

# MODULAR CATEGORIES OF DIMENSION $p^3m$ WITH $m$ SQUARE-FREE

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**ABSTRACT.** We give a complete classification of modular categories of dimension  $p^3m$  where  $p$  is prime and  $m$  is a square-free integer relatively prime to  $p$ . When  $p$  is odd, all such categories are pointed. For  $p = 2$  one encounters modular categories with the same fusion ring as orthogonal quantum groups at certain roots of unity, namely  $SO(2m)_2$ . We also classify the more general class of modular categories with the same fusion rules as  $SO(2N)_2$  with  $N$  odd.

## 1. INTRODUCTION

Weakly integral modular categories, i.e. modular categories of integral FP-dimension, arise in a number of settings such as quantum groups at certain roots of unity [23], equivariantizations of Tambara-Yamagami categories [16], and by gauging pointed modular categories [1, 3, 8]. Several key conjectures characterizing weakly integral categories are found in the literature, e.g. the Property-F Conjecture [23, Conjecture 2.3], the Weakly Group-Theoretical Conjecture [14, Question 2], and the Gauging Conjecture [1, Abstract]. In particular, it is known [27] that the braid group representations associated with the weakly integral modular categories  $SO(M)_2$  factor over finite groups for all  $M$ ; that is, these categories have property  $F$ .

We will assume throughout that all objects in our categories have positive dimensions. This is not a significant loss of generality; see Remark 2.1. In the spirit of [18], we call any modular category with the same fusion rules as the  $4M$ -dimensional modular category  $SO(M)_2$  with  $M$  even an *even metaplectic* modular category.<sup>1</sup> Our main results are summarized as follows:

**Theorem.** *If  $\mathcal{C}$  is a non-pointed modular category of dimension  $p^3m$  where  $p$  is a prime and  $m$  is a square-free integer relatively prime to  $p$ , then  $p = 2$  and one of the following is true:*

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<sup>1</sup>In fact they require unitarity. It is not known if this is a strictly stronger assumption than positivity of dimensions.

- (i)  $\mathcal{C}$  is a Deligne product of an even metaplectic modular category of dimension  $8\ell$  and a pointed  $\mathbb{Z}_k$ -cyclic modular category with  $k$  odd, or
- (ii)  $\mathcal{C}$  is the Deligne product of a Semion modular category with a modular category of dimension  $4m$  (see [5]).

Furthermore, we extend the work of [5, Theorem 3.1] characterizing modular categories of dimension  $p^2m$  to include the case  $p$  odd: in this case such categories are pointed (see Theorem 4.5). In particular, all modular categories of dimension  $p^2m$  and  $p^3m$  satisfy the Weakly Group-Theoretical Conjecture [14, Question 2] and Gauging Conjecture [1]. Moreover, we obtain the following result similar to that of [1, Theorem 3.1]:

**Theorem** (Theorem 3.8). *If  $\mathcal{C}$  is an even metaplectic modular category of dimension  $8N$ , with  $N$  an odd integer, then  $\mathcal{C}$  is a gauging of the particle-hole symmetry of a  $\mathbb{Z}_{2N}$ -cyclic modular category. Moreover, there are exactly  $2^{r+2}$  inequivalent even metaplectic modular categories of dimension  $8N$  for  $N = p_1^{k_1} \cdots p_r^{k_r}$ , with  $p_i$  distinct odd primes.*

## 2. PRELIMINARIES

In this section we recall results and notation that are presumably well-known to the experts. Further details can be found in [2, 12, 15].

A *premodular* category,  $\mathcal{C}$ , is a braided, balanced, fusion category. Throughout we will denote the isomorphism classes of simple objects of  $\mathcal{C}$  by  $X_a$  ordered such that  $X_0 = \mathbf{1}$  is the unit object. The set of isomorphism classes of simple objects will be denoted by  $\text{Irr}(\mathcal{C})$ . Duality in  $\mathcal{C}$  introduces an involution on the label set of the simple objects by  $X_a^* \cong X_{a^*}$ . The fusion matrices,  $N_{a,b}^c = (N_a)_{b,c}$ , provide the multiplicity of  $X_c$  in  $X_a \otimes X_b$ . These matrices are non-negative integer matrices and thus are subject to the Frobenius-Perron Theorem. The Frobenius-Perron eigenvalue of  $N_a$ ,  $d_a$  is called the *Frobenius-Perron dimension* of  $X_a$ , or FP-dimension for short. An object is said to be *invertible* if its FP-dimension is 1 and *integral* if its FP-dimension is an integer. The invertible and integral simple objects generate full fusion subcategories called the *pointed subcategory*,  $\mathcal{C}_{\text{pt}}$ , and the *integral subcategory*,  $\mathcal{C}_{\text{int}}$ . If  $\mathcal{C} = \mathcal{C}_{\text{int}}$ , then  $\mathcal{C}$  is said to be *integral*. The categorical FP-dimension is given by  $\text{FPdim}\mathcal{C} = \sum_a d_a^2$ . If  $\text{FPdim}\mathcal{C} \in \mathbb{Z}$ , then  $\mathcal{C}$  is said to be a *weakly integral category*. A weakly integral category that is not integral is called *strictly weakly integral*.

*Remark 2.1.* Let  $\mathcal{C}$  be any weakly integral modular category. By [15, Proposition 5.4] the underlying braided fusion category  $\mathcal{C}$  has a unique pivotal (in fact, spherical) structure so that each object in  $\mathcal{C}$  has positive dimension. Moreover, the spherical structures on the braided fusion category underlying any modular category are in 1-1 correspondence with simple objects  $a$  with  $a^{\otimes 2} \cong \mathbf{1}$ , and each spherical structure again yields a modular category [6, Lemma 2.4]. Thus a classification of some class of weakly integral modular categories with positive dimensions can be easily extended to a classification of all modular categories in that class. This motivates our assumption that the categorical and FP-dimensions coincide.

Important data for a premodular category are the  $S$ -matrix and  $T$ -matrix,  $S = (S_{a,b})$  and  $T = (\theta_a \delta_{a,b})$ . These matrices are indexed by the simple objects in  $\mathcal{C}$ , and the diagonal entries of the  $T$ -matrix are referred to as *twists*. These data

obey (see [2])  $S_{a,b} = S_{b,a}$ ,  $S_{0,a} = d_a$ ,  $\overline{S_{a,b}} = S_{a,b^*}$  and the *balancing* relation  $\theta_a \theta_b S_{a,b} = \sum_c N_{a^*,b}^c d_c \theta_c$ .

For  $\mathcal{D} \subset \mathcal{C}$  premodular categories the *centralizer* of  $\mathcal{D}$  in  $\mathcal{C}$  is denoted by  $C_{\mathcal{C}}(\mathcal{D})$  and is generated by the objects  $\{X \in \mathcal{C} \mid S_{X,Y} = d_X d_Y \forall Y \in \mathcal{D}\}$  (see [4]). The category  $C_{\mathcal{C}}(\mathcal{C})$  is called the *Müger center* of  $\mathcal{C}$  and is often denoted by  $\mathcal{C}'$ . Note that a useful characterization of a simple object being outside of the Müger center is that its column in the  $S$ -matrix is orthogonal to the first column of  $S$ . If  $\mathcal{C}' = \text{Vec}$ , then  $\mathcal{C}$  is said to be a *modular category*, whereas if  $\mathcal{C}' = \mathcal{C}$ , then  $\mathcal{C}$  is called *symmetric*. Symmetric categories are classified in terms of group data:

**Theorem 2.2** ([9]). *If  $\mathcal{C}$  is a symmetric fusion category, then there exists a finite group  $G$  such that  $\mathcal{C}$  is equivalent to the super-Tannakian category  $\text{Rep}(G, z)$  of super-representations (i.e.  $\mathbb{Z}_2$  graded) of  $G$  where  $z \in Z(G)$  is a distinguished central element with  $z^2 = 1$  acting as the parity operator.*

Observe that  $\mathcal{C} \cong \text{Rep}(G, z)$  with  $z = e \in G$  if and only if the  $\mathbb{Z}_2$ -grading is trivial so that  $\text{Rep}(G, z) = \text{Rep}(G)$ . In this case we say  $\mathcal{C}$  is Tannakian, and otherwise we say it is non-Tannakian. The smallest non-Tannakian symmetric category is  $\text{Rep}(\mathbb{Z}_2, 1)$ , which we will denote by  $\text{sVec}$  when equipped with the (unique) structure of a ribbon category with  $\dim(\chi) = 1$  for a non-trivial object  $\chi$ .

