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Nonlinear integrable couplings of a generalized super Ablowitz-Kaup-Newell-Segur hierarchy and its super bi-Hamiltonian structures

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In this paper, a new generalized 5×5 matrix spectral problem of Ablowitz-Kaup-Newell-Segur type associated with the enlarged matrix Lie superalgebra is proposed, and its corresponding super soliton hierarchy is established. The super variational identities are used to furnish super Hamiltonian structures for the resulting super soliton hierarchy.

KEYWORDS
Lie superalgebras, generalized super AKNS hirearchy, super integrable couplings, super bi-Hamiltonian structures

1 | INTRODUCTION

It is known that super integrable systems provide interesting and important models in the supersymmetry theory. Supersymmetry originated in the 1970s when physicists have proposed simple models with supersymmetric colors in string models and mathematical physics. After that, Wess and Zumino[1] applied supersymmetry to the four-dimensional space-time. Unfortunately, the supersymmetry partners of any particle have not been found so far, and it is generally believed that this symmetry is spontaneous rupture. To unify two kinds of particles with different spin and statistical properties—Boson and Fermion—theoretical physicists proposed the concept of hyperspace in the study of unified field theory and quantum field theory. Inspired by this, mathematicians developed the super analysis, hypergeometric, and superalgebra.

Because of the importance of supersymmetry in physics (especially in the exploration of the relationship between supersymmetric conformal field and chord theory), which have aroused growing attentions by many mathematicians and physicists to study of super integrable systems associated with Lie superalgebra, many classical soliton equations have been extended to be the super completely integrable system. For example, the super Ablowitz-Kaup-Newell-Segur (AKNS) hirearchy,[2-10] super Dirac hierarchy,[4,11-13] super Kaup-Newell (KN) hierarchy,[14-16] and others.[17-26] Among those, Hu[27] and Ma[4] have made a great contribution. Hu[27] proposed the super-trace identity at the first time, which is an effective tool to constructing super Hamiltonian structures of super integrable systems. Ma gave the proof of the super-trace identity in 2008, and a lot of the super Hamiltonian structures of the super integrable systems are established by using of the super-trace identity.[4] Recently, the study of integrable couplings of the well-known integrable hierarchy associated with enlarging matrix Lie superalgebra has also aroused growing attentions by many mathematicians and physicists.[28-34] Meanwhile, the research of super integrable couplings generalizes the classical integrable couplings theory and provides clues toward complete classification of super integrable systems.[35-41]

In literature,[30] You considered that an enlarged super AKNS matrix spectral problem is given by

$$\phi_x = M\phi, M = \begin{pmatrix} \lambda & p & 0 & r & \alpha \\ q & -\lambda & s & 0 & \beta \\ 0 & 0 & \lambda & p+r & 0 \\ 0 & 0 & q+s & -\lambda & 0 \\ \beta & -\alpha & -\beta & \alpha & 0 \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix}, u = \begin{pmatrix} p \\ q \\ \alpha \\ \beta \\ r \\ s \end{pmatrix}, \tag{1}$$

where λ is the spectral parameter; p, q, r, and s are even potentials, and α and β are odd ones. Take $\alpha = \beta = 0$, the hierarchy (1) reduces to a nonlinear integrable couplings of the classical AKNS hierarchy,[35] whose super Hamiltonian structure is furnished by super-trace identity. Recently, Shen et al.[42] considered a generalized spatial spectral problem of AKNS integrable coupling as follows:

$$\phi_x = U\phi, U = \begin{pmatrix} \lambda+\omega & p & 0 & r \\ q & -\lambda-\omega & s & 0 \\ 0 & 0 & \lambda+\omega & p \\ 0 & 0 & q & -\lambda-\omega \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, u = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}, \tag{2}$$

where $\omega = \epsilon(ps + qr)$; λ is the spectral parameter; and p, q, r, and s are commuting variables. Obviously, when $\epsilon = 0$, this generalized spatial spectral problem (2) is reduced to a new case of AKNS integrable couplings,[43] whose bi-Hamiltonian structures were constructed by using the component-trace identity in.[42] Inspired by those spatial spectral problems, in this paper, we would like to construct nonlinear super integrable couplings of a generalized super AKNS hirearchy.

