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A SPARSE COLLOCATION APPROACH FOR OPTIMAL FEEDBACK CONTROL FOR SPACECRAFT ATTITUDE MANEUVERS

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In this paper, sparse collocation approach is used to develop optimal feedback control laws for spacecraft attitude maneuvers. The effective collocation process is accomplished by utilizing the recently developed Conjugate Unscented Transformation to provide a minimal set of collocation points. In conjunction with the minimal cubature points, an l_1 norm minimization technique is employed to optimally select the appropriate basis functions from a larger complete dictionary of polynomial basis functions. Finite time attitude regulation problem with terminal constraint is considered. Numerical simulations involve asymmetric spacecraft equipped with four reaction wheels.

INTRODUCTION

Spacecraft attitude maneuvers play an important role in the success of many space missions and they result in nonlinear optimal control problem. Extensive research has been carried out to develop semi-analytical and numerical solutions to optimal open loop and feedback attitude control problems [1–10]. Application of the variational principles, [11] in conjunction with the Pontryagin's principle [12], typically yields a two-point boundary value problem for optimal state and the control law. The main shortcoming of the open loop solutions to optimal control problems is their sensitivity to the initial conditions and the unstructured perturbations, including modeling errors and exogenous disturbance inputs. Feedback solutions provide an attractive alternative to the optimal open-loop solutions in that they are fundamentally conceived from Bellman's principle of optimality, [13] and are robust to variations in initial conditions and external disturbances. The feedback control solutions can be determined by solving a nonlinear hyperbolic partial differential equation called the Hamilton Jacobi Bellman (HJB) equation [1, 14] over the domain of interest, with specified boundary conditions for some or all initial states.

While the quest for a unified solution approach to the time dependent and asymptotic HJB equation remains a holy-grail for a general dynamical system with arbitrary functions as performance indices and terminal manifolds, researchers have worked on various methods to provide local solutions to this important problem [2–10]. Approaches like $\theta - D$ method [4] and its variants [3] employ quasi-linearization style strategy, where the value function is updated recursively, as dictated by the ordering parameter. An alternative solution methodology emerges by developing a series expansion of the value functional and writing the resulting optimal control law as a function of the high order

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nonlinear feedback gains, following a process originally developed by Albrecht [6, 7]. This process was recently generalized to specialize the feedback control process to reach terminal manifolds at a specified terminal time by Vadali and Sharma [8]. *While the series solution methodology was shown to be useful, the curse of dimensionality renders the method to only provide local solutions. This shortcoming is shared by all other solutions to the problem.*

An alternative consideration of the solution of the HJB equation is to solve the PDE directly. In addition to the finite difference approach, the method of characteristics, the Galerkin method [15–17], the finite element approach [18] , Collocations methods [19–21] and Level set methods [22, 23] are also used to approximate the solution to HJB equation. The high computational cost involved in these methods for higher dimensions limit the applicability of these methods for many practical engineering problems. Furthermore, all aforementioned approaches assume a structure for the optimal feedback control, which is unknown in the general problem.

In this work, recently developed sparse collocation methods [24–26] have been used to develop optimal feedback control laws for spacecraft attitude maneuvers. The solution process involves the finite series expansion of the value function in terms of suitable polynomial basis functions. The coefficient and order of the finite series expansion for the value function are determined by exactly satisfying the HJB equation at the collocation points. The main challenge in the development of any collocation method lies in choosing appropriate collocation points and the basis functions. In one-dimensional system, the Gaussian quadrature points along with Lagrange interpolation polynomials provide the optimal choice for collocation points along with minimal order basis functions. However, the Gaussian quadrature methods suffer from curse of dimensionality since the number of quadrature points in an n-dimensional space are constructed from the tensor product of one dimensional quadrature points. On the other hand, the total degree of multivariate Lagrange interpolation polynomials constructed from the tensor product of one dimensional interpolation polynomials, grow quickly with number of multi-dimensional collocation points. The higher order polynomial basis functions are less desirable due to Gibbs phenomenon [27]. The construction of multivariate interpolation polynomials from a given set of collocation points is an active area of research. In Ref. [28], an algorithm is proposed to construct a minimal degree interpolation polynomial for the given set of points in general multi-dimensional space. The least degree interpolation polynomial is generated by first constructing the Vandermonde matrix [28] from the given points, with each row corresponding to one point. The columns are formed from the monomials of increasing order. Gauss elimination with partial pivoting is applied to the Vandermonde matrix, where the partial pivoting process follows special rules as outlined in [28]. This algorithm produces the least degree interpolation polynomial for the given set of points and function values, which can in turn be used for collocation. However, this process can become computationally expensive with dimension and as the number of points increase. Sparse collocation method utilizes recently developed non-product quadrature scheme known as Conjugate Unscented Transform (CUT) methodology [29, 30] to alleviate the effect of *curse of dimensionality* by providing *minimal set of cubature points* in a multi-dimensional space. Rather than using tensor products as in Gauss quadrature, the CUT approach judiciously selects special structures to extract symmetric quadrature points constrained to lie on specially defined axes. These new sets of so-called sigma points are guaranteed to exactly evaluate expectation integrals involving polynomial functions with significantly fewer points. Furthermore, the recent advances in sparse approximation are utilized to formulate the interpolation polynomials directly in the multidimensional space for the chosen collocation points. The handshake of CUT approach with sparse approximation tools provide the foundation of sparse collocation methods to solve the multivariate PDE like HJB equation.