An invertible object in  $\mathcal{C}$  that generates a Tannakian category (i.e.  $\text{Rep}(\mathbb{Z}_2)$ ) is referred to as a *boson*, while an invertible object in  $\mathcal{C}$  generating  $\text{sVec}$  is referred to as a *fermion*. By [20, Lemma 5.4] we find that if  $\text{sVec} \subset \mathcal{C}'$  for some premodular category  $\mathcal{C}$  and  $\chi$  is the generator of  $\text{sVec}$ , then  $\chi \otimes Y \not\cong Y$  for all simples  $Y$  in  $\mathcal{C}$ . Bosons are useful through a process known as de-equivariantization, which we will discuss shortly.

The Semion and Ising categories will appear frequently in the sequel in factorizations of modular categories. The Semion categories, denoted by  $\text{Sem}$ , are modular categories with the same fusion rules as  $\text{Rep}(\mathbb{Z}_2)$ ; see [26] for details. The Ising categories are rank 3 modular categories with simple objects:  $\mathbf{1}$ ,  $\psi$  (a fermion), and  $\sigma$  (the Ising anyon) of dimension  $\sqrt{2}$ . The key fusion rules are  $\psi^2 = \mathbf{1}$  and  $\sigma^2 = \mathbf{1} + \psi$ , while the modular datum can be found in [26]. There are exactly 8 inequivalent Ising categories. They are the smallest examples of *generalized Tambara-Yamagami categories*, i.e. non-pointed fusion categories with the property that the tensor product of any two non-invertible simple objects is a direct sum of invertible objects. In fact, by [24] any *modular* generalized Tambara-Yamagami category is a (Deligne) product of an Ising category and a pointed modular category. Characterizations of the Ising categories can be found in [5, 12]. In Lemmas 4.7, 4.8 and 4.9 we find new conditions implying a category must contain an Ising category.

A *grading* of a fusion category  $\mathcal{C}$  by a finite group  $G$  is a decomposition of the category as direct sum  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ , where the components are full abelian subcategories of  $\mathcal{C}$  indexed by the elements of  $G$ , such that the tensor product maps  $\mathcal{C}_g \times \mathcal{C}_h$  into  $\mathcal{C}_{gh}$ . The *trivial component*  $\mathcal{C}_e$  (corresponding to the unit of the group  $G$ ) is a fusion subcategory of  $\mathcal{C}$ . The grading is called *faithful* if  $\mathcal{C}_g \neq 0$ , for all  $g \in G$ , and in this case all the components are equidimensional with  $|G| \dim \mathcal{C}_g = \dim \mathcal{C}$ .

Every fusion category  $\mathcal{C}$  is faithfully graded by the universal grading group  $\mathcal{U}(\mathcal{C})$ , and every faithful grading of  $\mathcal{C}$  is a quotient of  $\mathcal{U}(\mathcal{C})$ . Furthermore, the trivial component under the universal grading is the *adjoint subcategory*  $\mathcal{C}_{\text{ad}}$ , the full fusion subcategory generated by the objects  $X \otimes X^*$ , for  $X$  simple. The universal grading was first studied in [17], and it was shown that if  $\mathcal{C}$  is modular, then  $\mathcal{U}(\mathcal{C})$  is

canonically isomorphic to the character group of the group  $G(\mathcal{C})$  of isomorphism classes of invertible objects in  $\mathcal{C}$ .

When  $\mathcal{C}$  is a weakly integral fusion category there is another useful grading called the  $GN$ -grading first studied in [17]:

**Theorem 2.3** ([17, Theorem 3.10]). *Let  $\mathcal{C}$  be a weakly integral fusion category. Then there is an elementary abelian 2-group  $E$ , a set of distinct square-free positive integers  $n_x$ ,  $x \in E$ , with  $n_0 = 1$ , and a faithful grading  $\mathcal{C} = \bigoplus_{x \in E} \mathcal{C}(n_x)$  such that  $\dim(X) \in \mathbb{Z}\sqrt{n_x}$  for each  $X \in \mathcal{C}(n_x)$ .*

Notice the trivial component of the  $GN$ -grading is the integral subcategory  $\mathcal{C}_{\text{int}}$ .

Our approach to classification relies upon *equivariantization* and its inverse functor *de-equivariantization* (see for example [12]), which we now briefly describe. An *action* of a finite group  $G$  on a fusion category  $\mathcal{C}$  is a strong tensor functor  $\rho : \underline{G} \rightarrow \text{End}_{\otimes}(\mathcal{C})$ . The  $G$ -equivariantization of the category  $\mathcal{C}$  is the category  $\mathcal{C}^G$  of  $G$ -equivariant objects and morphisms of  $\mathcal{C}$ . When  $\mathcal{C}$  is a fusion category over an algebraically closed field  $\mathbb{K}$  of characteristic 0, the  $G$ -equivariantization,  $\mathcal{C}^G$ , is a fusion category with  $\dim \mathcal{C}^G = |G| \dim \mathcal{C}$ . The fusion rules of  $\mathcal{C}^G$  can be determined in terms of the fusion rules of the original category  $\mathcal{C}$  and group-theoretical data associated to the group action [7]. De-equivariantization is the inverse to equivariantization. Given a fusion category  $\mathcal{C}$  and a Tannakian subcategory  $\text{Rep}(G) \subset \mathcal{C}$ , consider the algebra  $A = \text{Fun}(G)$  of functions on  $G$ . Then  $A$  is a commutative algebra in  $\mathcal{C}$ . The category of  $A$ -modules on  $\mathcal{C}$ ,  $\mathcal{C}_G$ , is a fusion category called a  $G$ -de-equivariantization of  $\mathcal{C}$ . We have  $|G| \dim \mathcal{C}_G = \dim \mathcal{C}$ , and there are canonical equivalences  $(\mathcal{C}_G)^G \cong \mathcal{C}$  and  $(\mathcal{C}^G)_G \cong \mathcal{C}$ . An important property of the de-equivariantization is that if the Tannakian category in question is  $\mathcal{C}'$ , then  $\mathcal{C}_G$  is modular and is called the *modularization* of  $\mathcal{C}$  [4, 20].

One may also construct modular categories from a given modular category  $\mathcal{C}$  with an action of a finite group  $G$  by the *gauging* introduced and studied in [3] and [8]. Gauging is a 2-step process. The first step is to extend the modular category  $\mathcal{C}$  to a  $G$ -crossed braided fusion category  $\mathcal{D} \cong \bigoplus_{g \in G} \mathcal{D}_g$  such that  $\mathcal{D}_e = \mathcal{C}$  with a  $G$ -braiding; see [12]. The second step is to equivariantize  $\mathcal{D}$  by  $G$ . One important remark is that there are certain obstructions to the first step [13], while the second is always possible.

