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Planning of parking enforcement patrol considering drivers' parking payment behavior

Chao Lei, Qian Zhang, Yanfeng Ouyang*

Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

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ABSTRACT

This paper focuses on improving the effectiveness of parking enforcement patrol by optimizing the schedule of visit at each parking lot and the routing plan of patrol vehicles. Meanwhile, individual parking driver makes his/her parking payment decision based on knowledge of the patrol visit frequencies. Game-theoretic models are proposed to capture the interactions among the parking enforcement agency and parking drivers. We first develop a discrete formulation of the problem in the form of a mixed-integer program and propose a Lagrangian relaxation based solution approach. For large-scale instances, we also develop a continuum approximation model that can be reduced to a simpler non-linear optimization problem. A series of numerical experiments are conducted to show that, for small problem instances, both modeling approaches can yield reasonable solutions, although the continuum approximation approach is able to produce a solution within a much shorter time. For large-scale instances, the discrete model incurs prohibitive computational burdens, while the continuum approximation approach still provides a near-optimum solution effectively. We also discuss impacts of various system parameters, as well as the performance of different policy options (e.g., whether to allow multiple parking tickets to be issued to a vehicle with a long time of parking violation).

1. Introduction

As the demand for parking continues to grow especially in urban areas, it has become a major challenge to enhance the efficiency and sustainability of parking services. Many cities have started to implement various parking management strategies (e.g., parking pricing, reservation) to improve the efficiency of their parking systems. For any of these strategies to be effective, parking enforcement has to be established to reduce or eliminate parking violations (Litman, 2006). However, this is not a trivial task. According to NYC OpenData (2016), in the fiscal year 2015 alone, 7.3 million parking violation tickets were issued in New York City, and 45% of them were due to overtime parking. As parking violations waste already limited parking resources, it is essential to strengthen parking enforcement management.

In most urban areas, parking is not free. A driver needs to pay a certain amount of money, usually at the time of arrival, to secure a parking space for a chosen length of time. However, in many systems, especially with parking meters, the drivers often do not know the exact length of needed parking duration when he or she is making the payment. In many other situations, the parking duration is actually random due to unexpected delays or distractions. A parked vehicle is considered

* Corresponding author.
E-mail address: yfouyang@illinois.edu (Y. Ouyang).

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to be in the violation state once the actual parking duration exceeds the paid parking time, and a ticket will be issued to such a vehicle when a patrol officer visits the parking lot. The fine associated with a parking ticket is usually much larger than the parking price. As such, a driver's parking payment decision shall be affected by the likelihood of encountering an officer, or in other words the schedule/frequency of patrol activities (Adiv and Wang, 1987; Elliot and Wright, 1982). Although the drivers may not possess accurate knowledge of the exact patrol schedule, it is not uncommon that they are aware of the approximate patrol frequency at a specific parking lot, especially for those drivers who use the parking lot on a daily basis. Accordingly, when the patrol frequency is low, a knowledgeable driver may incline to be more opportunistic and pay less money for parking; in contrast, drivers at a frequently patrolled parking lot would tend to be more conservative.

In practice, the parking enforcement agency usually divides the operating hours into multiple time periods and sends out officers to visit various parking lots in a periodic and repetitive manner. For a single period, a patrol officer departs from the base location at the beginning of the period, and visits a sequence of parking lots according to a routing plan. When the officer arrives at a parking lot, he or she checks the status of all parked vehicles to see if there is any violation. Once the officer finds a violation, he or she needs to spend some time on processing the violation, such as taking photos of the vehicle and issuing a ticket. After processing all the violations in the parking lot, the officer proceeds to the next location. Finally, the officer needs to return to the base location before the end of this period. From the perspective of the parking enforcement agency, it would be ideal if all the parking lots can be patrolled sufficiently frequently such that no parking violation exists. However, the agency may have only limited patrol resources to use. In addition, as mentioned above, there is also a time duration limit imposed on each single patrol route. As such, in order to construct an optimal or at least feasible routing plan, the agency needs to figure out the length of time that is needed for processing violations in each parking lot (which is called service time in this paper), and yet the travel time in-between parking lots. The travel time is considered to be deterministic and known, while the service time directly depends on the number of violations that is processed en route.

Parking enforcement patrol is one type of patrol activities routinely conducted by law enforcement agencies. Related resource allocation, positioning or routing problems have been explored in similar contexts. For instance, police patrol planning has been extensively studied in the urban environment (Larson, 1972; Chaiken and Dormont, 1978a; 1978b). In these models, the deployed police patrol vehicles were typically modeled as servers in a queuing system; such models capture the reactive nature of the police force (i.e., the primary mode of police operation is to respond to calls for service). Birge and Pollock (1989) extended the police patrol problem to the rural environment, and Taylor et al. (1985) proposed an integer nonlinear goal programming model which focused on maintaining a visible police presence on highways. Another example of patrol problems is the well-known art gallery and illumination problem, which deals with the positioning and deployment of guards in art galleries or museums (Urrutia, 2000).

The parking agency's problem shares some similarities with the periodic vehicle routing problem (PVRP) in which a set of given customers (each with a known visit frequency requirement) are repeatedly visited over multiple time periods, and different routes are constructed in each time period to serve the customers collectively (Christofides and Beasley, 1984). The PVRP is a variant of the classic vehicle routing problem and has been widely applied to many practical contexts, e.g., waste collection (Beltrami and Bodin, 1974), elevator maintenance and repair (Blakeley et al., 2003) and vending machine replenishment (Rusdiansyah and Tsao, 2005). The readers are referred to Campbell and Wilson (2014) for more details on PVRP. Since patrol frequency is part of the agency's decision, the closest literature might be that on PVRP with service choice (PVRP-SC), which is an extension of the PVRP that allows each customer's visit frequency/schedule to be a decision variable. Francis et al. (2006) studied the PVRP-SC in the context of interlibrary book delivery, and proposed a mixed-integer model and an exact solution method. A survey of related variants, formulations, and solution methods can be found in Francis et al. (2008).

The challenge is that, instead of knowing the exact demand in advance (as we normally assume for PVRP-SC and other patrol problems), the demand in the parking enforcement patrol problem (i.e., depending on the number of parking violations in each parking lot) is not only stochastic but also dependent on the agency's patrol decision. That is, the likelihood of a vehicle being in violation is directly related to the driver's payment decision, and this payment decision is affected by the agency's patrol frequency. Such bi-directional relationships between the number of parking violations (as well as the service time) and the agencys patrol routing plans, therefore, imposes an additional layer of complexity on the already difficult PVRP-SC type problems.

In this paper, we propose two game-theoretic mathematical models to capture the interactions between the agency's patrol decision and drivers' parking payment decision. The parking enforcement agency determines the patrol schedule at each parking lot and the routing plan of the available patrol vehicles, while taking into consideration the fact that each parking driver makes his/her optimal parking payment decision in accordance with the patrol frequencies. We model the agency's patrol planning problem as a variant of PVRP-SC and handle the driver's parking payment problem as a variant of the news-vendor problem. As the driver's parking payment problem can be solved in closed form, the parking enforcement patrol problem can be transformed into a single-level mathematical program. We first develop a discrete formulation as well as a Lagrangian relaxation based solution algorithm. To facilitate the solution process for large-scale instances, we also develop a continuum approximation (CA) formulation that can be reduced to a non-linear optimization problem. The performance of the proposed models and algorithms is tested through a series of numerical experiments. It is shown that the both the discrete and CA approach can produce reasonable solutions, though the computation time for the CA approach is much shorter. The CA approach demonstrates to be an effective alternative to avoid prohibitive computational difficulty in solving the discrete models, especially for large-scale instances. We discuss impacts of various system parameters (e.g., parking demand distribution, ratio between ticket fine and parking price) and also draw managerial insights on the impact of different ticketing policies (e.g., whether to allow a violator to receive multiple tickets for one single long-duration violation).

This remainder of the paper is organized as follows. In Section 2, a discrete mathematical model, as well as a Lagrangian relaxation-based algorithm, are developed for the parking enforcement patrol problem. Then, a continuum approximation formulation and its corresponding solution method are presented in Section 3. Section 4 presents the numerical results for the performance of the proposed models and solution approaches. Section 5 concludes the paper and discusses possible directions of future research.

2. Discrete model

The aforementioned parking enforcement patrol problem is modeled as a discrete mathematical program, which contains both the agency's optimization problem and the parking drivers' payment problem. The agency needs to decide: i) what level of patrol schedule or frequency should be enforced for each parking lot, and ii) how to deploy limited patrol vehicles to maximize the effectiveness of the patrol. Drivers know about the patrol frequencies (but not the exact schedule) and adjust payment decisions accordingly to minimize the total cost (including the parking payment and expected violation penalty).

2.1. User's problem

We first consider a generic driver who decides to pay for parking at a generic parking lot with a perceived patrol frequency R/T, where R denotes the total number of patrol visits within a time horizon T. Let τ denote the needed duration of parking for a vehicle in the parking lot, which is a random variable drawn from a cumulative distribution function $F(\tau)$. Assume $\bar{\tau}$ is its expectation. Let p denote the length of time that a driver decides to pay. Note that p, τ and T all have the same unit of time. Since $(\tau - p)^+$ is the length of time during which the vehicle is in violation and T/R is the headway of two consecutive officer visits, $\frac{(\tau-p)^+}{T/R}$ defines the expected number of times that the vehicle is caught in violation. The agency follows a "multiple-ticket policy;" i.e., a vehicle receives a ticket every time that it is caught in violation, regardless of how many tickets it has already received. As such, the expected parking violation fine for an individual violator is $\gamma\frac{R}{T}(\tau - p)^+$, where γ denotes the fine per ticket. The fine per ticket γ is much greater than the parking price per time unit β, i.e., $\gamma \gg \beta$. The optimal payment decision can thus be determined by solving the following news-vendor problem:

$$\min_p \quad \mathbf{E}_\tau\left[\beta p + \gamma \tfrac{R}{T}(\tau - p)^+\right], \tag{1}$$

$$\text{s.t.} \quad p \geq 0, \tag{2}$$

where $(\tau - p)^+ =: \max\{\tau - p, 0\}$.

