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A Conjugate Unscented Transform-Based Scheme for Optimal Control with Terminal State Constraints

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***Abstract*—In this paper, sparse collocation approach is used to develop optimal feedback control laws for Optimal Control Problems (OCPs) involving the terminal state constraints. The effective collocation process is accomplished by utilizing the recently developed Conjugate Unscented Transformation to provide a minimal set of collocation points. In conjunction with the minimal cubature points, an l_1 norm minimization technique is employed to optimally select the appropriate basis functions from a larger complete dictionary of polynomial basis functions. Finite time attitude regulation problem with terminal constraint is considered. Numerical simulations demonstrates the effectiveness of the proposed approach in derving the feedback laws for terminally constrained OCPs.**

I. Introduction

While a significant attention has been paid to derive numerical solutions for the HJB equation such as the Galerkin method [1],Collocations methods [2] and Level set methods [3], solving the HJB equation with terminal constraint remains a significant challenge. This is due to the fact that the presence of terminal constraint prohibits the application of many numerical methods to solve the terminally constrained HJB equation. This is due to the fact that the value function at the terminal time is defined only on the constraint surface rather than over the whole state space. Furthermore, the high computational cost involved in these methods for higher dimensions limit the applicability of these methods for many practical engineering problems. Finally, all aforementioned approaches assume a structure for the optimal feedback control, which is unknown in the general problem.

Unlike the regulator problem, few solution methods exist to solve the terminally-constrained HJB. For a linear system with a quadratic cost (LQ system), and a linear terminal penalty, one may employ the "sweep method" to determine the optimal feedback control law [4]. Refs. [5], [6] presents an extension of Bryson's sweep method for nonlinear system by considering a series solution for the value function in both the state and Lagrange Multiplier variables [5], [6]. The sensitivity of the value function with respect to the Lagrange multiplier corresponding to terminal state constraints is exploited to derive so called "gains" equations. However, the series expansion solution involve the vector series inversion for the Lagrange multiplier, which can be computationally expansive as the state dimension increases [5]. Finally, the assembly of the gain ODEs can be a cumbersome process and typically require the use of a symbolic toolbox [5] or manipulation of high order tensors.

In this work, recently developed sparse collocation methods [7], [8], [9] have been used to develop optimal feedback control laws for OCP with terminal constraints. The solution process involves the finite series expansion of the value function in terms of suitable polynomial basis functions. The coefficient and order of the finite series expansion for the value function are determined by exactly satisfying the HJB equation at the collocation points. The main challenge in the development of any collocation method lies in choosing appropriate collocation points and the basis functions. The number of collocation points for conventional methods like Gaussian quadrature methods increases exponentially with the state dimension and hence suffer from *curse of dimensionality*. Furthermore, the order of interpolating polynomial functions increases combinatorial leading to Gibbs phenomenon [10]. Sparse collocation method utilizes recently developed non-product quadrature scheme known as Conjugate Unscented Transform (CUT) methodology [11], [12] to alleviate the effect of *curse of dimensionality* by providing *minimal set of cubature points* in a multi-dimensional space. Furthermore, the recent advances in sparse approximation are utilized to formulate the interpolation polynomials directly in the multidimensional space for the chosen collocation points. The handshake of CUT approach with sparse approximation tools provide the foundation of sparse collocation methods to solve the multivariate PDE like HJB equation.

This work extends the framework of sparse collocation methods to determine the optimal feedback control law for the terminally-constrained OCP. A method is sought where knowledge of the domain of the Lagrange Multipliers is not required, and the optimal basis functions are chosen from a larger, complete dictionary of polynomial basis functions. The link between the co-states of the open-loop solutions and the gradient of the value function is exploited to determine a feedback control solution. Numerical examples are considered to demonstrate the efficacy of the proposed approach.