In the following, we outline the sparse collocation method which will be used to derive optimal feedback control laws for spacecraft attitude maneuvers. Finite time attitude control problem with terminal time constraint is considered. For attitude control purposes, the spacecraft is assumed to be equipped with reaction wheel.

PROBLEM STATEMENT

Let us consider a spacecraft with four reaction wheels and hence the angular momentum of the system $\mathbf{H}$ will be sum of the spacecraft and wheels angular momentum.

$$\mathbf{H} = I\boldsymbol{\omega} + C^T J\boldsymbol{\Omega} \tag{1}$$

where, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ represent the spacecraft angular velocity vector and reaction wheel angular velocity vector, respectively. I is the spacecraft inertia matrix and J is the wheels axial moment of inertia matrix. The matrix C represents the orientation of the wheels axis to spacecraft body frame. In the absence of the external torques, the angular momentum will be conserved and the governing Euler equation of motion can be written as follows:

$$\dot{\mathbf{H}} = I\dot{\boldsymbol{\omega}} + C^T J\dot{\boldsymbol{\Omega}} + [\tilde{\boldsymbol{\omega}}]\,\mathbf{H} = 0 \tag{2}$$

where

$$[\tilde{\boldsymbol{\omega}}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3}$$

Finally, the system dynamic equation can be written in the following form:

$$I\dot{\boldsymbol{\omega}} = -\,[\tilde{\boldsymbol{\omega}}]\,\mathbf{H} - \mathbf{u} \tag{4}$$

The design of optimal spacecraft attitude maneuver involves the solution of the following optimal control problem:

$$\min_{\mathbf{u}(t)} J = \frac{1}{2}\int_0^{t_f} (\boldsymbol{\omega}^T Q\boldsymbol{\omega} + \mathbf{u}^T R\mathbf{u})dt \tag{5}$$

$$Subject\ to: \quad I\dot{\boldsymbol{\omega}} = -\,[\tilde{\boldsymbol{\omega}}]\,\mathbf{H} - \mathbf{u} \qquad \boldsymbol{\omega}(0) \in [-\ell, \ell] \tag{6}$$

$$\psi[\boldsymbol{\omega}(t_f), t_f] = 0 \tag{7}$$

Q is positive semidefinite matrix and R is positive definite matrix. It is assumed that t_f is known. We are interested in computing the feedback control law for control vector $\mathbf{u}$, which require the solution of the the Hamilton Jacobi Bellman (HJB) equation for the optimal value function V:

$$\frac{\partial V}{\partial t} + \min_{\mathbf{u}} \left\{ \frac{1}{2}\boldsymbol{\omega}^T Q\boldsymbol{\omega} + \frac{1}{2}\mathbf{u}^T R\mathbf{u} + \frac{\partial V}{\partial \boldsymbol{\omega}}^T \left[-I^{-1}([\tilde{\boldsymbol{\omega}}]\,\mathbf{H} + \mathbf{u})\right] \right\} = 0 \tag{8}$$

$$\forall \boldsymbol{\omega} \in [-\ell, \ell] \,, \quad V(0,t) = 0 \,, \quad V(\boldsymbol{\omega}(t_f), t_f) = 0 \;\; on \, \psi \left[\boldsymbol{\omega}(t_f), t_f\right] = 0 \tag{9}$$

Notice that due to the existence of the terminal constraint, the final boundary condition of value function is only satisfied on the terminal manifold and is undefined outside the support of terminal manifold [1].

The optimal control law through the application of Pontryagin's maximum principle is related to the optimal value function by the following expression:

$$\mathbf{u}(\boldsymbol{\omega}, t) = R^{-1} I^{-T} \frac{\partial V}{\partial \boldsymbol{\omega}} \tag{10}$$

