A PROJECTIVE VARIETY WITH DISCRETE, NON-FINITELY GENERATED AUTOMORPHISM GROUP

JOHN LESIEUTRE

Abstract. We construct a projective variety with discrete, non-finitely generated automorphism group. As an application, we show that there exists a complex projective variety with infinitely many non-isomorphic real forms.

1. Introduction

Suppose that X is a projective variety over a field $\mathbf{K}$, with $\overline{\mathbf{K}}$ an algebraic closure. The set of automorphisms of X can be given the structure of a $\mathbf{K}$-scheme by realizing it as an open subset of $\mathrm{Hom}(X, X)$. In general, $\mathrm{Aut}(X)$ is locally of finite type, but it may have countably many components. Write $\mathrm{Aut}^0(X)$ for the connected component of the identity, and $\pi_0(\mathrm{Aut}(X)) = (\mathrm{Aut}(X)/\mathrm{Aut}^0(X))_{\overline{\mathbf{K}}}$ for the group of geometric components. When $\mathbf{K}$ is the field of complex numbers, $\pi_0(\mathrm{Aut}(X))$ is simply the group of components of $\mathrm{Aut}(X)$, sometimes denoted $\mathrm{Aut}(X)^\sharp$. We will say that the group of automorphisms of X is *discrete* if $H^0(X, TX) = 0$, which implies that $\mathrm{Aut}^0(X)$ is trivial.

Examples.

(1) Let $X = \mathbb{P}^r$. Then $\mathrm{Aut}(X) \cong \mathrm{Aut}^0(X) \cong \mathrm{PGL}_{r+1}(\mathbf{K})$, and $\pi_0(\mathrm{Aut}(X))$ is trivial.
(2) Let E be an elliptic curve over $\mathbf{K}$. Then $\pi_0(\mathrm{Aut}(E \times E))$ contains $\mathrm{GL}_2(\mathbf{Z})$ and hence is an infinite group.
(3) Let X be a very general hypersurface of type $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with $\mathbf{K} = \mathbf{C}$. Then X is a K3 surface, and the covering involutions associated to the three projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$ generate $\pi_0(\mathrm{Aut}(X)) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ [6].

According to a result of Brion [5], any connected algebraic group over a field of characteristic 0 can be realized as $\mathrm{Aut}^0(X)$ for some smooth, projective variety. In contrast, very little seems to be known in general about the component group $\pi_0(\mathrm{Aut}(X))$.

Our first result is the following.

Theorem 1. *Suppose that $\mathbf{K}$ is a field of characteristic 0, or that $\mathbf{K}$ is a field of characteristic $p > 0$, not algebraic over $\mathbf{F}_p$. Then there exists a smooth, geometrically simply connected, 6-dimensional, projective variety X over $\mathbf{K}$ for which $\pi_0(\mathrm{Aut}(X))$ is not finitely generated.*

The question of finite generation of $\pi_0(\mathrm{Aut}(X))$ has been raised several times in various arithmetic [16, 1] and geometric [5, 8, 22] contexts.

The automorphism group owes its arithmetic interest in part to its close relation with the forms of a variety over extension fields. If X is a variety over a field $\mathbf{K}$, and if $\mathbf{L}$ is a Galois extension of $\mathbf{K}$, then an $\mathbf{L}/\mathbf{K}$-*form* of X is a variety X' over $\mathbf{K}$ for which $X_{\mathbf{L}} \cong X'_{\mathbf{L}}$. The set of $\mathbf{L}/\mathbf{K}$-forms of X is in bijection with the Galois cohomology set $H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}}))$, and we will construct a variety with infinitely many $\mathbf{L}/\mathbf{K}$-forms by exhibiting a variety for which $\mathrm{Aut}(X_{\mathbf{L}})$ is pathological.

Theorem 2. *Suppose that* $\mathbf{K}$ *is a field of characteristic* 0*, or that* $\mathbf{K}$ *is a field of characteristic* $p > 0$*, not algebraic over* $\mathbf{F}_p$*. Let* $\mathbf{L}/\mathbf{K}$ *be a separable quadratic extension. Then there exists a 6-dimensional, projective* $\mathbf{K}$*-variety* X' *with infinitely many* $\mathbf{L}/\mathbf{K}$*-forms.*

In the case $\mathbf{K} = \mathbf{R}$ and $\mathbf{L} = \mathbf{C}$, we obtain an example of a variety with infinitely many non-isomorphic real structures.

The component group $\pi_0(\mathrm{Aut}(X))$ is an algebraic analog of the mapping class group $\pi_0(\mathrm{Diff}(M))$ of a smooth manifold M. In general, the mapping class group is not finitely generated, with an example provided by tori in dimension at least five [15]. However, according to a theorem of Sullivan, if $\dim M \geq 5$ and M is simply connected, then $\pi_0(\mathrm{Diff}(M))$ is finitely generated [21]. This contrasts with our example, which is simply connected and has real dimension 12 if $K = \mathbf{C}$.

The group $\pi_0(\mathrm{Aut}(X))$ is always finitely generated in a number of simple situations, although even then the group can be quite complicated. If X is a K3 surface, the group of automorphisms is always finitely generated [20], but there are examples in which the group of automorphisms is not even commensurable with an arithmetic group [22, Corollary 6.2]. Some other interesting automorphism groups of K3 surfaces have been studied by Baragar [2].

If X is not projective, the group of automorphisms may not even have the structure of a locally finite type scheme. Blanc and Dubouloz have exhibited affine surfaces over any uncountable field for which the normal subgroup $\mathrm{Aut}(S)_{\mathrm{alg}}$ of $\mathrm{Aut}(S)$ generated by all algebraic subgroups can not be generated by any countable family of these subgroups, and for which $\mathrm{Aut}(S)/\mathrm{Aut}(S)_{\mathrm{alg}}$ contains a free group on uncountably many generators [3].

Before giving our example, we sketch the technique. If X is a variety and Z is a closed subscheme of X with $\operatorname{codim} Z > 1$, then the automorphisms of X that lift to automorphisms of the blow-up $\mathrm{Bl}_Z(X)$ are precisely those that map Z to itself (not necessarily fixing Z pointwise). Our approach, roughly speaking, is to find a variety X with a subscheme Z so that the stabilizer $\mathrm{Stab}(Z) \subset \mathrm{Aut}(X)$ is not finitely generated, and then to pass to the blow-up $\mathrm{Bl}_Z(X)$ to obtain a variety realizing $\mathrm{Stab}(Z)$ as an automorphism group. There are three main difficulties. The first is to find X and Z for which the stabilizer of Z in $\mathrm{Aut}(X)$ is not finitely generated. The second is to prove it: in general it is very difficult to be sure that one knows the full group $\mathrm{Aut}(X)$ (cf. [18]). The third is to ensure that $\mathrm{Bl}_Z(X)$ does not have any automorphisms other than those lifted from X.

2. The construction

In what follows, let $\mathbf{K}$ be an infinite field, not necessarily algebraically closed, and let $\overline{\mathbf{K}}$ be an algebraic closure. Where not otherwise qualified, a "variety" is a variety over $\mathbf{K}$, and a "point" is a $\mathbf{K}$-point. Let $N^1(X)_{\mathbf{R}} = N^1(X) \otimes \mathbf{R}$ denote the finite-dimensional vector space of numerical classes of divisors on X. Given a subvariety $V \subset X$, write

$$\mathrm{Aut}(X; V) = \{\phi \in \mathrm{Aut}(X) : \phi(V) = V\}.$$

Step 1: Automorphisms of surfaces with prescribed action on a curve

If z_1, z_2, z_3, and z_4 are four distinct points in $\mathbb{P}^1$, there is a unique involution $\imath : \mathbb{P}^1 \to \mathbb{P}^1$ with $\imath(z_1) = z_2$ and $\imath(z_3) = z_4$, which is defined over $\mathbf{K}$. Figure 1 shows how this map can be constructed geometrically when $\mathbb{P}^1$ is embedded as a conic in $\mathbb{P}^2$. This involution is defined even if $z_1 = z_2$ or $z_3 = z_4$; in that case, the construction is the same except that we draw the tangent to the conic at z_1 or z_3. However, if $\operatorname{char} \mathbf{K} = 2$ and both $z_1 = z_2$ and $z_3 = z_4$, the projection from q induces an inseparable degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$; we will take care to avoid this situation.

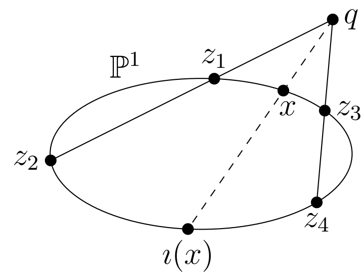

FIGURE 1. Geometric construction of $\imath$

Given an ordered 5-tuple $P = (p_1, p_2, p_3, p_4, p_5)$ of distinct points in $\mathbb{P}^1$, let $\Gamma_P \subset \mathrm{PGL}_2(\mathbf{K})$ be the subgroup generated by the involutions $\imath_{ij,kl} : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying $\imath_{ij,kl}(p_i) = p_j$ and $\imath_{ij,kl}(p_k) = p_l$, where i, j, k and l are distinct indices. For a given configuration P, there are 15 such involutions, and the group Γ_P depends only on the unordered set of points p_i.