### 3. EVEN METAPLECTIC CATEGORIES OF DIMENSION $8N$ , WITH $N$ AN ODD INTEGER

In this section we consider even metaplectic modular categories. A general understanding of these categories will facilitate a classification of modular categories of dimension  $8N$  for any odd square-free integer  $N$ . Metaplectic categories were defined and studied in [1]. To conform with their notation we define:

**Definition 3.1.** An **even metaplectic modular category** is a modular category  $\mathcal{C}$  with positive dimensions that is Grothendieck equivalent to (i.e. has the same fusion rules as)  $SO(2N)_2$ , for some integer  $N \geq 1$ .

*Remark 3.2.* There are significant differences between  $N$  odd and even; see [23]. For our results only the case  $N$  odd plays a role. Notice the case  $N = 1$  is degenerate, corresponding to  $SO(2)_2$ , which has fusion rules like  $\mathbb{Z}_8$ . Our results carry over to this (pointed) case with little effort, so we include it.

Throughout the remainder of this section,  $\mathcal{C}$  will denote an even metaplectic category with  $N$  an odd integer. In particular,  $\mathcal{C}$  has dimension  $8N$ . In this case it follows from the definition that  $\mathcal{C}$  has rank  $N + 7$  [23]. While the dimension and rank of  $\mathcal{C}$  are straightforward, the fusion rules strongly depend on the parity of  $N$ . While the full fusion rules and  $S$ -matrix can be found in [23, Subsection 3.2], we review here some of the fusion rules that will be relevant in later proofs. The group of isomorphism classes of invertible objects of  $\mathcal{C}$  is isomorphic to  $\mathbb{Z}_4$ . We will denote by  $g$  a generator of this group and will abuse notation referring to the invertible objects as  $g^k$ . The only non-trivial self-dual invertible object is  $g^2$ . In  $\mathcal{C}$ , there are  $N - 1$  self-dual simple objects,  $X_i$  and  $Y_i$ , of dimension 2. Furthermore, one may order the 2-dimensional objects such that  $X_1$  generates  $\mathcal{C}_{\text{ad}}$  and  $Y_1$  generates  $\mathcal{C}_{\text{int}}$ . The remaining four simples in  $\mathcal{C}$ ,  $V_i$ , have dimension  $\sqrt{N}$ .  $\mathcal{C}$  has the following fusion rules:

- $g \otimes X_a \simeq Y_{\frac{N+1}{2}-a}$ , and  $g^2 \otimes X_a \simeq X_a$ , and  $g^2 \otimes Y_a \simeq Y_a$  for  $1 \leq a \leq (N - 1)/2$ .
- $X_a \otimes X_a = \mathbf{1} \oplus g^2 \oplus X_{\min\{2a, N-2a\}}$ ;  $X_a \otimes X_b = X_{\min\{a+b, N-a-b\}} \oplus X_{|a-b|}$  ( $a \neq b$ ).
- $V_1 \otimes V_1 = g \oplus \bigoplus_{a=1}^{\frac{N-1}{2}} Y_a$ .
- $gV_1 = V_3$ ,  $gV_3 = V_4$ ,  $gV_2 = V_1$ ,  $gV_4 = V_2$  and  $g^3V_a = V_a^*$ ,  $V_2 = V_1^*$ ,  $V_4 = V_3^*$ .

We remark that it is immediately clear that an even metaplectic modular category of dimension  $8N$  with  $N$  odd is prime (not the Deligne product of two modular subcategories):  $V_1$  is a tensor generator, hence cannot reside in any proper fusion subcategory.

For  $\text{SO}(2N)_2$  we order the simple objects by their highest weights as follows:  $\mathbf{0}, 2\lambda_1, 2\lambda_{N-1}, 2\lambda_N, \lambda_1, \dots, \lambda_{N-2}, \lambda_{N-1} + \lambda_N, \lambda_{N-1}, \lambda_N, \lambda_{N-1} + \lambda_1, \lambda_N + \lambda_1$ . The correspondence with the notation used in this paper is

$$\mathbf{1}, g^2, g, g^3, Y_1, X_1, \dots, Y_{\frac{N-1}{2}}, X_{\frac{N-1}{2}}, V_1, V_2, V_3, V_4.$$

The  $S$ - and  $T$ -matrices have the following forms:

$$S = \begin{pmatrix} A & B & C \\ B^t & D & 0 \\ C^t & 0 & E \end{pmatrix}, \quad T = \text{diag} \left( 1, 1, i^N, i^N, \dots, (-1)^a e^{-a^2 \pi i / 2N}, \dots, \theta_\epsilon, \theta_\epsilon, -\theta_\epsilon, -\theta_\epsilon \right)$$

where  $\theta_\epsilon = e^{\pi i \frac{2N-1}{8}}$ , and  $A$ ,  $E$ , and  $C$  are  $4 \times 4$  matrices,  $B$  is a  $4 \times (N - 1)$  matrix and  $D$  is  $(N - 1) \times (N - 1)$ . To give their explicit forms we first define a  $\mathbb{C}$ -valued function:

$$G(N) = \begin{cases} (\frac{-1}{N})\sqrt{N} & \text{if } N \equiv 1 \pmod{4}, \\ i(\frac{-1}{N})\sqrt{N} & \text{if } N \equiv 3 \pmod{4}, \end{cases}$$

where  $(\frac{-1}{N})$  is the Jacobi symbol, and set  $\alpha = \overline{G(N)}\theta_\epsilon^2$  and  $\beta = i^{2N-1}\theta_\epsilon^2 G(N)$ . Then:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad B = 2 \begin{pmatrix} 1 & \dots & 1 \\ -1 & \dots & (-1)^a \dots 1 \\ -1 & \dots & (-1)^a \dots 1 \end{pmatrix}, \quad C = \sqrt{N} \begin{pmatrix} \frac{-1}{i^N} & \frac{1}{i^N} & \frac{-1}{i^N} & \frac{1}{i^N} \\ \frac{-1}{i^N} & \frac{1}{i^N} & \frac{-1}{i^N} & \frac{1}{i^N} \\ \frac{-1}{i^N} & \frac{1}{i^N} & \frac{-1}{i^N} & \frac{1}{i^N} \end{pmatrix},$$

$$E = \begin{pmatrix} \bar{\alpha} & \alpha & \beta & \bar{\beta} \\ \alpha & \bar{\alpha} & \bar{\beta} & \beta \\ \beta & \bar{\beta} & \bar{\alpha} & \alpha \\ \bar{\beta} & \beta & \alpha & \bar{\alpha} \end{pmatrix}, \quad D_{a,b} = 4 \cos\left(\frac{\pi ab}{N}\right), \quad 1 \leq a, b \leq N - 1.$$

Finally, we note that in terms of the simple objects the basis in which  $S$  and  $T$  is expressed is ordered such that the first two columns of  $A$  correspond to invertibles in  $\mathcal{C}_{\text{ad}}$ , and the even indexed columns of  $B$  correspond to the 2-dimensional objects in  $\mathcal{C}_{\text{ad}}$ .

**Lemma 3.3.** *If  $\mathcal{C}$  is an even metaplectic modular category, then the unique non-trivial self-dual invertible object is a boson.*

*Proof.* For  $N = 1$ , we have the degenerate case where  $\mathcal{C}$  is Grothendieck equivalent to  $\text{Rep}(\mathbb{Z}_8)$  with twists  $\theta_j = e^{j^2\pi i/8}$ . There is a unique non-trivial self-dual object,  $j = 4$ , and the corresponding twist is  $\theta_4 = e^{16\pi i/8} = 1$ ; thus it is a boson.