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II. PROBLEM STATEMENT

The main objective of this research is to develop a numerical framework to solve the terminally-constrained OCP:

$$\min_{\mathbf{u}(t)} \quad J = S(\mathbf{x}(t_f), t_f) + \int_0^{t_f} L(\mathbf{x}(t), \mathbf{u}(\mathbf{x}, \tau))\, d\tau \tag{1}$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t),\ \ \mathbf{x}(0) \in \Omega_0,\ \ \boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right] = 0 \tag{2}$$

where $\mathbf{x} \in \mathcal{R}^d$, $\mathbf{u} \in \mathcal{R}^p$, $\Omega_0 \subset \mathcal{R}^d$, and $\boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right] \in \mathcal{R}^q$ with $q \leq d$. It is assumed that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and $L(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}$. Finally, it is assumed that the final time t_f is known. This problem differs from the regulator problem in the sense that the terminal constraints *must* be satisfied at the final time [4]. We seek to derive optimal feedback control laws for the aforementioned OCP.

According to the Pontraygin's Maximum Principle (PMP) [4], the HJB equation governs the evolution of the value function for the terminally-constrained OCP and the optimal control laws are related to the jacobian of the value function.

$$\frac{\partial V}{\partial t} = -\min_{\mathbf{u}} \left\{ L(\mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial \mathbf{x}}^T \left[\mathbf{f}(\mathbf{x}(t), t) + \mathbf{g}(\mathbf{x}(t), t)\mathbf{u}(\mathbf{x}, t)\right] \right\} \tag{3}$$

$$V(\mathbf{0}, t) = 0,\ \ V(\mathbf{x}(t_f), t_f) = S(\mathbf{x}(t_f), t_f) + \boldsymbol{\nu}^T\boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right]$$

$$\mathbf{u}(\mathbf{x}, t) = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{g}^T(\mathbf{x})\frac{\partial V}{\partial \mathbf{x}} \tag{4}$$

Eq. (3) is the terminally-constrained HJB, which provides both necessary and sufficient conditions for optimality of the value function. The inclusion of the terminal constraints simply alters the boundary condition on the HJB. Since the terminal constraints must be satisfied, the terminal (boundary) condition on the value function is valid only on the manifold defined by $\boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right] = 0$. The value function attains an infinite value outside of this manifold [4]. $\boldsymbol{\nu}$ is the Lagrange multiplier to take into account the terminal state constraints. The terminal Lagrange multipliers are necessary to track the sensitivity of the value function with respect to the terminal constraint, so as to make the feedback control process continually aware of the manifold to reach at the given time. Notice that the solution process in this case becomes more complicated as the newly introduced Lagrange multiplier, $\boldsymbol{\nu}$ is an additional unknown. In this work, an expansion of the value function in terms of state and Lagrange variables is performed and a convex optimization problem is defined to compute unknown coefficients of this series expansion. Furthermore, the fact that the $\boldsymbol{\nu}$ is constant along an optimal trajectory is exploited to solve for its unknown value. In the following section, a description of proposed methodology is presented.

III. PROPOSED METHODOLOGY

A. Optimal Solution Matching Equations

Motivated by the appearance of the Lagrange Multipliers in the boundary condition, and by solutions of linear systems and those in Ref. [13], it can be stated that the value function is now a function of the state vector, as well as the Lagrange Multiplier vector $V^*(\mathbf{x}, \boldsymbol{\nu})$.

$$V(\mathbf{x}, \boldsymbol{\nu}, t) = \sum_{i=1}^{m} c_i(t)\phi_i(\mathbf{x}, \boldsymbol{\nu}) = \mathbf{c}^T(t)\Phi(\mathbf{x}, \boldsymbol{\nu}) \tag{5}$$

where $c_i(t)$ are unknown time-varying coefficients, and $\phi_i(\mathbf{x}, \boldsymbol{\nu})$ are known basis functions in the state and the Lagrange Multiplier. Substituting the value function expansion into the HJB yields:

$$\begin{aligned} e(\mathbf{x}, \boldsymbol{\nu}, t) &= \Phi(\mathbf{x}, \boldsymbol{\nu})^T\dot{\mathbf{c}} + \mathbf{f}^T(\mathbf{x}(t), t)\frac{\partial \Phi(\mathbf{x}, \boldsymbol{\nu})}{\partial \mathbf{x}}^T\mathbf{c}(t) + \mathbf{x}^T\mathbf{Q}\mathbf{x} \\ &- \frac{1}{4}\mathbf{c}^T(t)\frac{\partial \Phi(\mathbf{x}, \boldsymbol{\nu})}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}(t), t)\mathbf{R}^{-1}\mathbf{g}^T(\mathbf{x}(t), t)\frac{\partial \Phi(\mathbf{x}, \boldsymbol{\nu})}{\partial \mathbf{x}}^T\mathbf{c}(t) \\ \mathbf{c}^T(t_f)\Phi(\mathbf{x}_f, \boldsymbol{\nu}) &= S(\mathbf{x}(t_f), t_f) + \boldsymbol{\nu}^T\boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right] \end{aligned} \tag{6}$$

Notice that the error term, $e(\boldsymbol{\omega}, \boldsymbol{\nu}, t)$ is a result of the truncation of the value function series. Generally, the method of weighted residuals such as Galerkin transcription or collocation methods are used to solve for unknown coefficient in the solution domain. However, the presence of terminal constraint prohibits the application of weighted residual methods to solve the terminally constrained HJB equation. This is due to the fact that the state-space domain is being *mapped* to the domain of the terminal constraints over time as shown in Fig. 1, i.e., as $t_0 \to t_f$, $\mathcal{R}^d \to \mathcal{R}^q$. The sudden change in the state space domain poses the difficulty in the transcription process as the value function at the terminal time is defined only on the constraint surface rather than over the whole state space. In case, the terminal constraint corresponds to the specified value for the terminal state, the value function at terminal time is defined only at singleton point in the state space. Furthermore, the Lagrange multipliers are constant along an optimal trajectory, which is specified by an initial condition on the state variable. Hence, they are function of the state variable rather than time.

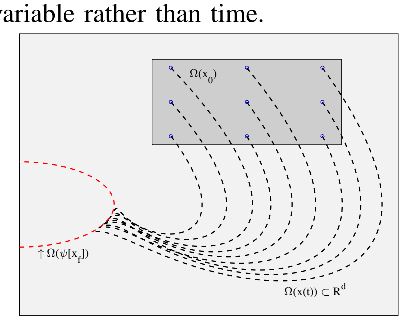


Fig. 1: Terminal Constraint Simulation

The aforementioned issues have prevented the direct application of many numerical methods to the terminally-constrained OCP. The groundwork for the proposed methodology develops from the relationship between first-order necessary conditions and characteristic curves of the terminally-constrained HJB. From Ref. [4], the first-order necessary conditions are:

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H(.)}{\partial \mathbf{x}},\ \ \dot{\mathbf{x}}(t) = \frac{\partial H(.)}{\partial \boldsymbol{\lambda}},\ \ \frac{\partial H(.)}{\partial \mathbf{u}} = 0 \tag{7}$$

$$\frac{\partial S(\mathbf{x}_f, t_f)}{\partial \mathbf{x}_f} + \boldsymbol{\nu}^T\frac{\partial \boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right]}{\partial \mathbf{x}_f} = \boldsymbol{\lambda}(t_f) \tag{8}$$

where the Hamiltonian is defined as:

$$H = L(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{9}$$

The PMP provides the relationship between the co-states and the optimal value function:

$$\frac{\partial V^*(\mathbf{x}(t), t)}{\partial \mathbf{x}} = \boldsymbol{\lambda}^*(t) \tag{10}$$

