Let $\tilde{w}^*$ be the associated automorphism of E_∞. Define $A^w \subset \mathrm{Aut}_k(E_\infty)$ be the subgroup generated by A^{PQ} and $\tilde{w}^*$. We give the subgroup $A^w \subset A$ the induced topology from A.

Remark 6.5. The notation A^w is *a priori* ambiguous as it seems to depend on a choice of lift $\tilde{w}$. Pick a second lift $\tilde{w}'$ of w. Then $\tilde{w}\tilde{w}'$ fixes Z as w was an involution. In particular, $\tilde{w}^*\tilde{w}'^* \in \mathrm{Gal}(E_\infty/k(Z)) \subset A^{PQ}$, so A^w is independent of the choice of lift of w.

Corollary 6.6. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be a symmetric correspondence of curves over k without a core with symmetry w. Suppose Z is hyperbolic or $k \cong \mathbb{F}$ and let $\tilde{w}$ be a lift of the symmetry to W_∞. Then A^w and hence A acts transitively on the oriented edges of $\mathcal{G}_{gen}$.*

Proof. Corollary 5.16 says that A^{PQ} acts transitively on $\mathcal{G}_{gen}$ subject to the constraint that the colors of the vertices are preserved. The automorphism $\tilde{w}^* \in \mathrm{Aut}(E_\infty)$ swaps the points P and Q. By conjugating we get that A^w acts transitively on the edges of $\mathcal{G}_{gen}$, in the usual sense of remembering the endpoints. □

Corollary 6.7. *Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be a symmetric correspondence of curves over k without a core with symmetry w. Suppose Z is hyperbolic or $k \cong \mathbb{F}$. Then A^{PQ} is an normal subgroup of index 2 inside of A^w.*

Proof. Conjugating by $\tilde{w}^*$ swaps G_P and G_Q and hence stabilizes A^{PQ}. Therefore A^{PQ} is normal inside of A^w. Moreover, $(\tilde{w}^*)^2 \in A^{PQ}$, so A^w/A^{PQ} is of order 2. □

Definition 6.8. Let (G, A) be a pair where G is a connected graph and A is a group of automorphisms of G. (G, A) is said to be (sharply) s-transitive if A acts (sharply) transitively on all s-arcs. (G, A) is said to be ∞-transitive if it is s-transitive for all $s \geq 1$.

In this language, under the hypotheses of Corollary 6.6 the pair $(\mathcal{G}_{gen}, A)$ is 1-transitive.

Theorem 6.9. *(Tutte) Let G be a connected trivalent graph, A a group of automorphisms of G, and s a positive integer. If (G, A) is s-transitive and not $s+1$-transitive, then (G, A) is sharply s-transitive.*

Proof. The proof is exactly the same as in 7.72 in Tutte's book Connectivity in Graphs [26]. Alternatively, see Djoković and Miller [8], Theorem 1, for exactly this statement. □

Lemma 6.10. *Let $X \leftarrow Z \rightarrow X$ be a symmetric type (3,3) correspondence of curves over k without a core with symmetry w. Suppose Z is hyperbolic or $k \cong \mathbb{F}$. Then the pair $(\mathcal{G}_{gen}, A^w)$ is ∞-transitive and $\mathcal{G}_{gen}$ is a tree.*

Proof. Suppose $\mathcal{G}_{gen}$ had a cycle. The graph $\mathcal{G}_{gen}$ is infinite by Proposition 5.10. Then the pair $(\mathcal{G}_{gen}, A^w)$ is 1-transitive, so there exists a positive n such that $(\mathcal{G}_{gen}, A^w)$ is n-transitive but not $n+1$-transitive. Therefore, to prove $\mathcal{G}_{gen}$ is a tree it suffices to prove that the pair $(\mathcal{G}_{gen}, A^w)$ is ∞-transitive.

Suppose $(\mathcal{G}_{gen}, A^w)$ was not ∞-transitive. Then there exists a positive integer n such that $(\mathcal{G}_{gen}, A^w)$ is n-transitive but not $n+1$-transitive because the graph is infinite, connected and 1-transitive. Theorem 6.9 implies that the pair $(\mathcal{G}_{gen}, A^w)$ is then sharply n-transitive, i.e. there exists a unique

automorphism in A^w sending any n-arc to any other n-arc. Therefore any automorphism in A^w that fixes any given n-arc must be the identity automorphism. To any n-arc R I can associate the field E_R which is the field generated by the images of the points and edges inside of E_∞ as in Definition 5.3. Pick the n-arc R through P so that E_R is a finite extension of $P(k(X))$. Note that E_∞ is Galois over E_R. The group $\mathrm{Gal}(E_\infty/E_R)$ acts faithfully on $\mathcal{G}_{gen}$ and fixes R. As $(\mathcal{G}_{gen}, A^w)$ is sharply n-transitive, the group $\mathrm{Gal}(E_\infty/E_R)$ acts trivially on $\mathcal{G}_{gen}$. Therefore $E_R = E_\infty$ is a finite extension $k(Z)$, Galois over both $k(X)$ and $k(Y)$, which is a contradiction. □

Lemma 6.10 poses the following refinement to Question 5.17 on whether or not $\mathcal{G}_{gen}$ is a tree.

Question 6.11. *Let $X \leftarrow Z \rightarrow X$ be a minimal, symmetric, étale correspondence of curves over k without a core. Is the pair $(\mathcal{G}_{gen}, A^w)$ ∞-transitive?*

We may use Question 6.11 to pose a refinement of Question 3.16

Question 6.12. *Let $X \leftarrow Z \rightarrow X$ be a minimal, symmetric, étale correspondence of projective curves over k without a core. Does the pair $(\mathcal{G}_{gen}, A^w)$ "look like" the action of SL_2 over a local field on its building?*

7. Specialization of Graphs and Special Orbits

Given a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ over a field k, we have defined an undirected 2-colored graph $\mathcal{G}_{gen}^{full}$, the full generic graph, using an algebraically closed overfield Ω. In this section we define the $\mathcal{G}_{phys}^{full}$, the full physical graph, which will be an undirected 2-colored graph, using $\overline{k}$. The goal of this section is to speculate on the behavior of "specialization maps" $s_{\tilde{z}} : \mathcal{G}_{gen} \rightarrow \mathcal{G}_{phys,z}$; informally, if we think of $\mathcal{G}_{gen}$ as the "graph of generic dynamics", this map specializes to the graph associated to the dynamics of a physical point $z \in Z(\overline{k})$.

Definition 7.1. Given a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ of curves over k, the *full physical graph* $\mathcal{G}_{phys}^{full}$ is the following 2-colored graph. The edges are the points $z \in Z(\overline{k})$, the blue vertices are the points $X(\overline{k})$ and the red vertices are the points $Y(\overline{k})$. Adjacent to $z : \mathrm{Spec}(\overline{k}) \rightarrow Z$ is the blue vertex $f \circ z \in X(\overline{k})$ and the red vertex $g \circ z \in Y(\overline{k})$. Given a choice of $z \in Z(\overline{k})$, we denote by the $\mathcal{G}_{phys,z}$ the connected component of $\mathcal{G}_{phys}^{full}$ that contains z.