The objective function (1) minimizes the expected total parking cost for an individual driver, including the parking payment and expected parking violation fine. Constraint (2) states that the paid length of time is nonnegative.[1] This news-vendor type model can be optimally solved if R and $F(\tau)$ are known. The optimal solution, denoting as p^*, can be derived in closed form as the following:

$$p^* = \begin{cases} F^{-1}\left(1 - \dfrac{\beta T}{\gamma R}\right), & \text{if } \dfrac{R}{T} > \dfrac{\beta}{\gamma}; \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $F^{-1}(\cdot)$ is the inverse of the cumulative distribution function. The condition $\frac{R}{T} > \frac{\beta}{\gamma}$ in (3) indicates that a driver would pay for parking only if the fine per unit time is larger than the meter payment cost per unit time; otherwise, the driver would be better off paying the fine. Moreover, the probability for the vehicle to be in parking violation, denoted by P^{vio}, can be derived as follows:

$$P^{\text{vio}} = \Pr(\tau > p^*) = \begin{cases} 1 - F(p^*) = \dfrac{\beta T}{\gamma R}, & \text{if } \dfrac{R}{T} > \dfrac{\beta}{\gamma}; \\ 1, & \text{otherwise}. \end{cases} \tag{4}$$

The above formula for P^{vio} provides a theoretical explanation for the intuitive fact that increasing the patrol frequency (R) and fine-to-price ratio (γ/β) can help deter the occurrence of overtime parking. We can also observe another interesting fact that under the multiple-ticket policy, the probability for a parking vehicle to be in the violation state is independent of the probability distribution of parking duration.

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[1] It is well known that variants to the news-vendor model could allow meter time limits (e.g., 2 h or 10 h) to be imposed and yet still yield closed-form solutions.

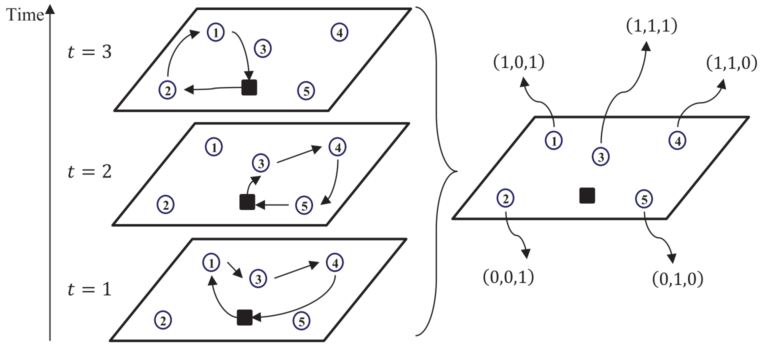


Fig. 1. Illustration of patrol schedule and routes.

Now we consider the alternative case of a "single-ticket policy," i.e., at most one ticket can be issued to a same vehicle in violation during the horizon T. In this case, the parking users model can be modified into the following variant:

$$\min_{p} \ \mathbf{E}_{\tau}\left[\beta p + \gamma \cdot \min\left\{\frac{R}{T}(\tau - p)^{+}, 1\right\}\right]$$
$$\text{s.t.} \quad p \geq 0. \tag{5}$$

The objective function (5) can be written equivalently as:

$$\int_{0}^{\infty}\left(\beta p + \gamma \cdot \min\left\{\frac{R}{T}(\tau - p)^{+}, 1\right\}\right) dF(\tau) = \beta p + \frac{\gamma R}{T}\int_{p}^{p+\frac{R}{T}}(\tau - p)dF(\tau) + \gamma\int_{p+\frac{R}{T}}^{+\infty} dF(\tau). \tag{6}$$

Then, the first-order derivative of (6) with respect to p is

$$\beta - \frac{\gamma R}{T}\left[F\left(p + \frac{T}{R}\right) - F(p)\right].$$

It shall be easy to verify that, in the degenerated case $\frac{R}{T} \leq \frac{\beta}{\gamma}$, (6) is a monotonically nondecreasing function of $p \in [0, +\infty)$, and the parking payment model (5) reaches its optimal solution at $p = 0$. Otherwise, when $\frac{R}{T} > \frac{\beta}{\gamma}$, (6) will increase first and decrease for $p \in [0, p^*]$, and then increase again for $p \in (p^*, \infty)$; the optimal solution of (5) is either at the boundary point (i.e., $p = 0$), or at $p = p^*$, whereas p^* solves the following implicit equation:

$$F\left(p + \frac{T}{R}\right) - F(p) = \frac{\beta T}{\gamma R}. \tag{7}$$

Even though we can no longer get a closed-form solution to (5), it can still be solved numerically, and the probability for a vehicle to be in violation (which now depends on the probability distribution of parking duration) can be computed as well. Such probabilities can be computed for all possible patrol schedules and serve as input data to the agencys optimization model in the next section.

2.2. Agency's problem

The parking enforcement agency manages an urban area with a set of spatially distributed parking lots $\mathcal{V}$, each of which contains sufficiently many parking spaces. The agency decides the patrol routing plan within the long time horizon T, which can be further divided into a finite set of discrete time periods $\mathcal{T}$, each with length H. Every patrol route would start and end at a depot, which we denote the depot by index 0. We consider each parking lot as an $M/G/\infty$ queuing system, in which the arrival of parking vehicles follows a Poisson process with rate λ_j, and the service time for per vehicle (i.e., parking duration) follows a general distribution with expectation value $\overline{\tau_j}$. Further, we assume that the parking system is in the stationary state over the entire time horizon, such that the number of vehicles parked in each parking lot per time period follows a same probability distribution. It has been shown in the literature that under these assumptions, the number of parking vehicles in lot j per time period, N_j, should follow a Poisson distribution (Newell, 1966).

For each parking lot, the combination of time periods during which it is visited defines its patrol schedule. We denote $\mathcal{S}$ as the set of all possible patrol schedules, indexed by s. A schedule $s \in \mathcal{S}$ and a time period $t \in \mathcal{T}$ are connected by a binary parameter a_{st}, where $a_{st} = 1$ if period t is part of schedule s, or $a_{st} = 0$ otherwise. Hence, a generic schedule s can be represented by a vector of binary parameters in the form of $\left(a_{s,t=1}, a_{s,t=2}, \ldots, a_{s,t=|\mathcal{T}|}\right)$. For example, in Fig. 1, parking lot $j = 1$ is patrolled in period 1 and 3, which means that its patrol schedule is (1, 0, 1). We can see that each parking lot in Fig. 1 has a different patrol schedule. Since each possible vector indicates a schedule, the total number of distinct schedules $|S| = 2^{|\mathcal{T}|}$. For schedule $s \in \mathcal{S}$, the associated average patrol frequency is $\frac{R_s}{T} = \frac{\sum_{t \in \mathcal{T}} a_{st}}{T}$. Note that multiple patrol schedules may have the same average patrol frequency; e.g., parking lots $j = 1$ and $j = 4$ in Fig. 1. Based on (4), the value of P_s^{vio}, i.e., the probability that a vehicle is in violation if it is patrolled according to schedule s, can be computed for each schedule; i.e., $\forall s \in \mathcal{S}$,

$$P_s^{vio} = \begin{cases} \dfrac{\beta T}{\gamma \sum_{t\in\mathcal{T}} a_{st}}, & \text{if } \dfrac{\sum_{t\in\mathcal{T}} a_{st}}{T} > \dfrac{\beta}{\gamma}; \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Given the probability distribution of N_j and the definition of P_s^{vio} , the probability distribution for the number of violations per time period in lot j under schedule s, M_{js}, can be obtained according to Proposition 1.

Proposition 1. *If parking lot $j \in \mathcal{V}$ is patrolled according to schedule $s \in \mathcal{S}$, the number of violations per time period in this lot follows a Poisson distribution with mean $P_s^{vio}\lambda_j\bar{\tau}_j$.*

Proof. Each of the parked vehicles in lot $j \in \mathcal{V}$ in a time period, independently of each other, would be in the violation state with probability P_s^{vio} if this lot is patrolled according to schedule $s \in \mathcal{S}$. The number of parked vehicles in the lot follows a Poisson distribution. According to Proposition 5.2 in Ross (2014), the number of violations per time period in lot j is also an independent Poisson random variable with mean $P_s^{vio}\lambda_j\bar{\tau}_j$. This completes the proof. □

The parking enforcement agency needs to decide the type of patrol schedule to be assigned to each parking lot, and the patrol routes in each time period that fulfill the schedule at all lots. We denote $\mathcal{K}$ as the set of patrol vehicles, indexed by k. Denote h_{ij} as the travel time on arc $(i, j) \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of arcs within the network, and φ denotes the service time for processing a violation. The summation of total travel time and service time within a single route should not exceed H. We denote y_{sj} as the scheduling assignment variable, which equals to 1 if parking lot $j \in \mathcal{V}$ is visited according to schedule $s \in \mathcal{S}$, or 0 otherwise. For example, in the example shown in Fig. 1, $y_{s,j=1} = 1$ for schedule $s = (1, 0, 1)$, and $y_{s,j=1} = 0$ for any $s \neq (1, 0, 1)$. Combining parameter P_s^{vio} with decision variable y_{sj}, we can express the probability for a vehicle in parking lot j to be in violation as $\sum_s P_s^{vio} y_{sj}$, whereas $\sum_s y_{sj} = 1, \forall j$ and $y_{sj} \in \{0, 1\}$, $\forall s, j$. Let x_{tkij} denote the routing decision variable, whose value equals to 1 if patrol vehicle $k \in \mathcal{K}$ travels through arc (i, j) in period $t \in \mathcal{T}$, or 0 otherwise. The binary decision variable v_{stkj} links the routing and the scheduling decision together. It equals to 1 if parking lot $j \in \mathcal{V}$ is patrolled according to schedule $s \in \mathcal{S}$ and meanwhile it is visited by patrol vehicle $k \in \mathcal{K}$ in period $t \in \mathcal{T}$, or 0 otherwise.