Our starting point is a classical construction, a de Jonquières involution of degree 3 [12, §7.2.3]. Suppose that $C \subset \mathbb{P}^2$ is a smooth plane cubic, and that p is a point on C. Given a point z in $\mathbb{P}^2$, let ℓ_{pz} be the line connecting p and z. Over $\overline{\mathbf{K}}$, this line meets C at p and two additional points x and y. If x and y are distinct, there is a unique involution of $\ell_{\overline{\mathbf{K}}}$, defined over $\mathbf{K}$, which fixes the points x and y. Then $\imath(z)$ is defined to be the image of z under this involution. In what follows, we will use a different description of one such map, in which the cubic does not explicitly appear.

Theorem 3. *Suppose that P is a configuration of five distinct points in $\mathbb{P}^1$. There exists a smooth rational surface S containing a smooth rational curve $C \cong \mathbb{P}^1$ such that*

(1) the group $\mathrm{Aut}(S)$ is discrete;
(2) the subgroup $\mathrm{Aut}(S; C)$ has finite index in $\mathrm{Aut}(S)$;
(3) the image of $\rho : \mathrm{Aut}(S; C) \to \mathrm{Aut}(C)$ contains Γ_P;
(4) if $\phi : S \to S$ is an automorphism fixing a point on C, then $\phi(C) = C$.

Proof. Let $L_0, \ldots, L_5$ be six lines in $\mathbb{P}^2$ intersecting at 15 distinct points $p_{ij} = L_i \cap L_j$, and suppose that for any partition of the lines into three sets of two, the three pairwise intersections are not collinear. Let S be the blow-up of $\mathbb{P}^2$ at these 15 points, with exceptional divisor E_{ij} over p_{ij}. Write R for a partition of the six lines into three sets of two, with one of the three pairs distinguished. Given such a labelling, denote by $L_{R,1}$ and $L_{R,2}$ the members of the distinguished pair, by $L'_{R,1}$ and $L'_{R,2}$ the members of a second pair, and by $L''_{R,1}$ and $L''_{R,2}$ those of the third. Let o_R be the point of intersection of $L_{R,1}$ and $L_{R,2}$.

If ℓ is a general line passing through the point o_R, it meets the lines $L'_{R,1}$, $L'_{R,2}$, $L''_{R,1}$, and $L''_{R,2}$ at four distinct points. There is a unique involution on ℓ which exchanges the first and second points, and exchanges the third and fourth points. We define $\imath_R : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to be the birational involution defined in this way on any line ℓ through o_R. Under the assumption that three pairwise intersections of the L_i are never collinear, we will see that this map has five points of indeterminacy, o_R itself and the four points $q_{ij} = L'_{R,i} \cap L''_{R,j}$, where i and j are either 1 or 2. For the lines between o_R and q_{ij}, the construction in Figure 1 degenerates: two of the points (for example z_1 and z_3) coincide, and the point q in turn coincides with both. In particular, it lies on the conic, and projection from q does not yield a degree 2 morphism.

Let $\pi_R : S_R \to \mathbb{P}^2$ be the blow-up at these five points. We argue next that $\imath_R$ lifts to a biregular involution of S_R; the strategy is to carry out the construction illustrated in Figure 1 simultaneously for all lines in the pencil of lines through o_R. Projection from o_R realizes $\mathrm{Bl}_{o_R}\mathbb{P}^2$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$. The blow-up $\mathrm{Bl}_{o_R}\mathbb{P}^2$ then embeds in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathrm{Sym}^2(\mathcal{O} \oplus \mathcal{O}(1))$, and the image of a line through o_R is a conic contained in the corresponding fiber of $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. The four lines L'_{R_i} and L''_{R_j} determine four sections of $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$, all with image contained in $\mathrm{Bl}_{o_R}\mathbb{P}^2$. In each fiber $F \cong \mathbb{P}^2$, form the lines through $L'_{R,1} \cap F$ and $L'_{R,2} \cap F$, and through $L''_{R,1} \cap F$ and $L''_{R,2} \cap F$. Taking the fiberwise intersections of these lines, we obtain a section $Q : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$, corresponding to the point q of Figure 1.

The image of Q meets $\mathrm{Bl}_{o_R}\mathbb{P}^2$ precisely in the four fibers where some $L'_{R,i}$ and $L''_{R,j}$ intersect, at the points q_{ij}. Let $\mathcal{Q} \subset \mathcal{E}$ be the subbundle corresponding to Q. Fiberwise projection from the image of Q determines a morphism $c : \mathrm{Bl}_{Q(\mathbb{P}^1)}\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}/\mathcal{Q})$, where the image is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let S_R be the strict transform of $\mathrm{Bl}_{o_R}\mathbb{P}^2$ on $\mathrm{Bl}_{Q(\mathbb{P}^1)}\mathbb{P}(\mathcal{E})$, so that S_R is isomorphic to the blow-up of $\mathbb{P}^2$ at the five points o_R and q_{ij}. Write E_{o_R} for the exceptional divisor of S_R above o_R, and $E_{q_{ij}}$ for the exceptional divisors above the points q_{ij}.

The restriction $c|_{S_R}$ is a finite morphism of degree 2; because the lines $L'_{R,1}$ and $L'_{R,2}$ are distinct, as are $L''_{R,1}$ and $L''_{R,2}$, this morphism is separable (in fact, because of the hypothesis that $L_{R,1} \cap L_{R,2}$, $L'_{R,1} \cap L'_{R,2}$, and $L''_{R,1} \cap L''_{R,2}$ are not collinear, the restriction to any fiber is separable). The corresponding covering involution is precisely the map $\imath_R$, and so S_R provides a resolution of this involution as claimed.

The map $\imath_R$ preserves each of the lines $L_{R,i}$, while exchanging the two lines $L'_{R,i}$ and the two lines $L''_{R,j}$. It also preserves the pencil of lines through o_R and the canonical class K_{S_R}. Note too that the image of $E_{q_{ij}}$ under $\imath_R$ is the strict transform of the line from o_R to q_{ij}, with class $H - E_{o_R} - E_{q_{ij}}$. This completely characterizes the action of $\imath_R^* : N^1(S_R)_{\mathbf{R}} \to N^1(S_R)_{\mathbf{R}}$, which we record for later use. With respect to the basis given by $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, E_{o_R}, and the $E_{q_{ij}}$, the matrix for $\imath_R^*$ is

$$\imath_R^* = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Observe next that $\imath_R$ lifts to an involution on S, the surface obtained by blowing up all 15 points of intersection of the six lines L_i. The five points $L_{R,1} \cap L_{R,2}$ and $L'_{R,i} \cap L''_{R,j}$ are already blown up on S_R. The eight points $L_{R,i} \cap L'_{R,j}$ and $L_{R,i} \cap L''_{R,j}$ are exchanged in four pairs of two, so $\imath_R$ lifts to the blow-up at these eight additional points (for example, $L_{R,1} \cap L'_{R,1}$ is exchanged with $L_{R,1} \cap L'_{R,2}$). At last, the two points $L'_{R,1} \cap L'_{R,2}$ and $L''_{R,1} \cap L''_{R,2}$ are both fixed, and so $\imath_R$ lifts to the blow-up S.

The rational surface S claimed by the theorem can now be constructed by choosing the lines in special position. Fix a line $C = L_0 \subset \mathbb{P}^2$, and choose five other lines $L_1, \dots, L_5$ so that $L_i \cap C = p_i$, where the p_i are the points of the configuration P. Since the field $\mathbf{K}$ is infinite, for general choices of the L_i, the fifteen points of intersection are distinct and three pairs of lines never have collinear intersections. The involution $\imath_{ij,kl} : C \to C$ is realized as the restriction of $\imath_R : S \to S$ for some labelling R: let m be the unique index which does not appear among i, j, k, and l, and take $L_{R,1} = C$, $L_{R,2} = L_m$, $L'_{R,1} = L_i$, $L'_{R,2} = L_j$, $L''_{R,1} = L_k$, $L''_{R,2} = L_l$. The involution $\imath_{ij,kl}$ on C is then the restriction of the automorphism $\imath_R : S \to S$ fixing C, proving claim (3).

To check (1), note that a section in $H^0(S, TS)$ descends to a section of $H^0(\mathbb{P}^2, T\mathbb{P}^2)$ vanishing at the fifteen blown-up points, and $\mathrm{Aut}^0(S)$ can thus be identified with an algebraic subgroup of $\mathrm{Aut}(\mathbb{P}^2)$, fixing the points. Since four of the points are linearly general, such a group is trivial. It follows that $H^0(S, TS) = 0$ and $\mathrm{Aut}^0(S)$ is trivial.