The substitution of equation (10) in equation (8) leads to the following PDE for the solution of the value function:

$$\frac{\partial V}{\partial t} - \frac{\partial V^T}{\partial \boldsymbol{\omega}} I^{-1} \left[\tilde{\omega}\right] H + \frac{1}{2} \boldsymbol{\omega}^T Q \boldsymbol{\omega} - \frac{1}{2} \frac{\partial V^T}{\partial \boldsymbol{\omega}} I^{-1} R^{-1} I^{-T} \frac{\partial V}{\partial \boldsymbol{\omega}} = 0, \;\; V(\boldsymbol{\omega}(t_f), t_f) = \boldsymbol{\nu}^T \psi[\boldsymbol{\omega}(t_f), t_f] \tag{11}$$

where ν is the Lagrange multiplier to take into account the terminal state constraints. The terminal Lagrange multipliers are necessary to track the sensitivity of the value function with respect to the terminal constraint, so as to make the feedback control process continually aware of the manifold to reach at the given time. Notice that the solution process in this case becomes more complicated as the newly introduced Lagrange multiplier, ν is an additional unknown. A key result can be arrived at by considering the sensitivity of the augmented cost function, $V_a = V(\mathbf{x}(t_0)) + \nu^T \left(\psi(\mathbf{x}_{t_f}) - \psi_f\right)$ with respect to Lagrange multiplier vector, ν. It can be derived that if all of the necessary conditions are satisfied, then the gradient of augmented cost function with respect to ν along the optimal trajectory is given by:

$$\frac{\partial V_a}{\partial \nu} = \frac{\partial V(\mathbf{x}(t_0))}{\partial \nu} + \left(\psi(\mathbf{x}_{t_f}) - \psi_f\right) = 0 \tag{12}$$

Because the terminal conditions are also satisfied as part of the necessary conditions and because the initial time is arbitrary, the preceding result simply results in $\frac{\partial V(\mathbf{x}(t),t)}{\partial \nu} = 0$. This extra condition was exploited by Sharma and Vadali to find the value of ν [8]. This important and relatively unknown result has similarities to Bryson's sweep method. However, the series expansion solution involve the vector series inversion for the Lagrange multiplier which can be computationally expansive as the state dimension increases.

PROPOSED METHODOLOGY

In this work, we exploit recently develop sparse collocation method to solve the HJB equation in presence of terminal constraints. According to this approach [24–26], a series expansion for the value function is assumed. Motivated by the appearance of the Lagrange Multipliers in the boundary condition, it can be stated that the value function is now a function of the state vector, as well as the Lagrange multiplier vector:

$$V(\boldsymbol{\omega}, \boldsymbol{\nu}, t) = \sum_{i=1}^{m} c_i(t) \phi_i(\boldsymbol{\omega}, \boldsymbol{\nu}) = \mathbf{c}^T(t) \Phi(\boldsymbol{\omega}, \boldsymbol{\nu}) \tag{13}$$

where, $c_i(t)$ are unknown coefficients and $\phi_i(.)$'s are known basis functions of $\boldsymbol{\omega}$ and $\boldsymbol{\nu}$.The substitution of approximated value function in the HJB equation leads to the following error equation:

$$\begin{aligned} e(\boldsymbol{\omega},\boldsymbol{\nu},t) = \Phi(\boldsymbol{\omega},\boldsymbol{\nu})^T\dot{\mathbf{c}} - \left(I^{-1}\left[\tilde{\omega}\right]\mathbf{H}\right)^T \frac{\partial\Phi(\boldsymbol{\omega},\boldsymbol{\nu})^T}{\partial\boldsymbol{\omega}}\mathbf{c}(t) + \frac{1}{2}\boldsymbol{\omega}^T Q\boldsymbol{\omega} \\ - \frac{1}{2}\mathbf{c}^T(t)\frac{\partial\Phi(\boldsymbol{\omega},\boldsymbol{\nu})}{\partial\boldsymbol{\omega}} I^{-1}R^{-1}I^{-T}\frac{\partial\Phi(\boldsymbol{\omega},\boldsymbol{\nu})^T}{\partial\boldsymbol{\omega}}\mathbf{c}(t) \end{aligned} \tag{14}$$

$$\mathbf{c}^T(t_f)\Phi(\boldsymbol{\omega}_f,\boldsymbol{\nu}) = \boldsymbol{\nu}^T\psi\left[\boldsymbol{\omega}(t_f),t_f\right] \tag{15}$$

Notice that the error term, $e(\boldsymbol{\omega},\boldsymbol{\nu},t)$ is a result of the truncation of the value function series. Generally, method of weighted residuals such as Galerkin transcription or collocation methods are used to solve for unknown coefficient in the solution domain. However, the presence of terminal constraint prohibit the application of weighted residual methods. This is due to the fact that the value function at the terminal time is defined only on the constraint surface rather than over the whole state space. In case, the terminal constraint corresponds to the specified value for the terminal state, the value function at terminal time is defined only at singleton point in the state space. Furthermore, the Lagrange multipliers are constant along an optimal trajectory, which is specified by an initial condition on the state variable. Hence, they are function of the state variable rather than time. Hence, in this work we exploit characteristic solution of the HJB equation to construct the optimal value function [31].