The agency may bear different types of goals while planning the parking enforcement patrol. If the perspective is to improve the social benefit, the agency could attempt to minimize the expected total costs associate with all parking violations during the planning horizon, i.e.,

$$\sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{V}} \gamma P_s^{vio} \lambda_j \bar{\tau}_j y_{sj}. \tag{9}$$

Here we use the fine per ticket to convert violation counts to monetary values. Alternatively, the agency may want to maximize the total expected revenue from parking meter payments, i.e.,

$$\sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{V}} \beta p_{sj}^* \lambda_j \bar{\tau}_j y_{sj}, \tag{10}$$

and/or the expected total fine revenue, i.e.,

$$\sum_{s\in\mathcal{S}/\{s_0\}} \sum_{j\in\mathcal{V}} \gamma R_s P_s^{vio} \lambda_j \bar{\tau}_j y_{sj} = \sum_{s\in\mathcal{S}/\{s_0\}} \sum_{j\in\mathcal{V}} \gamma R_s \frac{\beta T}{\gamma R_s} \lambda_j \bar{\tau}_j y_{sj} = \sum_{j\in\mathcal{V}} \beta T \lambda_j \bar{\tau}_j \left(1 - y_{s_0 j}\right), \tag{11}$$

where s_0 denotes the schedule that $\sum_{t\in\mathcal{T}} a_{s_0 t} = 0$, and the second equality holds because $P_s^{vio} = \frac{\beta T}{\gamma R_s}$. As such, maximizing the total collected fines will lead to minimizing the expected parking demand that is never patrolled during the planning horizon.

The agency's problem can be formulated as the following integer program, where the three possible objectives are combined with relative weights ω_1, ω_2, ω_3.

$$\min \quad \omega_1 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{V}} \gamma P_s^{vio} \lambda_j \bar{\tau}_j y_{sj} - \omega_2 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{V}} \beta p_{sj}^* \lambda_j \bar{\tau}_j y_{sj} - \omega_3 \sum_{s\in\mathcal{S}/\{s_0\}} \sum_{j\in\mathcal{V}} \gamma R_s P_s^{vio} \lambda_j \bar{\tau}_j y_{sj} \tag{12}$$

$$\text{s.t.} \sum_{j\in\mathcal{V}\cup\{0\}} x_{tkij} - \sum_{j\in\mathcal{V}\cup\{0\}} x_{tkji} = 0, \quad \forall i \in \mathcal{V}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{13}$$

$$\sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{V}} x_{tkij} \le 1, \quad \forall i \in \mathcal{V} \cup \{0\}, t \in \mathcal{T}, \tag{14}$$

$$\sum_{j\in\mathcal{V}} x_{tk0j} = \sum_{j\in\mathcal{V}} x_{tkj0} \le 1, \quad \forall k \in \mathcal{K}, t \in \mathcal{T}, \tag{15}$$

$$\sum_{i,j\in\mathcal{Q}} x_{tkij} \le |\mathcal{Q}| - 1, \quad \forall \mathcal{Q} \subseteq \mathcal{V}, \forall k \in \mathcal{K}, t \in \mathcal{T}, \tag{16}$$

$$\Pr\left\{ \sum_{(i,j)\in\mathcal{A}} h_{ij}x_{tkij} + \varphi \sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} M_{sj}v_{stkj} \le H \right\} \ge P_\alpha, \forall k \in \mathcal{K}, t \in \mathcal{T}, \tag{17}$$

$$\sum_{s\in\mathcal{S}} v_{stkj} = \sum_{i\in\mathcal{V}\cup\{0\}} x_{tkij}, \quad \forall j \in \mathcal{V}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{18}$$

$$\sum_{k\in\mathcal{K}} v_{stkj} = a_{st}y_{sj}, \quad \forall j \in \mathcal{V}, s \in \mathcal{S}, t \in \mathcal{T}, \tag{19}$$

$$\sum_{s\in\mathcal{S}} y_{sj} = 1, \quad \forall j \in \mathcal{V}, \tag{20}$$

$$x_{tkij}, v_{stkj}, y_{sj} \in \{0, 1\}, \quad \forall (i, j) \in \mathcal{A}, j \in \mathcal{V}, s \in \mathcal{S}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{21}$$

The objective function (12) is to minimize the expected total violations and meanwhile maximize the expected total revenue from meter payments and fines. Constraints (13)–(16) are typical vehicle flow balance constraints and subtour elimination constraints. Constraints (17) are the chance-constrained constraints for the route time restrictions, enforcing a minimum probability P_α for the time limit constraints to hold. Constraints (18) represent the connection between decision variable v and x, while constraints (19) present the connection between the scheduling decision y and routing related decision v. Constraints (20) ensure that each parking lot is assigned to a specific schedule. Constraints (21) define the binary variables.

The number of parking violations that patrol vehicle k visits in period t is $\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} M_{sj}v_{stkj}$. It follows a Poisson distribution with mean $\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj}$, because each M_{sj} is a Poisson random variable. According to the Central Limit Theorem, when the number of parking violations is larger, the Poisson distribution can be closely approximated by a normal distribution with equal mean and variance, i.e.,

$$\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} M_{sj}v_{stkj} \sim \mathcal{N}\left(\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj}, \sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj} \right).$$

Then, the deterministic equivalent of the chance constraints (17) can be written as:

$$\sum_{(i,j)\in\mathcal{A}} h_{ij}x_{tkij} + \varphi \left(\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj} + \alpha \sqrt{\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj}} \right) \le H, \forall k \in \mathcal{K}, t \in \mathcal{T}, \tag{22}$$

where $\alpha = \Phi^{-1}(P_\alpha)$ and $\Phi^{-1}(\cdot)$ represents the inverse of a standard normal p.d.f. function.

In the above formulation, the information from the user's level is incorporated into the agency's level through parameter P_s^{vio}. In this way, the agency's patrol planning problem and drivers' parking payment problem are integrated into a single-level model.

2.3. Lagrangian relaxation solution approach

We develop a Lagrangian relaxation (LR) algorithm (Fisher, 1985) to solve the above discrete model. If we relax constraints (19) and incorporate them into the objective function with properly defined Lagrangian multipliers, the schedule related variables (y) and routing related variable (x and v) can be completely separated into two subproblems that can be solved independently. Denote ρ_{tsj} as the Lagrangian multiplier associated with each one of the relaxed constraints (19), and $\rho = \{\rho_{tsj}\}$. The objective function of the relaxed problem can be written as follows:

$$\sum_{j\in\mathcal{V}} \left(\omega_1 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \gamma P_s^{\text{vio}}\lambda_j\bar{\tau}_j y_{sj} - \omega_2 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \beta p_{sj}^*\lambda_j\bar{\tau}_j y_{sj} \right.$$
$$\left. - \omega_3 \sum_{s\in\mathcal{S}/\{s_0\}} \gamma R_s P_s^{\text{vio}}\lambda_j\bar{\tau}_j y_{sj} - \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \rho_{stj}a_{st}y_{sj} \right) + \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \sum_{j\in\mathcal{V}} \sum_{k\in\mathcal{K}} \rho_{stj}v_{stkj}.$$

For a given vector ρ, the relaxed problem can be decomposed into two groups of independent subproblems: schedule assignment subproblem and routing subproblems. It is clear that the schedule assignment subproblem can be further decomposed into a set of subproblems (SP1-j) for all $j \in \mathcal{V}$, as follows:

$$\text{(SP1-}j\text{)} \min \quad \omega_1 \sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}} \gamma P_s^{\text{vio}}\lambda_j\bar{\tau}_j y_{sj} - \omega_2 \sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}} \beta p^*_{sj}\lambda_j\bar{\tau}_j y_{sj} - \omega_3 \sum_{s\in\mathcal{S}/\{s_0\}} \gamma R_s P_s^{\text{vio}}\lambda_j\bar{\tau}_j y_{sj} - \sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}} \rho_{stj} a_{st} y_{sj}$$

$$\text{s.t.} \quad \sum_{s\in\mathcal{S}} y_{sj} = 1,$$

$$y_{sj} \in \{0,1\}, \quad \forall s \in \mathcal{S}.$$

Similarly, the routing subproblem can be decomposed into a set of subproblems (SP2-t) for all $t \in \mathcal{T}$, as follows:

$$\text{(SP2-}t\text{)} \min \quad \sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{V}}\sum_{k\in\mathcal{K}} \rho_{stj} v_{stkj}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{V}\cup\{0\}} x_{tkij} - \sum_{j\in\mathcal{V}\cup\{0\}} x_{tkji} = 0, \quad \forall i \in \mathcal{V}\cup\{0\}, k \in \mathcal{K},$$

$$\sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{V}} x_{tkij} \le 1, \quad \forall i \in \mathcal{V}\cup\{0\},$$

$$\sum_{j\in\mathcal{V}} x_{tk0j} = \sum_{j\in\mathcal{V}} x_{tkj0} \le 1, \quad \forall k \in \mathcal{K},$$

$$\sum_{i,j\in\mathcal{Q}} x_{tkij} \le |\mathcal{Q}| - 1, \quad \forall \mathcal{Q} \subseteq \mathcal{V}, \forall k \in \mathcal{K},$$

$$\sum_{(i,j)\in\mathcal{A}} h_{ij} x_{tkij} + \varphi\left(\sum_{j\in\mathcal{V}}\sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj} + \alpha\sqrt{\sum_{j\in\mathcal{V}}\sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj}}\right) \le H, \ \forall k \in \mathcal{K},$$

$$\sum_{s\in\mathcal{S}} v_{stkj} = \sum_{i\in\mathcal{V}\cup\{0\}} x_{tkij}, \quad \forall j \in \mathcal{V}, k \in \mathcal{K},$$

$$x_{tkij}, v_{stkj}, \quad \forall (i,j) \in \mathcal{A}, j \in \mathcal{V}, s \in \mathcal{S}, k \in \mathcal{K}.$$

