ANAGRAM-FREE CHROMATIC NUMBER IS NOT PATHWIDTH-BOUNDED[☿]

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Abstract. The anagram-free chromatic number is a new graph parameter introduced independently by Kamčev, Łuczak, and Sudakov [1] and Wilson and Wood [5]. In this note, we show that there are planar graphs of pathwidth 3 with arbitrarily large anagram-free chromatic number. More specifically, we describe $2n$-vertex planar graphs of pathwidth 3 with anagram-free chromatic number $\Omega(\log n)$. We also describe kn vertex graphs with pathwidth $2k-1$ having anagram-free chromatic number in $\Omega(k \log n)$.

1 Introduction

A string $s = s_1, \ldots, s_{2k}$ is called an *anagram* if $s_1, \ldots, s_k$ is a permutation of $s_{k+1}, \ldots, s_{2k}$. For a graph G, a c-colouring $\varphi : V(G) \to \{1, \ldots, c\}$ is *anagram-free* if, for every odd-length path $v_1, v_2, \ldots, v_{2k}$ in G, the string $\varphi(v_1), \ldots, \varphi(v_{2k})$ is not an anagram. The *anagram-free chromatic number* of G, denoted $\pi_\alpha(G)$, is the smallest value of c for which G has an anagram-free c-colouring.

Answering a long-standing question of Erdős and Brown, Keränen [2] showed that the path P_n on n vertices has an anagram-free 4-colouring. A straightforward divide-and-conquer algorithm applied to any n-vertex graph of treewidth k yields an anagram-free $O(k \log n)$-colouring. The same divide-and-conquer algoritm, applied to graphs that exclude a fixed minor gives an anagram free $O(\sqrt{n})$-colouring [1]. An interesting variant of this divide-and-conquer algorithm is used by Wilson and Wood [5] to obtain anagram-free $(4k+1)$-colourings of trees of pathwidth k. On the negative side, Kamčev, Łuczak, and Sudakov [1] and Wilson and Wood [5] have shown that there are trees—even binary trees—with arbitrarily large anagram-free chromatic number. These results, and some others, are summarized in Table 1.

All of the examples of graphs having large anagram-free chromatic number are graphs with large pathwidth [3]. Therefore, an obvious question is whether anagram-free chromatic number is pathwidth-bounded, i.e., can $\pi_\alpha(G)$ be upper bounded by some function of the pathwidth $\mathrm{pw}(G)$ of G? Such a result seems plausible, for two reasons:

1. pathwidth is a measure of how path-like a graph is and Keränen showed that paths have anagram-free 4-colourings; and
2. the result of Wilson and Wood [5] shows that $\pi_\alpha(T) \le 4\,\mathrm{pw}(T) + 1$ for every tree, T.

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Graph class	Bounds	Reference
Paths	$\pi_\alpha(G) = 4$	[2, Theorem 1]
Graphs of treewidth k	$\pi_\alpha(G) \in O(k \log n)$	folklore
Graphs excluding a minor of size h	$\pi_\alpha(G) \in O(h^{3/2} n^{1/2})$	[1, Proposition 1.2]
Trees	$\pi_\alpha(G) \in \Omega(\log n / \log\log n)$	[5, Theorem 3]
Trees of pathwidth k	$k \le \pi_\alpha(G) \le 4k+1$	[5, Theorem 5]
Trees of radius r	$r \le \pi_\alpha(G) \le r+1$	[5, Theorem 4]
Binary trees	$\pi_\alpha(G) \in \Omega(\sqrt{\log n / \log\log n})$	[1, Proposition 1.1]
4-regular graphs	$\pi_\alpha(G) \in \Omega(\sqrt{n}/\log n)$	[1, Proposition 3.1]
d-regular graphs	$\pi_\alpha(G) \in \Omega(n)$	[1, Theorem 1.3]
Subdivisions of graphs	$\pi_\alpha(G) \le 8$	[4, Theorem 6]
Planar graphs	$\pi_\alpha(G) \in O(\sqrt{n})$	[1, Corollary 2.3]
Planar graphs of maximum degree 3	$\pi_\alpha(G) \in \Omega(\log n / \log\log n)$	[1, Proposition 2.4] [5, Theorem 1]
Planar graphs of pathwidth 3	$\pi_\alpha(G) \in \Omega(\log n)$	Theorem 1
Graphs of pathwidth $k > 3$	$\pi_\alpha(G) \in \Omega(k \log n)$	Theorem 2

Table 1: Bounds on anagram-free chromatic number. Upper bounds apply to all graphs in the class. Lower bounds apply to some graphs in the class.

