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Research Article

Generalized Matrix Exponential Solutions to the AKNS Hierarchy

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Generalized matrix exponential solutions to the AKNS equation are obtained by the inverse scattering transformation (IST). The resulting solutions involve six matrices, which satisfy the coupled Sylvester equations. Several kinds of explicit solutions including soliton, complexiton, and Matveev solutions are deduced from the generalized matrix exponential solutions by choosing different kinds of the six involved matrices. Generalized matrix exponential solutions to a general integrable equation of the AKNS hierarchy are also derived. It is shown that the general equation and its matrix exponential solutions share the same linear structure.

1. Introduction

Many nonlinear models are studied and shown to possess hierarchies, and recursion operators play a crucial role in constructing hierarchies of soliton equations [1]. Associated with the variational derivative, recursion operators have been developed to formulate the Hamiltonian structures proving the integrability of soliton hierarchies [2, 3]. Recursion operators also have a tight correlation with one-soliton solutions. For example, the Korteweg-de Vries (KdV) hierarchy

$$u_t = T^n u_x, \quad n = 0, 1, 2, \ldots, \tag{1}$$

possesses the following one-soliton solution:

$$u = \frac{k^2}{2}\operatorname{sech}^2\frac{k\left(x + k^{2n}t + \delta\right)}{2}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $T = \partial^2 + 4u + 2u_x\partial^{-1}$ is the recursion operator of the KdV hierarchy, δ is a constant, and $\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$. Notice that the dispersion relation of (2)

$$kx + k^{2n+1}t \tag{3}$$

is linked closely with the order of the recursion operator T.

Since the discovery of scattering behavior of solitons [4] and the IST [5], solitons have received much attention. The IST has been also well developed and widely used to solve nonlinear equations [1, 6]. It can be used to solve not only normal soliton equations but also unusual soliton equations such as equations with self-consistent sources [7], nonisospectral equations [8, 9], and equations with steplike finite-gap backgrounds [10] and on quasi-periodic backgrounds [11]. Recently, Ablowitz and Musslimani developed the IST for the integrable nonlocal nonlinear Schrödinger (NLS) equation [12].

The Sylvester equation

$$AM - MB = C \tag{4}$$

is one of the most well-known matrix equations. It appears frequently in many areas of applied mathematics and plays a central role, in particular, in systems and control theory, signal processing, filtering, model reduction, image restoration, and so on. In recent years, it has been used to solve soliton equations [13, 14] successfully. The method based on the

Sylvester equation is also known as Cauchy matrix approach [14–18], which is actually a by-product of direct linearization approach first proposed by Fokas and Ablowitz in 1981 [15] and developed to discrete integrable systems by Nijhoff et al. in early of 1980s [16].

In the process of solving soliton equations, it can yield fantastic results by using matrices properly. Successful examples are the Wronskian technique and the IST. Ma et al. had introduced the matrix element in Wronskian determinants when they used the Wronskian technique to solve soliton equations [19–23]. They obtained various kinds of solutions such as soliton, rational, Matveev, and complexiton solutions. In 2006, Aktosun and Van Der Mee proposed a modified inverse scattering transformation (MIST) [24]. They expressed the scattering data of spectral problems by three matrices, A, B, C, and proved that the matrices A, B, C satisfy the Sylvester equation. The advantage of the method is that it can get more kinds of solutions and its process is simpler than the traditional IST [24–27].

In this paper, we would like to consider the AKNS hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -q \\ r \end{pmatrix} \quad (n = 0, 1, 2, \ldots), \tag{5}$$

with the help of MIST, where

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Many integrable systems can be reduced from it such as the modified KdV, sine-Gordon, nonlinear Schrödinger, and nonlocal nonlinear Schrödinger system. There have had already a lot of researches on AKNS hierarchy for it is a representative integrable hierarchy [21, 22, 28]. In this paper, we will construct the soliton, complexiton, and Matveev solutions to the first nonlinear equation of (5). We will show that the linear relation (3) exists not only in one-soliton solutions but also in multisoliton, complexiton, and Matveev solutions.

The paper is organized as follows: In Section 2, we will review the recovered potentials of the AKNS spectral problem by IST. In Section 3, we will obtain the coupled Sylvester equations and generalized matrix exponential solutions to the AKNS equation. In Section 4, some different types of explicit solutions will be constructed. In Section 5, generalized matrix exponential solutions will be given to the AKNS hierarchy. The relationship between the recursion operator and the solutions of the AKNS hierarchy will be discussed also. We conclude the paper in Section 6.

2. Preparation

To make the paper self-contained, we first briefly recall the Lax integrability of the isospectral AKNS hierarchy and its Gel'fand-Levitan-Marchenko (GLM) equations.