If  $N \geq 2$ , consider  $X_i$  one of the 2-dimensional simple objects in  $\mathcal{C}_{\text{ad}}$  and recall that  $g^2 \otimes X_i = X_i$ . Of course,  $g^2 \in \mathcal{C}_{\text{ad}}$ . Since  $\mathcal{C}$  is modular we know that  $C_{\mathcal{C}}(\mathcal{C}_{\text{ad}}) = \mathcal{C}_{\text{pt}}$  [17]. In particular,  $g^2$  is in  $C_{\mathcal{C}_{\text{ad}}}(\mathcal{C}_{\text{ad}})$ . Thus  $g^2$  is not a fermion by [20, Lemma 5.4].  $\square$

It follows that such a category,  $\mathcal{C}$ , contains a Tannakian category equivalent to  $\text{Rep}(\mathbb{Z}_2)$ .

**Definition 3.4.** A *cyclic modular category* is a modular category that is Grothendieck equivalent to  $\text{Rep}(\mathbb{Z}_n)$  for some integer  $n$ . When the specific value of  $n$  is important we will refer to such a category as a  $\mathbb{Z}_n$ -cyclic modular category.

For more details regarding cyclic modular categories see [1, Section 2].

**Lemma 3.5.** *If  $\mathcal{C}$  is an even metaplectic modular category of dimension  $8N$  with  $N$  odd, then  $\mathcal{C}_{\mathbb{Z}_2}$  is a generalized Tambara-Yamagami category of dimension  $4N$ . In particular, the trivial component of  $\mathcal{C}_{\mathbb{Z}_2}$  is a cyclic modular category of dimension  $2N$ .*

*Proof.* As we noted previously, by Lemma 3.3,  $\langle g^2 \rangle \cong \text{Rep}(\mathbb{Z}_2)$  is a Tannakian subcategory of  $\mathcal{C}$ . In particular we can form the de-equivariantization  $\mathcal{C}_{\mathbb{Z}_2}$  which is a  $\mathbb{Z}_2$ -crossed braided fusion category of dimension  $4N$ . In particular, the trivial component of  $\mathcal{C}_{\mathbb{Z}_2}$  under the  $\mathbb{Z}_2$ -crossed braiding is modular of dimension  $2N$  [12, Proposition 4.56(ii)].

Since  $g^2$  fixes the 2-dimensional objects of  $\mathcal{C}$ , these 2-dimensionals give rise to pairs of distinct invertibles in  $\mathcal{C}_{\mathbb{Z}_2}$ . This produces  $2(N - 1)$  invertibles in  $\mathcal{C}_{\mathbb{Z}_2}$ . On the other hand,  $g^2 \otimes 1 = g^2$  and  $g^2 \otimes g = g^3$ . This leads to two more invertibles in  $\mathcal{C}_{\mathbb{Z}_2}$ . Since the non-integral objects are moved by  $g^2$ , we can conclude that  $\mathcal{C}_{\mathbb{Z}_2}$  has only two non-integral objects which each have dimension  $\sqrt{N}$ . In particular,  $\mathcal{C}_{\mathbb{Z}_2}$  is a generalized Tambara-Yamagami category, and the trivial component of  $\mathcal{C}_{\mathbb{Z}_2}$  is a pointed modular category with  $2N$  invertible objects [19]. To conclude that this pointed category is cyclic we apply the proof of [1, Lemma 3.4] *mutatis mutandis*.  $\square$

**Remark 3.6.** By the previous lemmas, every even metaplectic category with  $N$  odd can be obtained as a  $\mathbb{Z}_2$ -gauging of a cyclic modular category of dimension  $2N$ . We refer the reader to [3] and [8] for a precise definition of gauging and its properties.

So to classify even metaplectic categories with  $N$  odd, we must understand  $\mathbb{Z}_2$  actions by braided tensor autoequivalences of a cyclic modular category of dimension  $2N$ . Note that a  $\mathbb{Z}_n$ -cyclic modular category with  $n$  odd has a fixed-point-free

$\mathbb{Z}_2$  action by braided tensor autoequivalences associated to the  $\mathbb{Z}_n$  group automorphism  $j \mapsto n - j$ . We refer to this automorphism as the *particle-hole symmetry*. In fact, it is shown in [1] that for  $n = p^a$  an odd prime power, this is the only non-trivial  $\mathbb{Z}_2$  action by braided tensor autoequivalences. By decomposing a  $\mathbb{Z}_n$ -cyclic modular category into its prime power Deligne tensor factors, we see that every  $\mathbb{Z}_2$  action by braided tensor autoequivalences on a  $\mathbb{Z}_n$  cyclic modular category (for  $n$  odd) is obtained by choosing either particle-hole symmetry or the identity on each  $\mathbb{Z}_{p^a}$  cyclic modular Deligne factor. Of course particle-hole symmetry for  $\mathbb{Z}_n$  corresponds to choosing particle-hole symmetry on each Deligne factor. On the other hand, the Semion (or its conjugate) has exactly one non-trivial  $\mathbb{Z}_2$  action by braided tensor autoequivalences corresponding to the non-trivial element of  $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$  [3].

For a  $\mathbb{Z}_{2N}$  cyclic modular category ( $N$  odd) let us define the particle-hole symmetry to be the  $\mathbb{Z}_2$  action by braided tensor autoequivalences corresponding to the non-trivial tensor autoequivalence on the Semion factor and ordinary particle-hole symmetry on the odd  $\mathbb{Z}_N$  factor.

*Remark 3.7.* Let  $\mathcal{C}$  be a  $\mathbb{Z}_{2N}$ -cyclic modular category with  $N$  odd. It follows from [3, 8] that if we choose a  $\mathbb{Z}_2$  action by braided tensor autoequivalences on  $\mathcal{C}$  that restricts to the identity on some  $\mathbb{Z}_{p^a}$ -cyclic modular subcategory  $\mathcal{P}_{p^a}$  of  $\mathcal{C}$  (including  $p = 2, a = 1$ ), then the corresponding gauging  $\mathcal{C}_G^{\times, G}$  will have a Deligne factor of  $\mathcal{P}_{p^a}$ . Indeed, in the extreme case of  $N = 1$ , if we gauge Sem by the trivial  $\mathbb{Z}_2$  action we obtain  $\text{Sem} \boxtimes \text{Rep}(D^\omega \mathbb{Z}_2)$  for some cocycle  $\omega$ . Indeed, gauging Vec by the trivial  $\mathbb{Z}_2$  action produces the factor on the right.

**Theorem 3.8.** *If  $\mathcal{C}$  is an even metaplectic category of dimension  $8N$ , with  $N$  an odd integer, then  $\mathcal{C}$  is a gauging of the particle-hole symmetry of a  $\mathbb{Z}_{2N}$ -cyclic modular category. Moreover, for  $2N = 2p_1^{k_1} \cdots p_r^{k_r}$ , with  $p_i$  distinct odd primes, there are exactly  $2^{r+2}$  many inequivalent even metaplectic modular categories of this form.*

*Proof.* By Remark 3.6, each even metaplectic modular category, with  $N$  odd, is obtained as a  $\mathbb{Z}_2$ -gauging of a  $\mathbb{Z}_{2N}$ -cyclic modular category. However, since an even metaplectic modular category is prime, Remark 3.7 implies that it can only be obtained by gauging particle-hole symmetry. This proves the first statement.