This link can also be obtained by direct application of the Method of Characteristics to the HJB. The first-order necessary conditions, and the co-state dynamics, are characteristic solutions of the HJB. The optimal open-loop solutions allow one to determine gradients of the optimal value function *along an optimal trajectory*. Eq. (10) provides the basis for the entire framework for the proposed method. The substitution of value function Eq. (5) in Eq. (11) leads to the relationship between unknown coefficients and the co-state vector:

$$\boldsymbol{\lambda}^*(t|\mathbf{x}^*(t), \boldsymbol{\nu}) = \frac{\partial \Phi(\mathbf{x}, \boldsymbol{\nu})}{\partial \mathbf{x}}^T \mathbf{c}(t) \tag{11}$$

The co-states are not explicit functions of the states and/or the Lagrange Multipliers; they are determined for a *given* optimal trajectory. Hence, one can aim to determine unknown coefficients $\mathbf{c}(t)$ by solving N distinct optimal trajectories corresponding to N unique initial conditions, $\mathbf{x}_{0_i}$, $i = 1, 2, \ldots, N$. This leads to following matching conditions to find unknown coefficients $\mathbf{c}(t)$:

$$\boldsymbol{\lambda}_i^*(t) = \frac{\partial V^*(\mathbf{x}_i^*(t), \boldsymbol{\nu})}{\partial \mathbf{x}} = \frac{\partial \Phi(\mathbf{x}_i^*(t), \boldsymbol{\nu}(\mathbf{x}_{0_i}))}{\partial \mathbf{x}}^T \mathbf{c}(t),$$
$$i = 1, 2, \ldots, N \tag{12}$$

Eq. (12) is the main equation used to develop a numerical framework to determine feedback control laws for the terminally-constrained system. Discretizing the time span into n samples, leads to following system of equations at each time instant (say t_k):

$$\mathbf{A}^{(j)}(t_k)\mathbf{c}_k = \mathbf{B}^{(j)}(t_k), \quad j = 1, 2, \ldots, d \tag{13}$$

where:

$$\mathbf{A}_i^{(j)}(t_k) = \frac{\partial \Phi(\mathbf{x}_i^*(t_k), \boldsymbol{\nu}(\mathbf{x}_{0_i}))}{\partial x^{(j)}}^T \tag{14}$$

$$\mathbf{B}_i^{(j)}(t_k) = \lambda_i^{(j)*}(t_k) \tag{15}$$

where $\lambda_i^{(j)*}(t_k)$ is the j^{th} co-state at time t_k from initial condition $\mathbf{x}_{0_i}$.

Eq. (13) provides a linear system of equations to determine the gradient of the value function and thus, the feedback control law in the presence of terminal state constraints. The procedure can be outlined as follows:

1) Generate N initial conditions such that $\mathbf{x}_{0_i} \in \Omega_0$.
2) Solve the open-loop TPBVP for each $\mathbf{x}_{0_i} \in \Omega_0$ to obtain $\mathbf{x}_i^*(t)$, $\boldsymbol{\lambda}_i^*(t)$, and $\boldsymbol{\nu}(\mathbf{x}_{0_i})$.
3) Solve Eq. (12) for $\mathbf{c}(t)$ along all optimal trajectories to determine the gradient of the value function.

The system of ODEs given by Eq. (13) is linear in the coefficient vector, and is of dimension DN. The main challenge lies in solving the unknown coefficients through aforementioned method lies in choosing appropriate initial condition samples (or collocation point) and the basis functions. This is due to the fact that the number of collocation points and polynomial basis functions would not be the same for a general system.

For a scalar systems, the optimal choice is Gaussian quadrature points in conjunction with Lagrange interpolation polynomials. For N collocation points, a $N-1$ degree interpolation polynomial is required. As the dimensionality is increased, a tensor product is required for both the collocation points and the interpolation polynomials, e.g. a fourth-order interpolation polynomial with five quadrature points in $1-D$ becomes an eighth order interpolation polynomial with 25 quadrature points in $2-D$. A depiction of this procedure is available in Fig. 2(a) and Fig. 2(b). The resulting increase in polynomial order results in incorrect interpolation, and large oscillations at the domain boundaries, a consequence known as the Gibbs phenomenon [14]. Thus the determination of such a polynomial, and the required collocation points, is not a trivial process.