It is well known that the AKNS hierarchy has the following Lax pairs [29]:

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \tag{7a}$$

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{7b}$$

where $\phi = (\phi_1, \phi_2)^T$, k is a spectral parameter, and $q = q(t, x)$, $r = r(t, x)$ are potential functions. We assume that $q(x, t)$ and $r(x, t)$ are smooth functions of variables t and x, and their derivatives of any order with respect to x vanish rapidly as $x \to \infty$. The compatibility condition, zero curvature equation

$$M_t - N_x + [M, N] = 0, \tag{8a}$$

with boundary conditions

$$N|_{(q,r)=(0,0)} = \begin{pmatrix} -\frac{1}{2}(2ik)^n & 0 \\ 0 & \frac{1}{2}(2ik)^n \end{pmatrix}, \tag{8b}$$

can yield the isospectral AKNS hierarchy (5). The first two nonlinear equations in the AKNS hierarchy are

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \end{pmatrix}, \tag{9a}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}. \tag{9b}$$

Next we mainly follow the notions and results given in [1, 6].

Lemma 1. *If the read potentials $(q(x, t), r(x, t))^T$ satisfy*

$$\int_{-\infty}^{\infty} \left|x^j q(x)\right| dx < +\infty,$$
$$\int_{-\infty}^{\infty} \left|x^j r(x)\right| dx < +\infty, \tag{10}$$
$$(j = 0, 1),$$

the spectral problem (7a) has a group of Jost solutions $\phi(x, k)$, $\overline{\phi}(x, k)$, $\psi(x, k)$, and $\overline{\psi}(x, k)$ which are bounded for all values of x and also enjoy the following asymptotic behaviors:

$$\phi(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx},$$
$$\overline{\phi}(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx},$$
$$(x \longrightarrow \infty),$$

$$\psi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx},$$
$$\overline{\psi}(x,k) \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{ikx}, \quad (x \longrightarrow -\infty). \tag{11}$$

In the usual manner, we define the scattering coefficient by

$$\psi(x,k) = a(k)\,\overline{\phi}(x,k) + b(k)\,\phi(x,k), \tag{12a}$$
$$\overline{\psi}(x,k) = -\overline{a}(k)\,\phi(x,k) + \overline{b}(k)\,\overline{\phi}(x,k), \tag{12b}$$

where

$$\overline{a}(k)\,a(k) + b(k)\,\overline{b}(k) = 1. \tag{13}$$

Furthermore, one can give $\phi(x,k)$, $\overline{\phi}(x,k)$ by the integral representation

$$\phi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} + \int_x^\infty K(x,y)\,e^{iky}dy, \tag{14a}$$
$$\overline{\phi}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} + \int_x^\infty \overline{K}(x,y)\,e^{-iky}dy, \tag{14b}$$

where $K(x,y) = (K_1(x,y), K_2(x,y))^T$ and $\overline{K}(x,y) = (\overline{K}_1(x,y), \overline{K}_2(x,y))^T$ are column vectors.

In order for (14a) and (14b) to be valid, it is necessary that

$$q(x) = -2K_1(x,x),$$
$$r(x) = -2\overline{K}_2(x,x). \tag{15}$$

Definition 2. If k_j and $\overline{k}_j$ are single roots of $a(k)$ and $\overline{a}(k)$, respectively, there exist c_j and $\overline{c}_j$ such that

$$2\int_{-\infty}^{\infty} c_j^2 \phi_1(x,k_j)\,\phi_2(x,k_j) = 1,$$
$$2\int_{-\infty}^{\infty} \overline{c}_j^2 \overline{\phi}_1(x,\overline{k}_j)\,\overline{\phi}_2(x,\overline{k}_j) = 1. \tag{16}$$

c_j and $\overline{c}_j$ are named the normalization constants for the eigenfunctions $\phi(x,k_j)$ and $\overline{\phi}(x,\overline{k}_j)$. Accordingly, $c_j\phi(x,k_j)$ and $\overline{c}_j\overline{\phi}(x,\overline{k}_j)$ are named the normalization eigenfunctions.

Definition 3. One named the set

$$\left\{ k\ (\mathrm{Im}\,k = 0),\ R(k) = \frac{b(k)}{a(k)},\ \overline{R}(k) = \frac{\overline{b}(k)}{\overline{a}(k)};\right.$$
$$k_j\ (\mathrm{Im}\,k_j > 0),\ \overline{k}_m\ (\mathrm{Im}\,\overline{k}_m < 0),\ c_j,\ \overline{c}_m,\ j = 1,2, \tag{17}$$
$$\left.\ldots, l,\ m = 1,2,\ldots,\overline{l}\right\}$$

to be the scattering data for the spectral problem (7a).