To count the inequivalent even metaplectic categories of dimension  $8N$  with  $N$  odd, we note that for each of the  $r + 1$  prime divisors of  $2N = 2p_1^{k_1} \cdots p_r^{k_r}$  there are exactly two cyclic modular categories (see [1, Section 2]). Gauging the particle-hole symmetry leads to an additional choice of  $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ , yielding a total of  $2^{r+2}$  choices.  $\square$

#### 4. MODULAR CATEGORIES OF DIMENSION $8m$ , WITH $m$ ODD SQUARE-FREE INTEGER

Metaplectic categories are a large class of dimension  $8m$  weakly integral modular categories for  $m$  odd and square-free. In this section we show that the only non-metaplectic modular categories of dimension  $8m$  are pointed or products of smaller categories. In order to accomplish this we will first determine the structure of the pointed subcategory. This will enable us to show that the integral subcategory is Grothendieck equivalent to  $\text{Rep}(D_{4m})$ . This will be sufficient to establish that  $\mathcal{C}$  is metaplectic if it is prime. Along the way we will resolve the case that  $\mathcal{C}$  has dimension  $p^3m$  for  $p$  an odd prime and  $m$  a square-free integer.

We begin by considering integral categories of dimension  $p^2m$  and  $p^3m$ . This will allow us to quickly reduce to the case of  $8m$ .

**Lemma 4.1.** *Let  $p$  be a prime and  $m$  a square-free integer such that  $\gcd(m, p) = 1$ . If  $\mathcal{C}$  is a modular category of dimension  $p^k m$  with  $p^k \mid \dim \mathcal{C}_{\text{pt}}$ , then  $\mathcal{C}$  is pointed.*

*Proof.* First note that  $\dim \mathcal{C}_{\text{ad}} = \frac{p^k m}{\dim \mathcal{C}_{\text{pt}}}$ . Since  $p^k \mid \dim \mathcal{C}_{\text{pt}}$  we know that  $\dim \mathcal{C}_{\text{ad}} \mid m$ . On the other hand,  $\mathcal{C}_{\text{ad}} \subset \mathcal{C}_{\text{int}}$  and all simples in  $\mathcal{C}_{\text{int}}$  have dimension  $p^j$ ,  $j \geq 0$ . Thus the dimensions of simples in  $\mathcal{C}_{\text{ad}}$  are 1 since  $\gcd(p, m) = 1$  and  $p \nmid \dim \mathcal{C}_{\text{ad}}$ , thus  $\mathcal{C}_{\text{ad}} \subset \mathcal{C}_{\text{pt}}$ . In particular,  $\mathcal{C}$  is nilpotent and is given by  $\mathcal{C}_{p^k} \boxtimes \mathcal{C}_{q_1} \boxtimes \cdots \boxtimes \mathcal{C}_{q_\ell}$  where  $q_j$  are primes such that  $m = q_1 q_2 \cdots q_\ell$  and  $\mathcal{C}_n$  is a modular category of dimension  $n$ ; see [11, Theorem 1.1]. By the classification of prime dimension modular categories we know  $\mathcal{C}_{q_1} \boxtimes \cdots \boxtimes \mathcal{C}_{q_\ell}$  is pointed, but  $p^k$  does not divide its dimension. Thus  $\mathcal{C}_{p^k}$  must be pointed.  $\square$

For an integral modular category of dimension  $p^2m$  with  $p$  and  $m$  as in the above lemma, we have that  $p^2m = \dim \mathcal{C} = |\mathcal{C}_{\text{pt}}| + ap^2$ , where  $a$  is the number of simple objects of dimension  $p$ . Thus we have the following corollary.

**Corollary 4.2.** *If  $p$  is prime,  $m$  is a square-free integer with  $\gcd(m, p) = 1$  and  $\mathcal{C}$  is an integral modular category with dimension  $p^2m$ , then  $\mathcal{C}$  is pointed.*

**Lemma 4.3.** *Any integral modular category  $\mathcal{C}$  of dimension  $p^4$  or  $p^3m$  where  $p$  is a prime and  $m$  is a square-free integer relatively prime to  $p$  is pointed.*

*Proof.* The case  $m = 1$  (i.e.  $p^3$ ) is well-known and easily verified using the grading and  $|\mathcal{U}(\mathcal{C})| = \text{rank} \mathcal{C}_{\text{pt}}$ .

First we consider  $\dim \mathcal{C} = p^4$ . By [10, Lemma 4.9]  $\mathcal{C}$  is a Drinfeld center  $\mathcal{Z}(\text{Vec}_G^\omega) \cong \text{Rep}(D^\omega G)$  for some group  $G$  of order  $p^2$ . By [25] such a  $D^\omega G$  is commutative; i.e.  $\mathcal{C}$  is pointed.

Assume that  $\mathcal{C}$  is not pointed. Since  $\mathcal{C}$  is modular we know that the simples have dimensions 1 and/or  $p$ . In particular, there is an integer  $b$  such that  $p^3m = \dim \mathcal{C}_{\text{pt}} + p^2b$  and thus  $p^2 \mid \dim \mathcal{C}_{\text{pt}}$ . By Lemma 4.1 we may assume  $p^3$  does not divide  $\dim \mathcal{C}_{\text{pt}}$ . In particular, there exists an integer  $k$  such that  $k \mid m$  and  $\dim \mathcal{C}_{\text{pt}} = p^2k$ . Next let  $\mathcal{C}_i$  denote the components of the universal grading of  $\mathcal{C}$ . Then  $pm/k = \dim \mathcal{C}_i = a_i + b_i p^2$  where  $a_i$  is the number of invertibles in component  $\mathcal{C}_i$  and  $b_i$  is the number of dimension  $p$  objects in  $\mathcal{C}_i$ . So we can conclude that  $p \mid a_i$  and  $a_i \neq 0$  for all  $i$ . Thus  $\dim \mathcal{C}_{\text{pt}} = \sum_i a_i \geq \sum_i p = p \dim \mathcal{C}_{\text{pt}}$ , an impossibility.  $\square$

Next we note that the condition that  $\mathcal{C}$  is integral in the previous statements is vacuous if  $p$  is odd. This is made explicit by the following lemma.

**Lemma 4.4.** *If  $\mathcal{C}$  is a strictly weakly integral modular category, then  $4 \mid \dim \mathcal{C}$ .*

*Proof.* Coupling the weakly integral grading of [17] and the fact that  $\mathcal{C}$  is strictly weakly integral, we have  $2 \mid \dim \mathcal{C}_{\text{pt}} \mid \dim \mathcal{C}_{\text{int}} \mid \dim \mathcal{C}/2$ .  $\square$

Lemma 4.4, Corollary 4.2, and [5, Theorem 3.1] resolve the case of  $\mathcal{C}$  a weakly integral modular category of dimension  $p^2m$  with  $m$  square-free and coprime to  $p$ . We have:

**Theorem 4.5.** *If  $p$  is a prime,  $m$  is a square-free integer coprime to  $p$ , and  $\mathcal{C}$  is a non-pointed modular category of dimension  $p^2m$ , then  $p = 2$  and one of the following is true:*

- (i)  $\mathcal{C}$  contains an object of dimension  $\sqrt{2}$  and is equivalent to a Deligne product of an Ising modular category with a cyclic modular category, or
- (ii)  $\mathcal{C}$  contains no objects of dimension  $\sqrt{2}$  and is equivalent to a Deligne product of a  $\mathbb{Z}_2$ -equivariantization of a Tambara-Yamagami category over  $\mathbb{Z}_k$  and a cyclic modular category of dimension  $n$  where  $1 \leq n = m/k \in \mathbb{Z}$ .