The main challenge in determining a solution to the system of equations in Eq. (13) is that the choice of the initial condition samples and the basis dictionary directly affect the performance of the proposed method. To relax the burden of the curse of dimensionality, a minimal set of initial condition samples, and a minimal polynomial expansion of the value function are desired.

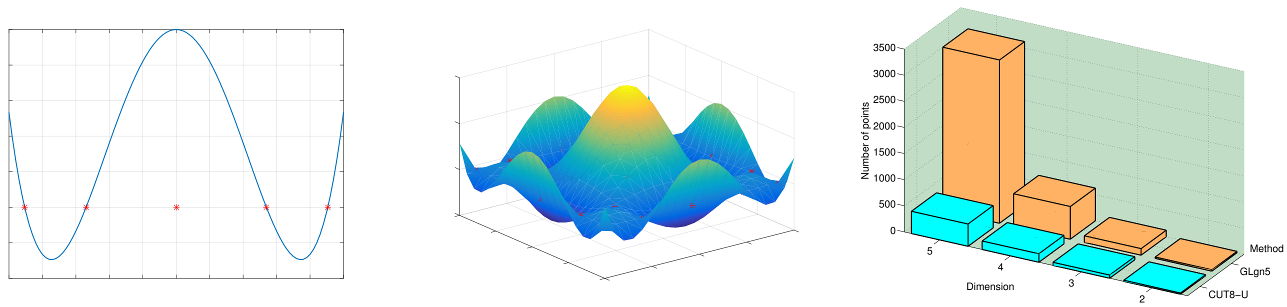
B. Generation of Initial Condition Samples

Depending upon number of basis functions and number of collocation points, there are three possible scenarios with regard to the solution of Eq. (13):

1) **Over-Determined System:** # of basis functions $(m) <$ Dimensionality times # of collocation points $(DN) \rightarrow$ No Solution!
2) **Square System:** # of basis functions $(m) =$ Dimensionality times # of collocation points $(DN) \rightarrow$ Unique Solution!
3) **Under-Determined System:** # of basis functions (m) $>$ Dimensionality times # of collocation points (DN) $\rightarrow$ Infinitely Many Solutions!

The additional design freedom offered in the case of the under-determined system due to the presence of the redundant basis functions manifests itself as a lack of uniqueness in choosing the appropriate polynomial basis function set and will be exploited in this work.

In one-dimensional system, the Gaussian quadrature points along with Lagrange interpolation polynomials provide the optimal choice for collocation points along with minimal order basis functions. However, the Gaussian quadrature methods suffer from *curse of dimensionality* since the number of quadrature points in general n-dimensional space are constructed from the tensor product of one dimensional quadrature points [15]. Minimal cubature rules have been developed in an effort to accurately compute multi-dimensional



(a) 4^{th} Order Lagrange Poly: $1-D$

(b) 8^{th} Order Lagrange Poly: $2-D$

(c) Comparison of Quadrature Points - 9^{th} Order Accuracy

Fig. 2: Lagrange Interpolation Polynomials & CUT vs. Conventional Quadrature Methods

integrals while avoiding the exponential increase in the required points. An extensive analysis of these cubature rules is available in [16]. For this research, the recently developed Conjugate Unscented Transform (CUT) is employed to generate the necessary collocation points [15].