The purpose of this note, however, is to show that the result of Wilson and Wood can not be strengthened even to planar graphs of pathwidth 3 and maximum degree 5. (Here and throughout, $\log x = \log_2 x$ denotes the binary logarithm of x.)

Theorem 1. *For every $n \in \mathbb{N}$, there exists a $2n$-vertex planar graph of pathwidth 3 and maximum degree 5 whose anagram-free chromatic number is at least $\log(n+1)$.*

Theorem 2. *For every $n \in \mathbb{N}$ and every integer $k \ge 3$, there exists a kn-vertex graph of pathwidth $2k-1$ and maximum degree $3k-1$ whose anagram-free chromatic number is at least $(k-2)\log(n/3)$.*

These two results show that the straightforward divide-and-conquer algorithm using separators gives asymptotically worst-case optimal colourings for graphs of pathwidth k and graphs of treewidth k.

2 Proof of Theorem 1

Let $s \in \Sigma^*$ be a string over some alphabet Σ. For each $a \in \Sigma$, we let $n_a(s)$ denote the number of occurences of a in s. We say that s is *even* if $n_a(s)$ is even for each $a \in \Sigma$. The following lemma says that strings with no even substrings must use an alphabet of at least logarithmic size.

Lemma 1. *If $s = s_0, \ldots, s_{2n-1} \in \Sigma^{2n}$ and $|\Sigma| < \log(n+1)$, then s contains a non-empty even substring $s_{2i}, \ldots, s_{2j-1}$ for some $0 \le i < j \le n$.*

Proof. For any string $q \in \Sigma^*$, we define the *parity vector* $P(q) = \langle n_a(q) \bmod 2 : a \in \Sigma \rangle$ and observe that q is even if and only if $P(q) = \langle 0, \ldots, 0 \rangle$. Furthermore, for two strings p and q, the parity vector of their concatenation pq is equal to the xor-sum (i.e., modulo 2 sum) of their parity vectors:

$$P(pq) = P(p) \oplus P(q) \ .$$

Define the strings $t_0, \ldots, t_n$, where t_0 is the empty string and, for each $i \in \{1, \ldots, n\}$, define $t_i = s_0, \ldots, s_{2i-1}$.

Now consider the parity vectors $P(t_0), P(t_1), \ldots, P(t_n)$. Each of these $n+1$ vectors is a binary string of length $|\Sigma| < \log(n+1)$ therefore, there must exist two indices $i, j \in \{0, \ldots, n\}$ with $i < j$ such that $P(t_i) = P(t_j)$. However,

$$P(t_j) = P(t_i) \oplus P(s_{2i}, \ldots, s_{2j-1})$$

and since $P(t_i) = P(t_j)$, this implies that $P(s_{2i}, \ldots, s_{2j-1}) = \langle 0, \ldots, 0 \rangle$ and $s_{2i}, \ldots, s_{2j-1}$ is even, as required. □

The next lemma says that if we split an even string into consecutive pairs, then we can can colour one element of each pair red and the other blue in such a way that the resulting red and blue multisets are exactly the same.

Lemma 2. *Let $s = s_0, \ldots, s_{2r-1} \in \Sigma^{2r}$ be an even string. Then there exists a binary sequence $v_0, \ldots, v_{r-1}$ such that the string $s_v = s_{0+v_0}, s_{2+v_1}, \ldots, s_{2(r-1)+v_{r-1}}$ has $n_a(s_v) = n_a(s)/2$ for all $a \in \Sigma$.*

Proof. Suppose for the sake of contradiction that the lemma is not true, and let s be the shortest counterexample. For $v \in \{0,1\}^r$, let $s_{\overline{v}} = s_{0+1-v_0}, s_{2+1-v_1}, \ldots, s_{2(r-1)+1-v_{r-1}}$ be the complement of s_v. Let $v \in \{0,1\}^r$ be the binary vector that minimizes

$$\sum_{a \in \Sigma} |n_a(s_v) - n_a(s_{\overline{v}})| \ . \qquad (1)$$

Since s is a counterexample to the lemma, (1) is greater than zero.