We next check claim (2), that the subgroup $\mathrm{Aut}(S; C)$ has finite index in $\mathrm{Aut}(S)$. This is a consequence of the fact that S is a Coble rational surface [13], [7]: the linear system $|-2K_S|$ has a unique element, the union of the strict transforms of the six lines L_i. Indeed, each line satisfies $-2K_S \cdot L_i = -4$, and so must be contained in the base locus of $|-2K_S|$. An automorphism preserves the anticanonical class, so the six lines are permuted by any element of $\mathrm{Aut}(S)$, giving rise to a map $\mathrm{Aut}(S) \to \mathsf{S}_6$. The subgroup $\mathrm{Aut}(S; C)$ is the preimage of the subgroup of permutations fixing C, and so of finite index.

Claim (4) follows similarly: any automorphism of S permutes the components of $|-2K_S|$,. Since these six curves are disjoint on S, if a point on C is fixed by an automorphism ϕ, it must be that $\phi(C) = C$ as well. □

Remark 1. Consider the four lines through the point p_{05} given by L_0, L_5, $L(p_{05}, p_{14})$, and $L(p_{05}, p_{23})$, which define four points in $\mathbb{P}T_{p_{05}}(\mathbb{P}^2) \cong \mathbb{P}^1$. It will later be convenient to assume that these points are distinct and there is no automorphism of this $\mathbb{P}^1$ which fixes the first two points while exchanging the third and fourth; this will be the case for general choices of the five lines even after the intersections with L_0 are prescribed.

Remark 2. For an alternative construction of the involutions $\imath_R$, one can partition the six lines L_i into two sets of three, and consider the pencil of cubics spanned by the two triangles. This determines an elliptic fibration $S'_R \to \mathbb{P}^1$, where S'_R is obtained by blowing up the nine points of the base locus of the pencil. Choosing a distinguished line from each set of three gives a section of the fibration (provided by the exceptional divisor above the point of intersection), and the fiberwise map $z \mapsto -z$ with respect to this section determines a biregular involution of $S_{R'}$, which can be checked to lift to S.

Step 2: Specializing the configuration P

We now exhibit configurations $P = (p_1, p_2, p_3, p_4, p_5)$ of points in $\mathbb{P}^1$ for which the group Γ_P contains two particular transformations with a common fixed point. Fix projective coordinates on C, let the affine coordinate z represent the point $[z, 1]$, and write ∞ for the point $[1, 0]$.

Definition 1. A configuration P of five distinct points in $\mathbb{P}^1$ is *suitable* if the group Γ_P contains two elements

$$\tau = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},$$

where a is nonzero and either

(1) $\operatorname{char} \mathbf{K} = 0$ and b^{-1} is not an algebraic integer;
(2) $\operatorname{char} \mathbf{K} = p > 0$ and b is not algebraic over $\mathbf{F}_p$.

This assumption means that the abelian group $\mathbf{Z}[\frac{1}{b}]$ (in characteristic 0) or $\mathbf{F}_p[\frac{1}{b}]$ (in characteristic p) is not finitely generated.

Lemma 4.

(1) *Suppose that* $\mathbf{K}$ *has characteristic* 0. *Then the configuration*
$$(p_1, p_2, p_3, p_4, p_5) = (0, 1, 2, 3, \infty)$$
is suitable, as Γ_P *contains the two elements*
$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) *Suppose that* $\mathbf{K}$ *has characteristic* $p > 0$ *and is not algebraic over* $\mathbf{F}_p$. *Let* t *be an element of* $\mathbf{K}$ *transcendental over* $\mathbf{F}_p$. *Then the configuration*
$$(p_1, p_2, p_3, p_4, p_5) = (0, 1, t, t+1, \infty)$$
is suitable, as Γ_P *contains the two elements*
$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We claim that in both cases we have $\tau = \imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45}$ and $\gamma = \imath_{15,34} \circ \imath_{15,24}$. Indeed,
$$(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(p_1) = (\imath_{24,35} \circ \imath_{12,34})(p_3) = \imath_{24,35}(p_4) = p_2,$$
$$(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(p_3) = (\imath_{24,35} \circ \imath_{12,34})(p_1) = \imath_{24,35}(p_2) = p_4,$$
$$(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(p_5) = (\imath_{24,35} \circ \imath_{12,34})(p_4) = \imath_{24,35}(p_3) = p_5.$$
In characteristic p, this yields $(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(0) = 1$, $(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(t) = t+1$, and $(\imath_{24,35} \circ \imath_{12,34} \circ \imath_{13,45})(\infty) = \infty$, and so the composition must be the automorphism τ given by $z \mapsto z + 1$. The same argument holds in characteristic 0, substituting 2 for t.

Similarly,
$$(\imath_{15,34} \circ \imath_{15,24})(p_1) = \imath_{15,34}(p_5) = p_1,$$
$$(\imath_{15,34} \circ \imath_{15,24})(p_2) = \imath_{15,34}(p_4) = p_3,$$
$$(\imath_{15,34} \circ \imath_{15,24})(p_5) = \imath_{15,34}(p_1) = p_5.$$
In characteristic p, this map sends 0 to 0, 1 to t, and ∞ to ∞, so it must be the automorphism γ given by $z \mapsto tz$. The same argument again holds in characteristic 0 after substituting 2 for t. □

In what follows, we fix a suitable configuration P and let S be a surface satisfying the conclusions of Theorem 3, so that the image of the restriction map $\rho : \mathrm{Aut}(S; C) \to \mathrm{Aut}(C)$ contains the elements τ and γ. Write p_∞ for the point on S corresponding to ∞ in our coordinates on C, and let $U \subset \mathrm{PGL}_2(\mathbf{K})$ be the subgroup comprising matrices of the form
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$
which correspond to parabolic Möbius transformations $z \mapsto z + a$ fixing ∞. The group U is abelian, isomorphic to $\mathbb{G}_a$. Let
$$G^+ = \rho^{-1}(U) = \{\phi \in \mathrm{Aut}(S; C) : \rho(\phi) \in U\} \subset \mathrm{Aut}(S; C)$$

be the subgroup of $\mathrm{Aut}(S;C)$ containing automorphisms whose restriction to C lies in U.

Lemma 5. *The subgroup $G^+ \subset \mathrm{Aut}(S;C)$ is not finitely generated.*

Proof. Since U is abelian and $\rho(G^+)$ is contained in U, the group $\rho(G^+)$ is abelian as well. For any positive n, the transformation

$$\gamma^{-n} \circ \tau \circ \gamma^n = \begin{pmatrix} 1 & ab^{-n} \\ 0 & 1 \end{pmatrix}$$

is contained in U. Since τ and γ both lie in $\mathrm{Im}(\rho) \subset \mathrm{Aut}(C)$ by the construction of Theorem 3, the elements $\gamma^{-n} \circ \tau \circ \gamma^n$ all lie in $\rho(G^+)$, and so $\rho(G^+)$ has a subgroup isomorphic to either $\mathbf{Z}\left[\frac{1}{b}\right]$ (in characteristic 0) or $\mathbf{F}_p\left[\frac{1}{b}\right]$ (in characteristic p). In either case, this group is not finitely generated, by hypothesis on b. Since $\rho(G^+)$ is abelian and has a non-finitely generated subgroup, it is not finitely generated either. A quotient of a finitely generated group is finitely generated, and we conclude that G^+ itself is not finitely generated. □

The following geometric characterization of elements of G^+ will prove useful. Let

$$\Delta_S : S \to S \times S$$

denote the diagonal map.

Lemma 6. *Suppose that $\phi : S \to S$ is an automorphism fixing p_∞. Then ϕ lies in G^+ if and only if $\mathrm{id}_S \times \phi : S \times S \to S \times S$ fixes the tangent direction $T_{\Delta_S(p_\infty)}(\Delta_S(C))$.*

Proof. By (4) of Theorem 3, it must be that $\phi(C) = C$. An automorphism fixing C and p_∞ lies in G^+ if and only if p_∞ is a fixed point of $\phi|_C$ with multiplicity 2, which is the case if and only if $\mathrm{id}_S \times \phi : S \times S \to S \times S$ fixes $\Delta_S(p_\infty)$ and the tangent direction $T_{\Delta_S(p_\infty)}(\Delta_S(C))$, so that $(\mathrm{id}_S \times \phi)(\Delta_S(C))$ is tangent to the diagonal at $\Delta_S(p_\infty)$. □

Remark 3. Let $\bar{\tau}$ and $\bar{\gamma}$ be automorphisms of S which restrict to C as τ and γ, as constructed in Theorem 3. Although the restrictions to C of the automorphisms $\bar{\mu}_m = \bar{\gamma}^{-m} \circ \bar{\tau} \circ \bar{\gamma}^m$ and $\bar{\mu}_n = \bar{\gamma}^{-n} \circ \bar{\tau} \circ \bar{\gamma}^n$ commute, these maps do not commute as automorphisms of S, and the map $\rho : \mathrm{Aut}(S;C) \to \mathrm{Aut}(C)$ is not injective. For example, with γ and τ as in Lemma 4, the commutator $[\bar{\mu}_0, \bar{\mu}_1]$ is an automorphism of S which restricts to C as the identity, and a straightforward if somewhat tedious computation of the action of the involutions $\imath_R$ on $N^1(S)_{\mathbf{R}}$ shows if P is as in Lemma 4, the induced map $[\bar{\mu}_0, \bar{\mu}_1] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation of degree 195,133 with first dynamical degree $\lambda_1 \approx 60{,}313$. This means that the n^{th} iterate of the transformation $[\bar{\mu}_0, \bar{\mu}_1]$ has degree roughly λ_1^n. It seems conceivable that G^+ is a free group on the countably many generators $\bar{\mu}_n$, though this is difficult to prove.