The CUT methodology avoids tensor products in the generation of quadrature points. Instead, the method exploits the structure of the domain to choose specially defined axes on which quadrature points are selected. The CUT points are developed using minimal cubature rules, and offer similar orders of accuracy as Gaussian quadratures with fewer required points. Figure 2(c) shows a comparison of the number of points required for CUT and Gauss-Legendre quadratures for similar accuracy, clearly illustrating the reduced growth exhibited by the CUT method. The order of accuracy of the CUT method is specified by the moment constraint equations, i.e. the highest order moment equation exactly satisfied using the CUT points. A detailed analysis of CUT is available in Refs. [15], [17], [18].

As discussed, the number of basis functions required can quickly outnumber that of the collocation points. Increasing the number of collocation points beyond the number of basis functions would require a subsequent increase in the number of basis functions required. If polynomial basis functions are utilized, the growth is combinatorial, i.e. for k^{th} order polynomials in d-dimensional space, the required number of basis functions is $m = \binom{k+d}{d}$, resulting in the inclusion of higher-order polynomial terms, something that should be avoided due to the Gibbs phenomenon [14]. Thus, a procedure is needed to optimally select the appropriate basis functions with respect to the minimal set of collocation points.

C. Basis Selection and Finite Horizon Solution Procedure

In this section, we focus on constructing the minimal order interpolation polynomial given the function value at interpolation points.

An iterative l_1 optimization routine is proposed to optimally select the required basis functions to obtain a minimal polynomial expansion of the co-states with respect to the fixed number of initial condition samples. In particular, the the linear system of equation Eq. (13) is solved by minimizing the l_1-norm of the coefficients. Ideally, l_0-norm of the coefficient vector is to be minimized but this leads to a non-convex optimization problem. On the other hand, l_1-norm is convex and provides a close approximation to l_0-norm cost function, by making the coefficients close to zero. Hence, an iterative l_1 norm optimization is used to find the minimal polynomial expansion for value function. The following optimization problem is then proposed to identify non-contributing basis functions:

$$\min_{\mathbf{c}_k} \quad ||\mathbf{W}\mathbf{c}_k||_1 \tag{16}$$

$$\text{subject to:} \quad \mathbf{A}(t_k)\mathbf{c}_k = \mathbf{B}(t_k) \tag{17}$$

Eq. (16) minimizes the l_1 norm of the coefficient vector at time t_k. The constraint given by Eq. (17) represents matching the co-states along each optimal trajectory. This approach is analogous to that developed in Ref. [19]. The initial weight matrix is arbitrary, and can be chosen to penalize higher order basis functions if necessary. Further, this algorithm allows for iteration if necessary, e.g. to enhance the sparsity of the solution when needed. One may choose to optimize the *increment* of the gain vector as opposed to minimizing the l_1 norm of the gain vector at each time:

$$\min_{\mathbf{c}_k} \quad ||\mathbf{W}(\mathbf{c}_k - \mathbf{c}_{k-1})||_1 \tag{18}$$

$$\text{subject to:} \quad \mathbf{A}(t_k)\mathbf{c}_k = \mathbf{B}(t_k) \tag{19}$$

The minimization of l_1 norm of the *increment* of the gain vector makes sure that gains vary smoothly over the time.

Notice that one can also use the same procedure to find an interpolation surface for Lagrange multipliers corresponding to terminal manifold, i.e., $\boldsymbol{\nu}$. Knowing that the Lagrange Multipliers are constant and distinct for a given optimal trajectory, and assuming that no neighboring extremal paths exist, i.e. the optimal trajectory for $\mathbf{x}_{0_i}$ is unique, we can write:

$$\nu_j(\mathbf{x}_0) = \mathbf{b}_j^T \Phi(\mathbf{x}_0), \quad \mathbf{x}_0 \in \Omega_0, \quad j = 1, 2, .., q \tag{20}$$

Example + Dimensionality	Number of CUT Points, N	Size of Basis Functions Dictionary, m
Ex. 1: $d = 2,\ q = 1, \rightarrow D = 3$	21	816
Ex. 2: $d = 3,\ q = 3, \rightarrow D = 6$	59	8008