For each $j \in \{0, \ldots, r-1\}$, let $x_j = s_{2j+v_j}$ and let $y_j = s_{2j+1-v_j}$ so that $s_v = x_0, \ldots, x_{r-1}$ and $s_{\overline{v}} = y_0, \ldots, y_{r-1}$. Since (1) is non-zero, there exists some j_1 such that $n_{x_{j_1}}(s_v) > n_{x_{j_1}}(s_{\overline{v}})$. This means that $n_{y_{j_1}}(s_v) \geq n_{y_{j_1}}(s_{\overline{v}})$, otherwise flipping[1] v_{j_1} would decrease (1) by two. Furthermore, $y_{j_1} \neq x_{j_1}$ since, otherwise, we could remove s_{2j} and s_{2j+1} from s and obtain a smaller counterexample, since the value of v_j has no effect on (1).

Refer to Figure 1. Let $a_1 = x_{j_1}$ and for $k = 2, 3, 4 \ldots$, define $a_k = y_{j_{k-1}}$ and define j_k to be any index such that $x_{j_k} = a_k$. Notice that that $n_{a_k}(s_v) \geq n_{a_k}(s_{\overline{v}})$ since, otherwise, flipping $v_{j_1}, \ldots, v_{j_{k-1}}$ would decrease the value of (1). Indeed, flipping $v_{j_1}, \ldots, v_{j_{k-1}}$ decreases $n_{a_1}(s_v)$ by one, increases $n_{a_k}(s_v)$ by one, and does not change $n_a(s_v)$ for any $a \in \Sigma \setminus \{a_1, a_k\}$. This implies that j_k is well-defined since $n_{a_k}(s_v) \geq n_{a_k}(s_{\overline{v}}) \geq 1$.

Since s is finite, there is some minimum value k such that $a_k = a_{k'}$ for some $k' < k$. This defines a sequence of indices $j_{k'}, \ldots, j_{k-1}$ such that

1. $a_{k'} = x_{j_{k'}} = y_{j_{k-1}} = a_k$;
2. $a_\ell = y_{j_{\ell-1}} = x_{j_\ell}$ for all $\ell \in \{k'+1, \ldots, k-1\}$.

[1] Here and throughout, flipping a binary variable b means changing its value to $1 - b$.

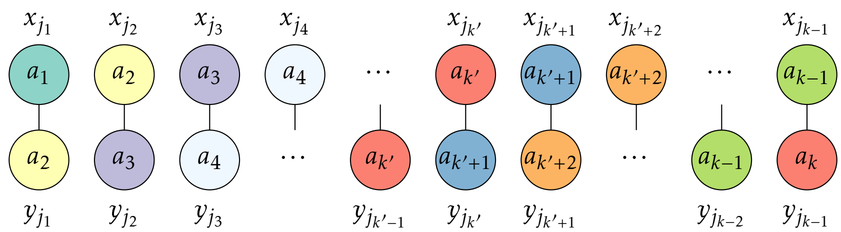

Figure 1: The proof of Lemma 2.

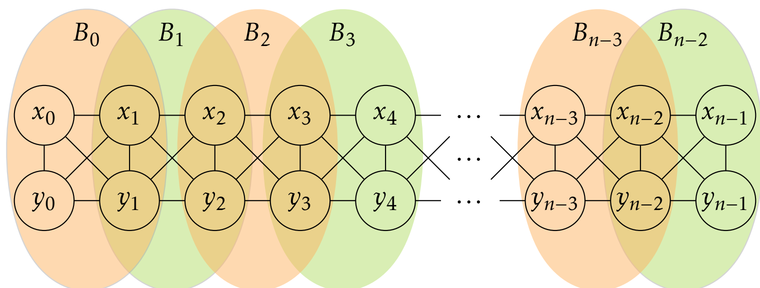

Figure 2: The graph G in the proof of Theorem 1.

In words, for each $\ell \in \{k', \dots, k\}$, each occurrence of a_ℓ in s_v is matched with a corresponding occurrence of a_ℓ in $s_{\overline{v}}$. We claim that this contradicts the minimality of s. Indeed, by removing $s_{2_{j_{k'}}}, s_{2j_{k'}+1}, s_{2_{j_{k'+1}}}, s_{2j_{k'+1}+1}, \dots, s_{2_{j_{k-1}}}, s_{2j_{k-1}+1}$ from s we obtain a smaller counterexample. □

Proof of Theorem 1. Let G be the graph with vertex set $V(G) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and with edge set

$$E(G) = \{x_i y_i : i \in \{0, \dots, n-1\}\} \cup \bigcup_{i=0}^{n-2} \{x_i y_{i+1}, x_{i+1} y_i\} \ .$$

The graph G has pathwidth 3 as can be seen from the path decomposition $B_0, \dots, B_{n-2}$ where $B_i = \{v_i, w_i, v_{i+1}, w_{i+1}\}$. See Figure 2. Although not immediately obvious from Figure 2, G is also planar—see Figure 3.