Applying Lemmas 4.1 and 4.3 we have:

**Corollary 4.6.** *If  $\mathcal{C}$  is a modular category of dimension  $8m$  with  $m$  an odd square-free integer, then either  $\mathcal{C}$  is pointed or  $\mathcal{C}$  is strictly weakly integral and  $8 \nmid |\mathcal{U}(\mathcal{C})|$ .*

So to study weakly integral modular categories of dimension  $p^3m$  it suffices to study strictly weakly integral modular categories of dimension  $8m$  by Lemmas 4.4 and 4.3.

**Lemma 4.7.** *Let  $G$  be a finite group and  $\mathcal{C}$  be the  $G$ -equivariantization of a fusion category  $\mathcal{D}$  which contains an Ising category. If  $\mathcal{C}$  has the property that every  $X \in \text{Irr}(\mathcal{C})$  such that  $\dim X \in \mathbb{Z}[\sqrt{2}]$  actually has dimension  $\sqrt{2}$ , then  $\mathcal{C}$  contains an Ising category.*

*Proof.* By [7], the objects in  $\mathcal{C}$  are pairs  $S_{X,\pi}$  where  $X$  is the  $G$ -orbit of a simple in  $\mathcal{D}$  and  $\pi$  is a projective representation of  $\text{Stab}_G(X)$ . Now let  $\sigma$  be the Ising object in  $\mathcal{D}$ . Then there exists a projective representation of  $\text{Stab}_G(\sigma)$  that is self-dual; denote it by  $\rho$ . Then  $S_{\sigma,\rho}$  is a simple in  $\mathcal{C}$  and is self-dual by [7]. Moreover,  $\dim S_{\sigma,\rho} = \sqrt{2} \dim \rho [G : \text{Stab}_G(X)]$ . However, our hypotheses regarding  $\mathcal{C}$  ensure that  $\dim S_{\sigma,\rho} = \sqrt{2}$ .  $\square$

The next two lemmas deal with the cases  $\dim \mathcal{C} = 8$  or  $16$ , i.e.  $m = 1, 2$ .

**Lemma 4.8.** *If  $\mathcal{C}$  is a dimension 8 strictly weakly integral premodular category, then  $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{D}$  where  $\mathcal{D}$  is a  $\mathbb{Z}_2$ -cyclic premodular category.*

*Proof.* Under the GN-grading (either  $(\mathbb{Z}/2\mathbb{Z})^2$  or  $\mathbb{Z}/2\mathbb{Z}$ ),  $\mathcal{C}_0 = \mathcal{C}_{\text{int}}$ , so  $\dim \mathcal{C}_{\text{int}} = 2$  or  $4$  (respectively). The first case would yield 3 objects of dimension  $\sqrt{2}$ , which is inconsistent with the GN-grading. Thus  $\dim \mathcal{C}_{\text{int}} = 4$ , and we conclude that  $\mathcal{C}_{\text{int}} = \mathcal{C}_{\text{pt}}$  and there are two distinct objects  $X$  and  $Y$  of dimension  $\sqrt{2}$ .

Observe that if *either* the universal grading group  $\mathcal{U}(\mathcal{C}) \cong (\mathbb{Z}/2\mathbb{Z})^2$  or  $X \cong X^*$ , then  $X^{\otimes 2} = \mathbf{1} \oplus z \in \mathcal{C}_{\text{ad}}$  with  $z \not\cong \mathbf{1}$  and  $z^* \cong z$ . Thus  $X, \mathbf{1}, z$  form a braided Ising category  $\mathcal{I}$  by [12]. Now it follows from a dimension count and [21, Cor. 7.8] that  $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{D}$  as claimed.

Thus it is enough to consider the case that  $X^* \cong Y$ . Then we have  $X \otimes Y \cong \mathbf{1} \oplus z$  for some self-dual  $z \in \mathcal{C}_{\text{ad}}$  with  $\theta_z^4 = 1$  (i.e.  $z$  is a boson, a fermion or a semion). If  $\theta_z = 1$ , then  $\langle z \rangle$  is Tannakian so that we may de-equivariantize  $\mathcal{C}$ . However, since  $(\mathbf{1} \oplus z) \otimes X \cong 2X$ , under the de-equivariantization functor  $F$  we have  $F(X) = X_1 \oplus X_2$  with  $\dim(X_i) = \frac{\sqrt{2}}{2}$  (see [22, Proposition 2.15]), which is not an algebraic integer. So  $\theta_z \neq 1$ . Now if the remaining two invertible objects satisfy  $\theta_a = \theta_b$ , the balancing equation implies that  $S_{z,b} = S_{z,a} = \frac{1}{\theta_z} \neq 1$ ; hence the Müger centers of both  $\mathcal{C}$  and  $\mathcal{C}_{\text{int}}$  are trivial. But by [21] we would then have  $\text{rank} \mathcal{C}_{\text{int}} = 4 \mid \text{rank} \mathcal{C} = 6$ , a contradiction. Thus we must have  $\theta_a \neq \theta_b$  (so  $a \not\cong b^*$ ) and  $\theta_z \neq 1$ . Now if either  $a$  or  $b$  is a boson we may de-equivariantize and then apply Lemma 4.7 to conclude that  $X$  generates an Ising category, contradicting  $X \not\cong X^*$ . So each of  $z, a$  and  $b$  are fermions or semions. Now applying balancing and the

fact that  $a \otimes X \cong X^* \cong b \otimes X$  we have  $S_{z,X} = \frac{d_X}{\theta_z} \neq d_X d_z$ ,  $S_{a,X} = \frac{d_X}{\theta_a} \neq d_X d_a$  and  $S_{b,X} = \frac{d_X}{\theta_b} \neq d_X d_b$  so that  $\mathcal{C}$  is modular. Now if any of  $a, b$  or  $z$  is a semion we may factor  $\mathcal{C}$  as a Deligne product  $\text{Sem} \boxtimes \mathcal{E}$ , again contradicting  $X \not\cong X^*$ . The only remaining possibility is that each of  $a, b$  and  $z$  is a fermion. From the balancing equation we compute the  $S$ -matrix of  $\mathcal{C}_{\text{int}}$  and find that  $S_{i,j} = -1$  for  $1 \neq i \neq j \neq 1$ , which implies that  $\mathcal{C}_{\text{int}}$  is modular, and we obtain the contradiction  $4 \mid 6$  as above.  $\square$

**Lemma 4.9.** *If  $\mathcal{C}$  is a strictly weakly integral modular category of dimension 16, then  $\mathcal{C}$  contains an Ising subcategory. In particular,  $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{D}$  where  $\mathcal{D}$  is Ising or a pointed modular category of dimension 4.*

*Proof.* The squares of the dimensions of the simple objects must divide 16, and so the possible simple object dimensions are 1, 2,  $\sqrt{2}$ , and  $2\sqrt{2}$ . In particular, the GN grading is  $\mathbb{Z}/2\mathbb{Z}$  and the dimension of the integral component is 8.

There are now five possibilities for the universal grading group:  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/4\mathbb{Z}$ ,  $(\mathbb{Z}/2\mathbb{Z})^2$ ,  $(\mathbb{Z}/2\mathbb{Z})^3$ , or  $\mathbb{Z}/8\mathbb{Z}$ . The first case is not possible by dimension count in the integral component.

In the case that the universal grading has order 8, each component has dimension 2. Then the integral components are rank 2 pointed categories and the non-integral components each have a single simple object of dimension  $\sqrt{2}$ . Then the category is a modular generalized Tambara-Yamagami category and  $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{D}$  where  $\mathcal{D}$  is a pointed modular category of dimension 4 by [24, Theorem 5.4].