The rest of this paper is organized as follows. In Section 2, we will review the Lie superalgebra sl(2,1) enlarged to the Lie superalgebra sl (4,1). In Section 3, we will construct a generalization of the super AKNS integrable coupling hierarchy from zero curvature equations, based on the above-mentioned generalized spatial spectral problem (1). In Section 4, the super bi-Hamiltonian form will be presented for the obtained super integrable couplings of the generalized super AKNS hierarchy by making use of the super-trace identity. For the sake of convenience, we will use the mathematical software Maple to deal with some complicated symbolic computations. And Section 5 is devoted to conclusions and discussions.

2 | ENLARGEMENT OF A LIE SUPERALGEBRA

In this section, we recalled the Lie superalgebra sl(2,1) enlarged to the Lie superalgebra sl(4,1).[28-34] Consider that the Lie superalgebra sl(2,1) is given by

$$E_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},E_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},E_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$
$$E_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},E_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix},$$

where E_1, E_2, E_3 are even elements and E_4, E_5 are odd ones; and $[.,.]$ and $[.,.]_+$ denote the commutator and the anticommutator, which satisfy the following operational relations:

$$\begin{aligned}&[E_1,E_2]=2E_2,[E_1,E_3]=-2E_3,[E_2,E_3]=E_1,\\&[E_1,E_4]=[E_2,E_5]=E_4,[E_1,E_5]=[E_4,E_3]=-E_5,[E_2,E_4]=[E_3,E_5]=0,\\&[E_4,E_4]_+=-2E_2,[E_5,E_5]_+=2E_3,[E_4,E_5]_+=[E_5,E_4]_+=E_1.\end{aligned}\tag{3}$$

Then the corresponding loop superalgebra is defined by

$$sl(2,1)=sl(2,1)\otimes\mathbb{C}[\lambda,\lambda^{-1}].\tag{4}$$

Let us enlarge the Lie superalgebra sl(2,1) to the Lie superalgebra sl(4,1) with a basis

$$e_1=\begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_2=\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},e_3=\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},$$
$$e_4=\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{pmatrix},e_5=\begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}e_6=\begin{pmatrix}0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}$$
$$e_7=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&-1&0&1&0\end{pmatrix}e_8=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&-1&0&0\end{pmatrix}.$$

Similarly, here, $e_1, e_2, e_3, e_4, e_5, e_6$ are even elements and e_7, e_8 are odd ones; $[.,.]$ and $[.,.]_+$ denote the commutator and the anticommutator, which satisfy the following operational relations:

$$\begin{aligned}&[e_1,e_2]=2e_2,[e_1,e_3]=-2e_3,[e_1,e_5]=-[e_2,e_4]=[e_4,e_5]=2e_5,[e_2,e_3]=e_1,\\&[e_1,e_6]=-[e_3,e_4]=[e_4,e_6]=-2e_6,[e_1,e_7]=[e_2,e_8]=e_7,[e_1,e_8]=-[e_3,e_7]=-e_8,\\&[e_2,e_6]=-[e_3,e_5]=[e_5,e_6]=e_4,[e_1,e_4]=[e_2,e_5]=[e_2,e_7]=[e_3,e_6]=[e_3,e_8]=0,\\&[e_4,e_7]=[e_4,e_8]=[e_5,e_7]=[e_5,e_8]=[e_6,e_7]=[e_6,e_8]=0\\&[e_7,e_8]_+=e_1-e_4,[e_7,e_7]_+=2e_5-2e_2,[e_8,e_8]_+=2e_3-2e_6.\end{aligned}\tag{5}$$

Define a loop superalgebra corresponding to the Lie superalgebra sl(4,1) and denote by

$$\mathrm{sl}(4,1)=\mathrm{sl}(4,1)\otimes\mathbb{C}[\lambda,\lambda^{-1}]=e_i\lambda^m,e_i\in\mathrm{sl}(4,1),i=1,2,\ \dots\ ,8;m=0,\pm1,\pm2,\ \dots\tag{6}$$