Lemma 4. *Given the scattering data for the spectral problem (7a) and*

$$F(x) = F_c(x) + F_d(x),$$
$$\overline{F}(x) = \overline{F}_c(x) + \overline{F}_d(x), \tag{18a}$$

where

$$F_c(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)\,e^{ikx}dk,$$
$$F_d(x) = \sum_{j=1}^{l} c_j^2 e^{ik_j x}, \tag{18b}$$
$$\overline{F}_c(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{R}(k)\,e^{-ikx}dk,$$
$$\overline{F}_d(x) = -\sum_{j=1}^{l} \overline{c}_j^2 e^{i\overline{k}_j x}, \tag{18c}$$

one has the GLM equations for the AKNS hierarchy

$$\overline{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^\infty K(x,z)\,F(z+y)\,dz = 0, \tag{19a}$$
$$K(x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{F}(x+y) - \int_x^\infty \overline{K}(x,z)\,\overline{F}(z+y)\,dz = 0. \tag{19b}$$

3. Generalized Matrix Exponential Solutions to the AKNS Equation

In this section, we will get generalized matrix exponential solutions to the AKNS equation (9a) via MIST.

From the previous section, we know that the potentials of spectral problem (7a) can be recovered by (15). For convenience, we denote $K_1(x,y)$ and $\overline{K}_2(x,y)$ by $K(x,y)$ and $\overline{K}(x,y)$, respectively, in this and the following sections.

Lemma 5. *The GLM equations (19a) and (19b) of the AKNS hierarchy have other forms:*

$$K(x,y) = \overline{F}(x+y) - \int_x^\infty \int_x^\infty ds\,dz K(x,s)\,F(s+z)\,\overline{F}(z+y),$$

$$\begin{aligned}&\overline{K}(x,y)\\&\quad=-F(x+y)\\&\qquad-\int_x^\infty\int_x^\infty ds\,dz\overline{K}(x,s)\,\overline{F}(s+z)\,F(z+y).\end{aligned}\tag{20}$$

Proof. Rewrite (19a) and (19b) in their component forms as

$$\overline{K}_1(x,y)+\int_x^\infty K_1(x,z)\,F(z+y)\,dz=0,\tag{21a}$$

$$\begin{aligned}&\overline{K}_2(x,y)+F(x+y)+\int_x^\infty K_2(x,z)\,F(z+y)\,dz\\&\quad=0,\end{aligned}\tag{21b}$$

$$\begin{aligned}&K_1(x,y)-\overline{F}_1(x+y)-\int_x^\infty \overline{K}_1(x,z)\,\overline{F}(z+y)\,dz\\&\quad=0,\end{aligned}\tag{21c}$$

$$K_2(x,y)-\int_x^\infty \overline{K}_2(x,z)\,\overline{F}(z+y)\,dz=0.\tag{21d}$$

Substituting (21a) and (21d) into (21c) and (21b) and replacing $K_1(x,y), \overline{K}_2(x,y)$ with $K(x,y), \overline{K}(x,y)$, respectively, we have

$$\begin{aligned}&K(x,y)\\&\quad=\overline{F}(x+y)\\&\qquad-\int_x^\infty\int_x^\infty ds\,dzK(x,s)\,F(s+z)\,\overline{F}(z+y),\\&\overline{K}(x,y)\\&\quad=-F(x+y)\\&\qquad-\int_x^\infty\int_x^\infty ds\,dz\overline{K}(x,s)\,\overline{F}(s+z)\,F(z+y).\end{aligned}\tag{22}$$

□

Lemma 6. *Let A and $\overline{A}$ be $l\times l$ constant matrices, B and $\overline{B}$ be $l\times 1$ constant column vectors, and C and $\overline{C}$ be $1\times l$ constant row vectors. Assume that A and $\overline{A}$ satisfy $\lim_{x\to\infty}e^{Ax}=0$ and $\lim_{x\to\infty}e^{-\overline{A}x}=0$, respectively, where 0 is the $l\times l$ matrix with all elements being zero. Then upon taking*

$$\begin{aligned}M&=\int_0^\infty e^{Az}B\overline{C}e^{-\overline{A}z}dz,\\\overline{M}&=\int_0^\infty e^{-\overline{A}z}\overline{B}Ce^{Az}dz,\end{aligned}\tag{23}$$

one has the following coupled Sylvester relations:

$$\begin{aligned}M\overline{A}-AM&=B\overline{C},\\\overline{A}\,\overline{M}-\overline{M}A&=\overline{B}C.\end{aligned}\tag{24}$$

Proof. We only prove the first relation, and the second relation can be proved similarly:

$$\begin{aligned}M\overline{A}-AM&=\int_0^\infty\left(e^{Az}B\overline{C}e^{-\overline{A}z}\overline{A}-Ae^{Az}B\overline{C}e^{-\overline{A}z}\right)dz\\&=-e^{Az}\,B\overline{C}e^{-\overline{A}z}\Big|_0^\infty=B\overline{C}.\end{aligned}\tag{25}$$

□

For convenience, we set

$$\begin{aligned}\Omega(x)&=e^{-\overline{A}x}\overline{M}e^{Ax},\\\overline{\Omega}(x)&=e^{Ax}Me^{-\overline{A}x},\\\Xi(t)&=e^{Tt},\\\overline{\Xi}(t)&=e^{-\overline{T}t},\end{aligned}\tag{26}$$

where T, $\overline{T}$ are $l\times l$ matrices, M, $\overline{M}$, A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$ are matrices defined in Lemma 6, and they enjoy the relations (24).