Now, consider some colouring $\varphi : V(G) \to \Sigma$ with $|\Sigma| < \log(n+1)$. Applying Lemma 1 to the string $s = \varphi(x_1), \varphi(y_1), \dots, \varphi(x_n), \varphi(y_n)$ we conclude that there is some $i < j$ such that $\varphi(x_i), \varphi(y_i), \dots, \varphi(x_j), \varphi(y_j)$ is even. By Lemma 2 and the symmetry between each x_i and y_i we can assume that $n_a(\varphi_(x_i), \dots, \varphi(x_j)) = n_a(\varphi(y_i), \dots, \varphi(y_j))$ for each $a \in \Sigma$. But then the path $x_i, \dots, x_j, y_j, y_{j-1}, \dots, y_i$ has a colour sequence $\varphi(x_i), \dots, \varphi(x_j), \varphi(y_j), \varphi(y_{j-1}), \dots, \varphi(y_i)$ that is an anagram. □

3 Proof of Theorem 2

Lemma 3. *For every sequence of sets $X_1, \dots, X_n \subseteq \Sigma$, each of size $k > 2$, with $|\Sigma| < (k-2)\log(n/3)$, there exists indices $1 \le i < j \le n$ and subsets $X'_i, \dots, X'_j$ such that, for each $\ell \in \{i, \dots, j\}$, $X'_\ell \subseteq X_\ell$, $|X'_\ell| \ge 2$ and, for each $a \in \Sigma$ the number of subsets in $X'_i, \dots, X'_j$ that contain a is even.*

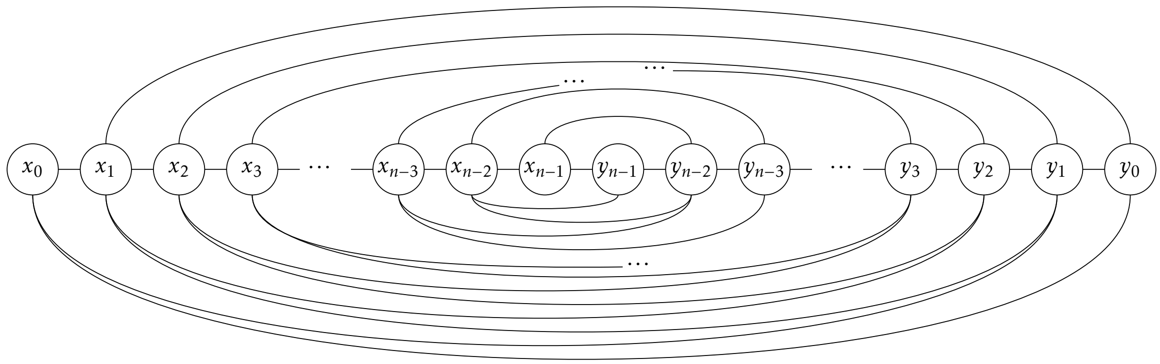

Figure 3: The graph G in the proof of Theorem 1 is planar and is even a 2-page graph.

Proof. For any $1 \le i \le j \le n$, let $\Sigma_{i,j} = \bigcup_{\ell=i}^{j} X_i$ and, for any $I \subset \Sigma_{i,j}$, let $N_{i,j}(I) = \{\ell \in \{i,\dots,j\} : X_\ell \cap I \neq \emptyset\}$. We distinguish between two cases.

Case 1: There is some pair of indices $1 \le i \le j \le n$ such that, for every $I \subseteq \Sigma_{i,j}$,

$$|N_{i,j}(I)| \ge |I|/(k-2) \ . \tag{2}$$

In this case we will show the existence of the desired sets $X'_i,\dots,X'_j$. Without loss of generality, assume $i = 1$, $j = n$, and define $N = N_{1,n}$.