The corresponding commutative and anticommutative relations are defined as

$$[e_i\lambda^m,e_j\lambda^n]=[e_i,e_j]\lambda^{m+n},\forall e_i,e_j\in\mathrm{sl}(4,1).\tag{7}$$

3 | NONLINEAR GENERALIZED SUPER INTEGRABLE COUPLINGS OF THE SUPER AKNS HIERARCHY

In this section, we shall construct nonlinear integrable couplings of a generalized super AKNS hierarchy from an enlarging matrix Lie superalgebra. Consider the following spatial spectral problem

$$\phi_x = M\phi, M = \begin{pmatrix} \lambda+h & p & 0 & r & \alpha \\ q & -\lambda-h & s & 0 & \beta \\ 0 & 0 & \lambda+h & p+r & 0 \\ 0 & 0 & q+s & -\lambda-h & 0 \\ \beta & -\alpha & -\beta & \alpha & 0 \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix}, u = \begin{pmatrix} p \\ q \\ \alpha \\ \beta \\ r \\ s \end{pmatrix}, \tag{8}$$

where $h = \mu(ps + qr + rs - 2\alpha\beta)$ with μ being an arbitrary even constant; λ is the spectral parameter; p, q, r, and s are even potentials; and α and β are odd potentials. Obviously, the spatial spectral problem (8) with $\mu = 0$ reduces to the standard nonlinear integrable couplings of super AKNS hierarchy case.[30]

To derive super integrable couplings of a generalized super integrable hierarchy associated with the spatial spectral problem (8), we must solve the stationary zero curvature equation

$$N_x = [M, N], \tag{9}$$

where

$$N = \begin{pmatrix} A & B & E & F & \rho \\ C & -A & G & -E & \delta \\ 0 & 0 & A+E & B+F & 0 \\ 0 & 0 & C+G & -A-E & 0 \\ \delta & -\rho & -\delta & \rho & 0 \end{pmatrix}, \tag{10}$$

in which the corresponding A, B, C, E, F, G are even elements and ρ, δ are odd elements.

Substituting M in (8) and N in (10) into Equation 9 yields

$$\begin{cases} A_x = pC - qB - \alpha\delta + \beta\rho, \\ B_x = 2\lambda B - 2pA - 2\alpha\rho + 2hB, \\ C_x = -2\lambda C + 2qA + 2\beta\delta - 2hC, \\ E_x = (p+r)G + rC - sB - (q+s)F - \alpha\delta - \beta\rho, \\ F_x = 2\lambda F - 2(p+r)E - 2rA + 2\alpha\rho + 2hF, \\ G_x = -2\lambda G + 2(q+s)E + 2sA - 2\beta\delta - 2hG, \\ \rho_x = \lambda\rho + p\delta - \alpha A - \beta B + h\rho, \\ \delta_x = -\lambda\delta + \beta A - \alpha C + q\rho - h\delta. \end{cases} \tag{11}$$

Choosing

$$\begin{aligned} A &= \sum_{m\geq 0}^{n} a_m\lambda^{-m}, B = \sum_{m\geq 0}^{n} b_m\lambda^{-m}, C = \sum_{m\geq 0}^{n} c_m\lambda^{-m}, E = \sum_{m\geq 0}^{n} e_m\lambda^{-m}, \\ F &= \sum_{m\geq 0}^{n} f_m\lambda^{-m}, G = \sum_{m\geq 0}^{n} g_m\lambda^{-m}, \rho = \sum_{m\geq 0}^{n} \rho_m\lambda^{-m}, \delta = \sum_{m\geq 0}^{n} \delta_m\lambda^{-m} \end{aligned} \tag{12}$$