Note that the subproblem SP1-j is actually a simple assignment problem that can be easily solved to optimality using current mixed-integer solver. The subproblem SP2-t is a variant of the team orienteering problem (Chao et al., 1996), in which a certain length limit is enforced for each route and not all customers have to be visited. We apply a conic optimization technique (Atamtürk and Narayanan, 2008) to handle the square root term in constraints (22). First, we introduce a new variable Ω_{kt}, which is defined as:

$$\Omega_{kt} := \sqrt{\sum_{j\in\mathcal{V}}\sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j v_{stkj}}, \forall k \in \mathcal{K}, t \in \mathcal{T}. \tag{23}$$

Given the fact that each v_{stkj} is a binary variable, it follows that $v_{stkj} = (v_{stkj})^2$. The above equations can thus be written in the form of second order cone constraints:

$$(\Omega_{kt})^2 = \sum_{j\in\mathcal{V}}\sum_{s\in\mathcal{S}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j (v_{stkj})^2, \forall k \in \mathcal{K}. \tag{24}$$

In this way, the routing subproblem is reformulated into a mixed-integer quadratic constrained program (MIQCP), which can be solved using commercial solver CPLEX. However, this MIQCP model is still NP-hard as it is far more complex than the classic orienteering problem. Moreover, as the MIQCP model needs to be solved for each $t \in \mathcal{T}$ in each iteration, solving it to optimality is time-consuming and unrealistic. Alternatively, we impose additional stopping criteria (e.g., near optimal gap value and computation time limit) such that the solution process for each routing subproblem may be terminated before reaching optimality. By solving the above relaxed subproblems, the obtained objective values constitute a lower bound to the original problem. The solution $\{\bar{v}_{stkj}\}$ and $\{\bar{y}_{sj}\}$ from subproblems SP1-j and SP2-t may violate the relaxed constraints (19). If this happens, we heuristically adjust the infeasible solution from the relaxed subproblems into a feasible one so as to obtain an upper bound to the original problem. In iteration l, the adjustment of solution is done as follows:

Step 1: For each parking lot $j \in \mathcal{V}$, find out schedule $s^0 \in \mathcal{S}$ such that $\sum_{s\in\mathcal{S}}\sum_{k\in\mathcal{K}} \bar{v}^l_{stkj} = a_{s^0t}, \forall t \in \mathcal{T}$;

Step 2: Modify the solution $\{\bar{y}^l_{sj}\}$ as $\bar{y}^l_{s^0j} = 1$ and $\bar{y}^l_{sj} = 0, \forall s \in \mathcal{S}/s^0$;

Step 3: Add constraints $\sum_{k\in\mathcal{K}} v_{stkj} = a_{st}\bar{y}^l_{sj}, \forall s \in \mathcal{S}, t \in \mathcal{T}, j \in \mathcal{V}$ to subproblem SP2-t, and check feasibility of the modified subproblem. If it is feasible, $UB^l = \min\{UB^{l-1}, \sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{V}} P_s^{\text{vio}}\lambda_j\bar{\tau}_j\bar{y}_{sj}\}$; otherwise, $UB^l = UB^{l-1}$, where UB^l is the best upper bound obtained so far.

After each iteration, we update the Lagrange multipliers according to the standard subgradient optimization procedure (Fisher, 1985). The initial values of the Lagrangian multipliers are set to zero. At the end of iteration l, the multipliers are updated as:

$$(\rho_{stj})^{l+1} = (\rho_{stj})^l + \beta^l \left(\sum_{k\in\mathcal{K}} v_{stkj} - a_{st} y_{sj} \right), \forall j \in \mathcal{V}, s \in \mathcal{S}, t \in \mathcal{T},$$

where the step size β_l is updated according to

$$\beta^l = \frac{\pi^l (UB^l - Z_{LB}^l)}{\sum_{j\in\mathcal{V}} \sum_{s\in\mathcal{S}} \sum_{t\in\mathcal{T}} \left(\sum_{k\in\mathcal{K}} v_{stkj} - a_{st} y_{sj} \right)^2}.$$

Here, Z_{LB}^l is the lower bound found in iteration l, and π is a control parameter whose value is initially set to 2 and then decreases by a factor of 0.5 whenever the optimality gap has failed to improve in several iterations.

3. Continuous approximation model

The discrete model proposed in the previous section is NP-hard, and would face difficulty handling large-scale instances. As an alternative, we propose a continuum approximation model to obtain approximate solutions to the parking enforcement patrol problem. The CA modeling technique has been established as a powerful tool for strategic and tactical decision-making for a wide range of large-scale logistic problems; e.g., see Erera (2000), Daganzo (2005), Ouyang and Daganzo (2006), Francis and Smilowitz (2006), Smilowitz and Daganzo (2007), Cui et al. (2010) and Ouyang et al. (2015). One important advantage of the CA model lies in its closed-form formulation, which not only yields managerial insights but also can be solved much more efficiently than the discrete counterpart.

3.1. Formulation

In the CA model, all discrete variables and parameters are approximated by continuous functions, and we assume that these approximate functions are smooth and slow-varying over the region $\mathscr{R}$. Let $\delta(x)$ denote the spatial density of parking lots near a point $x \in \mathscr{R}$ per unit area. Let $\psi(x)$ denote the density of parking demand near $x \in \mathscr{R}$, measured by the number of parking vehicles per unit area per time period. Furthermore, we let $\theta_s(x)$ denote the fraction of parking lots being patrolled according to schedule $s \in \mathcal{S}$ near a point $x \in \mathscr{R}$ and $\kappa_t(x)$ as the density of patrol vehicles used in time period $t \in \mathcal{T}$ near $x \in \mathscr{R}$ per unit area.

As it is possible that not all the parking lots are visited in each period, we introduce two auxiliary decision functions to describe the patrol routes. The first one is the spatial density of parking lots being visited in period $t \in \mathcal{T}$ near $x \in \mathscr{R}$, denoted by $\Delta_t(x)$ and measured also by the number of parking lots per unit area; i.e., $\Delta_t(x) := \sum_{s\in\mathcal{S}} a_{st}\delta(x)\theta_s(x)$. Another auxiliary decision function is denoted by $\Psi_t(x)$, which represents the density of parking violations in period $t \in \mathcal{T}$ around $x \in \mathscr{R}$ per unit area. Per our discussion in Section 2, $\Psi_t(x)$ is a random variable following a Poisson distribution with mean $\sum_{s\in\mathcal{S}} a_{st} P_s^{\text{vio}} \psi(x)\theta_s(x)$, which can be closely approximated by a normal distribution, i.e., $\Psi_t(x) \sim \mathcal{N}\left(\mu_t(x), \sigma_t^2(x)\right)$, where $\mu_t(x) = \sigma_t^2(x) = \sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st}\psi(x)\theta_s(x)$.

Based on the results of Daganzo (2005), the total travel distance can be divided into a line-haul distance from the depot to the vicinity of the customers and a local distance to visit each customer. Here we let $r(x)$ denote the distance from the depot to a point $x \in \mathscr{R}$. The line-haul travel time around $x \in \mathscr{R}$ in period $t \in \mathcal{T}$ is then derived as $2\upsilon r(x)\kappa_t(x)$, where υ denotes the inverse of traveling velocity. The total local travel distance around $x \in \mathscr{R}$ in period $t \in \mathcal{T}$ can be defined as $\hat{k}\upsilon(\Delta_t(x))^{\frac{1}{2}}$, where $\hat{k}$ is a dimensionless constant that depends on the distance metric (Daganzo, 2005). The complete CA formulation can be written as follows:

$$\begin{aligned} \min \quad & \omega_1 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \left(\int_{x\in\mathscr{R}} \gamma P_s^{\text{vio}} \psi(x)\theta_s(x) dx \right) - \omega_2 \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}} \left(\int_{x\in\mathscr{R}} \sum_{s\in\mathcal{S}} \beta p_s^*(x)\psi(x)\theta_s(x) dx \right) \\ & -\omega_3 \sum_{s\in\mathcal{S}/\{s_0\}} \left(\int_{x\in\mathscr{R}} \gamma R_s P_s^{\text{vio}} \theta_s(x) dx \right) \end{aligned} \tag{25}$$

$$\text{s.t.} \quad \sum_{s\in\mathcal{S}} \theta_s(x) = 1, \quad \forall x \in \mathscr{R}, \tag{26}$$

$$\int_{x\in\mathscr{R}} \kappa_t(x) dx \le K, \quad \forall t \in \mathcal{T}, \tag{27}$$

$$\Pr\left\{ \varphi \Psi_t(x) + 2\upsilon r(x)\kappa_t(x) + \upsilon\hat{k}(\Delta_t(x))^{\frac{1}{2}} \le H\kappa_t(x) \right\} \ge P_\alpha, \quad \forall t \in \mathcal{T}, x \in \mathscr{R}, \tag{28}$$

$$0 \le \theta_s(x) \le 1, \quad \forall s \in \mathcal{S}, x \in \mathscr{R}, \tag{29}$$

$$\kappa_t(x) \geq 0, \quad \forall t \in \mathcal{T}, x \in \mathscr{R}. \tag{30}$$

The objective function (25) can be interpreted similarly as (12). Constraints (26) enforces that all parking lots within each neighborhood must be assigned to a specific schedule. Constraints (27) impose a limit over the total number of patrol vehicles. Constraints (28) are the chance-constrained formulation of the time limit constraints, in which $\varphi\Psi_t(x)$ is the service time spent on dealing with violations in period $t \in \mathcal{T}$ around $x \in \mathscr{R}$, $2\upsilon r(x)\kappa_t(x) + \upsilon\hat{k}(\Delta_t(x))^{\frac{1}{2}}$ is the total travel time in period $t \in \mathcal{T}$ around $x \in \mathscr{R}$, and $H\kappa_t(x)$ is the time limit in period $t \in \mathcal{T}$ around $x \in \mathscr{R}$. Since $\Psi_t(x) \sim \mathcal{N}\left(\mu_t(x), \sigma_t^2(x)\right)$ and $\mu_t(x) = \sigma_t^2(x) = \sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st}\psi(x)\theta_s(x)$, the deterministic equivalent of constraints (28) can then be written as:

$$\varphi\left[\sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st}\psi(x)\theta_s(x) + \alpha\left(\sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st}\psi(x)\theta_s(x)\right)^{\frac{1}{2}}\right] + 2\upsilon r(x)\kappa_t(x) + \upsilon\hat{k}(\Delta_t(x))^{\frac{1}{2}} \leq H\kappa_t(x), \quad \forall t \in \mathcal{T}, x \in \mathscr{R}. \tag{31}$$

Constraints (29) and (30) specify non-negativity properties of functions $\theta_s(x)$ and $\kappa_t(x)$.