Remark 4. The kernel G of $\mathrm{Aut}(S;C) \to \mathrm{Aut}(C)$ is the subgroup of automorphisms which fix C pointwise, including the maps $[\bar{\mu}_m, \bar{\mu}_n]$ of the previous remark. It seems likely that G is not finitely generated; if this is the case, then by choosing a very general point q on C, we might obtain a rational surface $S' = \mathrm{Bl}_q S$ such that $\mathrm{Aut}(S')$ is isomorphic to G and is not finitely generated. However, it is not clear how to prove either that G is not finitely generated, or that the blow-up does not admit automorphisms other than those lifted from S.

Step 3: A variety with non-finitely generated $\mathrm{Aut}(X)$

We now construct a higher-dimensional variety X realizing G^+ as $\mathrm{Aut}(X)$. Although G^+ is not the stabilizer of any closed subscheme of S, it is the stabilizer of a closed subscheme of

$S \times S$ in the group of automorphisms of $S \times S$ of the form $\mathrm{id}_S \times \phi$: an automorphism ϕ lies in G^+ if and only if $\mathrm{id}_S \times \phi$ fixes both the point $\Delta_S(p_\infty) = (p_\infty, p_\infty)$ and the tangent direction $T_{\Delta_S(p_\infty)}(\Delta_S(C))$ (here Δ_S is again the diagonal map). Our variety X will be realized as a blow-up of $S \times S \times T$, where T is a surface of general type; taking the product with T makes it simpler to control automorphisms of blow-ups.

We begin with a definition that will sometimes enable us to show that a blow-up $\mathrm{Bl}_V X$ has no automorphisms except those lifted from X.

Definition 2. A smooth, projective variety X is $\mathbb{P}^r$*-averse* if every separable $\overline{\mathbf{K}}$-morphism $h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{\overline{\mathbf{K}}}$ is constant.

The term "separably $\mathbb{P}^r$-averse" might be more appropriate, but we omit the modifier in the interest of brevity. Note that if X is $\mathbb{P}^r$-averse, it is also $\mathbb{P}^s$-averse for any $s > r$. However, the property of $\mathbb{P}^r$-averseness is not a birational invariant. For example, an abelian surface S is $\mathbb{P}^1$-averse, but the blow-up of S at a point s is not $\mathbb{P}^1$-averse: there is a nonconstant morphism $\mathbb{P}^1_{\overline{\mathbf{K}}} \to S_{\overline{\mathbf{K}}}$ given by the inclusion of the exceptional divisor.

Lemma 7. *Let X be a smooth, projective variety and let $\pi : Y \to X$ be the blow-up of X at a smooth, equidimensional (but possibly non-connected) subvariety V, with exceptional locus E. Suppose that $\psi : Y \to Y$ is an automorphism of Y with $\psi(E) = E$ such that $\psi|_E$ permutes the fibers of $\pi|_E$. Then ψ descends to an automorphism $\phi : X \to X$ with $\phi(V) = V$.*

Proof. The composition $\pi \circ \psi$ contracts every fiber of $\pi|_E$. Since $\pi_* \mathcal{O}_Y = \mathcal{O}_X$, it follows from the rigidity lemma that ψ factors through π, inducing a map $\phi : X \to X$ [9, Lemma 1.15(b)]. An inverse to ϕ is obtained by applying the same argument to $\pi \circ \psi^{-1}$. Then $\phi(V) = \phi(\pi(E)) = \pi(\psi(E)) = \pi(E) = V$, and so the subvariety V is fixed. □

Lemma 8. *Suppose that X is a smooth, $\mathbb{P}^{r-1}$-averse variety of dimension n, and $V \subset X$ is a smooth, equidimensional subvariety of codimension r, with $r > 1$. Write $\pi : \mathrm{Bl}_V X \to X$ for the blow-up of V, with exceptional locus E. Then the map $\mathrm{Aut}(X; V) \to \mathrm{Aut}(\mathrm{Bl}_V X)$ is an isomorphism.*

Proof. We first observe that any nonconstant morphism $h : \mathbb{P}^{r-1}_{\overline{\mathbf{K}}} \to (\mathrm{Bl}_V X)_{\overline{\mathbf{K}}}$ must have image contained in a geometric fiber of $\pi|_{E_{\overline{\mathbf{K}}}}$. Indeed, $\pi \circ h : \mathbb{P}^{r-1}_{\overline{\mathbf{K}}} \to X_{\overline{\mathbf{K}}}$ must be constant since X is $\mathbb{P}^{r-1}$-averse.

Suppose that $\phi : \mathrm{Bl}_V X \to \mathrm{Bl}_V X$ is an automorphism, and let $h : \mathbb{P}^{r-1}_{\overline{\mathbf{K}}} \to (\mathrm{Bl}_V X)_{\overline{\mathbf{K}}}$ be the inclusion of a geometric fiber of $\pi|_{E_{\overline{\mathbf{K}}}}$. Then $\phi \circ h$ is a nonconstant morphism $\mathbb{P}^{r-1}_{\overline{\mathbf{K}}} \to (\mathrm{Bl}_V X)_{\overline{\mathbf{K}}}$, and so must be the inclusion of some fiber of $\pi|_{E_{\overline{\mathbf{K}}}}$. Thus ϕ permutes the fibers of $\pi|_{E_{\overline{\mathbf{K}}}}$, and so descends to an automorphism of X fixing $\pi(E) = V$ by Lemma 7. □

Lemma 9.

(1) *Suppose that X_1 and X_2 are $\mathbb{P}^r$-averse. Then $X_1 \times X_2$ is $\mathbb{P}^r$-averse.*
(2) *Suppose that X is $\mathbb{P}^r$-averse and $V \subset X$ is a smooth, equidimensional subvariety of codimension $s \leq r$. Then $\mathrm{Bl}_V X$ is $\mathbb{P}^r$-averse.*
(3) *Suppose that $r \geq 2$ and X is an r-dimensional variety which admits a surjective morphism to a variety V with $1 \leq \dim V < \dim X$. Then X is $\mathbb{P}^r$-averse.*

Proof. For (1), suppose that $h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{1,\overline{\mathbf{K}}} \times X_{2,\overline{\mathbf{K}}}$ is a separable morphism. Then the projections $p_1 \circ h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{1,\overline{\mathbf{K}}}$ and $p_2 \circ h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{2,\overline{\mathbf{K}}}$ must both be constant, so that h is constant. For (2), let $\pi : \mathrm{Bl}_V X \to X$ be the blow-up, and suppose that $h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{\overline{\mathbf{K}}}$ is a separable morphism. The map $\pi \circ h$ must be constant, and so if h is nonconstant, its image is contained in a fiber of $\pi|_{E_{\overline{\mathbf{K}}}}$. These fibers are isomorphic to $\mathbb{P}^{s-1}_{\overline{\mathbf{K}}}$, and since $s-1<r$, the map h must be constant. For (3), suppose that $h : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{\overline{\mathbf{K}}}$ is separable and nonconstant. The composite $j : \mathbb{P}^r_{\overline{\mathbf{K}}} \to X_{\overline{\mathbf{K}}} \to V_{\overline{\mathbf{K}}}$ must be constant, and so the image of h is contained in a fiber of j. But these fibers have dimension less than r, and so can not admit nonconstant maps from $\mathbb{P}^r_{\overline{\mathbf{K}}}$. □

We require one more simple lemma before proceeding to the construction.

Lemma 10. *Suppose that X is a smooth projective variety with $\operatorname{Aut}(X)$ discrete. There exists a smooth, geometrically connected divisor $W \subset X$ for which $\operatorname{Aut}(X;W)$ is trivial.*

Proof. Choose a very ample divisor W_0 on X, and let $\mathcal{W}$ denote the complete linear system $\mathbb{P}H^0(X, W_0)$. Let $\operatorname{Aut}(X, \mathcal{W}) \subset \operatorname{Aut}(X)$ denote the set of automorphisms preserving $\mathcal{W}$. There is a homomorphism $\operatorname{Aut}(X, \mathcal{W}) \to \operatorname{PGL}(H^0(X, W_0))$, which has trivial kernel: if $\phi : X \to X$ lies in $\operatorname{Aut}(X, \mathcal{W})$ and ϕ^* acts trivially on $H^0(X, W_0)$, then the restriction of ϕ to the image of X in $\mathbb{P}H^0(X, W_0))$ is also the identity. Consequently $\operatorname{Aut}(X, \mathcal{W})$ is a closed subgroup of $\operatorname{PGL}(H^0(X, W_0))$. Since $\operatorname{Aut}(X)$ is assumed discrete, $\operatorname{Aut}(X; \mathcal{W})$ is finite, and because the field $\mathbf{K}$ is infinite, a general element W of $\mathcal{W}$ is not fixed by any automorphisms. Since $\mathbf{K}$ is infinite, by Bertini's theorem there exists a W which is smooth and geometrically connected. □

Lemma 11. *There exists a smooth surface T over $\mathbf{K}$ for which:*

(1) the group $\operatorname{Aut}(T)$ is trivial;
(2) there exists a $\mathbf{K}$-point t on T;
(3) T is geometrically simply connected;
(4) T is not separably uniruled.