TABLE I
PARAMETERS FOR NUMERICAL EXAMPLES

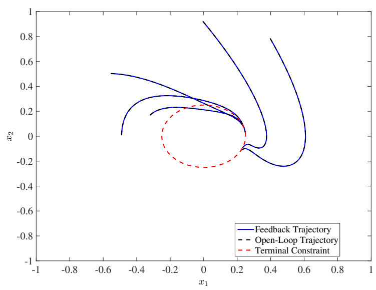


Fig. 3: Terminal Manifold Constraint - Trajectories

Therefore, for the chosen initial conditions, a series expression for $\boldsymbol{\nu}(\mathbf{x}_0),\ \forall \mathbf{x}_0 \in \Omega_0$ can be obtained by solving the following sparse approximation problem:

$$\min_{\mathbf{b}_j} \quad ||\mathbf{W}\mathbf{b}_j||_1 \tag{21}$$

$$\text{subject to:} \quad \Phi(\mathbf{x}_{0_i})^T \mathbf{b}_j = \nu_j(\mathbf{x}_{0_i}) \tag{22}$$
$$i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, q$$

In general, the dictionary of basis functions need not be the same as that of the value function. With the determination of the gradient of the value function and the Lagrange Multipliers discussed, the full solution algorithm can be detailed.

IV. Numerical Examples

This section details numerical examples to demonstrate the efficacy of the proposed approach. Examples and their chosen parameters are available in Table I. In all examples, the basis dictionary for the Lagrange Multiplier series includes polynomials up to 10^{th} order in the state variables.

A. Example 1: Linear System with Terminal Manifold Constraint

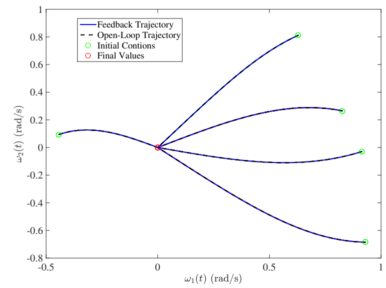
This example details a linear system driven to a terminal manifold in a fixed time interval:

$$\min_{\mathbf{u}(t)} J = \int_0^{t_f} (\mathbf{x}^T\mathbf{Q}\mathbf{x} + u)dt, \quad Q = \mathbf{I}_{2\times 2},\ t_f = 3$$
$$\text{s.t.: } \dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{g}u(\mathbf{x}, t), \quad \mathbf{x}_0 \in \Omega_0 = \{\mathbf{x}| -1 \leq \mathbf{x} \leq 1\}$$
$$\boldsymbol{\psi}\left[\mathbf{x}(t_f), t_f\right] = x_1^2 + x_2^2 - 0.25^2 = 0$$

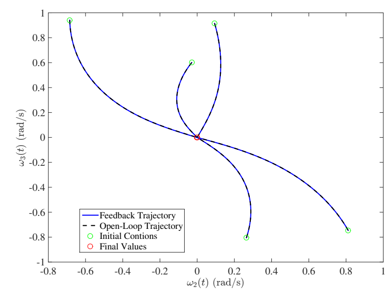
where the system matrices are given as:

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

The resulting TPBVPs for each CUT points are solved via a first-order gradient method to a tolerance of 1×10^{-7}. The computed feedback solution is validated against five random initial conditions. Fig. 3 depicts the resulting feedback and open-loop solutions for the terminal manifold problem. It can be seen that the proposed method accurately reproduces the optimal open-loop trajectories for the random initial conditions.