3.2. CA solution approach

To solve the CA model, we propose to apply the geographic decomposition technique in Francis and Smilowitz (2006). As parameters are assumed to vary slowly over the region $\mathscr{R}$, we can decompose the problem according to a set of separate subregions, and we assume that a subregion $\mathscr{A} \subseteq \mathscr{R}$ is small enough that all parameter functions are nearly constant inside this subregion. Meanwhile, a subregion should be large enough to accommodate at least one route. Denote the set of subregions as $\mathcal{Z}$ and $A_z = |\mathcal{Z}|$ as the area size of subregion $z \in \mathcal{Z}$. We further define V_z as the total number of parking lots in a subregion z, i.e., $V_z = \int_{x\in\mathscr{A}_z} \delta(x)dx$, and W_z as the total number of vehicles seeking parking in subregion z per time period, i.e., $W_z = \int_{x\in\mathscr{A}_z} \psi(x)dx$. We define decision variable θ_{sz} to represent the fraction of parking lots in subregion $z \in \mathcal{Z}$ being patrolled according to schedule $s \in \mathcal{S}$ and κ_{tz} as the number of patrol vehicles used in subregion $z \in \mathcal{Z}$ in period $t \in \mathcal{T}$. In this way, the CA formulation can be approximated as a summation of a series of local homogeneous problems (one for each subregion), as follows:

$$\min \quad \omega_1 \sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}}\sum_{z\in\mathcal{Z}} \gamma P_s^{\text{vio}} W_z\theta_{sz} - \omega_2 \sum_{t\in\mathcal{T}}\sum_{s\in\mathcal{S}}\sum_{z\in\mathcal{Z}} \beta p_{sz}^* W_z\theta_{sz} - \omega_3 \sum_{s\in\mathcal{S}/\{s_0\}}\sum_{z\in\mathcal{Z}} \gamma R_s P_s^{\text{vio}} W_z\theta_{sz} \tag{32}$$

$$\text{s.t.} \quad \sum_{s\in\mathcal{S}} \theta_{sz} = 1, \quad \forall z \in \mathcal{Z}, \tag{33}$$

$$\sum_{z\in\mathcal{Z}} \kappa_{tz} \leq K, \quad \forall t \in \mathcal{T}, \tag{34}$$

$$\varphi\left(\sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st} W_z\theta_{sz} + \alpha\left(\sum_{s\in\mathcal{S}} P_s^{\text{vio}} a_{st} W_z\theta_{sz}\right)^{\frac{1}{2}}\right) + \upsilon\hat{k}\left(\sum_{s\in\mathcal{S}} a_{st} A_z V_z\theta_{sz}\right)^{\frac{1}{2}} \leq [H - 2\upsilon r_z]\kappa_{tz}, \quad \forall t \in \mathcal{T}, z \in \mathcal{Z}, \tag{35}$$

$$0 \leq \theta_{sz} \leq 1, \quad \forall s \in \mathcal{S}, z \in \mathcal{Z}, \tag{36}$$

$$\kappa_{tz} \geq 0, \quad \forall t \in \mathcal{T}, z \in \mathcal{Z}. \tag{37}$$

The modified formulation can be solved using the non-linear optimization solver KNITRO (Artelys, 2016). Since the KNITRO solver only guarantees local optima, a multi-start mechanism is invoked to improve solution quality.

The continuous model is not meant to replace the discrete modeling approach; rather, it is considered as a complementary model that can be used to estimate system performance and generate guidelines to obtain feasible (and often near-optimum) discrete route designs (Daganzo and Newell, 1986). Given the number of customers to be served as well as the number of patrol vehicles being assigned to certain time and location, recipes for developing implementable vehicle routes have been proposed in the literature (Daganzo, 1984a; 1984b; Newell and Daganzo, 1986a; 1986b; Newell, 1986). Ouyang (2007) further proposed automated constructive heuristic algorithms that yield satisfying routing solutions for large instances.

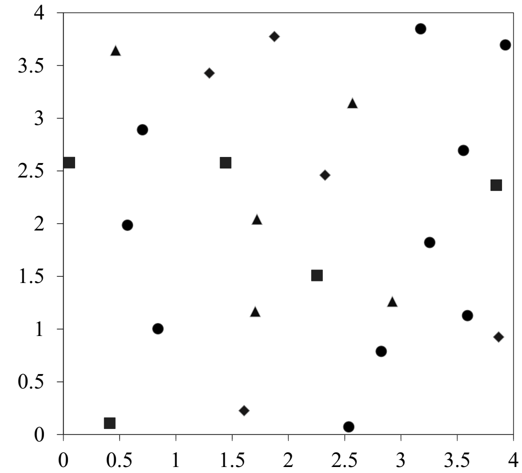


Fig. 2. Spatial distribution of parking lots.

Table 1
Comparison between discrete and CA models (10, 15, 20, 25-node).

# of parking lots	CPLEX			LR			CA		
	Upper bound	Lower bound	Gap	Upper bound	Lower bound	Gap	CPU time (s)	OBJ	Diff
10	15.5	10.0	35.5%	17.1	8.2	52.0%	59	16.9	9.0%
15	37.4	18.5	50.5%	38.7	16.6	57.1%	26	35.1	-6.1%
20	59.8	24.4	59.2%	53.9	25.2	53.2%	38	57.1	5.9%
25	97.2	31.3	67.8%	77.3	33.6	56.5%	33	80.4	4.0%

4. Numerical study

In this section, we present numerical experiments to: (i) test and compare the performance of both discrete and continuous models; (ii) show how the CA approach can be used as an effective decision support tool for various large-scale system settings; and (iii) obtain managerial insights on the impact of different agency objectives, as well as the performance of multiple-ticket vs. single-ticket policies. In the following tests, we consider a five-hour planning horizon and divide it into five identical time periods, such that $H = 60$ min. The service time per violation ϕ is 1 min, and the inverse of traveling velocity υ is 5 min/km. All distances are computed with Euclidean metric and measured in kilometers. We perform all numerical tests on a personal computer with 3.4GHz CPU and 8GB RAM.

4.1. Discrete vs. CA approach

To compare the performance of both models, we first run tests for small-scale homogeneous cases. The locations of all nodes (i.e., parking lots) are generated uniformly in a 4km × 4km square region, as shown in Fig. 2. Four test cases with an increasing number of parking lots (i.e., 10, 15, 20 and 25) are constructed. For example, the 10-node case only includes circles, while the 20-node case includes circles, triangles and diamonds. The average number of parked vehicles at each parking lot (i.e., $\lambda_j \bar{\tau}_j, \forall j$) is randomly drawn from a uniform distribution [20, 60]. Here we set $\frac{\gamma}{\beta} = 10, K = 1, \omega_1 = 1, \omega_2 = \omega_3 = 0$ and $P_\alpha = 0.8$. We solve the discrete model using both the proposed LR approach and a commercial MIP solver CPLEX. The computation time limit is set to be 8 h for each test instance. While applying the CA approach, we consider the entire region as one subregion.

Table 1 presents the best objective function values from the LR approach and the CPLEX solver, both after 8 h of computation, as well as the result and computation time from the CA model. Note that the upper bounds are in correspondence to feasible solutions, and hence the quality of the CA solution is compared to the best known upper bound. We can observe that when the test instances are small, the performance of the LR approach is similar to that of CPLEX. When the number of nodes exceeds 20, the LR approach outperforms CPLEX in terms of the gap between the upper and lower bounds. However, neither the LR approach nor the CPLEX solver could improve the lower bound of the discrete model, even for small-scale instances. The CA approach, on the other hand, always yields a solution that is close to the best-known solution from the discrete model, within merely 1 min for all test instances. Note that the CA approach is expected to perform better for large-scale problem instances, so the test results show strong promise.

To run tests for heterogeneous cases, the E-n33 dataset from Christofides and Eilon (1969) is used. To accommodate the data in the context of parking enforcement patrol, some modifications are made. The coordinates of the nodes are scaled

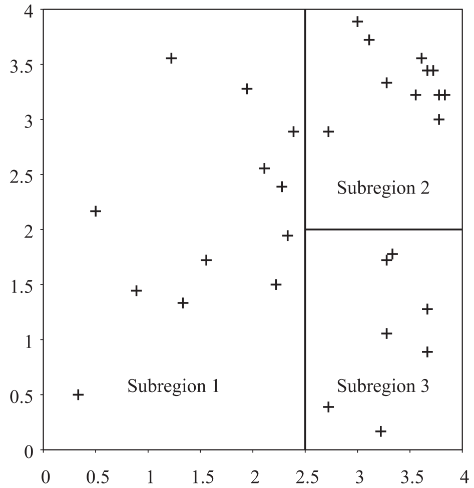


Fig. 3. Spatial distribution of parking lots (30-node).

Table 2
Comparison between discrete and CA models (30-node).