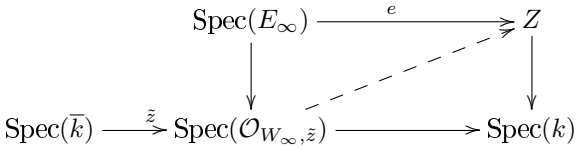
Recall the construction of $\mathcal{G}_{gen}$: pick an edge $PQ \in Z(\Omega)$ of $\mathcal{G}_{gen}^{full}$ and define $\mathcal{G}_{gen}$ to be the connected component of $\mathcal{G}_{gen}^{full}$ that contains PQ, suppressing the implicit PQ in the notation. The field $E_\infty \subset \Omega$ is the compositum of all of the points and edges of of $\mathcal{G}_{gen}$, thought of as subfields of Ω, by Corollary 5.8. Therefore, an edge e of $\mathcal{G}_{gen}$ yields an element of the set $Z(E_\infty)$. Similarly, a blue vertex v of $\mathcal{G}_{gen}$ yields an element of $X(E_\infty)$ and a red vertex w yields an element of $Y(E_\infty)$.

We spell out exactly what is fixed in the construction of a specialization map. First of all, assume the curves X, Y, and Z are proper over k: this is harmless as any correspondence of curves has a canonical compactification. Pick $z \in Z(\overline{k})$. Then pick a point $\tilde{z} \in W_\infty(\overline{k})$, a geometric point of the scheme W_∞, i.e. a compatible system of geometric points on the tower defining W_∞, lying over z. Taking the image of $\tilde{z}$ gives closed point of the scheme W_∞, and the ring $\mathcal{O}_{W_\infty,\tilde{z}}$ is a valuation ring because it is the filtered colimit of valuation rings. Moreover, the fraction field of $\mathcal{O}_{W_\infty,\tilde{z}}$ is E_∞. The choice of $\tilde{z} : \mathrm{Spec}(\overline{k}) \rightarrow W_\infty$ yields a morphism $\pi : \mathcal{O}_{W_\infty,\tilde{z}} \rightarrow \overline{k}$. We now construct the specialization map

$$s_{\tilde{z}} : \mathcal{G}_{gen} \rightarrow \mathcal{G}_{phys}^{full}$$

Let e be an edge of $\mathcal{G}_{gen}$. As discussed above, e yields an element of $Z(E_\infty)$. We want to describe $s_{\tilde{z}}(e_\infty)$, the image of e, in $\mathcal{G}_{phys,z}$. We have the following diagram; the dotted arrow exists uniquely

because the structure map $Z \to \mathrm{Spec}(k)$ is proper.



Composing $\tilde{z}$ with the dotted arrow, we get an element $\overline{e} \in Z(\overline{k})$. We set $s_{\tilde{z}}(e) = \overline{e}$. The exact same construction works with (red and blue) vertices, and the result is manifestly a map of graphs. Moreover, as $\mathcal{G}_{gen}$ is connected, so is the image.

Finally, we show that the edge $PQ \in Z(E_\infty)$ is sent to z. The inverse image of $\mathcal{O}_{W_\infty,\tilde{z}}$ under the map $PQ : k(Z) \hookrightarrow E_\infty$is the valuation ring R of $k(Z)$ corresponding to z. Therefore, when $e = PQ$, the above dotted arrow corresponds to the inclusion $R \hookrightarrow \mathcal{O}_{W_\infty,\tilde{z}}$. As $\tilde{z}$ lives over z, composing this inclusion with π yields $s_{\tilde{z}}(PQ) = z$, as desired. Therefore, we have constructed a map of graphs.

$$s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$$

Lemma 7.2. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be an étale correspondence of projective hyperbolic curves without a core over a field k. Then all of the specialization maps are surjective.*

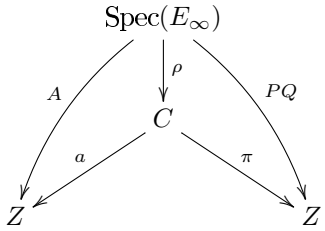
$$s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$$

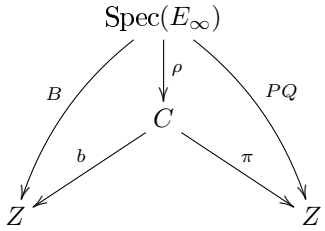
Proof. Because the correspondence is étale, each blue vertex of $\mathcal{G}_{phys}$ is adjacent to $d = \deg(f)$ edges and each red vertex is adjacent to $e = \deg(g)$ edges. It is therefore equivalent to show that no two adjacent edges of $\mathcal{G}_{gen}$ are sent to the same edge in $\mathcal{G}_{phys,z}$. Let A and B be two edges sharing the blue vertex p. We want to show that A and B are not sent to the same edge in $\mathcal{G}_{phys,z}$.

Recall that A and B yield elements of $Z(E_\infty)$ such that $f \circ A = f \circ B = p \in X(E_\infty)$. Proposition 5.11 implies that, after possibly enlarging k, there exists an irreducible curve C together with maps $\rho : \mathrm{Spec}(E_\infty) \to C$, $\pi : C \to Z$, and $a, b : C \to Z$ such that

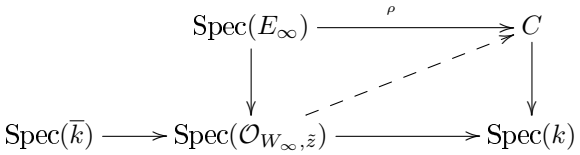
- $\pi \circ \rho = PQ$ considered as elements of $Z(E_\infty)$
- A and B factor through C via a and b.

In the language of Proposition 5.11, C is the curve associated to a field F of transcendence degree 1 over k, finite over $PQ(k(Z))$, that contains $A(k(Z))$ and $B(k(Z))$. More explicitly, we have the following factorizations:



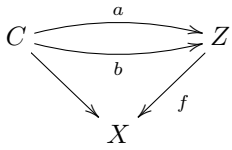


Moreover, the maps π, a, and b are all finite étale. Let us follow the specialization construction. Again, the dotted arrow exists because $C \to \mathrm{Spec}(k)$ is proper.



This diagram gives us a point $x \in C(\overline{k})$ by composition with the dotted arrow. If A and B are identified under the specialization map, $a(x) = b(x) \in Z(\overline{k})$. Now, $f \circ a = f \circ b$ because A and B shared the

vertex p, so we have the following diagram.



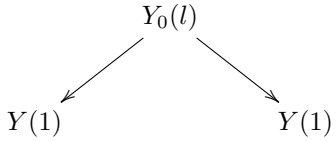
But C is irreducible and the maps a, b, and f are finite étale, so the assumption that $a(x) = b(x)$ implies that $a = b$ and hence $A = B$, as desired.

□

Remark 7.3. The graph $\mathcal{G}_{phys}$ helps describe the tower W_∞. In this remark, we suppose all morphisms are unramified at all points specified. For instance, let $\xi_Y \in W_Y(\overline{k})$ map to $z \in Z(\overline{k})$ which maps to $y \in Y(\overline{k})$. Then, as in Remark 4.9, there are naturally $\mathrm{Aut}(W_Y/Z)$ many maps from W_Y to Z and we can look at the images of ξ_Y under these maps. In this way, ξ_Y yields the graph of all edges coming out of y in $\mathcal{G}_{phys,z}$. More generally, a point $\xi_{YX\dots Y} \in W_{YX\dots Y}(\overline{k})$ which maps to $y \in Y(\overline{k})$ under the natural map yields the subgraph of $\mathcal{G}_{phys,z}$ with center y and radius n, where n is the number of letters in the string "$YX\dots Y$".