Setting

$$\begin{aligned}\Gamma&=I+\Omega(x)\,\Xi(t)\,\overline{\Omega}(x)\,\overline{\Xi}(t),\\\overline{\Gamma}&=I+\overline{\Omega}(x)\,\overline{\Xi}(t)\,\Omega(x)\,\Xi(t),\end{aligned}\tag{27}$$

if Γ and $\overline{\Gamma}$ are nondegenerate matrices, we have the following theorem.

Theorem 7. *If $AT=TA$ and $\overline{A}\,\overline{T}=\overline{T}\,\overline{A}$, the potentials of the AKNS spectral problem (7a) can be recovered as*

$$\begin{aligned}q(x,t)&=-2\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\,\Gamma^{-1}e^{-\overline{A}x}\overline{B},\\r(x,t)&=2Ce^{Ax}\Xi(t)\,\overline{\Gamma}^{-1}e^{Ax}B.\end{aligned}\tag{28}$$

Proof. Let $F_c(x)$ and $\overline{F}_c(x)$ be zero and $F_d(x)$ and $\overline{F}_d(x)$ be matrix exponential forms in (18a); that is,

$$\begin{aligned}F(x)&=F_d(x)=Ce^{Ax}B,\\\overline{F}(x)&=\overline{F}_d(x)=\overline{C}e^{-\overline{A}x}\overline{B}.\end{aligned}\tag{29}$$

Suppose that the time evolution of $F(x)$ and that of $\overline{F}(x)$ are

$$\begin{aligned}F(x,t)&=Ce^{Ax}\Xi(t)\,B,\\\overline{F}(x,t)&=\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\,\overline{B}.\end{aligned}\tag{30}$$

We may take

$$\begin{aligned}K(x,y)&=H(x,t)\,e^{-\overline{A}y}\overline{B},\\\overline{K}(x,y)&=\overline{H}(x,t)\,e^{Ay}B,\end{aligned}\tag{31}$$

accordingly, where $H(x,t)$, $\overline{H}(x,t)$ are $1\times l$ row vectors.

Substituting (30), (31) into the first equation of (20), we get

$$H(x,t) = \overline{C}e^{-\overline{A}x}\overline{\Xi}(t) - \int_x^\infty \int_x^\infty ds\,dz H(x,t) \cdot e^{-\overline{A}s}\overline{B}Ce^{A(s+z)}\Xi(t)B\overline{C}e^{-\overline{A}z}\overline{\Xi}(t). \tag{32}$$

That is,

$$H(x,t)\left(I + \int_x^\infty e^{-\overline{A}s}\overline{B}Ce^{As}ds\Xi(t)\int_x^\infty e^{Az}B\overline{C}e^{-\overline{A}z}dz\overline{\Xi}(t)\right) = \overline{C}e^{-\overline{A}x}\overline{\Xi}(t). \tag{33}$$

In the light of

$$\int_x^\infty e^{As}B\overline{C}e^{-\overline{A}s}ds = e^{Ax}Me^{-\overline{A}x}, \qquad \int_x^\infty e^{-\overline{A}s}\overline{B}Ce^{As}ds = e^{-\overline{A}x}\overline{M}e^{Ax}, \tag{34}$$

we have

$$H(x,t) = \overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}. \tag{35a}$$

Similarly, we can arrive at

$$\overline{H}(x,t) = -Ce^{Ax}\Xi(t)\overline{\Gamma}^{-1}. \tag{35b}$$

Finally, we find that the potentials of the AKNS spectral problem (7a) can be recovered as

$$q(x) = -2K(x,x) = -2\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}e^{-\overline{A}x}\overline{B}, \qquad r(x) = -2\overline{K}(x,x) = 2Ce^{Ax}\Xi(t)\overline{\Gamma}^{-1}e^{Ax}B, \tag{36}$$

by taking advantage of (15). □

Next we will get solutions to the AKNS equation (9a) from the recovered potentials (28).

Proposition 8. *The matrices $\overline{\Omega}(x)\overline{\Xi}(t)$ and $\Omega(x)\Xi(t)$ satisfy*

$$\Omega(x)\Xi(t)\overline{\Gamma}^{-1} = \Gamma^{-1}\Omega(x)\Xi(t), \tag{37a}$$

$$\overline{\Gamma}^{-1}\overline{\Omega}(x)\overline{\Xi}(t) = \overline{\Omega}(x)\overline{\Xi}(t)\Gamma^{-1}, \tag{37b}$$

respectively. Thus, the two matrices $\overline{\Gamma}^{-1}$ and Γ^{-1} are similar.