and comparing the coefficients of the same powers of λ in Equation 11, we have

$$\begin{cases} a_{m,x} = pc_m - qb_m + \alpha\delta_m + \beta\rho_m, \\ e_{m,x} = rc_m - sb_m + (p+r)g_m - (q+s)f_m - \alpha\delta_m - \beta\rho_m, \\ b_{m+1} = \frac{1}{2}b_{m,x} + pa_m + \alpha\rho_m - hb_m, \\ c_{m+1} = -\frac{1}{2}c_{m,x} + qa_m + \beta\delta_m - hc_m, \\ f_{m+1} = \frac{1}{2}f_{m,x} + ra_m + (p+r)e_m - \alpha\rho_m - hf_m, \\ g_{m+1} = -\frac{1}{2}g_{m,x} + sa_m + (q+s)e_m - \beta\delta_m - hg_m, \\ \rho_{m+1} = \rho_{m,x} + \alpha a_m + \beta b_m - p\delta_m - h\rho_m, \\ \delta_{m+1} = -\delta_{m,x} + \beta a_m - \alpha c_m + q\rho_m - h\delta_m, \end{cases} \tag{13}$$

which results in the recurrence relations

$$\begin{cases} (c_{m+1}, b_{m+1}, \delta_{m+1}, \rho_{m+1}, g_{m+1}, f_{m+1})^T = L(c_m, b_m, \delta_m, \rho_m, g_m, f_m)^T, \\ a_m = \partial^{-1}(rc_m - qb_m + \alpha\delta_m + \beta\rho_m), \\ e_m = \partial^{-1}(rc_m - sb_m - \alpha\delta_m - \beta\rho_m + (p+r)g_m - (q+s)f_m), \end{cases} \tag{14}$$

where the recursion operator L has the following form

$$L = \begin{pmatrix} L_1 & L_2 & 0 \\ L_3 & L_4 & 0 \\ L_5 & -L_2 & L_6 \end{pmatrix}, \tag{15}$$

with

$$\begin{aligned}
L_1 &= \begin{pmatrix} q\partial^{-1}p - \frac{1}{2}\partial - h & -q\partial^{-1}q \\ p\partial^{-1}p & -p\partial^{-1}q + \frac{1}{2}\partial - h \end{pmatrix}, L_2 = \begin{pmatrix} q\partial^{-1}\alpha + \beta & q\partial^{-1}\beta \\ p\partial^{-1}\alpha & p\partial^{-1}\beta + \alpha \end{pmatrix}, \\
L_3 &= \begin{pmatrix} \beta\partial^{-1}p - \alpha & -\beta\partial^{-1}q \\ \alpha\partial^{-1}p & -\alpha\partial^{-1}q + \beta \end{pmatrix}, L_4 = \begin{pmatrix} \beta\partial^{-1}\alpha - \partial - h & \beta\partial^{-1}\beta + q \\ \alpha\partial^{-1}\alpha - p & \alpha\partial^{-1}\beta + \partial - h \end{pmatrix}, \\
L_5 &= \begin{pmatrix} s\partial^{-1}p + (q+s)\partial^{-1}r & -s\partial^{-1}q - (q+s)\partial^{-1}s \\ r\partial^{-1}p + (p+r)\partial^{-1}r & -r\partial^{-1}q - (p+r)\partial^{-1}s \end{pmatrix}, \\
L_6 &= \begin{pmatrix} (q+s)\partial^{-1}(p+r) - \frac{1}{2}\partial - h & -(q+s)\partial^{-1}(q+s) \\ (p+r)\partial^{-1}(p+r) & -(p+r)\partial^{-1}(q+s) + \frac{1}{2}\partial - h \end{pmatrix}.
\end{aligned}$$