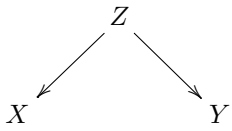
We will use this observation in Proposition 9.10 to show that if an *étale clump* exists, then Question 4.8 has an affirmative answer.

Consider the Hecke correspondence of open modular curves over $\mathbb{F}_p$



The graph $\mathcal{G}_{gen}$ is a tree. For $z \in Y_0(l)(\mathbb{F})$ an ordinary point, $\mathcal{G}_{phys,z}$ has at most one cycle. This follows from the work in David Kohel's thesis [13], summarized by Andrew Sutherland [25]. They call this structure an *Isogeny Volcano*. The cycle comes from the following fact: given an imaginary quadratic field $K/\mathbb{Q}$, there exists an elliptic curve $E/\mathbb{F}$ with multiplication by the maximal order $\mathcal{O}_K$. On the other hand, there are only finitely many supersingular points, and in fact Theorem 9.6 implies that if $\mathcal{G}_{phys,z}$ contains one supersingular point it contains all of them.

Definition 7.4. Given an étale correspondence of projective hyperbolic curves



over k without a core, we say a point $z \in Z(\overline{k})$ is *special* if there exists (equivalently for all) $\tilde{z} \in W_\infty(\overline{k})$ over z such that the map $s_{\tilde{z}} : \mathcal{G}_{gen} \to \mathcal{G}_{phys,z}$ is not an isomorphism. We say $z \in Z(\overline{k})$ is *generic* if the it is not special.

Question 7.5. *Let $X \leftarrow Z \to Y$ be an étale correspondence of projective curves over $\mathbb{F}_q$ without a core .*

1. *Is there always $z \in Z(\mathbb{F})$ that is generic?*
2. *Is there always a special point with unbounded orbit?*
3. *Suppose $\mathcal{G}_{gen}$ is free. For every point $z \in Z(\mathbb{F})$, is $\pi_1(\mathcal{G}_{phys,z})$ finitely generated? If $\mathcal{G}_{phys,z}$ is infinite, does $\mathcal{G}_{phys,z}$ have one cycle?*
4. *What is* $\lim_{n\to\infty} \frac{|z \in Z(\mathbb{F}_{q^n}) \text{ with } z \text{ generic}|}{|Z(\mathbb{F}_{q^n})|}$ *?*

8. Invariant Line Bundles and Invariant Sections

In this section we will need somewhat refined information about abelian varieties and finite group schemes over a field k. Our main reference is van der Geer and Moonen [28].

Definition 8.1. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over k. An *invariant line bundle* $\mathscr{L}$ on the correspondence is a triple $(\mathscr{L}_X, \mathscr{L}_Y, \phi)$ where $\mathscr{L}_X$ is a line bundle on X, $\mathscr{L}_Y$ is a line bundle on Y, and $\phi : f^*\mathscr{L}_X \to g^*\mathscr{L}_Y$ is an isomorphism of line bundles on Z. The *degree* of an invariant line bundle $\mathscr{L}$ on a correspondence of projective curves is $\deg(f^*\mathscr{L}_X) = \deg(g^*\mathscr{L}_Y)$ on Z. An isomorphism of invariant line bundles $i : \mathscr{L} \to \mathscr{L}'$ is a pair of isomorphisms $i_X : \mathscr{L}_X \to \mathscr{L}'_X$ and $i_Y : \mathscr{L}_Y \to \mathscr{L}'_Y$ that intertwine ϕ and ϕ' when pulled back to Z.

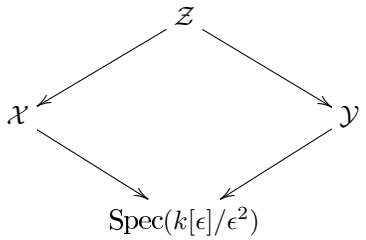
The cohomology of an invariant line bundle $\mathscr{L}$ is defined as follows.

$$H^i(\mathscr{L}) := \{(\xi_X, \xi_Y) \in H^i(X, \mathscr{L}_X) \oplus H^i(Y, \mathscr{L}_Y) | f^*(\xi_X) = \phi^* g^*(\xi_Y) \in H^i(Z, f^*\mathscr{L}_X)\}$$

The group $H^i(\mathscr{L})$ is naturally a finite-dimensional k vector space, and we let $h^i(\mathscr{L}) = \dim_k H^i(\mathscr{L})$. We call elements of $H^0(\mathscr{L})$ *invariant sections*. When it is especially clear from context, we omit the prefix "invariant".

In general, $\mathcal{O} = (\mathcal{O}_X, \mathcal{O}_Y, 1)$ is an invariant line bundle. Note that if the correspondence is étale, there is a natural invariant line bundle: $\Omega = (\Omega^1_X, \Omega^1_Y, \phi)$; here ϕ is the composition of the canonical isomorphism $f^*\Omega^1_X \to \Omega^1_Z$ and the inverse of the canonical isomorphism $g^*\Omega^1_Y \to \Omega^1_Z$. We call elements of $H^0(\Omega)$ *invariant differential 1-forms*.

Let $\mathscr{T}$ denote the dual of Ω. Then the first-order (equal-characteristic) deformation space of an étale correspondence of projective curves over k is $H^1(\mathscr{T})$, as we now explain. A first-order deformation is a diagram



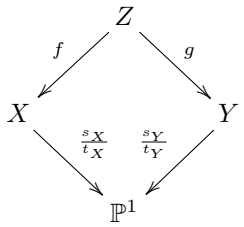
together with an identification of the special fiber with $X \leftarrow Z \rightarrow Y$. The first-order (equal-characteristic) deformation space of a smooth projective curve C/k is naturally isomorphic to $H^1(C, T_C)$. In particular, associated to such a diagram we obtain elements $\xi_X \in H^1(T_X)$, $\xi_Y \in H^1(T_Y)$, and $\xi_Z \in H^1(Z, T_Z)$ associated to $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ respectively. Now as $Z \to X$ is finite étale, any deformation $\tilde{X}$ of X naturally induces a deformation $\tilde{Z}$ of Z that maps to $\tilde{X}$. In the case of first-order deformations, this corresponds to the inclusion

$$H^1(X, T_X) \hookrightarrow H^1(Z, f^*T_X) \cong H^1(Z, T_Z)$$

Putting these facts together we see that $H^1(\mathscr{T})$ is naturally isomorphic to the first-order deformation space of an étale correspondence of projective curves.

Proposition 8.2. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over k without a core. Let $\mathscr{L}$ be an invariant line bundle on the correspondence. Then $h^0(\mathscr{L}) \leq 1$.*

Proof. If there were two linearly independent sections $s = (s_X, s_Y)$ and $t = (t_X, t_Y)$, then by taking their ratio we get a map to $\mathbb{P}^1$.