Although Proposition 8 is simple, we will use it many times in the following part of this paper.

Theorem 9. *If $T = (2A)^2$ and $\overline{T} = (2\overline{A})^2$ in $\Xi(t)$ and $\overline{\Xi}(t)$, respectively, the recovered the potentials (28) are solutions to the AKNS equation (9a).*

Proof. We only prove the first equation of (9a), and the second equation can be proved similarly.

On the one hand

$$\begin{aligned} q_t &= 8\overline{C}e^{-4\overline{A}^2t-\overline{A}x}\Gamma^{-1}\left[\Gamma\overline{A}^2 + \Omega(x)\Xi(t)A^2\overline{\Omega}(x)\overline{\Xi}(t) - \Omega(x)\Xi(t)\overline{\Omega}(x)\overline{\Xi}(t)\overline{A}^2\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B} \\ &= 8\overline{C}e^{-4\overline{A}^2t-\overline{A}x}\Gamma^{-1}\left[\overline{A}^2 + \Omega(x)\Xi(t)A^2\overline{\Omega}(x)\overline{\Xi}(t)\right] \cdot \Gamma^{-1}e^{-\overline{A}x}\overline{B}, \end{aligned} \tag{38}$$

and on the other hand

$$\begin{aligned} -q_{xx} + 2q^2r &= 2\overline{C}e^{-4\overline{A}^2t-\overline{A}x}\Gamma^{-1}\left[\left(\Gamma\overline{A}^2 + \overline{A}^2\Gamma\right) + 2\left(\Gamma_x + \Gamma\overline{A}\right)\Gamma^{-1}\left(\Gamma_x + \overline{A}\Gamma\right) - \Gamma_{xx}\right. \\ &\quad \left. + 8e^{-\overline{A}x}\overline{B}Ce^{4A^2t+Ax}\overline{\Gamma}^{-1}e^{Ax}B\overline{C}e^{-4\overline{A}^2t-\overline{A}x}\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B}. \end{aligned} \tag{39}$$

Through proper simplification, we have

$$-q_{xx} + 2q^2r = 8\overline{C}e^{-4\overline{A}^2t-\overline{A}x}\Gamma^{-1}\left[\overline{A}^2 + \Omega(x)\Xi(t)A^2\overline{\Omega}(x)\overline{\Xi}(t)\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B}. \tag{40}$$

□

4. Exact Solutions to the AKNS Equation

In this section, we will construct different kinds of explicit solutions to the AKNS equation (9a) by taking different kinds of A, B, C and $\overline{A}$, $\overline{B}$, $\overline{C}$.

(i) One-Soliton Solutions. Taking $A = k_1$, $B = b_1$, $C = c_1$ and $\overline{A} = k_2$, $\overline{B} = b_2$, $\overline{C} = c_2$, we get

$$q = \frac{-2(k_2-k_1)^2 b_2c_2e^{-4k_2^2t-2k_2x}}{(k_2-k_1)^2 + b_1b_2c_1c_2e^{4(k_1^2-k_2^2)t+2(k_1-k_2)x}}, \qquad r = \frac{2(k_2-k_1)^2 b_1c_1e^{4k_1^2t+2k_1x}}{(k_2-k_1)^2 + b_1b_2c_1c_2e^{4(k_1^2-k_2^2)t+2(k_1-k_2)x}}, \tag{41}$$

where $k_j, b_j, c_j \in \mathbb{C}$ $(j = 1, 2)$ and $\mathrm{Re}\,k_1 < 0$, $\mathrm{Re}\,k_2 > 0$.

(ii) Two-Soliton Solutions. Taking

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad C = \overline{C} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \tag{42}$$

we have

$$q = -\frac{16e^{2(-8t+x)}\left(9e^{12t} + 50e^{2x} + 1800e^{12t+8x} + 3600e^{32t+10x}\right)}{1 + 1800e^{8x} + 576e^{6(2t+x)} + 6400e^{10(2t+x)} + 1800e^{8(4t+x)}\left(1 + 8e^{8x}\right)},$$
$$r = \frac{16e^{4(4t+x)}\left[9 + 50e^{20t+2x}\left(1 + 72e^{8x} + 36e^{6(2t+x)}\right)\right]}{1 + 1800e^{8x} + 576e^{6(2t+x)} + 6400e^{10(2t+x)} + 1800e^{8(4t+x)}\left(1 + 8e^{8x}\right)}. \tag{43}$$