Upon choosing the initial conditions $a_0 = e_0 = 1, b_0 = c_0 = e_0 = f_0 = g_0 = \rho_0 = \delta_0 = 0$, all other $a_j, b_j, c_j, \rho_j, \delta_j (j \geq 1)$ can be worked out uniquely by the recurrence relations (14) and by using of symbolic computation software (Maple). We list the first three sets as follows:

$$\begin{aligned}
& a_1 = e_1 = 0, b_1 = p, c_1 = q, f_1 = p + r, g_1 = q + s, \rho_1 = \alpha, \delta_1 = \beta, \\
& a_2 = -\frac{1}{2}(pq + 2\alpha\beta), b_2 = \frac{1}{2}p_x - hp, c_2 = -\frac{1}{2}q_x - hq, \\
& e_2 = -\left(ps + qr + rs + \frac{1}{2}pq - \alpha\beta\right), f_2 = \frac{1}{2}p_x + \frac{1}{2}r_x - hp - hr, \\
& g_2 = -\frac{1}{2}q_x - \frac{1}{2}s_x - hq - hs, \rho_2 = \alpha_x - h\alpha, \delta_2 = -\beta_x - h\beta, \\
& a_3 = \frac{1}{4}(pq_x - p_x q) + \alpha\beta_x - \alpha_x\beta + h(pq + 2\alpha\beta), \\
& b_3 = \frac{1}{4}p_{xx} - \frac{1}{2}h_x p - hp_x - \frac{1}{2}(pq + 2\alpha\beta)p + \alpha\alpha_x + h^2 p, \\
& c_3 = \frac{1}{4}q_{xx} + \frac{1}{2}h_x q + hq_x - \frac{1}{2}(pq + 2\alpha\beta)q - \beta\beta_x + h^2 q, \\
& e_3 = \frac{1}{2}\left(ps_x - p_x s + q_x r - qr_x + rs_x - r_x s + \frac{1}{2}pq_x - \frac{1}{2}p_x q\right) \\
& \qquad - (\alpha\beta_x - \alpha_x\beta) + 2h\left(ps + qr + rs + \frac{1}{2}pq - \alpha\beta\right), \\
& f_3 = \frac{1}{4}p_{xx} + \frac{1}{2}r_{xx} - \frac{1}{2}h_x(p + 2r) - hp_x - 2hr_x - \frac{1}{2}(pq + 2\alpha\beta)r \\
& \qquad - \alpha\alpha_x + h^2(p + 2r) - (p + r)\left(\frac{1}{2}pq + ps + qr + rs - \alpha\beta\right), \\
& g_3 = \frac{1}{4}q_{xx} + \frac{1}{2}s_{xx} + \frac{1}{2}h_x(q + 2s) + hq_x + 2hs_x - \frac{1}{2}(pq + 2\alpha\beta)s \\
& \qquad + \beta\beta_x + h^2(q + 2s) - (q + s)\left(\frac{1}{2}pq + ps + qr + rs - \alpha\beta\right), \\
& \rho_3 = \alpha_{xx} - h_x\alpha - 2h\alpha_x - \frac{1}{2}(pq + 2\alpha\beta)\alpha + h^2\alpha + \frac{1}{2}p_x\beta + p\beta_x, \\
& \delta_3 = \beta_{xx} + h_x\beta + 2h\beta_x - \frac{1}{2}(pq + 2\alpha\beta)\beta + h^2\beta + \frac{1}{2}q_x\alpha + q\alpha_x.
\end{aligned}$$

Let us consider the spectral problem (8) with the following auxiliary spectral problem:

$$\phi_{t_n} = N^{(n)}\phi,$$

where

$$N^{(n)} = N_+^{(n)} + \Delta_n = \sum_{m=0}^{n} \begin{pmatrix} a_m & b_m & e_m & f_m & \rho_m \\ c_m & -a_m & g_m & -e_m & \delta_m \\ 0 & 0 & a_m + e_m & b_m + f_m & 0 \\ 0 & 0 & c_m + g_m & -a_m - e_m & 0 \\ \delta_m & -\rho_m & -\delta_m & \rho_m & 0 \end{pmatrix} \lambda^{n-m} + \Delta_n, \tag{16}$$

with Δ_n being the modification term. We set

$$\Delta_n = \begin{pmatrix} a & b & e & f & k \\ c & -a & g & e & l \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ l & -k & -l & k & 0 \end{pmatrix},$$