Define a bipartite graph H with vertex set $V(H) = \Sigma \cup \{1,\dots,n\}$ and edge set $E(H) = \{(a,i) : i \in \{1,\dots,n\}, a \in X_i\}$. We will show that $E(H)$ contains a subset E' such that each element $a \in \Sigma$ appears exactly once in E' and each element of $\{1,\dots,n\}$ appears at most $k-2$ times in E'. That is, E' defines a mapping $f : \Sigma \to \{1,\dots,n\}$ in which, for any $i \in \{1,\dots,n\}$, $|f^{-1}(i)| \le k-2$.

The existence of the mapping f establishes the lemma since we can start with $X'_i = X_i$ for all $i \in \{1,\dots,n\}$ and then, for each $a \in \Sigma$ that appears an odd number of times, we can remove a from the set $X'_{f(a)}$. When this process is complete each X'_i has size at least 2 and each $a \in \Sigma$ occurs in an even number of the sets $X'_1,\dots,X'_n$.

All that remains is to prove the existence of the edge set E', which we do using an augmenting paths argument like that used, for example, to prove Hall's Marriage Theorem. Consider an edge set $E' \subseteq E(H)$ that contains exactly one edge incident to each $a \in \Sigma$ and let $f : \Sigma \to \{1,\dots,n\}$ be the corresponding mapping. Then we define

$$\Phi(E') = \sum_{i=1}^{n} \max\{0, |f^{-1}(i)| - (k-2)\} \ .$$

Note that the set E' we hope to find has $\Phi(E') = 0$. Now, select some E' that minimizes $\Phi(E')$. If $\Phi(E') = 0$ then we are done, so assume by way of contradiction, that $\Phi(E') > 0$. Thus, there exists some index $i_0 \in \{1,\dots,n\}$ such that $|f^{-1}(i_0)| \ge k-1$ and therefore the set $\Sigma_0 = f^{-1}(i_0)$ has size at least $k-1$. Therefore,

$$|N(\Sigma_0)| \ge \left\lceil \frac{|\Sigma_0|}{k-2} \right\rceil \ge \left\lceil \frac{k-1}{k-2} \right\rceil = 2 \ .$$

In particular, $N(\Sigma_0) \setminus \{i_0\}$ is non-empty. Let $I_0 = \{i_0\}$ and observe that each $i_1 \in N(\Sigma_0) \setminus I_0$ must have $|f^{-1}(i_1)| \geq k-2$ since, otherwise we could replace the edge (a_1, i_0) with (a_1, i_1) in E' and this would decrease $\Phi(E')$. Let $I_1 = N(\Sigma_0)$ and let $\Sigma_1 = \bigcup_{i_1 \in I_i} f^{-1}(i_1)$. We have just argued that

$$|\Sigma_1| \geq |I_1|(k-2)+1$$

and therefore,

$$|N(\Sigma_1)| \geq \left\lceil \frac{|\Sigma_1|}{k-2} \right\rceil \geq \left\lceil \frac{|I_1|(k-2)+1}{k-2} \right\rceil \geq |I_1|+1 \ .$$

But now we can continue this argument, defining $I_j = N(\Sigma_{j-1})$ and $\Sigma_j = \bigcup_{i_j \in I_j} f^{-1}(i_j)$. Again, each $i_j \in I_j \setminus \bigcup_{\ell=1}^{j-1} I_\ell$ must have $|f^{-1}(i_j)| \geq k-2$, otherwise we can find a path $i_0, a_0, i_1, a_1, \ldots, a_{j-1} i_j$ and replace, in E', the edges $i_0 a_0, \ldots, i_{j-1} a_{j-1}$ with $a_0 i_1, a_1 i_2, \ldots, a_{j-1} i_j$ which would decrease $\Phi(E')$. In this way, we obtain an infinite sequence of subsets $I_0, \ldots, I_\infty \subseteq \{1, \ldots, n\}$ such that $|I_j| > |I_{j-1}|$. This is clearly a contradiction, since each $|I_j|$ is an an integer in $\{1, \ldots, n\}$.

Case 2: For every $1 \leq i < j \leq n$, there exists a set $I \subset \Sigma_{i,j}$ such that $|N_{i,j}(I)| < |I|/(k-2)$. In this case, we will show that $|\Sigma| \geq (k-2)\log(n/3)$.