The characteristic solution of the HJB equation comprised of the first-order necessary conditions, and the co-state dynamics. The co-state is defined as sensitivity of value function with respect to state, so the relation between optimal value function and co-state is:

$$\frac{\partial V(\boldsymbol{\omega}(t),t)}{\partial\boldsymbol{\omega}} = \boldsymbol{\lambda}(t) \tag{16}$$

The optimal open-loop solutions allow us to determine gradients of the optimal value function, i.e., co-state vector along an optimal trajectory. From the above, the co-state trajectories are determined for a single initial condition via solution of the Two-Point Boundary Value Problem (TPBVP). Through iterative numerical methods, one also obtains a Lagrange Multiplier value corresponding to this optimal trajectory. As stated, these methods provide only open-loop optimal control solutions, which must be determined for every desired initial condition. If open-loop solutions can be obtained over the entire state-space domain, then one can interpolate a optimal control law without resolving the TPBVP.

The substitution of equation (13) in equation (16) provides us a relationship between the co-state vector and unknown coefficients, $\mathbf{c}(t)$:

$$\boldsymbol{\lambda}(t|\,\boldsymbol{\omega}(t),\boldsymbol{\nu}) = \frac{\partial\Phi(\boldsymbol{\omega},t)^T}{\partial\boldsymbol{\omega}}\mathbf{c}(t) \tag{17}$$

Now, one can compute unknown coefficient vector $\mathbf{c}(t)$ by computing co-state vector for at least N initial conditions. This leads to the following constraint equation:

$$\boldsymbol{\lambda}_i(t) = \frac{\partial V(\boldsymbol{\omega}_i(t),\boldsymbol{\nu})}{\partial\boldsymbol{\omega}} = \frac{\partial\Phi(\boldsymbol{\omega}_i(t),\boldsymbol{\nu}(\boldsymbol{\omega}_{0_i}))^T}{\partial\boldsymbol{\omega}}\mathbf{c}(t) \qquad i = 1,\,2,\,\ldots,\,N \tag{18}$$

The equation (18) is the main equation which will provide a numerical framework to find the optimal feedback control law for a terminally constrained problem. By discretizing the time vector, one can write down the co-state matching conditions at each time step:

$$\lambda_i^{(j)}(t_k) = \frac{\partial \Phi(\boldsymbol{\omega}_i(t_k), \boldsymbol{\nu}(\boldsymbol{\omega}_{0_i}))^T}{\partial \omega_j} \mathbf{c}_k \qquad i = 1,\, 2,\, \ldots,\, N\,, \quad j = 1,\, 2,\, \ldots,\, d \tag{19}$$

This leads to the following set of linear equations at each time:

$$\mathbf{A}^{(j)}(t_k)\mathbf{c}_k = \mathbf{B}^{(j)}(t_k)\,, \qquad j = 1,\, 2,\, \ldots d \tag{20}$$

where:

$$\mathbf{A}_i^{(j)}(t_k) = \frac{\partial \Phi(\boldsymbol{\omega}_i(t_k), \boldsymbol{\nu}(\boldsymbol{\omega}_{0_i}))^T}{\partial \omega_j} \tag{21}$$

$$\mathbf{B}_i^{(j)}(t_k) = \lambda_i^{(j)*}(t_k) \tag{22}$$

After solving this system of equations for unknown coefficient $\mathbf{c}(t)$, one can compute optimal feedback law by using the following equation:

$$\mathbf{u}(\boldsymbol{\omega}, \boldsymbol{\nu}, t) = R^{-1} I^{-T} \frac{\partial \Phi(\boldsymbol{\omega}, \boldsymbol{\nu})^T}{\partial \omega} \mathbf{c}(t) \tag{23}$$

The whole process can be summarized in following steps:

1. Generate N initial condition in the desired domain, $\boldsymbol{\omega}_{0i} \in [-\ell, \ell]$.
2. Solve the open-loop TPBVP for all the N initial conditions from step 1 to obtain $\boldsymbol{\omega}_i(t)$, $\boldsymbol{\lambda}_i(t)$ and $\boldsymbol{\nu}(\boldsymbol{\omega}_{0_i})$.
3. Solve the equation (22) along all the optimal trajectories from step 2, to determine unknown coefficient vector $\mathbf{c}(t)$.

Generation of initial condition samples

The main challenge lies in choosing appropriate initial condition samples (or collocation point) and the basis functions. This is due to the fact that the number of collocation points and polynomial basis functions would not be the same for a general n-dimensional system. The growth of polynomial basis functions up to a fixed degree, d, is combinatorial in nature with the increase in state-space dimension. For a set of polynomial basis functions up to total degree d, the total number of polynomial basis functions is given as $m = \binom{n+d}{d}$, which will be different from the total number of collocation points (denoted by N) given by any of the method. When $m < N$, i.e. the number of collocation points are greater than the number of basis functions, the collocation process leads to an *over-determined* system of equations. The *over-determined* system typically does not possess sufficient degrees of freedom to accommodate the physics of the value function. An alternative approach would be to have $m > N$, i.e. fewer collocation points than the number of basis function. The collocation process in this case leads to an under-determined system of equations. This additional design freedom offered by the redundant basis functions manifests itself as a lack of