and substitute Equations 8 and 16 into the following zero curvature equation

$$M_{t_n} - N_x^{(n)} + [M, N^{(n)}] = 0, \tag{17}$$

where $n \geq 0$. Making use of Equation 11 yields

$$\begin{cases} h_{t_n} = a_x, b = c = e = f = g = k = l = 0, \\ p_{t_n} = b_{n,x} + 2pa_n + 2\alpha\rho_n - 2hb_n + 2pa = 2b_{n+1} + 2pa, \\ q_{t_n} = c_{n,x} - 2qa_n - 2\beta\delta_n + 2hc_n - 2qa = -2c_{n+1} - 2qa, \\ \alpha_{t_n} = \rho_{n,x} + \alpha a_n + \beta b_n - p\delta_n - h\rho_n + \alpha a = \rho_{n+1} + \alpha a, \\ \beta_{t_n} = \delta_{n,x} - \beta a_n + \alpha c_n - q\rho_n + h\delta_n - \beta a = -\delta_{n+1} - \beta a, \\ r_{t_n} = f_{n,x} + 2ra_n - 2\alpha\rho_n - 2(p+r)e_n - 2hf_n + 2ra = 2f_{n+1} + 2ra, \\ s_{t_n} = g_{n,x} - 2sa_n + 2\beta\delta_n - 2(q+s)e_n + 2hg_n - 2sa = -2g_{n+1} - 2sa, \end{cases} \tag{18}$$

which guarantees the following identity:

$$\begin{aligned} (ps + qr + rs - 2\alpha\beta)_{t_n} &= -2(rc_{n+1} - sb_{n+1} + (p+r)g_{n+1} - (q+s)f_{n+1} \\ &\quad - \alpha\delta_{n+1} - \beta\rho_{n+1}) = -2e_{n+1,x}. \end{aligned} \tag{19}$$

Choosing $a = -2\mu e_{n+1}$, we can obtain the following hierarchy:

$$u_{t_n} = \begin{pmatrix} p \\ q \\ \alpha \\ \beta \\ r \\ s \end{pmatrix}_{t_n} = \begin{pmatrix} 2b_{n+1} - 4\mu p e_{n+1} \\ -2c_{n+1} + 4\mu q e_{n+1} \\ \rho_{n+1} - 2\mu\alpha e_{n+1} \\ -\delta_{n+1} + 2\mu\beta e_{n+1} \\ 2f_{n+1} - 4\mu r e_{n+1} \\ -2g_{n+1} + 4\mu s e_{n+1} \end{pmatrix}, n \geq 0. \tag{20}$$