Before jumping into the messy details, we sketch an inductive proof that gives the main intuition for why $|\Sigma| \in \Omega(k \log n)$: There is some set $I_0 \subset \Sigma$ such that $N(I_0)$ partitions $\{1, \ldots, n\}$ into $O(|I|/k)$ intervals. One such interval $i_0, \ldots, j_0$ must have size $\Omega(nk/|I|)$. By induction on n, $|\Sigma_{i_0,j_0}| = \Omega(k \log(nk/|I|))$. But Σ_{i_0,j_0} is disjoint from I, so

$$|\Sigma| \geq |I| + \Omega(k \log(nk/|I|)) = |I| + \Omega(k \log n) - O(k \log(|I|/k)) = \Omega(k \log n) \ .$$

The messy details occur when $|I| = k-1$ since then the $|I|$ and $-O(k \log(|I|/k)$ terms are close in magnitude.

Let $n_0 = n$, $i_0 = 1$, $j_0 = n$, $\Sigma_0 = \Sigma$ and let $I_0 \subseteq \Sigma_0$ be such that $|N(I_0)| < |I|/(k-2)$. For each integer ℓ with $n_{\ell-1} \geq 1$, we define

1. i_ℓ and j_ℓ such that $i_{\ell-1} \leq i_\ell < j_\ell \leq j_{\ell-1}$, $\{i_\ell, \ldots, j_\ell\} \cap N_{i_{\ell-1}, j_{\ell-1}}(I_{\ell-1}) = \emptyset$, and $n_\ell = j_\ell - i_\ell + 1$ is maximized.

2. $I_\ell \subset \Sigma_{i_\ell, j_\ell}$ such that $|N_{i_\ell, j_\ell}(I_\ell)| < |I_\ell|/k$;

In words, $N_{i_{\ell-1}, j_{\ell-1}}(I_{\ell-1})$ partitions $i_{\ell-1}, \ldots, j_{\ell-1}$ into intervals and we choose i_ℓ and j_ℓ to be the endpoints of a largest such interval and recurse on that interval using a new set I_ℓ. Letting $y_\ell = |N_{i_\ell, j_\ell}(I_\ell)|$, observe that, for $\ell \geq 1$,

$$n_\ell \geq \frac{n_{\ell-1} - y_{\ell-1}}{y_{\ell-1}+1} > \frac{n_{\ell-1}}{y_{\ell-1}+1} - 1 \ .$$

By expanding the preceding equation we can easily show that

$$n_\ell \geq \frac{n}{\prod_{\tau=0}^{\ell-1}(y_\tau + 1)} - 2 \ .$$

Note that $n_{\ell+1}$ is defined until $n_\ell < 1$ so combining this with the preceding equation and taking logs yields

$$\sum_{\tau=0}^{\ell-1}(y_\tau + 1) > \log(n/3) \tag{3}$$

Finally, observe that the sets $I_0,\dots,I_{\ell-1}$ are disjoint, so

$$|\Sigma| \ge \sum_{\tau=0}^{\ell-1}|I_\tau| > \sum_{\tau=0}^{\ell-1}(k-2)y_\tau \quad . \tag{4}$$

Now, minimizing (4) subject to (3) and using the fact that each $y_\tau \ge 1$ is an integer shows that $|\Sigma| \ge (k-2)\log(n/3)$, as desired. (The minimum is obtained when $\ell = \log(n/3)$ and $y_1 = y_2 = \cdots = y_{\ell-1} = 1$.) □

Proof of Theorem 2. The pathwidth $2k-1$ graph, G, used in this proof is a natural generalization of the pathwidth 3 graph used in the proof of Theorem 1. The kn vertices of G are partitioned in subsets $V_1,\dots,V_n$, each size of size k. For each $i \in \{1,\dots,n\}$, V_i is a clique and, for each $i \in \{1,\dots,n-1\}$, every vertex in V_i is adjacent to every vertex in V_{i+1}. That this graph has pathwidth $2k-1$ can be seen from the path decomposition $B_1,\dots,B_{n-2}$ where each $B_i = \{V_i \cup V_{i+1}\}$.

Suppose we have some colouring $\varphi : V(G) \to \Sigma$, with $|\Sigma| < (k-2)\log(n/3)$. Define the sets $X_1,\dots,X_n$ where $X_i = \{\varphi(v) : v \in V_i\}$. By Lemma 3, we can find indices $i \in \{0,\dots,n-1\}$ and $r > 0$ and subsets $V'_1,\dots,V'_r$ such that, for each $\ell \in \{1,\dots,r\}$, $V'_\ell \subseteq V_{i+\ell}$, $|V'_\ell| \ge 2$, and such that each colour $a \in \Sigma$ appears in an even number of $V'_1,\dots,V'_r$.