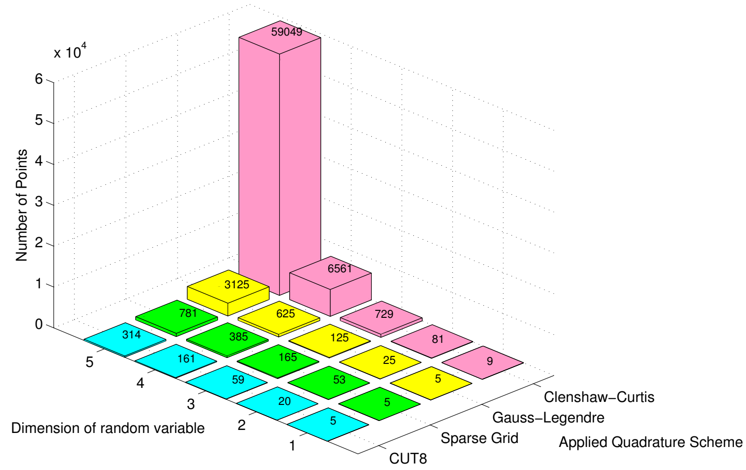

Figure 1. Comparison of Points - 9^{th} Order Accuracy.

uniqueness in choosing the appropriate polynomial basis function set and will be exploited in this work.

In one-dimensional system, the Gaussian quadrature points along with Lagrange interpolation polynomials provide the optimal choice for collocation points along with minimal order basis functions. However, the Gaussian quadrature methods suffer from *curse of dimensionality* since the number of quadrature points in general n-dimensional space are constructed from the tensor product of one dimensional quadrature points. Even for a moderate-dimension system involving, say, 6 state variables, the number of points is $5^6 = 15,625$ with only 5 points along each direction.

The sparse grid quadratures, and in particular Gauss-Legendre Smolyak (GLgnSM) quadrature method, take the sparse product of one dimensional quadrature rules and thus have fewer points than the equivalent Gaussian quadrature rules, but at the cost of introducing negative weights [32] which can further lead to numerical instability [33]. Note that the Gaussian quadrature rule is not minimal for $m \geq 2$, and there exist quadrature rules requiring fewer points in high dimensions [34]. In this work, recently developed CUT methodology is leveraged to break this *curse of dimensionality* and have a computationally efficient collocation scheme to solve the HJB equation in a multi-dimensional space.

The CUT approach [30, 35–37] can be considered an extension of the conventional UT method that satisfies additional higher order moment constraints. Rather than using tensor products as in Gauss quadrature, the CUT approach judiciously selects special structures to extract symmetric quadrature points constrained to lie on specially defined axes. These new sets of so-called sigma points are guaranteed to exactly evaluate expectation integrals involving polynomial functions with significantly fewer points. Figure 1 shows a comparison of the number of points required for CUT and Gauss-Legendre quadratures for similar accuracy, clearly illustrating the reduced growth exhibited by the CUT method. More details about the CUT methodology and its comparison with conventional quadrature rules can be found in Ref. [30, 35–38].

Optimal selection of basis functions

If one considers the construction of Lagrange interpolation polynomials from the tensor product of one dimensional Lagrange interpolation polynomials, it can be observed that the total degree of the resultant interpolation polynomial grows quickly even with few number of points. The higher order polynomial basis functions are not desirable due to Gibbs phenomenon [27]. In general, the appropriate set of basis or polynomial degree cannot be determined just from the number of points.

To overcome this difficulty, a sparse optimization based basis selection process is incorporated to select the basis that is best for the given set of points and dynamics of the system. In particular, the the linear system of equation (22) is solved by minimizing the l_1-norm of the coefficients. Ideally, l_0-norm of the coefficient vector is to be minimized but this leads to a non-convex optimization problem. On the other hand, l_1-norm is convex and provides a close approximation to l_0-norm cost function, by making the coefficients close to zero. Hence, an iterative l_1 norm optimization is used to find the minimal polynomial expansion for value function.

$$\min_{\mathbf{c}_k} \quad \|\mathbf{W}\mathbf{c}_k\|_1 \tag{24}$$

$$\text{subject to:} \quad \mathbf{A}(t_k)\mathbf{c}_k = \mathbf{B}(t_k) \tag{25}$$

Equation (24) minimizes the l_1 norm of $\mathbf{c}_k$ and equation (25) represents the collocation constraints. Intially, the weight w_i for i^{th} coefficient is chosen to be proportional to the total degree of the corresponding basis function. In successive iterations, the weight matrix is inversely proportion to the corresponding coefficient value to force the numerically small coefficients to be zero:

$$\mathbf{W} = diag[W_1,\ W_2,\ \ldots, W_m] \tag{26}$$

$$W_l^{(iter)} = \frac{1}{(c_l^{(iter-1)} + \varepsilon)}\ , \qquad l = 1,\ 2,\ \ldots,\ m \qquad \varepsilon = 0.1 \tag{27}$$

The same procedure can also be used to find an interpolating surface for the Lagrange multiplier $\boldsymbol{\nu}$.