(iii) Three-Soliton Solutions. Taking

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$
$$\overline{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$
$$\overline{B} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix},$$
$$\overline{C} = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}, \tag{44}$$

we have

$$q = \frac{-72e^{-4(4t+x)}\left(e^{12t} - 4e^{2x} - 144e^{6x} + 36e^{12t+8x}\right)}{720\cosh 12t - 648\cosh 2x - 1297\cosh 6x - 432\sinh 12t - 1295\sinh 6x},$$
$$r = \frac{72e^{4t-4x}\left(1 + 36e^{8x} - e^{2(6t+x)} - 36e^{12t+6x}\right)}{720\cosh 12t - 648\cosh 2x - 1297\cosh 6x - 432\sinh 12t - 1295\sinh 6x}. \tag{45}$$

(iv) Matveev Solutions. Taking

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\overline{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
$$\overline{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
$$C = \overline{C} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \tag{46}$$

we have

$$q = \frac{e^{-2(2t+x)}\left[22 + 32t + 8x + e^{2(6t+x)}(-6 + 32t + 4x)\right]}{4 + e^{4(6t+x)} + e^{2(6t+x)}\left[3 - 256t^2 - 96t(1+x) - 2x(5+4x)\right]},$$
$$r = \frac{2e^{4(4t+x)}\left[4 + 64t + 8x + e^{2(6t+x)}(3 + 16t + 4x)\right]}{4 + e^{4(6t+x)} + e^{2(6t+x)}\left[3 - 256t^2 - 96t(1+x) - 2x(5+4x)\right]}. \tag{47}$$

(v) Complexiton Solutions. Taking

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$
$$\overline{A} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$
$$\overline{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$
$$C = \overline{C} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \tag{48}$$

we have

$$q = \frac{32\cos(8t+2x) - 16\sin(8t+2x) - 4e^{2x}\left[2\cos(32t+4x) + \sin(32t+4x)\right]}{4e^{2x} + e^{6x} - e^{4x}\left[5\sin(24t+2x) + \sin(40t+6x)\right]},$$
$$r = \frac{4e^{4x}\left[2\cos(32t+4x) - \sin(32t+4x)\right] + 4e^{6x}\left[2\cos(8t+2x) + \sin(8t+2x)\right]}{4 + e^{4x} - e^{2x}\left[5\sin(24t+2x) + \sin(40t+6x)\right]}. \tag{49}$$

5. The Same Linear Relation That a General Equation and Its Solutions Share

In this section, we will construct a new general equation related to the AKNS hierarchy and get its generalized matrix exponential solutions. Furthermore, we find that the recursion operator and the solutions of the AKNS hierarchy have the same linear relation.

We consider a new general equation:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \theta(L)\begin{pmatrix} -q \\ r \end{pmatrix}, \tag{50}$$

where

$$\theta(L) = a_0 L^n + a_1 L^{n-1} + \cdots + a_{n-1}L + a_n \tag{51}$$

is a polynomial operator on L.

Let

$$\begin{pmatrix} q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}\left[\overline{A}^n + \Omega(x)\Xi(t)A^n\overline{\Omega}(x)\overline{\Xi}(t)\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B} \\ 2^{n+1}Ce^{Ax}\Xi(t)\overline{\Gamma}^{-1}\left[A^n + \overline{\Omega}(x)\overline{\Xi}(t)\overline{A}^n\Omega(x)\Xi(t)\right]\overline{\Gamma}^{-1}e^{Ax}B \end{pmatrix},$$
$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 2^n Ce^{Ax}\Xi(t)\overline{\Gamma}^{-1}e^{Ax}\left(A^{n-1}M - M\overline{A}^{n-1}\right)\overline{\Xi}(t)e^{-\overline{A}x}\Gamma^{-1}e^{-\overline{A}x}\overline{B} \\ 2^n\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}e^{-\overline{A}x}\left(\overline{M}A^{n-1} - \overline{A}^{n-1}\overline{M}\right)\Xi(t)e^{Ax}\overline{\Gamma}^{-1}e^{Ax}B \end{pmatrix}; \tag{52}$$

we have the following Lemma.

Lemma 10. *If $AT = TA$ and $\overline{A}\,\overline{T} = \overline{T}\,\overline{A}$, one obtains*

$$q_n + \partial q_{n-1} = 2q(u_n + v_n),$$
$$r_n - \partial r_{n-1} = -2r(u_n + v_n). \tag{53}$$