Hence there is a core. □

Question 8.3. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective curves over k without a core.*

- *If char$(k) = 0$, is $h^1(\mathscr{T}) = 0$?*
- *If char$(k) = p$, what is the maximal possible value of $h^1(\mathscr{T})$ in terms of the genera?*

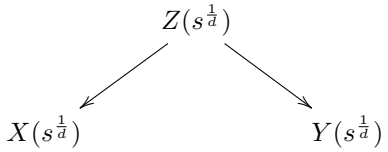
Remark 8.4. As noted in Remark 3.11, in characteristic 0 étale correspondences of projective curves without a core $X \leftarrow Z \rightarrow Y$ do not globally deform. However, it is still perhaps possible that they deform up to finite order: inside of the moduli space $\mathcal{M}$ of finite étale correspondences of projective curves with the given genera the moduli point $[X \leftarrow Z \rightarrow Y]$ is isolated by Corollary 4.13 but $\mathcal{M}$ might not be smooth at this point. Example 3.19 shows that in characteristic p, étale correspondences of projective curves without a core may in fact globally deform.

We will see that, in characteristic 0, there are no invariant sections of any non-trivial invariant line bundle on an étale correspondence without a core. However, they can exist in characteristic p. To better understand this, we briefly review the *cyclic cover trick* for smooth curves. Let T be a smooth curve over k, let $\mathscr{L}_T$ be a line bundle on T. Suppose $s \in H^0(T, \mathscr{L}_T^d)$ for some $d \in \mathbb{N}$ with $(d, \text{char}k) = 1$, such that s is not the power of a section of a smaller power of $\mathscr{L}_T$. Let $\mathscr{A}_s$ denote the following sheaf of algebras

$$\mathscr{A}_s = \mathcal{O}_T \oplus \mathscr{L}_T^{-1} \oplus \dots \mathscr{L}_T^{-(d-1)}$$

with multiplication given by the naive multiplication when possible and contraction with s when necessary. The condition that s is not the power of a section implies that $\mathscr{A}_s$ is an irreducible sheaf of algebras. We let $T(s^{\frac{1}{d}}) \rightarrow T$, the *$d^{th}$-cyclic cover of T by s*, be the normalization of $\text{Spec}_T \mathscr{A}_s$ equipped with the natural map to T. Then $T(s^{\frac{1}{d}})$ is a smooth curve over k. Then the pullback of $\mathscr{L}_T$ to $T(s^{\frac{1}{d}})$ has a (non-canonical) section, $s^{\frac{1}{d}}$, whose d^{th} power is s.

We remark that, by construction, the d^{th} cyclic cover of (T, s) is functorial. In particular, let $\mathscr{L}$ be an invariant line bundle on $X \leftarrow Z \rightarrow Y$ with $s \in H^0(\mathscr{L}^d)$ an invariant section, and suppose that s is not the power of any invariant section of a smaller power of $\mathscr{L}$. Then we may perform the cyclic cover trick to $(X \leftarrow Z \rightarrow Y, s)$ to obtain



The pullback of $\mathscr{L}$ to $X(s^{\frac{1}{d}}) \rightarrow Z(s^{\frac{1}{d}}) \rightarrow Y(s^{\frac{1}{d}})$ has a (non-canonical) invariant section, which we denote by $s^{\frac{1}{d}}$.

Example 8.5. Consider a Hecke correspondence of (open) modular curves over $\mathbb{F}$. Then the Hasse invariant H yields an invariant section of an invariant line bundle; if $p > 2$, this invariant line bundle is $\Omega^{\frac{p-1}{2}}$. Recall that the divisor of H is the supersingular locus. The Hasse invariant similarly exists on a Hecke correspondence of moduli spaces of fake elliptic curves. Therefore, there are examples of invariant sections of non-trivial invariant line bundles on étale correspondences of projective curves over $\mathbb{F}$ without a core.

In these cases, the "Igusa level structure" construction of Remark 3.18 is precisely the $(\frac{p-1}{2})^{st}$ cyclic cover construction associated to the invariant section H of $\Omega^{\frac{p-1}{2}}$. In particular, the induced correspondences of Igusa curves have an invariant differential form coming from "$H^{\frac{2}{p-1}}$". See Ullmer [27] for a brief introduction to the Hasse invariant and the Igusa construction and Chapter 1 of Katz [11] for a more thorough explication of modular forms.

Given a correspondence of projective curves, $X \leftarrow Z \rightarrow Y$, there are induced maps $f^* : Pic(X) \rightarrow Pic(Z)$ and $g^* : Pic(Y) \rightarrow Pic(Z)$ between the Picard schemes; both of these maps have finite (though not-necessarily reduced) kernel. Restricting, there are induced maps $f^* : Pic^0(X) \rightarrow Pic^0(Z)$ and $g^* : Pic^0(Y) \rightarrow Pic^0(Z)$. We denote by $f^*Pic^0(X) \cap g^*Pic^0(Y)$ the scheme-theoretic intersection of the image of these two maps in $Pic^0(Z)$; note that this group scheme need not be reduced in positive characteristic.

Definition 8.6. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of projective curves over k. The *Picard scheme of* $X \leftarrow Z \rightarrow Y$ is the closed subgroup scheme of $Pix(X) \times Pic(Y)$ given by

$$Pic(X \leftarrow Z \rightarrow Y) := \ker(Pic(X) \times Pic(Y) \xrightarrow{f^* - g^*} Pic(Z))$$

Similarly, $Pic^0(X \leftarrow Z \rightarrow Y) := \ker(Pic^0(X) \times Pic^0(Y) \xrightarrow{f^* - g^*} Pic^0(Z))$.

Remark 8.7. The scheme $Pic^0(X \leftarrow Z \rightarrow Y)$ need not be reduced in positive characteristic. As usual, if Z has a k-rational point, then $Pic(X \leftarrow Z \rightarrow Y)(k)$ is isomorphic the group of isomorphism classes of invariant line bundles on $X \leftarrow Z \rightarrow Y$. Finally, $Pic(X \leftarrow Z \rightarrow Y)/Pic^0(X \leftarrow Z \rightarrow Y) \hookrightarrow \mathbb{Z}$ via the degree map on Z.

We note that $Pic(X \leftarrow Z \rightarrow Y) \rightarrow Pic(X)$ and $Pic(X \leftarrow Z \rightarrow Y) \rightarrow Pic(Y)$ both have finite kernels. Moreover,

$$Pic^0(X \leftarrow Z \rightarrow Y) \rightarrow f^*Pic^0(X) \cap g^*Pic^0(Y) \subset Pic^0(Z)$$

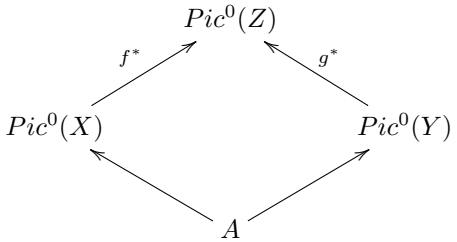
has finite kernel. The following theorem will be very useful for us.