In the case that the universal grading has order 4, dimension count reveals that there are four invertible objects, one object of dimension 2, and four objects of dimension  $\sqrt{2}$ . Moreover,  $\mathcal{C}_{\text{ad}} = \mathcal{C}_{\text{pt}}$ . Since there is only one simple object of dimension 2, it is self-dual. Moreover,  $\mathcal{C}_{\text{int}}$  is a braided Tambara-Yamagami category, and therefore  $G(\mathcal{C}) \cong U(\mathcal{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , by [28, Theorem 1.2]. Then the fusion subcategory generated by the adjoint component and one non-integral component has dimension 8. Then, by Lemma 4.8, it contains Ising and the result follows.  $\square$

**Proposition 4.10.** *If  $\mathcal{C}$  is a strictly weakly integral modular category of dimension  $8m$  where  $m$  is an odd square-free integer, then one of the following is true:*

- (i)  $\mathcal{C}$  is a Deligne product of a prime modular category of dimension  $8\ell$  and a pointed  $\mathbb{Z}_k$ -cyclic modular category with  $k$  odd, or
- (ii)  $\mathcal{C} = \text{Sem} \boxtimes \mathcal{D}$ , where  $\text{Sem}$  is a Semion category and  $\mathcal{D}$  is a strictly weakly integral modular category of dimension  $4m$  (see [5]).

*Proof.* If  $\mathcal{C}$  is prime, then we are in case (i) with  $k = 1$ . If  $\mathcal{C}$  is not prime, then it must factor into a Deligne product of modular categories [21, Theorem 4.2]. By Lemma 4.4 we can conclude that  $\mathcal{C} = \mathcal{C}_1 \boxtimes \mathcal{C}_2$ ,  $\mathcal{C}_1$  is strictly weakly integral, and  $\mathcal{C}_2$  is cyclic, pointed and non-trivial. The result now follows from earlier work [5, 26]: if  $\dim \mathcal{C}_2$  is odd, then we are in case (i), otherwise  $2 \mid \dim \mathcal{C}_2$ , and hence  $\mathcal{C}$  contains a Semion category [1].  $\square$

We have now reduced to the case that  $\mathcal{C}$  is a prime strictly weakly integral modular category of dimension  $8m$  with  $m$  an odd square-free integer, and we assume  $\mathcal{C}$  has this form in what follows.

**Proposition 4.11.** *Suppose  $\mathcal{D}$  is a non-symmetric premodular category that is Grothendieck equivalent to  $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ . If  $\mathcal{D}$  contains the symmetric category  $\text{Rep}(\mathbb{Z}_2)$ , then  $\mathcal{D}$  contains a semion category.*

*Proof.* Denote the simple objects in  $\mathcal{D}$  by  $1, g, h, gh$ , and  $g$  generates the symmetric category  $\text{Rep}(\mathbb{Z}_2)$ . Then examining the balancing relations for  $S_{h,h}, S_{g,h}, S_{gh,gh}$ , and  $S_{h,gh}$  we see that  $\theta_h = \theta_{gh}$ ,  $S_{h,h} = S_{gh,gh} = S_{h,gh} = \theta_h^2$ . Since  $\mathcal{D}$  is not symmetric but  $\langle g \rangle$  is, we can conclude that  $S_{h,h} \neq 1$ . However,  $\mathcal{D}$  is self-dual, and hence  $S_{h,h}$  is a real root of unity. In particular,  $S_{h,h} = -1$  and  $\theta_h = \pm i$ . Consequently,  $h$  generates a Semion subcategory.  $\square$

**Corollary 4.12.** *Let  $\mathcal{U}(\mathcal{C}) \cong \mathbb{Z}/4\mathbb{Z}$ .*

*Proof.* By Lemma 4.1 we know that 8 does not divide  $\dim \mathcal{C}_{\text{pt}}$ . Moreover, by invoking the universal grading and examining the dimension equation for  $\mathcal{C}_{\text{ad}}$  modulo 4 we see that there exists an odd integer  $k$  such that  $\dim \mathcal{C}_{\text{pt}} = 4k$ . If  $k > 1$ , there exists an odd prime  $p$  dividing  $k$  and a  $p$ -dimensional category  $\mathcal{D} \subset \mathcal{C}_{\text{pt}}$ .

Since  $\mathcal{C}$  is prime we know that  $\mathcal{D}$  is symmetric and Tannakian. If we de-equivariantize  $\mathcal{C}$  by  $\mathbb{Z}_p$ , the trivial component of the resulting  $\mathbb{Z}_p$ -graded category has dimension  $8m/p^2$  [12, Proposition 4.56(i)]. This is not possible as  $p$  is odd and  $m$  is square-free. Thus  $\dim \mathcal{C}_{\text{pt}} = 4$ .

By dimension count we can deduce that  $\dim(\mathcal{C}_{\text{ad}})_{\text{pt}} = 2$ . Since  $\mathcal{C}_{\mathcal{C}}\mathcal{C}_{\text{pt}} = \mathcal{C}_{\text{ad}}$  we can deduce that  $\mathcal{C}_{\text{pt}}$  is not symmetric. However,  $\mathcal{C}$  is prime and so  $\text{Vec} \neq \mathcal{C}_{\mathcal{C}_{\text{ad}}}(\mathcal{C}_{\text{ad}}) \subset (\mathcal{C}_{\text{ad}})_{\text{pt}}$  and so  $\mathcal{C}_{\mathcal{C}_{\text{ad}}}(\mathcal{C}_{\text{ad}}) = (\mathcal{C}_{\text{ad}})_{\text{pt}}$ . Letting  $g$  be a generator of this Müger center and  $X$  a 2-dimensional object in  $\mathcal{C}_{\text{ad}}$ , by dimension count  $g$  is a subobject of  $X \otimes X^*$ . In particular,  $g$  fixes  $X$  and thus  $\theta_g = 1$  by [20, Lemma 5.4].

By applying Proposition 4.11 to  $\mathcal{D} = \mathcal{C}_{\text{pt}}$  and invoking the primality of  $\mathcal{C}$  we have that  $\mathcal{C}_{\text{pt}} \not\cong \text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ .  $\square$

Henceforth, let  $g$  be a generator of  $\mathcal{C}_{\text{pt}}$  and  $\mathcal{C}_{g^k}$  the component of the universal grading of  $\mathcal{C}$  corresponding to the simple  $g^k \in \mathcal{C}_{\text{pt}}$ . Furthermore, note that  $\mathcal{C}_1$  and  $\mathcal{C}_{g^2}$  each contain exactly two invertible objects. Finally, we will denote by  $X_1, X_2, \dots, X_n$  the simple 2-dimensional objects in  $\mathcal{C}_{\text{ad}}$  and by  $Y_{i=g \otimes X_i}$  the simple 2-dimensional objects in  $\mathcal{C}_{g^2}$ .

*Remark 4.13.* Under this notation  $(\mathcal{C}_{\text{ad}})_{\text{pt}} = \langle g^2 \rangle$  is a 2-dimensional Tannakian category.