(a) $\omega_1 - \omega_2$



(b) $\omega_2 - \omega_3$

Fig. 4: Spin Stabilization - State Comparison

B. Example 2: Spacecraft Detumbling

This example details the detumbling of a spacecraft in a fixed time interval as seen in Ref. [13]:

$$\min_{\mathbf{u}(t)} J = \int_0^{t_f} (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})dt, \ \ \mathbf{Q} = 50\mathbf{I}_{3\times 3}, \ \ \mathbf{R} = \mathbf{I}_{3\times 3}$$
$$\text{s.t.: } \dot{\boldsymbol{\omega}} = -\mathbf{J}^{-1}\boldsymbol{\omega} \times \mathbf{J}\omega + \mathbf{g}\mathbf{u}(\mathbf{x}, t), \quad \boldsymbol{\omega}(0) \in \Omega_0, t_f = 2sec.$$
$$\boldsymbol{\psi}\left[\boldsymbol{\omega}(t_f), t_f\right] = [\omega_1(t_f), \omega_2(t_f), \omega_3(t_f)] = [0, 0, 0]$$

where the system matrices are given as:

$$\mathbf{J} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 8 \end{bmatrix} kg \cdot m^2, \quad \mathbf{g} = \mathbf{J}^{-1} \tag{24}$$

The resulting TPBVPs for each CUT points are solved via BVP4C in Matlab to a tolerance of 1×10^{-10}. For validation purposes, five random initial conditions are generated within $-1 \leq \boldsymbol{\omega} \leq 1$, and the resulting optimal trajectories are determined via the obtained feedback solution as well as by solving the TPBVP in Matlab.

For this example, the states are driven to a zero terminal condition and thus, the basis functions in the state variables

$\boldsymbol{\omega}_0$	J_{OL}	J_{FB}
$[0.6294, 0.8116, -0.7460]$	136.8305	136.8335
$[0.8268, 0.2647, -0.8049]$	132.6676	132.6802
$[-0.4430, 0.0938, 0.9150]$	76.2426	76.2557
$[0.9298, -0.6848, 0.9412]$	201.8735	201.7884
$[0.9143, -0.0292, 0.6006]$	129.7323	129.7545

TABLE II
COST COMPARISON

approach zero as $t \to t_f$. Therefore, as $\boldsymbol{\omega} \to 0$, the resulting gains of the pure state basis functions tend towards infinity to match the co-state trajectories. Because of this, the l_1 optimization will, at some point in the time interval, switch to the basis functions of the Lagrange Multipliers.

Fig. 4 depicts the resulting feedback and open-loop trajectories for the spacecraft detumbling problem. It can be seen that the proposed method accurately reproduces the optimal open-loop trajectories for the random initial conditions. It can be seen that the feedback solutions accurately represent the open-loop solutions. Finally, in all cases, the behavior is significantly different from the Spacecraft Spin Regularization in Ref. [20] which requires approximately 20 $sec.$ to regulate. This demonstrates that prior knowledge of the regulator problem has no bearing on the behavior of the solution of the terminally-constrained problem. The two results are vastly different and thus, one cannot assume any linkage between the two solutions.

Table II shows the open-loop J_{OL} and feedback J_{FB} costs for the controls generated. It can be seen that the two cost values are similar in magnitude, with both methods performing adequately. The open-loop cost is slightly lower than that of the feedback solution for most cases examined.

V. CONCLUSIONS

This research proposed a numerical framework for determining optimal feedback control laws for terminally-constrained systems. The link between Dynamic Programming and open-loop solutions is exploited to obtain a global feedback solution. These open-loop solutions are *characteristic curves* of the HJB, i.e. the Hamiltonian dynamics can be obtained from applying the Method of Characteristics to the HJB directly. Thus by matching these characteristic curves, one obtains a global expression for the gradient of the value function. For systems affine in the control variable, an explicit expression for the feedback control law can then be obtained.

A collocation-based scheme is developed to intelligently select the initial conditions for which the open-loop optimal solutions are determined. A sparsity-enhancing l_1 optimization routine is employed to obtain a sparse approximation for the gradient of the value function, and thus, the feedback control solution. It is noted that since a finite number of trajectories are generated, the resulting feedback control may be sub-optimal. However, as shown by numerical examples, the proposed methodology performs admirably in representing open-loop optimal solutions.

VI. ACKNOWLEDGEMENT

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