Theorem 8.8. *Let A be an abelian variety over a field k and let $G \hookrightarrow A$ be a closed subgroup scheme. Then the connected reduced group subscheme $G^0_{red} \hookrightarrow A$ is an abelian subvariety.*

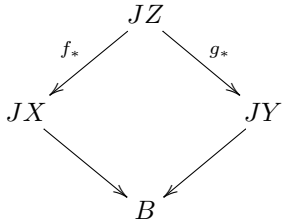
Proof. This is Proposition 5.31 in [28]. □

Lemma 8.9. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of projective curves over k without a core. Then $Pic(X \leftarrow Z \rightarrow Y)$ has no positive-dimensional abelian subvarieties. In particular, $Pic^0(X \leftarrow Z \rightarrow Y)$ is a finite group scheme over k.*

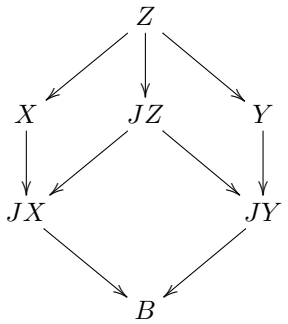
Proof. We may suppose all of the curves have genus at least 1. To prove the first statement, we show that there is no abelian variety A with finite maps fitting into the following diagram



By dualizing, this is equivalent to showing that there is no abelian variety B with non-constant surjective maps fitting into the following diagram



(While $Pic^0(Z)$ is canonically principally polarized, we write the dual as JZ to remember the Albanese functoriality.) Suppose such a B fitting into the diagram existed. We will prove that the correspondence has a core. Choose a point $z \in Z(k)$ (extend k if necessary) and let $x = f(z)$, $y = g(z)$. Then we have Abel-Jacobi maps which yield a morphism of *correspondences*:



i.e. the above diagram commutes. Moreover, under the Abel-Jacobi map, Z spans JZ as a group and likewise with X and Y. Therefore the induced maps $X \to B$ and $Y \to B$ are non-constant. In particular, their image in B is a curve; therefore $X \leftarrow Z \to Y$ has a core.

We now prove that $Pic^0(X \leftarrow Z \to Y)$ is finite. If $Pic^0(X \leftarrow Z \to Y)$ were not finite, then it would be a positive-dimensional group subscheme of $Pic^0(X) \times Pic^0(Y)$. Then $A = Pic^0(X \leftarrow Z \to Y)^0_{red}$ is a closed, reduced, connected subgroup scheme of an abelian variety over k and is hence an abelian variety by Theorem 8.8. □

Corollary 8.10. *Let $X \leftarrow Z \to Y$ be an étale correspondence of projective hyperbolic curves over k without a core. Let $\mathscr{L}$ be an invariant line bundle of positive degree. Then there exists $j, k \in \mathbb{N}$ with $\mathscr{L}^j \cong \Omega^k$.*

Proof. The set of degrees of invariant line bundles is a subgroup of $\mathbb{Z}$, so $\Omega^{-n} \otimes \mathscr{L}^m$ has degree 0 for some $m, n \in \mathbb{N}$. As our correspondence doesn't have a core, $\Omega^{-n} \otimes \mathscr{L}^m$ is torsion by Lemma 8.9. Therefore there exists $j, k \in \mathbb{N}$ such that $\mathscr{L}^j \cong \Omega^k$. □

Corollary 8.10 shows that, for étale correspondences of projective curves without a core, Ω plays a special role. We will now see several striking consequences of Lemma 8.9 and Corollary 8.10 in characteristic 0.

Corollary 8.11. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of projective curves over k without a core. Suppose char$(k) = 0$. Then $f^*H^1(X, \mathcal{O}_X) \cap g^*H^1(Y, \mathcal{O}_Y) = 0$ inside of $H^1(Z, \mathcal{O}_Z)$ and $f^*H^0(X, \Omega^1_X) \cap g^*H^1(Y, \Omega^1_Y) = 0$ inside of $H^0(Z, \Omega^1_Z)$.*

Proof. The vector space $H^1(X, \mathcal{O}_X)$ is the tangent space at the identity of $Pic^0(X)$. Moreover, the vector space $f^*H^1(X, \mathcal{O}_X) \cap g^*H^1(Y, \mathcal{O}_Y)$ is the tangent space at the identity of $f^*Pic^0(X) \cap g^*Pic^0(Y)$, a closed subgroup of $Pic^0(Z)$. As the characteristic is 0, $f^*Pic^0(X) \cap g^*Pic^0(Y)$ is reduced and hence the connected component of the identity of $f^*Pic^0(X) \cap g^*Pic^0(Y)$ is an abelian variety. Lemma 8.9 implies that this abelian variety has dimension 0 and hence $f^*H^1(X, \mathcal{O}_X) \cap g^*H^1(Y, \mathcal{O}_Y) = 0$.

By the Lefschetz principle, we may suppose $k \cong \mathbb{C}$. If C is a smooth projective complex curve, $H^1_{sing}(C(\mathbb{C}), \mathbb{C}) \cong H^0(C, \Omega^1_C) \oplus H^1(C, \mathcal{O}_C)$ and $\overline{H^1(C, \mathcal{O}_C)} = H^0(C, \Omega^1_C)$ by Hodge symmetry. Therefore

$$f^* H^0(X, \Omega^1_X) \cap g^* H^0(Y, \Omega^1_Y) = \overline{f^* H^1(X, \mathcal{O}_X) \cap g^* H^1(Y, \mathcal{O}_Y)}$$

inside of $H^1_{sing}(Z(\mathbb{C}), \mathbb{C})$. The fact that $\dim f^* H^1(X, \mathcal{O}_X) \cap g^* H^1(Y, \mathcal{O}_Y) = 0$ implies the result. □

Corollary 8.12. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of projective curves over $\mathbb{C}$ without a core. Then $f^* H^1_{sing}(X, \mathbb{Z}) \cap g^* H^1_{sing}(Y, \mathbb{Z}) = 0$ inside of $H^1_{sing}(Z, \mathbb{Z})$.*

Proof. This is immediate from Corollary 8.11 and the fact that pulling back H^1_{sing} under f and g induces a morphism of integral Hodge structures. □

Corollary 8.13. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective curves over k without a core. Suppose $char(k) = 0$. Let $\mathscr{L}$ be a non-trivial invariant line bundle. Then $h^0(\mathscr{L}) = 0$.*

Proof. We may suppose $\deg \mathscr{L} > 0$. Then there exists $j, k \in \mathbb{N}$ such that $\mathscr{L}^j \cong \Omega^k$ by Corollary 8.10. It therefore suffices to prove that no positive power of Ω has a section.

Suppose $s \in H^0(\Omega^k)$ is not the power of any smaller-degree invariant pluricanonical form on $X \leftarrow Z \rightarrow Y$. Then we may apply the cyclic-cover trick to obtain an étale correspondence

$$\begin{array}{ccc} & Z(s^{\frac{1}{k}}) & \\ \swarrow & & \searrow \\ X(s^{\frac{1}{k}}) & & Y(s^{\frac{1}{k}}) \end{array}$$

with an invariant differential form. This contradicts Corollary 8.11. □

We know, via the example of the Hasse invariant (see Example 8.5), that Corollary 8.13 is false in characteristic p. By examining the argument of Proposition 8.11, we see that the characteristic 0 hypothesis is used twice. First, we used that fact that all group schemes are reduced to argue that $h^1(\mathcal{O}) = 0$. Second, we used the Lefschetz principle and Hodge theory, namely $H^0(X, \Omega^1_X) = \overline{H^1(X, \mathcal{O}_X)}$, to relate $h^1(\mathcal{O})$ to $h^0(\Omega)$.