Proof. We only prove the first equation. For

$$\begin{aligned}
\partial q_{n-1} &= \partial\left\{2^n\overline{C}e^{-\overline{A}x}\left[\overline{A}^{n-1}\overline{\Xi}(t)\right.\right. \\
&\quad + \overline{\Xi}(t)\Gamma^{-1}\Omega(x)\Xi(t)A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t) \\
&\quad \left.\left. - \overline{\Xi}(t)\Gamma^{-1}\Omega(x)\Xi(t)\overline{\Omega}(x)\overline{\Xi}(t)\overline{A}^{n-1}\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B}\right\} \\
&= 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}\left[\overline{A}^n - \overline{A}\Gamma^{-1}\overline{A}^{n-1}\right. \\
&\quad - \overline{A}^{n-1}\Gamma^{-1}\overline{A} - \Omega(x)\Xi(t)A\overline{\Omega}(x)\overline{\Xi}(t)\Gamma^{-1}\overline{A}^{n-1} \\
&\quad - \overline{A}^{n-1}\Gamma^{-1}\Omega(x)\Xi(t)A\overline{\Omega}(x)\overline{\Xi}(t) - \Omega(x)\Xi(t) \\
&\quad \cdot A\overline{\Omega}(x)\overline{\Xi}(t)\Gamma^{-1}\Omega(x)\Xi(t)A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t) \\
&\quad \left. - \overline{A}\Gamma^{-1}\Omega(x)\Xi(t)A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t) - \Omega(x)\Xi(t)\right. \\
&\quad \cdot A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t)\Gamma^{-1}\Omega(x)\Xi(t)A\overline{\Omega}(x)\overline{\Xi}(t) - \Omega(x) \\
&\quad \cdot \Xi(t)A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t)\Gamma^{-1}\overline{A} + \Omega(x)\Xi(t)A^n\overline{\Omega}(x) \\
&\quad \left.\cdot \overline{\Xi}(t)\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B},
\end{aligned} \tag{54}$$

we have

$$\begin{aligned}
q_n + \partial q_{n-1} &= 2^{n+1}\overline{C}e^{-(2\overline{A})^n t - \overline{A}x}\Gamma^{-1}\left\{\left[\overline{A}\Omega(x)\Xi(t)\right.\right. \\
&\quad \left. - \Omega(x)\Xi(t)A\right]\overline{\Gamma}^{-1}\overline{\Omega}(x)\overline{\Xi}(t)\overline{A}^{n-1} \\
&\quad + \left[\overline{A}^{n-1}\Omega(x)\Xi(t) - \Omega(x)\Xi(t)A^{n-1}\right] \\
&\quad \cdot \overline{\Gamma}^{-1}\left[\overline{\Omega}(x)\overline{\Xi}(t)\overline{A} - A\overline{\Omega}(x)\overline{\Xi}(t)\right] \\
&\quad \left. - \left[\overline{A}\Omega(x)\Xi(t) - \Omega x\Xi(t)A\right]\overline{\Gamma}^{-1}A^{n-1}\overline{\Omega}(x)\overline{\Xi}(t)\right\} \\
&\quad \cdot \Gamma^{-1}e^{-\overline{A}x}\overline{B} = 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\Gamma^{-1}e^{-\overline{A}x}\left[\left(\overline{A}\,\overline{M}\right.\right. \\
&\quad \left. - \overline{M}A\right)e^{Ax}\Xi(t)\overline{\Gamma}^{-1}e^{Ax}\left(M\overline{A}^{n-1} - A^{n-1}M\right)\overline{\Xi}(t) \\
&\quad + \left(\overline{A}^{n-1}\overline{M} - \overline{M}A^{n-1}\right)e^{Ax}\Xi(t)\overline{\Gamma}^{-1}e^{Ax}\left(M\overline{A}\right.
\end{aligned}$$

$$
\begin{aligned}
&- AM\big)\,\overline{\Xi}(t)\Big]\, e^{-\overline{A}x}\Gamma^{-1}e^{-\overline{A}x}\overline{B} = 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\\
&\cdot\Gamma^{-1}e^{-\overline{A}x}\left[\overline{B}Ce^{Ax}\Xi(t)\,\overline{\Gamma}^{-1}e^{Ax}\left(M\overline{A}^{n-1}-A^{n-1}M\right)\right.\\
&\cdot\overline{\Xi}(t)+\left(\overline{A}^{n-1}\overline{M}-\overline{M}A^{n-1}\right)\Xi(t)\\
&\left.\cdot\, e^{Ax}\overline{\Gamma}^{-1}e^{Ax}B\overline{C}\,\overline{\Xi}(t)\right]e^{-\overline{A}x}\Gamma^{-1}e^{-\overline{A}x}\overline{B}.
\end{aligned}
\tag{55}
$$

From (28), we know that the conclusion is right. □

Theorem 11. *If $AT = TA$ and $\overline{A}\,\overline{T} = \overline{T}\,\overline{A}$, one has*

$$
\begin{pmatrix} q_n(x,t)\\ r_n(x,t)\end{pmatrix} = L^n\begin{pmatrix} -q(x,t)\\ r(x,t)\end{pmatrix}. \tag{56}
$$

Proof. We use the mathematical induction to prove the theorem.