Proof. According to [17, Table 1], the hypersurface T in $\mathbb{P}^3$ defined by $x_0^5 + x_0x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3^5$ is smooth and has trivial automorphism group in any characteristic other than 2 or 5. This surface has the $\mathbf{K}$-point $[0,1,0,0]$. Since T is a smooth hypersurface in $\mathbb{P}^3$, it follows from the Lefschetz hyperplane theorem [14, XII, Cor. 3.5] that T is geometrically simply connected. At last, T is of general type, and hence is not separably uniruled.

In characteristic 5, we take T defined by $x_0^7 + x_0x_1^6 + x_1x_2^6 + x_2x_3^6 + x_3^7$, while in characteristic 2, the surface defined by $x_0^4x_1 + x_1^5 + x_2^5 + x_0x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3^5$ suffices [17]. □

Note that if we work over $\mathbf{K} = \mathbf{C}$, then any very general hypersurface in $\mathbb{P}^3$ of degree at least 4 has the required properties.

Take $X_0 = S \times S \times T$. The variety X will be constructed by a sequence of four blow-ups of X_0. In each case, the blow-up satisfies the hypotheses of Lemma 8, so we may identify its automorphism group with a subgroup of $\operatorname{Aut}(X_0)$.

Lemma 12. *Let $X_0 = S \times S \times T$. Fix a point s on S and a divisor W on S with $\operatorname{Aut}(S;W)$ trivial, as in Lemma 10. Choose three distinct smooth, geometrically connected curves C_1, C_2, and C_3 in T, and a point t on C_3 which does not lie on C_1 or C_2.*

(1) The variety X_0 is $\mathbb{P}^r$-averse for any $r \geq 2$. The automorphisms of X_0 are of the form $\operatorname{Aut}(S \times S) \times \mathrm{id}_T$.
(2) Let $\pi_1 : X_1 \to X_0$ be the blow-up of X_0 along $s \times S \times C_1$. The variety X_1 is $\mathbb{P}^r$-averse for any $r \geq 3$. The automorphisms of X_1 are all lifts of $\operatorname{Aut}(S;s) \times \operatorname{Aut}(S) \times \mathrm{id}_T$.

(3) Let $\pi_2 : X_2 \to X_1$ be the blow-up along the strict transform of $W \times p_\infty \times C_2$. The variety X_2 is $\mathbb{P}^r$-averse for $r \geq 4$. The automorphisms of X_2 are given by $\mathrm{id}_S \times \mathrm{Aut}(S; p_\infty) \times \mathrm{id}_T$.

(4) Let $\pi_3 : X_3 \to X_2$ be the blow-up along the strict transform of $p_\infty \times p_\infty \times C_3$. Then X_3 is $\mathbb{P}^r$-averse for $r \geq 5$, and the automorphisms of X_3 are of the form $\mathrm{id}_S \times \mathrm{Aut}(S; p_\infty) \times \mathrm{id}_T$.

(5) Let E_3 be the exceptional divisor of $\pi_3 : X_3 \to X_2$. Then the strict transform of $\Delta_S(C) \times t$ meets E_3 at a single point u. Let $\pi_4 : X_4 \to X_3$ be the blow-up at u. The automorphism group of X_4 is isomorphic to $\mathrm{id}_S \times G^+ \times \mathrm{id}_T$.

Proof. We treat the blow-ups in order.

(1) To show that X_0 is $\mathbb{P}^r$-averse for $r \geq 2$, it suffices to check that S and T are both $\mathbb{P}^2$-averse, according to the first part of Lemma 9. For T this follows since T is not separably uniruled, while S admits a surjective morphism to a curve and so it is $\mathbb{P}^2$-averse by (3) of Lemma 9.

Suppose that $\chi : X_0 \to X_0$ is an automorphism. Let $p_3 : X_0 \to T$ be the third projection. We first claim that χ must satisfy $p_3 \circ \chi = p_3$. Indeed, consider the separable map $p_3 \circ \chi : X_0 = S \times S \times T \to T$. Since S is rational, if this map does not factor through the projection to T, then T is separably uniruled, contradicting the choice of T from Lemma 11. Since $\mathrm{Aut}(T)$ is trivial, the map χ must preserve every fiber of p_3, and so $\chi : S \times S \times T \to T$ is an automorphism defined over T.

The group $\mathrm{Aut}(S \times S)$ is discrete, since $H^0(S \times S, TS \times TS) = H^0(S, TS) \oplus H^0(S, TS) = 0$. Consequently every automorphism of X_0 is of the form $\phi \times \mathrm{id}$, where ϕ is an automorphism of $S \times S$, and the group $\mathrm{Aut}(X_0)$ can be identified with $\mathrm{Aut}(S \times S) \times \mathrm{id}_T$.

(2) The center of the blow-up π_1 has codimension 3, so it follows from part (1) and Lemma 9 that X_1 is $\mathbb{P}^r$-averse for $r \geq 3$. According to Lemma 8, since X_0 is $\mathbb{P}^2$-averse, $\mathrm{Aut}(X_1)$ is the stabilizer of $s \times S \times C_1$ in $\mathrm{Aut}(X_0)$, which is isomorphic to the stabilizer of $s \times S$ in $\mathrm{Aut}(S \times S)$.

We claim that an element ϕ of $\mathrm{Aut}(S \times S)$ fixes $s \times S$ only if it is of the form $\phi_1 \times \phi_2$, where ϕ_1 is in $\mathrm{Aut}(S; s)$ and ϕ_2 is in $\mathrm{Aut}(S)$. Indeed, if ϕ fixes one fiber of $p_1 : S \times S \to S$, it must permute the fibers, and so induces an automorphism $\phi_1 : S \to S$ on the base with $p_1 \circ \phi = \phi_1 \circ p_1$. Then $(\phi_1^{-1} \times \mathrm{id}_S) \circ \phi$ is an automorphism of $S \times S$ defined over p_1. This must be given by a map $\mathrm{id}_S \times \phi_2 : S \times S \to S \times S$, since $\mathrm{Aut}(S)$ is a 0-dimensional scheme, and so ϕ is of the form $\phi_1 \times \phi_2$, where ϕ_1 fixes s.

(3) Since X_1 is $\mathbb{P}^r$-averse for $r \geq 3$ and X_2 is the blow-up of X_1 at a center of codimension 4, it follows that X_2 is $\mathbb{P}^r$-averse for $r \geq 4$. Lemma 8 implies that the automorphisms of X_2 are all lifts of automorphisms of X_1 fixing $W \times p_\infty \times C_2$, whether or not s lies on W. The automorphisms of X_1 are all of the form $\phi_1 \times \phi_2 \times \mathrm{id}_T$, and since $\mathrm{Aut}(S; W)$ is trivial, this stabilizer is exactly $\mathrm{id}_S \times \mathrm{Aut}(S; p_\infty) \times \mathrm{id}_T$.

(4) We have seen that X_2 is $\mathbb{P}^4$-averse, and X_3 is the blow-up of X_2 at a center of codimension 5. It follows that X_3 is $\mathbb{P}^r$-averse for $r \geq 5$, and the automorphisms of X_3 are lifts of automorphisms of X_2 that fix $p_\infty \times p_\infty \times C_3$. Every automorphism of X_2 fixes $p_\infty \times p_\infty \times C_3$, and so the automorphisms of X_3 are again given by $\mathrm{id}_S \times \mathrm{Aut}(S; p_\infty) \times \mathrm{id}_T$.

(5) The centers of the blow-ups π_1 and π_2 are both disjoint from the fiber $S \times S \times t$, since t lies on neither C_1 nor C_2, while the center of the blow-up π_3 meets $S \times S \times t$ at the single point $p_\infty \times p_\infty \times t$. As a result, $\Delta_S(C) \times t$ meets E_3 at one point u, as claimed; this point u corresponds to the tangent direction of the diagonal embedding of C at the point p_∞. The

restriction of $\pi_3 \circ \pi_2 \circ \pi_1$ to the strict transform of $S \times S \times t$ is the blow-up at the point $p_\infty \times p_\infty \times t$.

Since X_3 is $\mathbb{P}^5$-averse and the center of π_3 has codimension 6, $\mathrm{Aut}(X_4)$ is isomorphic to the stabilizer of u in $\mathrm{Aut}(X_3)$. These are exactly the automorphisms $\mathrm{id}_S \times \phi \times \mathrm{id}_T$ of X_3 that fix the tangent direction $T_{\Delta(p_\infty)}(\Delta_S(C)) \times t$. According to Lemma 6, these are exactly the lifts of automorphisms of the form $\mathrm{id}_S \times G^+ \times \mathrm{id}_T$. □

This completes the construction.