Next, label the vertices in $V'_1,\dots,V'_r$ red and blue as follows. If $|V'_i|$ is even, then label half its vertices red and half its vertices blue, arbitrarily. Let $Q_1,\dots,Q_t$ denote the subsequence of $V'_1,\dots,V'_r$ consisting of only sets of odd size (so the vertices in $Q_1,\dots,Q_t$ are not labelled red or blue yet). Then, for odd values of i, label $\lceil |Q_i|/2 \rceil$ vertices of Q_i red and the remaining blue. For even values of i label $\lfloor |Q_i|/2 \rfloor$ vertices of Q_i red and the remaining blue. Observe that, since $\sum_{i=1}^{r}|V'_i|$ is even, t is also even, so exactly half the vertices in $\bigcup_{i=1}^{r} V'_i$ are red and half are blue.

Now, consider the following perfect bichromatic matching of $\bigcup_{i=1}^{r} V'_i$: In every set V'_i of even size we match each red vertex in V'_i with a blue vertex in V'_i. In each odd size set Q_i, we match $\lfloor |Q_i|/2 \rfloor$ red vertices with blue vertices leaving one vertex v_i unmatched. This leaves t unmatched vertices $v_1,\dots,v_t$ and these vertices alternate colour between red and blue. To complete the matching, we match v_{2i} with v_{2i-1} for each $i \in \{1,\dots,t/2\}$.

Now, treat this matching as a long string $s = x_1, y_1, \dots, x_q, y_q$ where each $x_i = \varphi(v_i)$, each $y_i = \varphi(w_i)$, and each (v_i, w_i) is a matched pair of vertices. Now, applying Lemma 2 to s, we obtain two sets of vertices $V = \{v'_1,\dots,v'_q\}$ and $W = \{w'_1,\dots,w'_q\}$ such that, for each $a \in \Sigma$, $n_a(\varphi(v_1),\dots,\varphi(v_q)) = n_a(\varphi(w_1),\dots,\varphi(w_q))$. Thus, all that remains is to show that G contains a path P whose first half is some permutation of V and whose second half is some permutation of W. But this is obvious, because, for each $i \in \{1,\dots,r\}$, V'_i contains at least one vertex of V and at least one vertex of W. Thus, the path P first visits all the vertices of $V \cap V'_1$ followed by all the vertices of $V \cap V'_2$, and so on until visiting all the vertices in $V \cap V'_r$. Next, the path returns and visits all the vertices in $W \cap V'_r$, $W \cap V'_{r-1}$, and so on back to $W \cap V_1'$. The existence of the path P shows that no colouring of G with fewer than $(k-2)\log(n/3)$ colours is anagram-free, so $\pi_\alpha(G) \geq (k-2)\log(n/3)$. □

4 Remarks

We have show that anagram-free chromatic number is not pathwidth-bounded, even for planar graphs. The graph we use in the proof of Theorem 1 is a 2-page graph; it has a book embedding using two pages. Outerplanar graphs have a book embedding using a single page. Is anagram-free chromatic number pathwidth-bounded for outerplanar graphs? We do not even know if the $2 \times n$ grid has constant anagram-free chromatic number.

References

[1] Nina Kamčev, Tomasz Łuczak, and Benny Sudakov. Anagram-free colorings of graphs. *Combinatorics, Probability and Computing*, pages 1–20, 2017. Online first edition published August 2017. arXiv:1606.09062, doi:10.1017/S096354831700027X.

[2] Veikko Keränen. Abelian squares are avoidable on 4 letters. volume 623 of *Lecture Notes in Computer Science*, pages 41–52, Berlin, 1992. Springer. doi:10.1007/3-540-55719-9_62.

[3] Neil Robertson and P. D. Seymour. Graph minors. I. Excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983. doi:10.1016/0095-8956(83)90079-5.

[4] Tim E. Wilson and David R. Wood. Anagram-free colourings of graph subdivisions, 2017. arXiv:1708.09571.

[5] Tim E. Wilson and David R. Wood. Anagram-free graph colouring, 2017. arXiv:1607.01117.