We further investigate the failure of Proposition 8.13 in characteristic p. To do this, we briefly recall a few facts about (commutative) finite group schemes. Let k be a field of characteristic p and let G be a finite group scheme over k. We denote by G^0 the connected component of the identity. There is the connected-étale sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow G^{\acute{e}t} \rightarrow 1$$

which splits if k is perfect. The space of invariant differentials on G, a k-vector space denoted by $\omega_{G/k}$, may be identified with the cotangent space at the origin of G (see 3.14, 3.15 of [28]). We denote by $G[p]$ the p-torsion of G. Then the embedding $G[p] \hookrightarrow G$ induces an isomorphism on the level of (co)tangent spaces at the identity (e.g. see the proof of 4.47 of *loc. cit.*)

Remark 8.14. The nomenclature "invariant differential" is slightly overloaded; we use this phrase to refer to a (left-) invariant differential form on a group scheme. When we use "invariant differential form", we mean a section of $H^0(\Omega)$ on an étale correspondence. We trust that this is not too confusing.

We record the following fact, surely well-known, for lack of a reference.

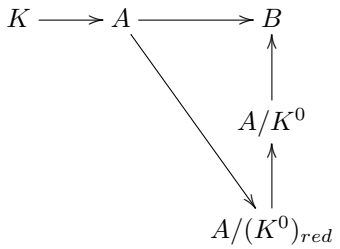
Lemma 8.15. *Let $f : A \rightarrow B$ be a surjective morphism of abelian varieties over a field k. Then f is separable if and only if the pullback map $f^* : H^0(B, \Omega^1_B) \rightarrow H^0(A, \Omega^1_A)$ is injective.*

Proof. Let $K = \ker(f)$ be the kernel of f. Then we have the following inclusion of group schemes

$$(K^0)_{red} \subset K^0 \subset K$$

Now, $(K^0)_{red}$ is a closed, reduced, connected subgroup scheme of an abelian variety over k; hence it is an abelian variety by Theorem 8.8. Therefore $A/(K^0)_{red}$ exists as an abelian variety (section 9.5 of

Polishchuk [23] or Example 4.40 in [28].) Similarly, A/K^0, a quotient of $A/(K^0)_{red}$ by the finite group scheme $K^0/(K^0)_{red}$ exists as an abelian variety. We have the following commutative diagram



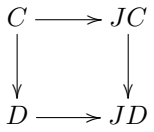
where the right vertical arrows are isogenies. In particular $A/K^0 \to B$ is a separable isogeny and $A/(K^0)_{red} \to A/K^0$ is a purely inseparable isogeny. By looking at tangent spaces, we see K^0 is non-reduced if and only if the pullback map $H^0(B, \Omega^1_B) \to H^0(A/(K^0)_{red}, \Omega^1_{A/(K^0)_{red}})$ is not injective. On the other hand the short exact sequence of abelian varieties over k

$$0 \to (K^0)_{red} \to A \to A/(K^0)_{red} \to 0$$

shows that the pullback map $H^0(A/(K^0)_{red}, \Omega^1_{A/(K^0)_{red}}) \to H^0(A, \Omega^1_A)$ is injective. Therefore $f^* : H^0(B, \Omega^1_B) \to H^0(A, \Omega^1_A)$ is injective if and only if K^0 is reduced, i.e. if and only if f is a separable morphism. □

Corollary 8.16. *Let $f : C \to D$ be a generically separable, finite morphism of projective curves over k. Then $f_* : JC \to JD$ is separable.*

Proof. Choose an element c of $C(k)$ (after possibly extending k) and let $d = f(c)$. Then we have the following commutative diagram

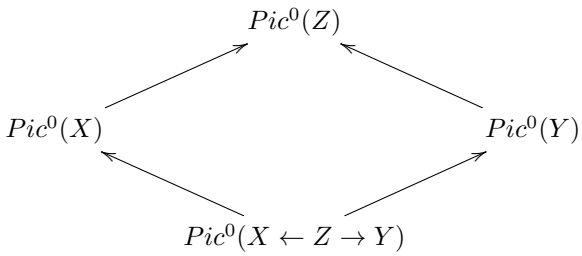


where the horizontal arrows are the Abel-Jacobi maps associated to c and d respectively. Pulling back along these Abel-Jacobi maps yields isomorphisms $H^0(JC, \Omega^1_{JC}) \to H^0(C, \Omega^1_C)$ and $H^0(JD, \Omega^1_{JD}) \to H^0(D, \Omega^1_D)$, *compatible with pulling back along f and f_**. As f was assumed to be generically separable, we obtain that

$$(f_*)^* : H^0(JD, \Omega^1_{JD}) \to H^0(JC, \Omega^1_{JC})$$

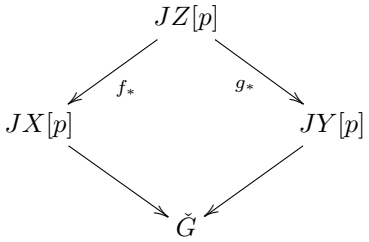
is injective. Now apply Lemma 8.15. □

Let k be a field of characteristic p and let $X \leftarrow Z \to Y$ be a correspondence of projective curves over k without a core. Suppose $Pic^0(X \leftarrow Z \to Y)$ is non-trivial. We have the following diagram



Let $G = Pic^0(X \leftarrow Z \rightarrow Y)[p]$. Take p-torsion and apply Cartier duality to obtain the diagram:

(8.1)



Now, if A is any abelian variety over k, the natural inclusion $A[p] \hookrightarrow A$ induces an isomorphism on the level of invariant differentials: $H^0(A, \Omega^1_A) \cong \omega_{A[p]/k}$. On the other hand, pulling back differential 1-forms under $f_* : JZ \rightarrow JX$ and $g_* : JZ \rightarrow JY$ is an injective operation by Corollary 8.16. Therefore pulling back invariant differentials is injective:

$$\omega_{JX[p]/k} \hookrightarrow \omega_{JZ[p]/k}$$

$$\omega_{JY[p]/k} \hookrightarrow \omega_{JZ[p]/k}$$

Pick $z \in Z(k)$ and set $x = f(z)$, $y = g(z)$. Then using the compatible Abel-Jacobi maps, we obtain that the following two vector spaces are isomorphic.

$$\{(\eta_X, \eta_Y) \in H^0(X, \Omega^1_X) \oplus H^0(Y, \Omega^1_Y) | f^*\eta_X = g^*\eta_Y\} \cong$$

$$\{(s,t) \in \omega_{JX[p]/k} \oplus \omega_{JY[p]/k} | f^*s = g^*t\}$$

Corollary 8.17. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective curves over k without a core. Then*

- $h^0(\Omega) = \dim_k\{(s,t) \in \omega_{JX[p]/k} \oplus \omega_{JY[p]/k} | f^*s = g^*t\}$
- *If the map $T_eJX[p] \rightarrow T_e\check{G}$ is non-zero, then $h^0(\Omega) = 1$.*
- *The dimension of the image of $T_eJX[p] \rightarrow T_e\check{G}$ is no greater than 1.*