NUMERICAL RESULTS

This section represents the numerical simulation results to validate the proposed method. For simulation purposes, following values are assumed for spacecraft and reaction wheel parameters. All values are in SI units if not specified otherwise.

$$\mathbf{I} = \begin{bmatrix} 87.212 & -0.2237 & -0.2237 \\ -0.2237 & 86.067 & -0.2237 \\ -0.2237 & -0.2237 & 114.562 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix}$$

$$\mathbf{J} = diag[0.05,\ \ 0.05,\ \ 0.05,\ \ 0.05]$$

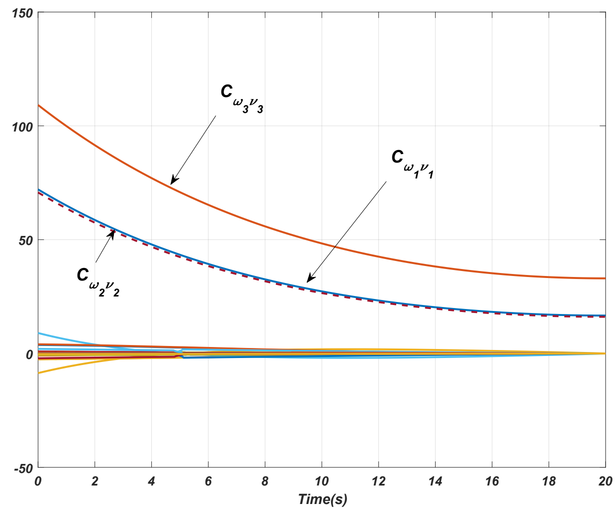

Figure 2. Non-Zero Coefficients vs. Time.

The assumed cost function parameters for numerical simulation are listed below:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{Q} = \mathbf{I} \qquad t_f = 0$$

The terminal constraint manifold has been defined as:

$$\psi\left[\boldsymbol{\omega}(t_f), t_f\right] = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{t_f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The initial condition domain is considered to be $\boldsymbol{\omega}(0) \in [-5, 5] deg/s$. It is assumed that the reaction wheels at $t = 0$ are in rest. Both the value function and Lagrange multipliers expansion includes polynomial basis functions up to order 10^{th} resulting in a total of 8008 basis functions. According to the CUT8, there are a total of $N = 59$ collocation points. For each collocation point, the TPBVP is solved by MATLAB in-built function BVP4c with tolerance of 10^{-7}. The time domain $t \in [0, 20]$ is discretized into 500 equal increments. The sparse approximation procedure outlined in previous section is used to solve for the unknown coefficient vector. Figure 2 shows the plot of non-zero coefficients as a function of time. At a given time, on average only 174 coefficients (out of 8008) are non-zero.

It should be noted that the coefficients of the quadratic terms such as $\omega_1\nu_1, \omega_2\nu_2, \omega_3\nu_3$ dominate other non-zero coefficients and hence, one can conclude that a linear controller is a good approximation of the nonlinear controller. For validation purposes, the feedback solution is also compared to open-loop solution corresponding to five random conditions (other than collocation points). Table 1 shows the initial conditions and the optimal cost value corresponding to both the open loop and closed loop solutions. It can be observed that the closed-loop cost values matches very well with open loop cost values providing a basis of optimism in support of the proposed approach.

Table 1. Open Loop Cost vs. Closed Loop Cost

Initial Condition(deg/s)	Open Loop Cost Value	Closed Loop Cost Value
$[-3.5810, -0.7850, 4.1597]$	5.1173	5.1178
$[2.9221, 4.5951, 1.5584]$	4.1755	4.1752
$[-4.6410, 3.4893, 4.3373]$	7.9899	7.9891
$[1.7876, 2.5783, 2.4293]$	2.3793	2.3792
$[-1.0771, 1.5527, -3.2888]$	7.9899	7.9891

Furthermore, Figures 3-7 shows the error between open-loop and closed loop solutions for optimal state and control trajectories. As expected, the closed loop solution matches very well with the open-loop solution.