Proof of Theorem 1. Let $X = X_4$ be as in Lemma 12. The variety X is smooth, projective and geometrically simply connected, since it is a blow-up of $S \times S \times T$ where S is a rational surface and T is smooth and geometrically simply connected. The group $\mathrm{Aut}(X)$ is isomorphic to G^+, which is not finitely generated according to Lemma 5. □

3. A variety with many forms

We now show how the construction of the previous section can be adapted to give an example of a $\mathbf{K}$-variety with infinitely many $\mathbf{L}/\mathbf{K}$-forms even when $\mathbf{L}/\mathbf{K}$ is a finite extension.

If $\mathbf{L}/\mathbf{K}$ is a Galois extension, a standard descent argument shows that the $\mathbf{L}/\mathbf{K}$-forms of X are classified by the Galois cohomology $H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}}))$ [19, III.§1, Proposition 5]. In many settings, this set is finite. Indeed, according to a theorem of Borel and Serre [4, Théorème 6.1], if $\mathbf{K} = \mathbf{R}$ and $\pi_0(\mathrm{Aut}(X_{\mathbf{C}}))$ is an arithmetic group, then the set of $\mathbf{C}/\mathbf{R}$-forms of X is finite; this includes nearly all varieties for which the group of automorphisms is known. The set of $\mathbf{C}/\mathbf{R}$-forms is also finite when X is a minimal surface of non-negative Kodaira dimension [10, Appendix D, pg. 233].

Our example of a variety with infinitely many forms is obtained by an additional blow-up of the variety X constructed in Section 2.

Lemma 13. *Suppose that $\mathbf{L}/\mathbf{K}$ is a separable quadratic extension, and that X is a smooth, projective variety over $\mathbf{K}$. Suppose that there is a finite-index subgroup $G' \subset \mathrm{Aut}(X_{\mathbf{L}})$ which contains infinitely many conjugacy classes of involutions and on which $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$ acts trivially. Then the variety X has infinitely many $\mathbf{L}/\mathbf{K}$-forms.*

Proof. The forms of X are classified by the set $H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}}))$. Because the action of $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$ on G' is trivial, $H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), G')$ is the set of conjugacy classes of involutions in G', which is infinite by assumption. There is an exact sequence

$$H^0(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}})/G') \to H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), G') \to H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}})).$$

Here $\mathrm{Aut}(X_{\mathbf{L}})/G'$ should be interpreted as the set of left-conjugacy classes of G' rather than a group, but the sequence is nevertheless exact [19, III.§5, Proposition 36]. Since G' has finite index in $\mathrm{Aut}(X_{\mathbf{L}})$, the leftmost set is finite, whence $H^1(\mathrm{Gal}(\mathbf{L}/\mathbf{K}), \mathrm{Aut}(X_{\mathbf{L}}))$ is infinite, as claimed. □

Remark 5. Concretely, suppose that $\mathbf{L} = \mathbf{C}$ and $\mathbf{K} = \mathbf{R}$, and that $\mathrm{Aut}(X_{\mathbf{C}}) = \mathrm{Aut}(X_{\mathbf{R}})$. If $c : X_{\mathbf{C}} \to X_{\mathbf{C}}$ is the antiholomorphic involution determined by complex conjugation, then $X_{\mathbf{R}}$ can be recovered as the fixed locus of c. For any involution ϕ in $\mathrm{Aut}(X_{\mathbf{R}})$, the composite $c \circ \phi$ defines another antiholomorphic involution, giving rise to another real form on the fixed locus. Two involutions ϕ and ϕ' define equivalent real structures on $X_{\mathbf{C}}$ if and only if they are conjugate in $\mathrm{Aut}(X_{\mathbf{R}})$.

Our argument will rely on some specific details from the construction in the proof of Theorem 3; for the remainder of Section 3 the surface S is taken to be the specific example constructed in the proof of Theorem 3, rather than an arbitrary surface satisfying its conclusions.

Fix the configuration P of Lemma 4 (depending on the characteristic), and maintain the notation introduced in the proof of Theorem 3, labelling the six lines as $L_0, \ldots, L_5$, with L_0 the curve C. Let $p_{ij} = L_i \cap L_j$, and write p_i for the point p_{0i}. The lines L_i are chosen so that the intersections of L_1, L_2, L_3, L_4 and L_5 with L_0 (with respect to affine coordinates) are given by

$$(p_1, p_2, p_3, p_4, p_5) = \begin{cases} (0, 1, 2, 3, \infty) & \text{if } \operatorname{char} \mathbf{K} = 0, \\ (0, 1, t, t+1, \infty) & \text{if } \operatorname{char} \mathbf{K} > 0. \end{cases}$$

We will consider the following subgroups of $\operatorname{Aut}(S; C)$:

(1) $G^+ \subset \operatorname{Aut}(S; C)$, the subgroup of automorphisms restricting to L_0 as $z \mapsto z + a$;
(2) $G^\pm \subset \operatorname{Aut}(S; C)$, the subgroup of automorphisms restricting to L_0 as either $z \mapsto z + a$ or $z \mapsto -z + a$;
(3) $G^\pm_{\text{ev}} \subset G^\pm$, the subgroup of automorphisms which fix the two lines L_0 and L_5 as well as the curves $L_1 \cup L_4$ and $L_2 \cup L_3$.

Recall that every automorphism of S must permute the six lines L_i since their union is the unique member of $|-2K_S|$; an automorphism lies in $G^\pm_{\text{ev}}$ if it fixes L_0 and L_5 and either fixes or exchanges the members of the two other pairs. In particular, $G^\pm_{\text{ev}}$ has finite index in $G^\pm$.

Let $s_0 : S \to S$ be the involution of S determined by the marking with L_0, L_5 the distinguished pair, and L_1, L_4 and L_2, L_3 the other two pairs: the automorphism s_0 fixes the two distinguished lines L_0 and L_5, and exchanges L_1 with L_4 and L_2 with L_3. This map restricts to L_0 in such a way that it exchanges p_1 with p_4 and p_2 with p_3; thus the restriction is an involution $z \mapsto c - z$, where $c = 3$ if $\operatorname{char} \mathbf{K} = 0$ or $c = t + 1$ if $\operatorname{char} \mathbf{K} > 0$. It follows that s_0 lies in the subgroup $G^\pm_{\text{ev}}$.

Figure 2 shows the important curves in $\mathbb{P}^2$ acted on by the map s_0. The two dashed lines are exchanged, as are the two heavily dotted lines. The pencil of lines passing through the point p_5 is preserved by s_0. The strict transforms of the two lightly dotted lines through p_5 are (-1)-curves on S with classes $H - E_{05} - E_{14}$ and $H - E_{05} - E_{23}$, which will appear later. By the generality assumption of Remark 1, these two lines are distinct.

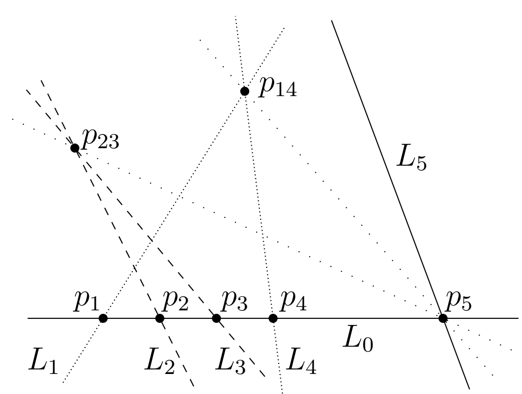

FIGURE 2. The involution s_0

Let $\tilde{s}_0 : X_3 \to X_3$ be the automorphism of X_3 induced by $\mathrm{id}_S \times s_0 \times \mathrm{id}_T$. Observe that if $\operatorname{char} \mathbf{K} \neq 2$, the point $\tilde{s}_0(u)$ is distinct from u: $\mathrm{id}_S \times s_0$ acts on $T_{\Delta_S(p_\infty)}(S \times S)$ by the linear transformation $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which fixes $T_{\Delta_S(p_\infty)}\Delta_S(C)$ if and only if $\operatorname{char} \mathbf{K} = 2$.

We now construct a variety dominating X and on which $\tilde{s}_0$ lifts to an automorphism. If $\operatorname{char} \mathbf{K} \neq 2$, take X' to be the blow-up of X at the two points u and $\tilde{s}_0(u)$. If $\operatorname{char} \mathbf{K} = 2$, take $X' = X$; the involution $\tilde{s}_0$ already lifts to an automorphism of X and no blow-up is needed.