Proof. The first part follows from the above discussion. If the map $T_eJX[p] \rightarrow T_e\check{G}$ is non-zero, then the pullback map $\omega_{\check{G}/k} \rightarrow \omega_{JX[p]/k} \hookrightarrow \omega_{JZ[p]/k}$ has non-zero image. By the commutativity of Diagram 8.1 there exists a pair $(s,t) \in \omega_{JX[p]/k} \oplus \omega_{JY[p]/k}$ such that the pullbacks to $\omega_{JZ[p]/k}$ agree. Hence there exists an invariant differential form on $X \leftarrow Z \rightarrow Y$. The dimension of the image of the map $\omega_{\check{G}/k} \rightarrow \omega_{JX[p]/k}$ is at most 1 because $h^0(\Omega) \leq 1$. In particular, the dimension of the image of $T_eJX[p] \rightarrow T_e\check{G}$ is at most 1. □

Question 8.18. *Let $X \leftarrow Z \rightarrow Y$ be an étale correspondence of projective curves over k without a core. Suppose $char(k) = p$. If $h^0(\Omega) = 1$, is the Cartier dual of $Pic^0(X \leftarrow Z \rightarrow Y)$ non-reduced?*

9. Clumps

Definition 9.1. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over a field k. A *clump* S is a finite set of $\overline{k}$ points $S \subset Z(\overline{k})$ such that $f^{-1}(f(S)) = g^{-1}(g(S)) = S$. An *étale clump* is a clump S such that f and g are étale at all points of S.

If $X \xleftarrow{f} Z \xrightarrow{g} Y$ has a core, then as in Remark 3.7 every $z \in Z(\overline{k})$ is contained in a clump. In the language of Remark 3.7, a clump is a *finite union of bounded orbits of geometric points.*

Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over k of type (d,e). Given an étale clump S, we now construct a natural invariant line bundle $\mathscr{L}(S)$ together with a one-dimensional subspace $V_S \subset H^0(\mathscr{L}(S))$ of invariant sections. (This line bundle may only be defined after a finite extension of k.) Think of S as an effective divisor on Z where all of the coefficients of the points are 1. Then f_*S is an effective divisor on X, all of whose coefficients are exactly d because f is étale at all points of S and has degree d. Therefore $\frac{1}{d}f_*S$ makes sense as an effective divisor on X; it is the divisor associated to the finite set $f(S) \subset X$. The associated line bundle $\mathscr{L}_X(S)$ on X comes equipped with a natural

one-dimensional space of sections $W_X \subset H^0(X, \mathscr{L}_X(S))$ with the following defining property: for any $w \in W_X$, $\mathrm{div}(w) = \frac{1}{d} f_* S$. Moreover, $f^* \mathscr{L}_X(S)$ is isomorphic to the line bundle associated with the divisor S. Similarly we obtain a line bundle $\mathscr{L}_Y(S)$ on Y with a natural one-dimensional space of sections W_Y. We set

$$\mathscr{L}(S) := (\mathscr{L}_X(S), \mathscr{L}_Y(S), \phi)$$

for any choice of isomorphism ϕ between the pullbacks. The vector space $H^0(\mathscr{L}(S))$ has a natural line V_S of invariant sections, given by $f^* W_X$ and $g^* W_Y$; in particular $h^0(\mathscr{L}(S)) \geq 1$.

Corollary 9.2. *Let $X \leftarrow Z \rightarrow Y$ be a étale correspondence of projective curves over k without a core. Suppose $\mathrm{char} k = 0$. Then there are no clumps.*

Proof. A clump S is automatically étale and hence yields a nontrivial invariant line bundle $\mathscr{L}(S)$ such that $h^0(\mathscr{L}(S)) \geq 1$. This contradicts Corollary 8.13. □

Remark 9.3. Corollary 9.2 shows that there is no direct analog of the supersingular locus in characteristic 0 for the following reason: Hecke orbits are big. This provides another conceptual reason why there is no canonical lift for supersingular elliptic curves.

Corollary 9.4. *A Hecke correspondence of compactified modular curves over $\mathbb{C}$ is ramified at at least one of the cusps.*

Proof. The cusps are a clump. Hecke correspondences are unramified on open modular curves; if the compactified correspondence were unramified at all of the cusps, then the cusps would form a clump on an étale correspondence of projective curves without a core, contradicting Corollary 9.2. □

Remark 9.5. The hypothesis of Corollary 9.2 implies that X, Y, and Z are Shimura curves by Theorem 3.10. This corollary was probably known, but we could not find a reference. Similarly, Corollary 9.4 admits a direct approach, but we find our method conceptually appealing.

Theorem 9.6. *Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a correspondence of curves over a field k without a core. There is at most one étale clump.*

Proof. It is harmless to compactify the correspondence, so we assume X, Y, and Z are all projective. Suppose there were two étale clumps, S and T. As in the discussion above, they give rise to positive invariant line bundles $\mathscr{L}(S)$ and $\mathscr{L}(T)$ together with lines $V_S \subset H^0(\mathscr{L}(S))$ and $V_T \subset H^0(\mathscr{L}(T))$. There exists $m, n \in \mathbb{N}$ such that $\mathscr{L}(S)^m \otimes \mathscr{L}(T)^{-n}$ has degree 0. Lemma 8.9 implies that $Pic^0(X \leftarrow Z \rightarrow Y)$ is a finite group scheme over k; in particular, $\mathscr{L}(S)^m \otimes \mathscr{L}(T)^{-n}$ is a torsion line bundle. Therefore there exists $j, k \in \mathbb{N}$ such that $\mathscr{L}(S)^j \cong \mathscr{L}(T)^k$.

The divisor of any element of $V_S^{\otimes j}$ is a positive multiple of S, and similarly the divisor of any element of $V_T^{\otimes k}$ is a positive multiple of T. In particular, if $S \neq T$, then the spaces $V_S^{\otimes j}$ and $V_T^{\otimes k}$ would be different lines inside of $H^0(\mathscr{L}(S)^j) \cong H^0(\mathscr{L}(T)^k)$. This would imply that $h^0(\mathscr{L}(S)^j) \geq 2$, contradicting Proposition 8.2. □

Question 9.7. *Let k be a field of characteristic p. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be an étale correspondence of projective curves over k without a core. Is there always a clump? Equivalently, is there always an invariant pluricanonical differential form?*

Remark 9.8. Theorem 9.6 generalizes the main theorem of Hallouin and Perret [10] (see the Introduction and Theorem 19 of *loc. cit.*), and the proof technique is completely different. In particular, they use the Perron-Frobenius theorem from spectral graph theory. We provide a detailed description of how to derive their result from ours.

Let $k \cong \mathbb{F}_q$ and let X be a smooth projective (geometrically irreducible) curve over k. Hallouin and Perret consider correspondences $\Gamma \subset X \times X$, with the assumption that Γ is absolutely irreducible and of type (d, d). Let $\mathcal{T}(X, \Gamma)$ be the sequence of curves $(C_n)_{n \geq 1}$ defined as follows:

$$C_n = \{(P_1, P_2, \ldots P_n) \in X^n | (P_i, P_{i+1}) \in \Gamma \text{ for each } i = 1 \ldots n-1\}$$

Let $\mathcal{G}_\infty(X,\Gamma)$, the *Geometric Graph*, be the graph whose vertices are the geometric points $X(\mathbb{F})$ and for which there is an oriented edge from $P \in X(\mathbb{F})$ to $Q \in X(\mathbb{F})$ if $(P,Q) \in \Gamma$. Theorem 19 of *loc. cit.* states that if the C_n are irreducible for all $n \geq 1$, then $\mathcal{G}_\infty(X,\Gamma)$ has *at most one* finite d-regular subgraph. As the correspondence is of type (d,d), every finite d-regular subgraph of $\mathcal{G}_\infty(X,\Gamma)$ induces an étale clump $S_\Gamma \subset \Gamma(\mathbb{F})$ with the following "symmetry" property: $\pi_1(S_\Gamma) = \pi_2(S_\Gamma)$. We call S_Γ a *symmetric étale clump* and set $S_X = \pi_1(S_\Gamma) = \pi_2(S_\Gamma)$.