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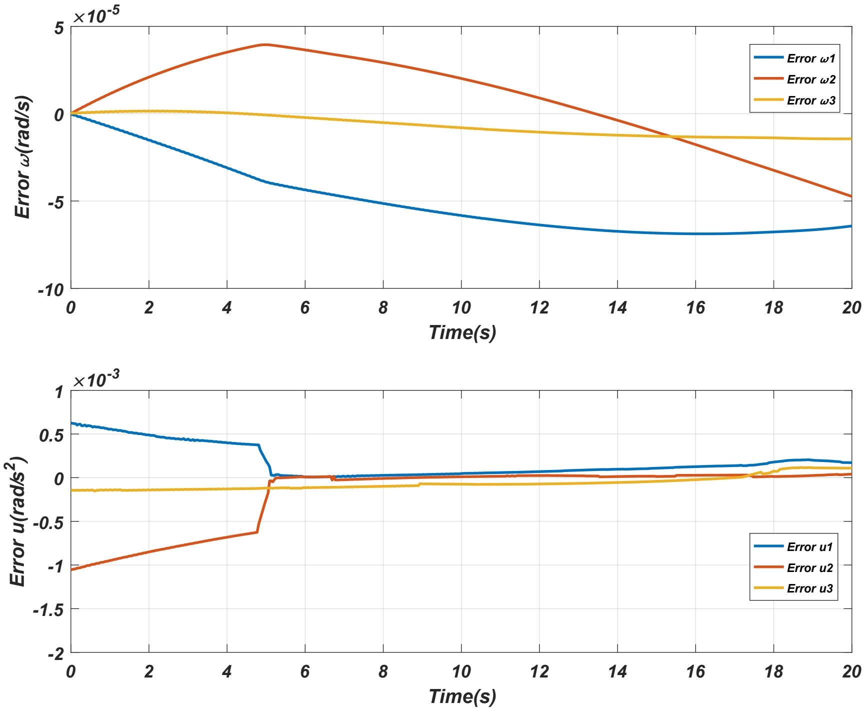

Figure 3. Error profiles between closed-loop and open-loop solution for initial condition # 1

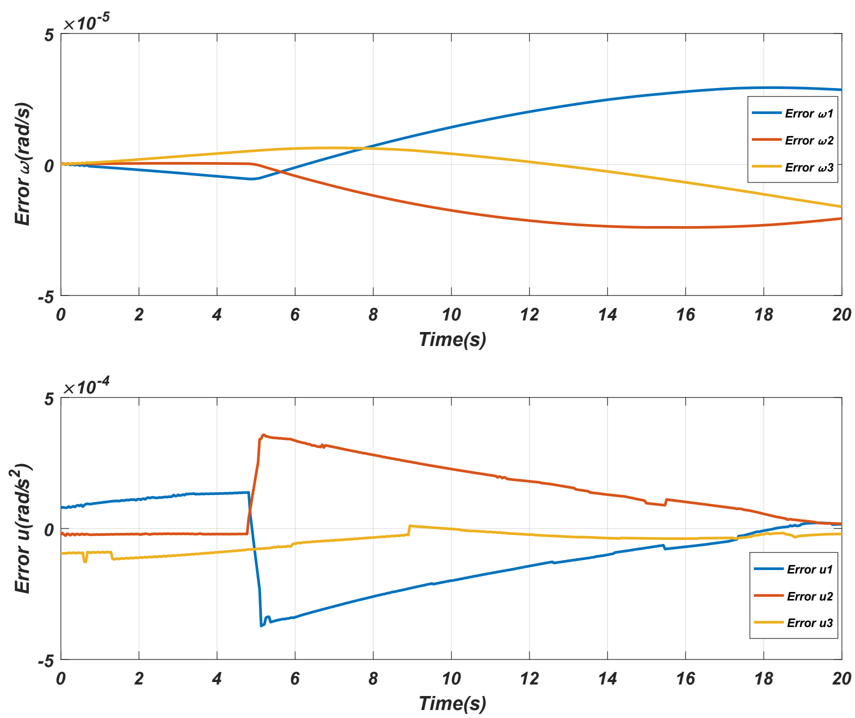

Figure 4. Error profiles between closed-loop and open-loop solution for initial condition # 2

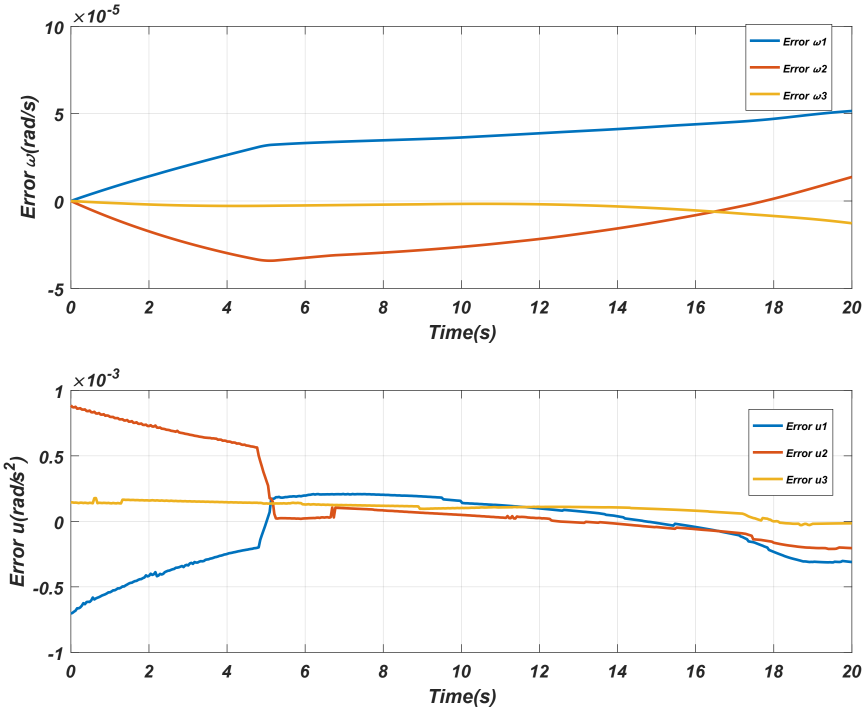

Figure 5. Error profiles between closed-loop and open-loop solution for initial condition # 3