When $n = 1$, we have

$$
\begin{aligned}
&\begin{pmatrix} q_1\\ r_1\end{pmatrix} = \begin{pmatrix} q_x\\ r_x\end{pmatrix}\\
&= \begin{pmatrix} 4\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\,\Gamma^{-1}\left[\overline{A}+\Omega(x)\,\Xi(t)\,A\overline{\Omega}(x)\,\overline{\Xi}(t)\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B}\\ 4Ce^{Ax}\Xi(t)\,\overline{\Gamma}^{-1}\left[A+\overline{\Omega}(x)\,\overline{\Xi}(t)\,\overline{A}\Omega(x)\,\Xi(t)\right]\overline{\Gamma}^{-1}e^{Ax}B\end{pmatrix},
\end{aligned}
\tag{57}
$$

and (56) is established.

Supposing that

$$
\begin{pmatrix} q_{n-1}\\ r_{n-1}\end{pmatrix} = L^{n-1}\begin{pmatrix} -q(x,t)\\ r(x,t)\end{pmatrix}, \tag{58}
$$

we obtain

$$
\begin{pmatrix} q_n\\ r_n\end{pmatrix} = L\begin{pmatrix} q_{n-1}\\ r_{n-1}\end{pmatrix}; \tag{59}
$$

that is,

$$
\begin{pmatrix} q_n\\ r_n\end{pmatrix} = \begin{pmatrix} -\partial q_{n-1}+2q\partial^{-1}\left(rq_{n-1}+qr_{n-1}\right)\\ \partial r_{n-1}-2r\partial^{-1}\left(rq_{n-1}+qr_{n-1}\right)\end{pmatrix}. \tag{60}
$$

Through tedious calculation, we arrive at

$$
\frac{\partial\left(u_n+v_n\right)}{\partial x} = qr_{n-1}+rq_{n-1}. \tag{61}
$$

By Lemma 10, we know that the first equality in (60) is right. Its second equality can be proved analogously. □

Corollary 12. *If $T = \theta(2A)$ and $\overline{T} = \theta(2\overline{A})$ in $\Xi(t)$ and $\overline{\Xi}(t)$, respectively, the recovered potentials (28) solve (50), where*

$$
\theta(A) = a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI \tag{62}
$$

is a polynomial matrix on A and I is an unit matrix.

Proof. When $\theta(2\overline{A}) = (2\overline{A})^n$, we have

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= 2^{n+1}\overline{C}e^{-(2\overline{A})^nt-\overline{A}x}\overline{A}^n\Gamma^{-1}e^{-\overline{A}x}\overline{B}\\
&\quad + 2\overline{C}e^{-(2\overline{A})^nt-\overline{A}x}\Gamma^{-1}\Gamma_t\Gamma^{-1}e^{-\overline{A}x}\overline{B},\\
&= 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\,\Gamma^{-1}\left[\Gamma\overline{A}^n\right.\\
&\quad + \Omega(x)\,\Xi(t)\,A^n\overline{\Omega}(x)\,\overline{\Xi}(t)\\
&\quad \left.- \Omega(x)\,\Xi(t)\,\overline{\Omega}(x)\,\overline{\Xi}(t)\,\overline{A}^n\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B},\\
&= 2^{n+1}\overline{C}e^{-\overline{A}x}\overline{\Xi}(t)\,\Gamma^{-1}\left[\overline{A}^n\right.\\
&\quad \left.+ \Omega(x)\,\Xi(t)\,A^n\overline{\Omega}(x)\,\overline{\Xi}(t)\right]\Gamma^{-1}e^{-\overline{A}x}\overline{B}.
\end{aligned}
\tag{63}
$$

That is, $\partial q/\partial t = q_t = q_n$. In the same way, we also can obtain $r_t = r_n$ when $\theta(2A) = (2A)^n$. Thus we have

$$
\begin{aligned}
\begin{pmatrix} q\\ r\end{pmatrix}_t &= a_0\begin{pmatrix} q_n\\ r_n\end{pmatrix} + a_1\begin{pmatrix} q_{n-1}\\ r_{n-1}\end{pmatrix} + \cdots + a_n\begin{pmatrix} -q\\ r\end{pmatrix}\\
&= \theta(L)\begin{pmatrix} -q\\ r\end{pmatrix}.
\end{aligned}
\tag{64}
$$

□

The corollary means that the operator and the solutions of the AKNS hierarchy enjoy the same linear structure.

6. Conclusions

To sum up, we have solved the AKNS hierarchy by the MIST. The MIST can determine solutions more directly and generate more diverse solutions than the traditional IST. Actually, such solutions can be also obtained by applying the Wronskian technique [19–23]. The arbitrariness of the matrices involved leads to the diversity of exact solutions.

A general integrable equation of the AKNS hierarchy was constructed and its matrix exponential solutions were obtained. The ways to generate the equation and its matrix exponential solutions are the same. They share the same linear algebraic structure, not only in the case of one-soliton solutions but also in the case of other interesting solutions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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