P_α	$\frac{\gamma}{\beta}$	K	CPU time (s)		Expected violation cost ($\times 10^3$)			
			LR	CA	LR(UB)	LR(LB)	CA	Diff (%)
0.8	30	1	27,330	60	37.4	6.5	39.3	5.2
0.8	30	2	2,827	43	6.5	6.5	6.5	0.0
0.8	30	3	2,386	55	6.5	6.5	6.5	0.0
0.8	15	1	27,864	38	87.9	13.1	93.3	6.1
0.8	15	2	29,427	37	29.6	13.1	30.9	4.6
0.8	15	3	3,113	35	13.1	13.1	13.1	0.0
0.8	10	1	25,994	55	119.7	19.5	124.5	4.0
0.8	10	2	25,364	43	76.5	19.5	79.7	4.1
0.8	10	3	3,181	37	33.2	19.5	35.1	5.8
0.9	30	1	28,824	69	44.4	6.5	46.4	4.4
0.9	30	2	3,113	54	6.5	6.5	6.5	0.0
0.9	30	3	2,307	64	6.5	6.5	6.5	0.0
0.9	15	1	28,156	36	93.0	13.1	96.2	3.3
0.9	15	2	25,571	41	38.3	13.1	36.9	-3.6
0.9	15	3	3,371	56	13.1	13.1	13.1	0.0
0.9	10	1	25,869	45	123.8	19.5	129.0	4.2
0.9	10	2	28,577	59	83.4	19.5	80.6	-3.3
0.9	10	3	28,592	31	38.4	19.5	36.2	-5.7

according to the following formula: horizontal coordinate $= (X - 255)/18$, vertical coordinate $= (Y - 375)/18$, where X and Y represent the original horizontal and vertical coordinates of the nodes. The demand for each node is scaled down as $\min\{\max\{L/15, 20\}, 60\}$, where L represents the original demand value. Then, we exclude node 1, 4 and 5 such that the remaining 30 nodes are included in a $[4 \times 4]$ square region. Now the dataset contains 30 nodes distributed across the region as shown in Fig. 3. We further assume that the depot is adjacent to the patrol area and the line-haul travel time from the depot to all parking lots is negligible. While solving the test case with the CA approach, we apply a so-called hierarchical clustering method to group the discrete parking lots into distinct clusters, and based on that determine the actual partition of subregions.[2] The main idea of hierarchical clustering is to build a hierarchy of clusters by maximizing the dissimilarity between clustered subsets of observations (Rokach and Maimon, 2005). In our analysis, the dissimilarity metric is based on the Euclidean distances among different parking lots. We also ensure that each subregion has a least one parking lot node. The result of clustering analysis indicates that the region in our test case should be partitioned into three subregions as shown in Fig. 3. The density of parking lots is 1.2/km^2 for subregion 1, 3.67/km^2 for subregion 2 and 2.33/km^2 for subregion 3. The parking demand densities for three regions are 59.4/km^2, 145.8/km^2 and 88.7/km^2, respectively.

We generate 18 test instances with the following parameter values $P_\alpha \in \{0.8, 0.9\}$, $\frac{\gamma}{\beta} \in \{30, 15, 10\}$ and $K \in \{1, 2, 3\}$. In the following tests, we assume the agency focuses on minimizing the expected total violations, i.e., $\omega_1 = 1, \omega_2 = \omega_3 = 0$. Table 2 presents the best objective function values obtained from both models, as well as the associated computation times.

[2] More discussion on other methods to translate discrete input data into continuous density representations can be found in Peng et al. (2014).

The test results show that the CA approach is a promising way to produce a good quality solution in a short computation time. We can observe that the CA approach can yield a solution which is less than 6% deviating from the best known solution of the discrete model. However, it usually takes more than 7 h to solve the discrete model, while the computation time for the CA model is no more than 60 s. Moreover, if the scale of the problem becomes much larger when the discrete model is expected to bear prohibitive burdens, the quality of CA solution can be even closer to that of the discrete solution.

4.2. CA performance for large instances

To test the performance of the CA approach for larger instances, we consider a 100 km^2 square region and partition it into 10×10 identical square cells. The test instances are grouped into spatially homogeneous and heterogeneous cases. In the homogeneous cases, we set $\frac{\gamma}{\beta} = 15$, $\omega_1 = 1$, $\omega_2 = \omega_3 = 0$, and $P_\alpha = 0.9$, and then generate 18 instances with $\delta(x) = \delta \in \{10, 20\}$, $\psi(x) = \psi \in \{150, 300, 600\}$, and $K \in \{10, 20, 30\}$. The computation times and objective function values from the CA model are summarized in the left part of Table 3.

For the heterogeneous cases, we consider the spatial variations in parking demand density and fine-to-price ratio. In the following, we let the parameters $\psi(x)$ and $\frac{\gamma(x)}{\beta(x)}$ be continuous functions that vary across space, defined as follows:

$$\psi(x) = \psi\left[1 + \eta \cos\left(\pi \frac{\min_{d\in\mathcal{D}}\|x, d\|}{\max_{\hat{x}\in\mathscr{R}, d\in\mathcal{D}}\|\hat{x}, d\|}\right)\right], \forall x \in \mathscr{R}, \tag{38}$$

$$\frac{\gamma(x)}{\beta(x)} = \frac{\gamma}{\beta}\left[1 - \zeta \cos\left(\pi \frac{\min_{d\in\mathcal{D}}\|x, d\|}{\max_{\hat{x}\in\mathscr{R}, d\in\mathcal{D}}\|\hat{x}, d\|}\right)\right], \forall x \in \mathscr{R}, \tag{39}$$

where $\mathcal{D}$ denotes the set of peak points (i.e., the places with the highest parking demand density or highest parking price) within region $\mathscr{R}$ and $\|x, d\|$ represents the Euclidean distance from point x to the nearest peak point d. The coordinate system is set up with the origin point located at the left bottom corner of the region. As the parking demand within a cell is aggregated to the center, we simply use the coordinates of the cell center to denote its location. Note that ψ and $\frac{\gamma}{\beta}$ control the average of parking demand density and fine-to-price ratio, while η and ζ control their variability. In what follows, we consider that there is only one peak point located at the center of the region unless stated otherwise. For example, based on (38), the parking demand density is high in the center area and decreases radially.

To test the effects of demand variations, 18 instances are generated by letting parameters take values from $\psi \in \{150, 300\}$, $\eta \in \{0, 0.5, 1\}$ and $K \in \{10, 20, 30\}$. In order to eliminate the potential interference from other parameters, we set $\frac{\gamma}{\beta} = 30$, $\zeta = 0$, $\delta = 10/\text{km}^2$, $\omega_1 = 1$, $\omega_2 = \omega_3 = 0$, and $P_\alpha = 0.9$ for all 18 instances. The computation times and objective function values are shown in the middle part of Table 3. Furthermore, we generate another 18 instances to test the effects of varying fine-to-price $\frac{\gamma}{\beta} \in \{30, 15\}$, $\zeta \in \{0, 0.5, 1\}$ and $K \in \{10, 20, 30\}$, whereas $\psi = 300/\text{km}^2$ and $\eta = 0$. The computational results are presented in the right part of Table 3.

We can observe that the computation time of the CA approach never exceeds 300 s for all test instances, which is ideal for quick decision-making. For any fixed values of the parking lot density and parking demand, the expected violations drastically increase when the fleet size of patrol vehicles is small. As such, the agency should be able to use the CA model to assess the size of patrol vehicle fleet needed to maintain patrol effectiveness at a certain target level.

For any given values of ψ and K, the more spatially heterogeneous the parking demand, the smaller the expected number of parking violations; this is shown in Table 3. As the spatial distribution of parking demand clusters more toward the peak demand point (as η grows larger), the agency tends to deploy more patrol vehicles toward the peak demand area (rather than patrolling the entire region indifferently); this is shown in Fig. 4.[3] In so doing, the expected total number of parking violations decreases as well.

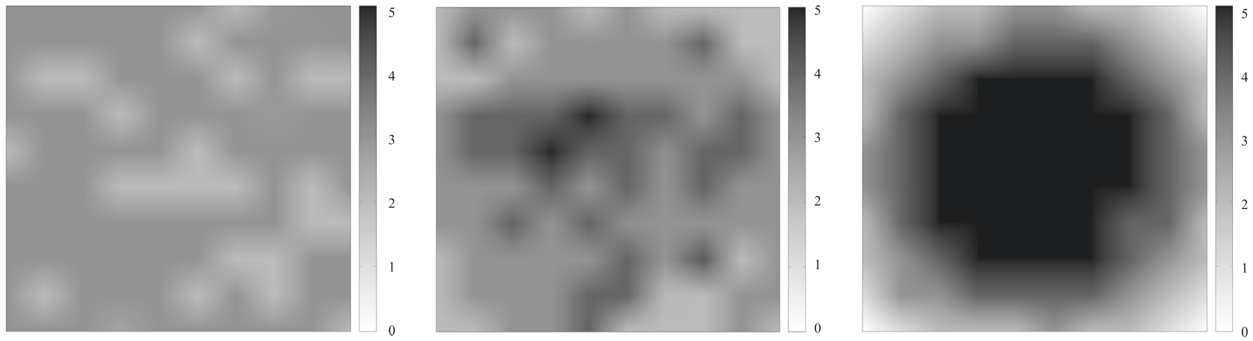
When the fine-to-price ratio varies over space, the expected parking violations decreases as well. This is reasonable. A low fine-to-price ratio implies a larger value of P^{vio} from (4). As such, in order to decrease P^{vio}, the agency deploys more patrol vehicles to areas with a lower fine-to-price ratio; this is shown in Fig. 5. In the surrounding area, even though the parked vehicles are patrolled relatively less frequently, the probability of violation is still small due to the high level of fine-to-price ratio.

We further conduct an analysis on the number and locations of the peak demand points. Three different cases are generated: case 1, one peak point at the center of the region (5,5); case 2, one peak point at (2.5,2.5); and case 3, two peak points at (2.5,2.5) and (7.5,7.5). We let $\psi = 300/\text{km}^2$, $\eta = 1$, $\delta = 10/\text{km}^2$, $\omega_1 = 1$, $\omega_2 = \omega_3 = 0$, $P_\alpha = 0.9$ and $K = 20$. The maps of the patrol frequency for all 3 test cases are shown in Fig. 6. For cases 1 and 2, when there is a single peak demand attraction area, the agency would patrol that peak demand region more frequently (such as more than three times within five hours), whereas visiting the other areas less frequently. This is intuitive, as the agency should prefer maintaining a high level of patrol frequency for these popular parking areas to reduce the occurrences of parking violation and maximize the utilization of parking resources. For case 3, the area that can be patrolled at a high level of frequency is much smaller. This is because there are only a limited number of patrol vehicles, such that the agency has to split the fleet and reduce the

[3] In Figs. 4–8, darker color in an area indicates more frequent patrols in that neighborhood, and vice versa.