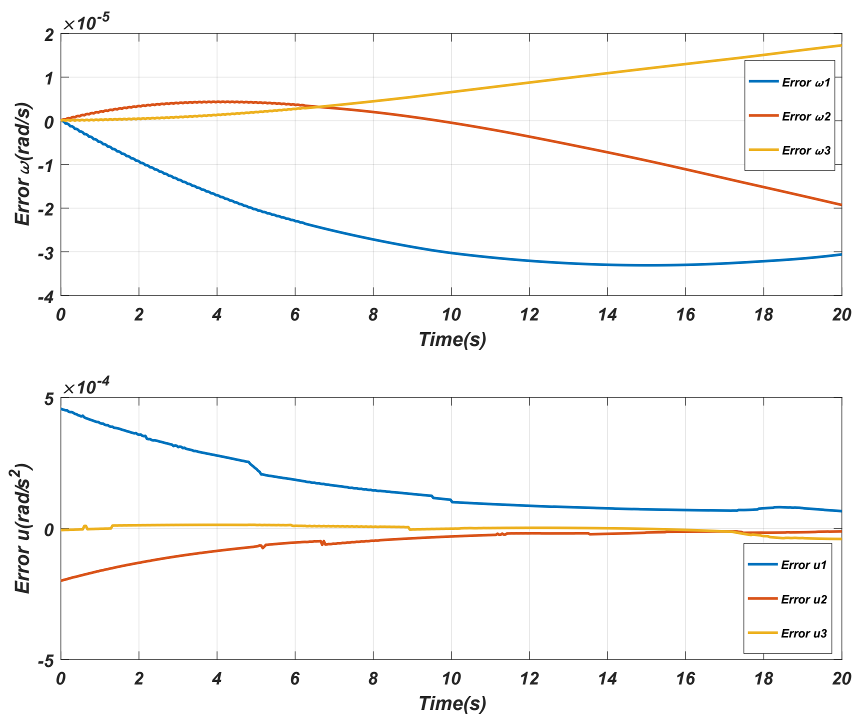

Figure 6. Error profiles between closed-loop and open-loop solution for initial condition # 4

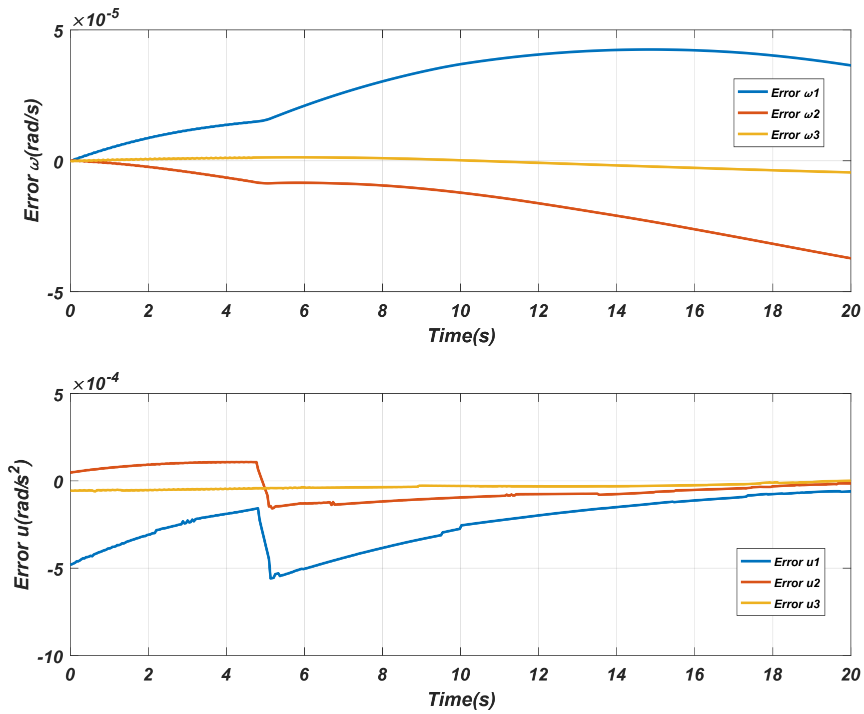

Figure 7. Error profiles between closed-loop and open-loop solution for initial condition # 5

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