**Lemma 4.14.** *All 2-dimensional simple objects in  $\mathcal{C}$  are self-dual.<sup>2</sup>*

*Proof.* Recall that under our notation,  $X_i$  are the simple 2-dimensional objects in  $\mathcal{C}_{\text{ad}}$ , and  $Y_i = gX_i$  are the 2-dimensional simple objects in  $\mathcal{C}_{g^2}$ . Then  $Y_i^* = g^3X_i^* = g(g^2X_i^*) = gX_i^*$ . So it suffices to show that  $X_i$  are self-dual. To this end, observe that  $X_i \otimes X_i^* = \mathbf{1} \oplus g^2 \oplus \tilde{X}_i$  for some 2-dimensional simple  $\tilde{X}_i$ . In particular,  $\mathbf{1}$  and  $g^2$  are simples in  $(\mathcal{C}_{\text{ad}})_{\text{ad}}$  and all 2-dimensionals in  $(\mathcal{C}_{\text{ad}})_{\text{ad}}$  are self-dual. Now suppose  $(\mathcal{C}_{\text{ad}})_{\text{ad}} \neq \mathcal{C}_{\text{ad}}$ . Then  $\mathcal{C}_{\text{ad}}$  has a non-trivial universal grading, the trivial component is  $(\mathcal{C}_{\text{ad}})_{\text{ad}}$  and it has dimension  $2+4k$  for some integer  $k$ . The remaining components

<sup>2</sup>Note added in proof: a clearer proof that  $\mathcal{C}_{\text{ad}}$  is self-dual is sketched as follows.  $\mathcal{C}_{\text{ad}}$  has Tannakian Müger center (generated by  $g^2$ ). The  $\mathbb{Z}_2$  de-equivariantization of  $\mathcal{C}_{\text{ad}}$  is a cyclic modular category  $\mathcal{C}(\mathbb{Z}_m, q)$  of dimension  $m$ . The only  $\mathbb{Z}_2$ -equivariantization of  $\mathcal{C}(\mathbb{Z}_m, q)$  with exactly two invertible objects comes from  $\mathbb{Z}_2$  acting by inversion on  $\mathbb{Z}_m$  and has self-dual fusion rules, namely those of  $\text{Rep}(D_{2m})$  (see [7]).

must also have this dimension, but consist entirely of 2-dimensional objects. This is not possible and so  $(\mathcal{C}_{\text{ad}})_{\text{ad}} = \mathcal{C}_{\text{ad}}$ .  $\square$

**Lemma 4.15.**  $\mathcal{C}_{\text{int}}$  is Grothendieck equivalent to  $\text{Rep}(\mathbb{Z}_m \rtimes \mathbb{Z}_4)$  and  $\mathcal{C}_{\text{ad}}$  is Grothendieck equivalent to  $\text{Rep}(D_{2m})$ .

*Proof.* Since  $\mathcal{C}_{\text{ad}}$  is a subcategory of  $\mathcal{C}_{\text{int}}$ , then  $C_{\mathcal{C}_{\text{int}}}(\mathcal{C}_{\text{int}}) \subset C_{\mathcal{C}_{\text{int}}}(\mathcal{C}_{\text{ad}}) = \text{Rep}\mathbb{Z}_2$ . Given that  $\mathcal{D}$  is prime,  $\mathcal{C}_{\text{int}}$  is not modular, and so its Müger center is  $\mathcal{C}_{\text{int}} \cong \text{Rep}\mathbb{Z}_2$  Tannakian. Thus  $(\mathcal{C}_{\text{int}})_{\mathbb{Z}_2}$  is pointed, modular, and of dimension  $2m$ . In particular,  $(\mathcal{C}_{\text{int}})_{\mathbb{Z}_2}$  is cyclically generated, i.e., the category is tensor generated by a single object. So by [12, Proposition 4.30(i)]  $\mathcal{C}_{\text{int}}$  must be cyclically generated. The remainder of the statement follows immediately from [23, Theorem 4.2 and Remark 4.4] and Lemma 4.14.  $\square$

To completely determine the Grothendieck class of  $\mathcal{C}$  we must determine  $X_i \otimes V_k$  and  $g \otimes V_k$  where  $V_k$  are the four non-integral objects.

**Corollary 4.16.**  $\mathcal{C}$  has four non-integral objects:  $V, gV, g^2V, g^3V$ . Moreover,  $V^* = g^3V$ . In particular, the GN-grading of  $\mathcal{C}$  is  $\mathbb{Z}_2$  and all non-integral objects have dimension  $\sqrt{m}$ .

*Proof.* We know that the GN-grading is given by a non-trivial abelian 2-group and hence is  $\mathbb{Z}/2\mathbb{Z}$ , by Lemma 4.1 and Corollary 4.12. In particular, since  $m$  is square-free, there is a square-free integer  $x$  such that the non-integral objects have dimension  $\sqrt{x}$  and  $2\sqrt{x}$ .

A straightforward application of the fusion symmetries and a dimension calculation reveals that  $g^2$  fixes all simples of dimension  $2\sqrt{x}$ . Using this fact, we can appeal to the balancing equation and find that  $S_{g^2, g^2X} = 2\sqrt{x}$  and  $S_{g^2, g^2Y} = \frac{\theta_Y}{\theta_{g^2Y}} \sqrt{x}$  for any simples  $X$  and  $Y$  of dimension  $2\sqrt{x}$  and  $\sqrt{x}$  respectively. However,  $g^2$  is self-dual and so  $\theta_Y/\theta_{g^2Y} = \pm 1$ . Now observe that the orthogonality of the  $g^2$  and  $\mathbf{1}$  columns of the  $S$ -matrix can only be satisfied if  $\theta_Y = -\theta_{g^2Y}$  and there are no objects of dimension  $2\sqrt{x}$ . In particular,  $g^2$  moves all simples of dimension  $\sqrt{x}$ .

Next, let  $V, W \in \mathcal{C}_g$  be simples of dimension  $\sqrt{x}$ . Moreover, without loss of generality we may assume that  $g$  is a subobject of  $V \otimes V$ . Then  $\mathbf{1}$  is a subobject of  $(g^3V) \otimes V$  and hence  $V^* = g^3V$ . Next note that by the universal grading and the parity of  $x$  we can deduce that either  $g$  or  $g^3$  is a subobject of  $V \otimes W$  (but not both). In the former case  $\mathbf{1}$  is a subobject of  $V \otimes g^3W$ , and in the latter case  $\mathbf{1}$  is a subobject of  $V \otimes gW$ . Thus  $g^3W = V^* = g^3V$  or  $gW = V^* = g^3V$ , according to the invertible subobject appearing in  $V \otimes W$ . Since  $W$  was arbitrary we can conclude that there are exactly four non-integral objects:  $V, gV, g^2V$ , and  $g^3V$ . Moreover, each of these objects has dimension  $\sqrt{m}$ .  $\square$

**Lemma 4.17.** Let  $X_i \otimes V \cong V \oplus g^2V$ .

*Proof.* By dimension count and the grading we have  $X_i \otimes V = V \oplus g^2V, 2V$ , or  $2g^2V$ . The latter two are not possible as  $g^2$  fixes  $X_i$  and hence must fix  $X_i \otimes V$ .  $\square$

**Theorem 4.18.** If  $\mathcal{C}$  is a non-pointed modular category of dimension  $p^3m$  where  $p$  is a prime and  $m$  is a square-free integer that is coprime to  $p$ , then  $p = 2$  and one of the following is true:

- (i)  $\mathcal{C}$  is a Deligne product of an even metaplectic modular category of dimension  $8\ell$  and a pointed  $\mathbb{Z}_k$ -cyclic modular category with  $k$  odd, or

- (ii)  $\mathcal{C}$  is the Deligne product of a Semion modular category with a modular category of dimension  $4m$  (see [5]).

*Proof.* By Lemmas 4.4 and 4.3, Proposition 4.10, and Lemma 4.8 it suffices to consider  $p = 2$ ,  $m$  odd and square-free, and  $\mathcal{C}$  prime. In this case we may apply Lemmas 4.17, 4.15, and Corollary 4.16 to conclude that  $\mathcal{C}$  is metaplectic.  $\square$

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