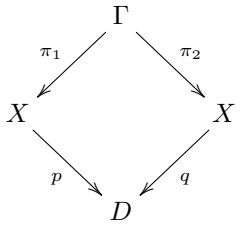
To understand their hypotheses, we first make the following definition. Let Ω be an algebraically closed field of transcendence degree 1 over k. Let $\mathcal{H}_{gen}^{full}$ be the following *directed graph*: the vertices are elements of $X(\Omega)$ and the edges are $\Gamma(\Omega)$. The source of an edge e is $\pi_1(e) \in X(\Omega)$ and the target of e is $\pi_2(e) \in X(\Omega)$. As usual, this graph is generally not connected and all connected components are isomorphic: we let $\mathcal{H}_{gen}$ be any connected component. Every vertex of the graph $\mathcal{H}_{gen}$ has in-degree and out-degree d. The hypothesis that C_n is irreducible for all n is equivalent to $\mathcal{H}_{gen}$ having *no directed cycles*. Note that this implies, but is not equivalent to, $\mathcal{H}_{gen}$ being infinite.

There is of course a surjective "collapsing" map $\mathcal{G}_{gen}^{full} \to \mathcal{H}_{gen}^{full}$ for a self correspondence $X \leftarrow \Gamma \to X$. One may make this a map of directed graphs by giving the following orientation to edges in the 2-colored graph $\mathcal{G}_{gen}^{full}$: an edge e between a blue vertex v and a red vertex w has the orientation $v \to w$. This map does *not necessarily* yield a surjective map $\mathcal{G}_{gen} \to \mathcal{H}_{gen}$; in particular, $\mathcal{G}_{gen}$ can be finite with $\mathcal{H}_{gen}$ infinite (e.g. see Elkies' Example 9.9.)

We now derive their result from ours. Let us assume, as they implicitly do, that $\mathcal{H}_{gen}$ has no directed cycles. There are two options:

- $X \leftarrow \Gamma \to X$ has no core (i.e. $\mathcal{G}_{gen}$ is infinite by Proposition 5.10)
- $X \leftarrow \Gamma \to X$ has a core (i.e. $\mathcal{G}_{gen}$ is finite by Proposition 5.10)

In the first case, Theorem 9.6 directly applies. In the second case, we will derive their theorem from ours. We first note that it is sufficient to prove the theorem after replacing Γ by its normalization, i.e. we may assume Γ is smooth. Call the coarse core D. We have the following diagram.



As D is the coarse core, Γ is the normalization of a component of $X \times_{p,D,q} X$. A symmetric étale clump S_Γ of $X \leftarrow \Gamma \to X$ yields unique étale clump S_X for the correspondence $D \leftarrow X \to D$. In particular, if we show that $D \leftarrow X \to D$ has at most one étale clump, we will have proven $X \leftarrow \Gamma \to X$ has at most one symmetric étale clump and we will have succeeded in deriving their theorem from ours.

We need only prove that $D \leftarrow X \to D$ has no core. This is where we use the irreducibility of all of the C_n. Note that C_n is birational to $\Gamma \times_{\pi_2,X,\pi_1} \Gamma \cdots \times_{\pi_2,X,\pi_1} \Gamma$ and $\lim_{n\to\infty} \deg(C_n \to D) = \infty$. On the other hand, Γ is birational to a component of $X \times_{p,D,q} X$. Therefore C_n is birational to an irreducible component

$$X \times_{p,D,q} X \times \cdots \times_{p,D,q} X$$

with increasing degree over D as $n \to \infty$. We now argue this cannot happen if $D \leftarrow X \to D$ had a core.

If $D \leftarrow X \to D$ has a core, we can find a curve $W \to X$ that is finite Galois over both compositions to D by Lemma 4.3. If E is any irreducible component of $X \times_{p,D,q} X \times \cdots \times_{p,D,q} X$, then

$$\deg(E \to D) \leq \deg(W \to D)$$

As the C_n are birational to irreducible of components of $X \times_{p,D,q} X \times \cdots \times_{p,D,q} X$ and $\deg(C_n \to D)$ goes to ∞ as $n \to \infty$, we see that $D \leftarrow X \to D$ has no core. Therefore Theorem 9.6 applies.

We remark that this argument only requires that there are components of C_n whose degree over D goes to ∞ as $n \to \infty$. In particular, we only need that $\mathcal{H}_{gen}$ is an infinite graph.

Example 9.9. Consider the symmetric modular correspondence $Y(1) \leftarrow Y_0(2) \rightarrow Y(1)$ over $\mathbb{F}$. Then points of the form $\{(P_1, P_2, P_1) | (P_1, P_2) \in Y_0(2)\}$ are an irreducible component of C_3. Therefore C_3 is not irreducible and their theorem does not directly apply. Note that $\mathcal{G}_{gen}$ is a tree, by direct computation or Lemma 6.10. However, one can massage the correspondence, à la Elkies [9], to obtain the one-clump theorem for this correspondence using their method: it is equivalent to prove that there is only one clump for the correspondence $Y_0(2) \leftarrow Y_0(4) \rightarrow Y_0(2)$. Here $Y_0(4)$ parametrizes pairs of elliptic curves equipped with a cyclic degree 4 isogeny between them $[E_1 \rightarrow E_2]$. This cyclic isogeny is uniquely the composition $E_1 \rightarrow E' \rightarrow E_2$, and the two maps to $Y_0(2)$ send this isogeny to $[E_1 \rightarrow E']$ and $[E' \rightarrow E_2]$ respectively. Note that this correspondence has a core: $Y(1)$, where $[E_1 \rightarrow E']$ and $[E' \rightarrow E_2]$ are both sent to $[E']$. Hallouin and Perret's theorem applies to this correspondence. This correspondence has the property that $\mathcal{G}_{gen}$ is finite (because there is a core) but $\mathcal{H}_{gen}$ is infinite. For more details, see Hallouin and Perret [10] or Section 2.5 of [15].

We describe a simple consequence of having a clump, providing a partial affirmative answer to Question 4.8.

Proposition 9.10. *Let $X \leftarrow Z \rightarrow Y$ be a correspondence of curves without a core with Z hyperbolic. If an étale clump exists, then the degree of the maximal "field of constants" of E_∞ is finite over k.*

Proof. If a étale clump exists, then all of the points of the clump are defined over a finite extension of fields k'/k. There are therefore k'-valued points of all of the curves $W_{YX\ldots Y}$, as in Remark 7.3. This implies that all of the $W_{YX\ldots Y}$ and hence W_∞ and E_∞ have field of constants contained in k'. The field of constants of E_∞ is then finite over k as desired. □

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