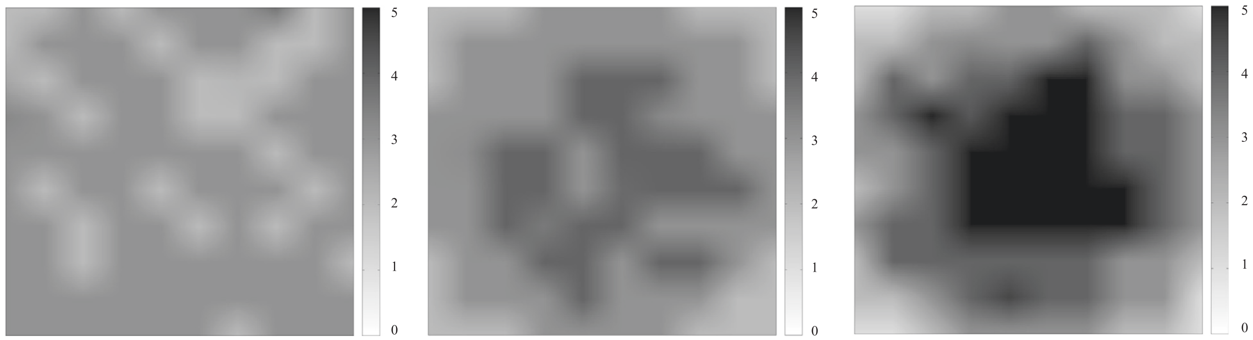
Table 3
CA results for homogeneous and heterogeneous cases.

Homogeneous parking demand density					Heterogeneous parking demand density					Heterogeneous fine-to-price ratio				
δ	ψ	K	CPU time (s)	Expected violation cost ($\times 10^3$)	ψ	η	K	CPU time (s)	Expected violation cost ($\times 10^3$)	γ/β	ζ	K	CPU time (s)	Expected violation cost ($\times 10^3$)
10	150	10	160.5	1,556.7	150	0.0	10	212.7	861.6	30	0.0	10	143	2,852.4
10	150	20	167	861.5	150	0.0	20	255.5	143.3	30	0.0	20	131.9	1,203.9
10	150	30	107.5	279.8	150	0.0	30	128	75.0	30	0.0	30	64.5	281.1
10	300	10	166.7	3,677.1	150	0.5	10	141.1	738.8	30	0.5	10	155.3	2,305.1
10	300	20	157.1	2,881.1	150	0.5	20	112.3	123.5	30	0.5	20	153.9	980.7
10	300	30	165.3	2,085.6	150	0.5	30	90.9	70.8	30	0.5	30	106.7	232.5
10	600	10	114.5	8,078.1	150	1.0	10	92.1	569.4	30	1.0	10	148.5	1,742.9
10	600	20	167.5	7,149.8	150	1.0	20	100	99.8	30	1.0	20	138.1	689.6
10	600	30	154.4	6,254.1	150	1.0	30	204.3	66.6	30	1.0	30	100.9	181.7
20	150	10	203.9	1,749.8	300	0.0	10	171.9	2,852.4	15	0.0	10	193.2	3,677.1
20	150	20	122.5	1,099.2	300	0.0	20	153.5	1,203.9	15	0.0	20	178.3	2,881.1
20	150	30	47.8	448.5	300	0.0	30	70.9	281.1	15	0.0	30	190.3	2,085.5
20	300	10	305	3,727.4	300	0.5	10	106.5	2,561.6	15	0.5	10	160	3,272.1
20	300	20	137.9	2,937.3	300	0.5	20	107.4	975.6	15	0.5	20	298.1	2,375.4
20	300	30	93.8	2,190.9	300	0.5	30	117.2	233.1	15	0.5	30	139.9	1,578.3
20	600	10	142.4	8,115.5	300	1.0	10	212.4	2,226.5	15	1.0	10	173.7	2,594.7
20	600	20	267.9	7,225.4	300	1.0	20	256.7	821.1	15	1.0	20	241.3	1,728.3
20	600	30	45.2	6,336.3	300	1.0	30	174.1	183.3	15	1.0	30	147.5	1,135.5



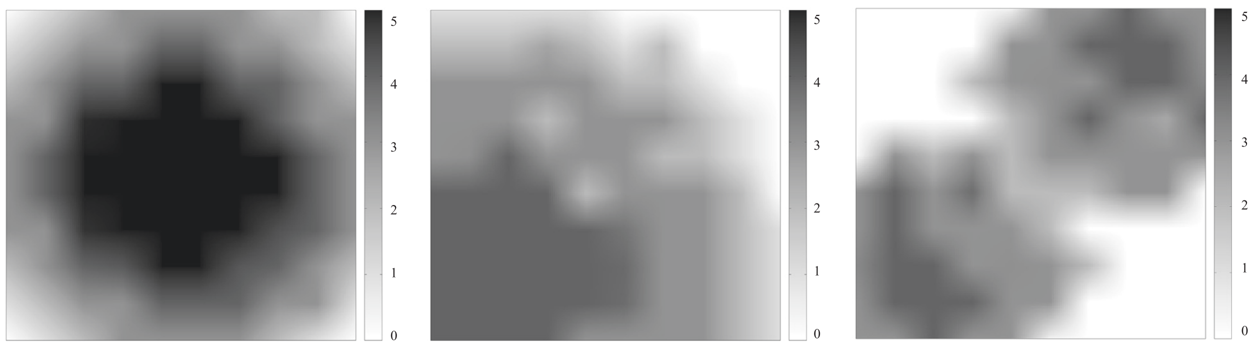
(a) $\eta = 0.0$ (b) $\eta = 0.5$ (c) $\eta = 1.0$

Fig. 4. Patrol frequency distribution under different values of η ($\psi = 300, K = 20$).



(a) $\zeta = 0.0$ (b) $\zeta = 0.5$ (c) $\zeta = 1.0$

Fig. 5. Patrol frequency distribution under different values of ζ ($\gamma/\beta = 15, K = 20$).



(a) Case 1 (b) Case 2 (c) Case 3

Fig. 6. Patrol frequency distribution under different spatial distributions of parking demand.

patrol frequency for both areas. In practice, if parking demand information within the patrol region is known, the above patrol frequency map can be generated easily from the CA approach, which enables the agency to assess the performance of the parking enforcement patrol system in a rather intuitive way.

4.3. Impact of agency objective

In the following, we run tests to compare the impact of different agency objectives. We generate eleven test cases, each with a different combination of weights ω_1, ω_2 and ω_3. Since meter payments and violation fines contribute to the revenues in a similar way, here we simply let $\omega_2 = \omega_3$. In these test cases, we assume that the parking durations of all vehicles in a cell follow a same normal distribution (truncated at 0) with mean $\overline{\tau} = 1$ and variance $\sigma_\tau^2 = 0.25$. The parking demand

Table 4
CA results for different combinations of objectives.

ω_1	ω_2	ω_3	Total objective function value	Expected violation cost	Expected payment revenue	Expected fine revenue
0.0	1.0	1.0	−356,595	186,675	223,806	132,789
0.1	0.9	0.9	−302,821	181,999	223,901	132,789
0.2	0.8	0.8	−248,643	183,693	223,939	132,789
0.3	0.7	0.7	−195,119	181,160	223,593	132,789
0.4	0.6	0.6	−138,581	186,858	223,561	131,981
0.5	0.5	0.5	−87,866	180,852	223,795	132,789
0.6	0.4	0.4	−32,788	182,806	223,389	132,789
0.7	0.3	0.3	20,080	181,544	223,841	132,829
0.8	0.2	0.2	74,457	182,153	223,538	132,789
0.9	0.1	0.1	128,022	181,889	223,987	132,789
1.0	0.0	0.0	181,594	181,594	223,694	132,789

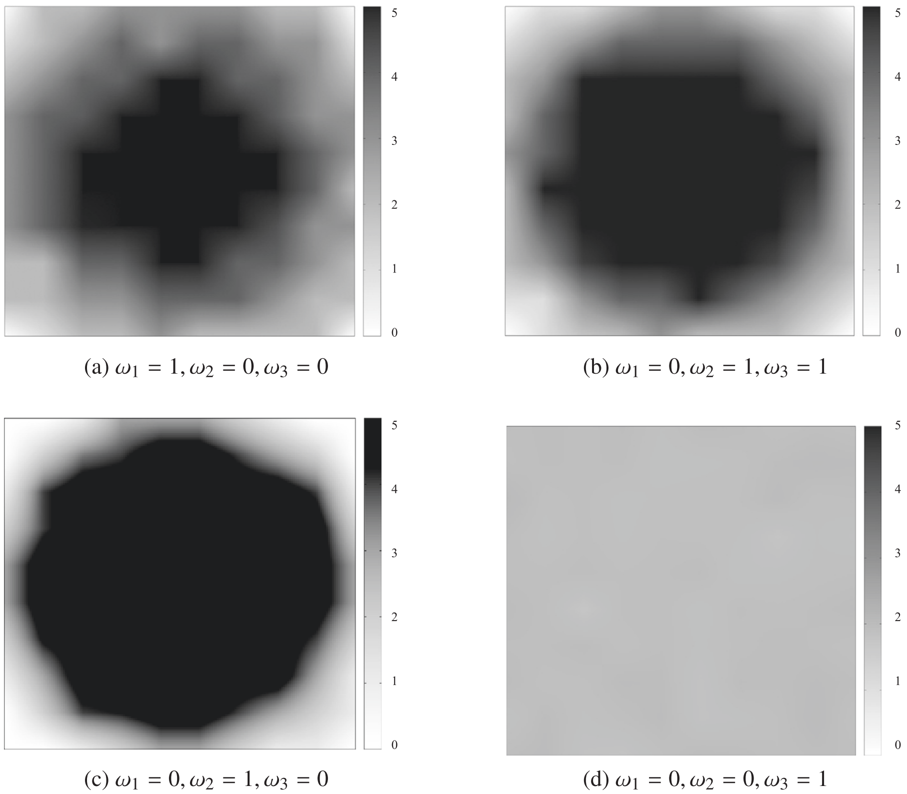


Fig. 7. Patrol frequency distribution under different weights of objective functions.

density is set to be heterogeneous with $\psi = 150\,/\text{km}^2$ and $\eta = 1$, and other parameters are $\delta = 10\,/\text{km}^2$, $\frac{\gamma}{\beta} = 15$, $\zeta = 0$, $K = 30$, and $P_\alpha = 0.9$. Table 4 shows the cost associated with expected total violations, the expected total meter payment, the expected total fine revenue, as well as the summation.