Lemma 14. *The variety X' satisfies $\mathrm{Aut}(X') \cong G^{\pm}$.*

Proof. In characteristic 2, $G^+ = G^{\pm}$ and we have already seen that $\mathrm{Aut}(X) \cong G^+$, so we may assume that $\operatorname{char} \mathbf{K} \neq 2$. According to Lemma 8, because X_3 is $\mathbb{P}^5$-averse, the automorphisms of X' are the stabilizer of $u \cup \tilde{s}_0(u)$. These are precisely the automorphisms S which are of either the form $z \mapsto z + a$ or $z \mapsto -z + a$, as required. □

Let $\bar{\gamma} : S \to S$ be an automorphism restricting to L_0 as $\gamma = (z \mapsto bz)$, where b is as in Lemma 4; if $\operatorname{char} \mathbf{K} = 0$ then $b = 2$, while if $\operatorname{char} \mathbf{K} > 0$, then $b = t$. The elements $s_n = \bar{\gamma}^{-n} \circ s_0 \circ \bar{\gamma}^n$ are all involutions, and the restriction of s_n to L_0 is the map $z \mapsto cb^{-n} - z$, which lies in $G^{\pm}$. Although the maps s_n are conjugate in $\mathrm{Aut}(S)$, they are conjugate by powers of $\bar{\gamma}$, and $\bar{\gamma}$ is not contained in $G^{\pm}$. We now work to show that the s_n indeed define distinct conjugacy classes in the subgroup $G^{\pm}$.

Lemma 15. *The $(+1)$-eigenspace of $s_0^* : N^1(S)_{\mathbf{R}} \to N^1(S)_{\mathbf{R}}$ has dimension 8. Moreover, the six classes*

$$
\begin{array}{rrrrrrrrrrrrrrrrr}
 & H & E_{01} & E_{02} & E_{03} & E_{04} & E_{05} & E_{12} & E_{13} & E_{14} & E_{15} & E_{23} & E_{24} & E_{25} & E_{34} & E_{35} & E_{45} \\
R_1 = (& 3 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \;), \\
R_2 = (& 3 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \;), \\
R_3 = (& 4 & 0 & -1 & -1 & 0 & -2 & -1 & -1 & -2 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \;), \\
R_4 = (& 4 & -1 & 0 & 0 & -1 & -2 & -1 & -1 & 0 & -1 & -2 & -1 & 0 & -1 & 0 & -1 \;), \\
R_5 = (& 5 & 0 & -1 & -1 & 0 & -3 & -1 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 \;), \\
R_6 = (& 5 & -1 & 0 & 0 & -1 & -3 & -1 & -1 & -2 & 0 & -2 & -1 & -1 & -1 & -1 & 0 \;).
\end{array}
$$

define elliptic fibrations on S which are invariant under s_0.

Proof. The dimension of the $(+1)$-eigenspace can be computed based on the geometric description of s_0 given in the proof of Theorem 3. The matrix $\imath_R^*$ for the action of s_0^* on a five-point blow-up S_R is given in the proof of Theorem 3 and has a two-dimensional $(+1)$-eigenspace. The other eleven exceptional divisors arise as four pairs exchanged by s_0^*, and two divisors fixed by s_0^*. The four pairs each contribute a one-dimensional subspace to the $(+1)$-eigenspace, as does each invariant divisor. The total dimension of the $(+1)$-eigenspace is then $2 + 4 + 2 = 8$.

Each R_i can be written as a sum of effective classes in two distinct ways:

$$
\begin{aligned}
R_1 &= L_0 + L_1 + L_4 + 2E_{01} + 2E_{04} + 2E_{14} \\
&= L_2 + L_3 + L_5 + 2E_{23} + 2E_{25} + 2E_{35} \\
R_2 &= L_1 + L_4 + L_5 + 2E_{14} + 2E_{15} + 2E_{45} \\
&= L_0 + L_2 + L_3 + 2E_{02} + 2E_{03} + 2E_{23}
\end{aligned}
$$

$$\begin{aligned}
R_3 &= L_2 + L_3 + 2E_{23} + 2(H - E_{05} - E_{14}) \\
&= L_0 + L_1 + L_4 + L_5 + 2E_{01} + 2E_{04} + 2E_{15} + 2E_{45} \\
R_4 &= L_1 + L_4 + 2E_{14} + 2(H - E_{05} - E_{23}) \\
&= L_0 + L_2 + L_3 + L_5 + 2E_{02} + 2E_{03} + 2E_{25} + 2E_{35} \\
R_5 &= L_2 + L_3 + L_5 + 2E_{25} + 2E_{35} + 2(H - E_{05} - E_{14}) \\
&= L_0 + L_1 + L_4 + 2E_{01} + 2E_{04} + 2(H - E_{05} - E_{23}) \\
R_6 &= L_0 + L_2 + L_3 + 2E_{02} + 2E_{03} + 2(H - E_{05} - E_{14}) \\
&= L_5 + L_1 + L_4 + 2E_{15} + 2E_{45} + 2(H - E_{05} - E_{23})
\end{aligned}$$

In each case, the two effective representatives have disjoint support, and so each R_i determines a basepoint-free linear system on S. For each i we have $R_i \cdot R_i = 0$ and $K_S \cdot R_i = 0$, so these define elliptic fibrations with the given divisors as reducible fibers.

It is also necessary to check that the R_i are invariant under s_0^*. In each case, the geometric description from the proof of Theorem 3 shows that s_0 permutes the components of the given reducible fibers. For example, the invariance of R_1 follows from the facts that $s_0(L_0) = L_0$, $s_0(L_1) = L_4$, $s_0(E_{01}) = E_{04}$, and $s_0(E_{14}) = E_{14}$. □

Lemma 16. *The class $H - E_{05}$ is the unique class D in $N^1(S)_{\mathbf{R}}$ for which:*

(1) D is contained in the $(+1)$-eigenspace of the involution $s_0^* : N^1(S)_{\mathbf{R}} \to N^1(S)_{\mathbf{R}}$.
(2) $D \cdot L_0 = D \cdot L_5 = 0$ and $D \cdot L_1 = D \cdot L_2 = D \cdot L_3 = D \cdot L_4 = 1$.
(3) D is nef.
(4) $D^2 = 0$.

Proof. The linear system $H - E_{05}$ contains the strict transforms on S of the pencil of lines through p_{05}. Since this pencil is preserved by s_0 and the linear system on S is basepoint-free, the claimed properties follow for the class $H - E_{05}$.

We next check that there are no other classes with these four properties. Suppose that D is such a class, and let $D' = D - (H - E_{05})$. Then D' lies in the $(+1)$-eigenspace of s_0^*, and $D' \cdot L_i = 0$ for each L_i; the set of D' satisfying these hypotheses is a linear subspace of $N^1(S)_{\mathbf{R}}$.

Since $L_1 - L_4$ and $L_2 - L_3$ lie in the (-1)-eigenspace, the space of D' which lie in the $(+1)$-eigenspace and satisfy $D' \cdot L_i = 0$ for each i is 4-dimensional. Each of the six classes R_i of Lemma 15 lies in this space, and so a basis is provided by R_1, R_2, R_3, and R_4; the other rays satisfy $R_5 = R_3 + R_4 - R_2$ and $R_6 = R_3 + R_4 - R_1$.

We wish to know when the class $D = (H - E_{05}) + a_1 R_1 + a_2 R_2 + a_3 R_3 + a_4 R_4$ is nef. Intersecting with the eight effective classes

$$\begin{array}{llll}
E_{14}, & E_{23}, & E_{35}, & E_{01}, \\
H - E_{05} - E_{14}, & H - E_{05} - E_{23}, & E_{15}, & E_{02},
\end{array}$$

we obtain the eight inequalities

$$\begin{array}{llll}
a_3 \geq 0, & a_4 \geq 0, & a_2 + a_3 \geq 0, & a_2 + a_4 \geq 0, \\
a_1 + a_2 + a_4 \geq 0, & a_1 + a_2 + a_3 \geq 0, & a_1 + a_4 \geq 0, & a_1 + a_3 \geq 0,
\end{array}$$

which all must hold if D is nef. Summing the eight inequalities we find that $a_1 + a_2 + a_3 + a_4 \geq 0$ with equality only if the left side of each is 0; it follows that $a_1 + a_2 + a_3 + a_4 = 0$ only if each a_i is 0. Any nonzero (a_1, a_2, a_3, a_4) satisfying the inequalities is thus a positive multiple of a solution with $a_1 + a_2 + a_3 + a_4 = 1$. Substituting $a_1 = 1 - a_2 - a_3 - a_4$, the four columns above yield the four constraints:

$$0 \leq a_3 \leq 1, \quad 0 \leq a_4 \leq 1, \quad -a_3 \leq a_2 \leq 1 - a_3, \quad -a_4 \leq a_2 \leq 1 - a_4.$$

These inequalities on (a_2, a_3, a_4) determine a compact three-dimensional polyhedron with six vertices $(a_2, a_3, a_4) = (0,0,0), (1,0,0), (0,1,0), (0,0,1), (-1,1,1), (0,1,1)$. The set of solutions to the original inequalities is then the four-dimensional cone spanned by the six classes

$$(a_1, a_2, a_3, a_4) = (1,0,0,0), \;\; (0,1,0,0), \;\; (0,0,1,0), \;\; (0,0,0,1), \;\; (0,-1,1,1), \;\; (-1,0,1,1).$$

The corresponding classes $a_1R_1 + a_2R_2 + a_3R_3 + a_4R_4$ in $N^1(S)_{\mathbf{R}}$ are precisely R_1, R_2, R_3, R_4, $-R_2 + R_3 + R_4 = R_5$ and $-R_1 + R_3 + R_4 = R_6$.