Interestingly, for different combinations of weights, there is no significant variation in each of the objective components. This implies that the agency's motivation to increase the revenue could actually be quite aligned with the goal of decreasing the total expected violations.

However, it should be noted that the patrol patterns under different objectives tend to be very different. As shown in Fig. 7(a), if the agency's objective is only to minimize the expected total violations (i.e., $\omega_1 = 1, \omega_2 = \omega_3 = 0$), the agency would strive to maintain a relatively high patrol frequency for as large an area as possible, and meanwhile patrol the high demand area more frequently. However, when the agency focuses on improve the expected total revenue (i.e., $\omega_1 = 0, \omega_2 = \omega_3 = 1$), a larger area will be patrolled at the highest frequency, as shown in Fig. 7(b). Meanwhile, the area that is patrolled at a rather low frequency would also increase, as compared with the case in Fig. 7(a).

To further gain managerial insights, we apply the CA approach to two more test cases: for one case, we maximize the expected total meter payment revenue only (i.e., $\omega_1 = 0, \omega_2 = 1, \omega_3 = 0$), while for the other case, we maximize the expected total fine revenue only (i.e., $\omega_1 = 0, \omega_2 = 0, \omega_3 = 1$). The resulted patrol frequency distributions are shown in Fig. 7(c)

Table 5
Comparison between multiple-ticket and single-ticket policy.

Ticket policy	σ_τ	P^{vio}						Expected violation cost ($\times 10^3$)
		$R=0$	$R=1$	$R=2$	$R=3$	$R=4$	$R=5$	
Multiple	–	1.0	0.3333	0.1667	0.1111	0.0833	0.0667	181.3
Single	0.5	1.0	0.3333	0.1667	0.1111	0.0834	0.0669	182.4
Single	1.0	1.0	0.3333	0.1669	0.1131	0.0879	0.0738	201.2
Single	1.5	1.0	0.3334	0.1707	0.1219	0.0999	0.0876	241.2

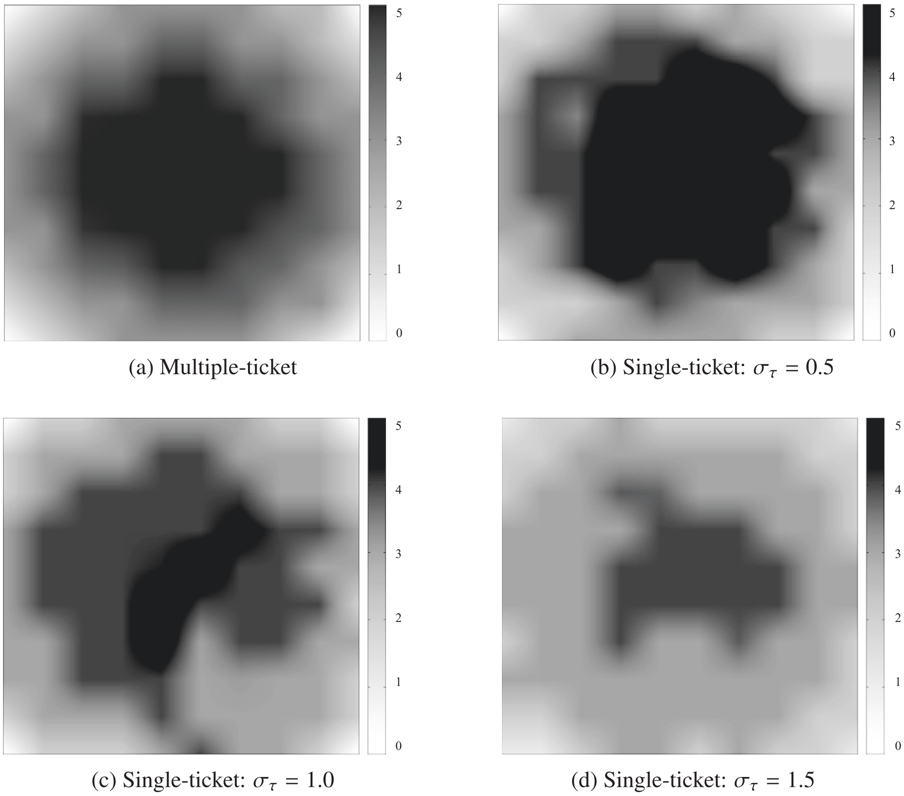


Fig. 8. Patrol frequency distribution under multiple-ticket and single-ticket policies.

and (d), respectively. Since the optimal paid parking duration is the inverse of the cumulative distribution function of the actual parking duration (see (3)), the parking payment would be highly sensitive to the patrol frequency. In other words, the parking payment may drastically decrease if the agency reduces the patrol frequency. Hence, in the aim of maximizing the payment revenue, the agency would rather ignore certain low demand area and try to maintain the highest level of patrol frequency for most of the high demand areas, as shown in Fig. 7(c). On the other hand, the fine revenue is much less sensitive to the patrol frequency. As shown in Fig. 7(d), in this case the agency would simply impose a uniform patrol frequency over the entire region. As mentioned in Section 2.2, maximizing the expected total fine revenue is equivalent to minimizing the expected parking demand that is never patrolled during the planning horizon. Thus, as long as the patrol frequency is larger than zero, the expected total fine revenue that agency obtains is unlikely to be affected by the patrol frequency.

4.4. Multiple-ticket vs. single-ticket policy

In the following, we run tests to compare the performance of different ticketing policies. Following the problem setting in the previous section, we assume that the parking durations of all vehicles in a cell follow a same normal distribution (truncated at 0) with mean $\overline{\tau}$ and variance σ_τ^2. Here we generate three different scenarios by fixing $\overline{\tau}$ while varying the value of σ_τ, i.e., $\mu_\tau = 2, \sigma_\tau \in \{0.5, 1.0, 1.5\}$. The parking demand density follows (38) with $\psi = 150\,/\text{km}^2$ and $\eta = 1$. Let $\delta = 10\,/\text{km}^2$, $\frac{\gamma}{\beta} = 15$, $\zeta = 0$, $K = 30$, $\omega_1 = 1$, $\omega_2 = \omega_3 = 0$, and $P_\alpha = 0.9$. Table 5 presents the results of P^{vio} values for different patrol frequencies and the objective function values. Also, Fig. 8 shows the distribution of patrol frequency for four different scenarios.

Under the same patrol frequency, a parking vehicle is likely to have a larger value of P^{vio} if the agency adopts the single-ticket policy rather than the multiple-ticket one. Moreover, the difference of P^{vio} between two policies grows even larger under higher patrol frequency. It is because that, no matter how large the patrol frequency is, the expected penalty cost for being in violation never exceeds γ for the single-ticket case. As such, drivers may incline to pay relatively less money for parking, which in turn leads to a larger probability of violation.

We can also observe that, by following the single-ticket policy, the system suffers from a much larger number of expected parking violations if there is a higher level of variability in drivers' parking durations. The reason is that the drivers would be less motivated to pay more money under the single-ticket policy as they know that the penalty cost would not exceed γ no matter how long they are in a violation state. This situation gets even worse when the variability of parking duration grows larger. Meanwhile, under the single-ticket policy, knowing that the effect of frequent patrols is quite limited, the agency would become reluctant to maintain a high level of patrol frequency. As such, as shown in Fig. 8(b)–(d), the agency tends to spread the patrol vehicles out to the entire region rather than focusing limited patrol resources to the high demand areas (as shown in Fig. 8(a)). More violations are expected to occur since a larger portion of parking areas are under a lower frequency of patrol. To this end, the multiple-ticket policy should be a better option if the agency aims at imposing a vigorous enforcement over parking.

5. Conclusion

This paper proposes a game-theoretic mathematical model to optimize the performance of parking enforcement patrol while taking into consideration the interactive relationships among the parking enforcement agency and parking drivers. The agency's patrol planning problem is modeled as a variant of PVRP-SC, while the driver's parking payment problem is modeled as a variant of the news-vendor problem. By solving the driver's parking payment model in closed form, these two interrelated problems can be modeled as a single-level mathematical program. We build up a discrete optimization model for the problem and develop a Lagrangian relaxation based algorithm as the solution approach. This model can only handle small size instances. Then, for the sake of large-scale instances, we propose a CA formulation and further reduce it to a non-linear optimization program that can be effectively solved using existing solvers. The numerical results show that both approaches can yield reasonable solutions for small problems, but the CA approach can produce the solution within a much shorter computation time. In addition, the CA approach is an effective way to avoid the prohibitive computational difficulty associated with solving large-scale instances. It has also been shown that the agency should adopt the multiple-ticket policy to implement a more effective enforcement to the parking system.

In the future, a number of research directions can be explored. By relaxing the restriction that the parking demand is stationary within the time horizon, we intend to examine the problem in which parking demand varies over time. We are also interested in considering more complex behaviors of parking drivers. For instance, we can relax the assumption that each driver is designated to a specific parking lot, and allow drivers to decide their own parking location (as per Ayala et al., 2011; Mackowski et al., 2015; He et al., 2015), as well as the amount of money to pay simultaneously. In addition, it would be interesting to investigate the gaming behavior between the two sides when the drivers only have inaccurate knowledge of patrol frequency.

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