As a result, the set of D satisfying hypotheses (1)–(3) is contained in the cone given by divisors of the form $(H - E_{05}) + \sum_{i=1}^{6} c_i R_i$ with $c_i \geq 0$. Since $(H - E_{05}) \cdot R_i = 2$ for each $1 \leq i \leq 6$, while $R_i \cdot R_j \geq 0$ for any i and j, and $(H - E_{05})^2 = 0$, we find that

$$\left((H - E_{05}) + \sum_{i=1}^{6} c_i R_i \right)^2 \geq 0,$$

with equality if and only if $c_i = 0$ for each i, so that $D = H - E_{05}$. □

Lemma 17. *The centralizer of s_0 in $G_{ev}^{\pm}$ is the two-element group $\{\mathrm{id}_S, s_0\}$.*

Proof. Suppose that $\phi : S \to S$ is an element of $G_{\mathrm{ev}}^{\pm}$ commuting with s_0. Then $\phi^* : N^1(S)_{\mathbf{R}} \to N^1(S)_{\mathbf{R}}$ must preserve the $(+1)$-eigenspace of s_0^*, and so $\phi^*(H - E_{05})$ lies in this eigenspace as well. If ϕ lies in $G_{\mathrm{ev}}^{\pm}$, then the intersection property (2) must be satisfied by $\phi^*(H - E_{05})$. Since ϕ^* also preserves the nef cone and the intersection form, $\phi^*(H - E_{05})$ in fact satisfies the hypotheses (1)–(4) of Lemma 16. It then follows from the lemma that $\phi^*(H - E_{05}) = H - E_{05}$ in $N^1(S)_{\mathbf{R}}$, and since $\mathrm{Pic}^0(S)$ is trivial, that ϕ must preserve the class $H - E_{05}$ in $\mathrm{Pic}(S)$. As a result, ϕ factors through the map $\lambda : S \to \mathbb{P}^1$ given by the basepoint-free linear system $|H - E_{05}|$.

In particular, ϕ permutes the singular fibers of λ. The fibers are the preimages on S of the lines in $\mathbb{P}^2$ passing through the point p_{05}, and the singular fibers are precisely those corresponding to lines which pass through p_{05} and any of the other 14 points which are blown up on S. Eight of those points lie on the two lines L_0 and L_5; the other six are the lines through the point p_{05} and a point p_{ij} for which neither i nor j is equal to 0 or 5. These six singular fibers are each unions of two (-1)-curves, with classes $H - E_{05} - E_{ij}$ and E_{ij}, arising as the strict transform of the line itself, and as the exceptional divisor of the blow-up.

Since by assumption ϕ lies in the subgroup $G_{\mathrm{ev}}^{\pm}$, it fixes the two curves L_0 and L_5 and either fixes or exchanges the members of the pairs L_1, L_4 and L_2, L_3. It must map E_{14} to another s_0-invariant (-1)-curve contained in a fiber of λ that has intersection 1 with both L_1 and L_4, and 0 with L_2 and L_3. From the description of the preceding paragraph, the only two such curves are E_{14} itself and the strict transform of the line from p_{05} to p_{23}, which has class $H - E_{05} - E_{23}$. However, under the generality hypothesis of Remark 1, there is no map that fixes L_0 and L_5 while exchanging the fibers containing these curves; consequently each of these fibers must be mapped to itself. This implies that ϕ fixes four fibers of the map λ, and since the base is $\mathbb{P}^1$, that ϕ maps every fiber of λ to itself.

Replacing ϕ with $\phi \circ s_0$ if necessary, we obtain an element commuting with s_0 which fixes the four curves L_0, L_5, L_1, and L_4, and either fixes the two curves L_2 and L_3 or exchanges them.

Suppose for now that ϕ exchanges the two sections L_2 and L_3. The fibers of λ are preserved, and so $\phi(E_{12})$ must be either E_{12} or the curve of class $H - E_{05} - E_{12}$. We have

$$L_1 \cdot \phi(E_{12}) = \phi(L_1) \cdot \phi(E_{12}) = L_1 \cdot E_{12} = 1,$$
$$L_2 \cdot \phi(E_{12}) = \phi(L_3) \cdot \phi(E_{12}) = L_3 \cdot E_{12} = 0.$$

Neither E_{12} nor $H - E_{05} - E_{12}$ has the required intersection properties for $\phi(E_{12})$: we have $L_2 \cdot E_{12} = 1$, while $L_1 \cdot (H - E_{05} - E_{12}) = 0$. We conclude that $\phi(L_2) = L_2$ and $\phi(L_3) = L_3$.

Thus ϕ must commute with the projection λ and fix the four sections L_1, L_2, L_3, and L_4. A general geometric fiber F of λ is a rational curve in the linear system $|H - E_{05}|$. The map ϕ fixes the four points of intersection of F with the sections listed, and so $\phi|_F$ must be the identity map. Since ϕ fixes a Zariski dense set of points on $S_{\overline{\mathbf{K}}}$, it must be the identity. As we may have previously replaced ϕ with $\phi \circ s_0$, we conclude that the centralizer is $\{\mathrm{id}_S, s_0\}$. □

Corollary 18. *The group $G^{\pm}_{ev}$ contains infinitely many conjugacy classes of involutions.*

Proof. Let $\bar{\gamma}$ be an automorphism of S restricting to $z \mapsto bz$ on L_0. Since any automorphism of S permutes the six lines, there exists some $N > 0$ for which the iterate $\bar{\gamma}^N$ maps each of the six lines L_i to itself. The map $s_{Nn} = \bar{\gamma}^{-Nn} \circ s_0 \circ \bar{\gamma}^{Nn}$ is an involution which restricts to L_0 as $z \mapsto cb^{-Nn} - z$, and since s_{Nn} induces the same permutation of the L_i as does s_0, it lies in the subgroup $G^{\pm}_{\mathrm{ev}}$.

We claim that no two distinct s_{Nm} and s_{Nn} are conjugate by an element of $G^{\pm}_{\mathrm{ev}}$. It suffices to show that s_0 is not conjugate to any s_{Nn}. If $s_{Nn} = \bar{\gamma}^{-Nn} \circ s_0 \circ \bar{\gamma}^{Nn} = \alpha \circ s_0 \circ \alpha^{-1}$ for some α, then $\bar{\gamma}^{Nn} \circ \alpha$ commutes with s_0. According to Lemma 17, either $\alpha = \bar{\gamma}^{-Nn}$ or $\alpha = \bar{\gamma}^{-Nn} \circ s_0$. Since neither $\bar{\gamma}^{-Nn}$ nor $\bar{\gamma}^{-Nn} \circ s_0$ is contained in $G^{\pm}_{\mathrm{ev}}$ for any nonzero value of n, the claim follows. □

Lemma 19. *Every automorphism of $X'_{\mathbf{L}}$ is defined over $\mathbf{K}$.*

Proof. Since $S_{\mathbf{L}}$ is constructed by blowing up $\mathbf{K}$-points in $\mathbb{P}^2$, its Picard group is generated by the classes of $\mathbf{K}$-divisors. The Galois action on $\mathrm{Pic}(S_{\mathbf{L}})$ is therefore trivial, and preserves the class of any (-1)-curve. Because each (-1)-curve is rigid in its numerical class, these curves are invariant under the conjugation map $c : S_{\mathbf{L}} \to S_{\mathbf{L}}$.

Suppose that $\phi : X'_{\mathbf{L}} \to X'_{\mathbf{L}}$ is any automorphism. Then ϕ is induced by some automorphism $\psi : S_{\mathbf{L}} \to S_{\mathbf{L}}$, and $c \circ \psi \circ c : S_{\mathbf{L}} \to S_{\mathbf{L}}$ is an automorphism which has the same action as ψ on the class of any (-1)-curve. Since $\mathrm{Pic}(S)$ is generated by classes of (-1)-curves defined over $\mathbf{K}$, ψ and $c \circ \psi \circ c$ have the same action on $\mathrm{Pic}(S)$. As $H^0(S_{\mathbf{L}}, TS_{\mathbf{L}}) = 0$, these two maps must coincide, so that $c \circ \psi = \psi \circ c$, and ψ is defined over $\mathbf{K}$. □

Proof of Theorem 2. We have $\mathrm{Aut}(X'_{\mathbf{L}}) \cong G^{\pm}$, and $G^{\pm}_{\mathrm{ev}}$ is a finite index subgroup of $G^{\pm}$ on which $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$ acts trivially. By Lemma 18, $G^{\pm}_{\mathrm{ev}}$ contains infinitely many conjugacy classes of involutions, and Theorem 2 then follows from Lemma 13. □

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607
Email address: jdl@uic.edu