When $n=2$ in Equation 20, we obtain the first nontrivial flow as follows

$$
\left\{
\begin{aligned}
p_{t_2} &= \tfrac{1}{2}p_{xx} - p^2q - 2p\alpha\beta + 2\alpha\alpha_x + \mu(pp_xq - pp_xs - 3prq_x - 3prs_x + pr_xq + pr_xs \\
&\quad -2p_xqr - 2p_xsr - p^2q_x - 3p^2s_x + 4p\alpha\beta_x - 4p\alpha_x\beta + 4p_x\alpha\beta) - 2\mu^2p(2p^2sq + 2pq^2r \\
&\quad +3q^2r^2 + 3s^2r^2 + 3p^2s^2 + 6ps^2r + 6sqr^2 + 8psqr) + 16\mu^2p\alpha\beta(ps + qr + sr + \tfrac{1}{2}pq), \\
q_{t_2} &= -\tfrac{1}{2}q_{xx} + pq^2 + 2q\alpha\beta + 2\beta\beta_x + \mu(pq_xq + pqs_x - 3qsp_x - 3qsr_x + qrq_x + qrs_x \\
&\quad -2psq_x - 2srq_x - q^2p_x - 3q^2r_x - 4q\alpha\beta_x + 4q\alpha_x\beta + 4q_x\alpha\beta) + 2\mu^2q(2p^2sq + 2pq^2r \\
&\quad +3q^2r^2 + 3s^2r^2 + 3p^2s^2 + 6ps^2r + 6sqr^2 + 8psqr) - 16\mu^2q\alpha\beta(ps + qr + sr + \tfrac{1}{2}pq), \\
\alpha_{t_2} &= \alpha_{xx} + \tfrac{1}{2}\alpha q_x + q\alpha_x - \tfrac{1}{2}pq\alpha + \mu(\tfrac{1}{2}\alpha qp_x - \tfrac{1}{2}\alpha pq_x - 2\alpha ps_x - 2\alpha rs_x \\
&\quad -2\alpha rq_x - 2ps\alpha_x - 2qr\alpha_x - 2sr\alpha_x + 2\alpha\alpha_x\beta) - \mu^2\alpha(2p^2sq + 2pq^2r \\
&\quad +3q^2r^2 + 3s^2r^2 + 3p^2s^2 + 6ps^2r + 6sqr^2 + 8psqr), \\
\beta_{t_2} &= -\beta_{xx} - \tfrac{1}{2}p_x\beta - p\beta_x + \tfrac{1}{2}pq\beta + \mu(\tfrac{1}{2}\beta pq_x - \tfrac{1}{2}\beta qp_x - 2\beta sp_x - 2\beta sr_x \\
&\quad -2\beta qr_x - 2ps\beta_x - 2qr\beta_x - 2sr\beta_x + 2\alpha\beta_x\beta) + \mu^2\beta(2p^2sq + 2pq^2r \\
&\quad +3q^2r^2 + 3s^2r^2 + 3p^2s^2 + 6ps^2r + 6sqr^2 + 8psqr), \\
r_{t_2} &= r_{xx} + \tfrac{1}{2}p_{xx} - p^2q + 2p\alpha\beta - 2\alpha\alpha_x - 2r^2s - 2qr^2 - 2p^2s - 4psr - 4pqr + \mu(-3psp_x \\
&\quad -2prq_x - pqr_x - qrp_x - 5prs_x - 5psr_x - 2srp_x - 4qrr_x - 4srr_x - 4r^2q_x \\
&\quad -4r^2s_x - p^2s_x + 4p_x\alpha\beta + 4r\alpha\beta_x - 4r\alpha_x\beta + 4r_x\alpha\beta) + 8\mu^2\alpha\beta(r^2q + r^2s - p^2s) \\
&\quad -2\mu^2(pq^2r^2 - p^3s^2 + 2r^3q^2 + 2r^3s^2 + 3ps^2r^2 + 4r^3qs + 4r^2psq), \\
s_{t_2} &= -s_{xx} - \tfrac{1}{2}q_{xx} + pq^2 - 2q\alpha\beta - 2\beta\beta_x + 2rs^2 + 2rq^2 + 2ps^2 + 4pqs + 4qrs\mu(-3qrq_x \\
&\quad -2srq_x - pqs_x - psq_x - 5qrs_x - 5qsr_x - 2sqp_x - 4pss_x - 4rss_x - 4s^2p_x \\
&\quad -4s^2r_x - q^2r_x + 4q_x\alpha\beta - 4s\alpha\beta_x + 4s\alpha_x\beta + 8s_x\alpha\beta) - 8\mu^2\alpha\beta(r^2s + p^2s - r^2q) \\
&\quad +2\mu^2(qp^2s^2 - q^3r^2 + 2s^3r^2 + 2s^3p^2 + 3qs^2r^2 + 4s^3pr + 4s^2prq),
\end{aligned}
\right.
\tag{21}
$$

whose Lax pair consists of M and $N^{(2)}$. M is defined by (8) and $N^{(2)}$

$$
N^{(2)} = \begin{pmatrix}
N_{11}^{(2)} & N_{12}^{(2)} & N_{13}^{(2)} & N_{14}^{(2)} & N_{15}^{(2)} \\
N_{21}^{(2)} & -N_{11}^{(2)} & N_{23}^{(2)} & -N_{13}^{(2)} & N_{25}^{(2)} \\
0 & 0 & N_{33}^{(2)} & N_{34}^{(2)} & 0 \\
0 & 0 & N_{43}^{(2)} & -N_{33}^{(2)} & 0 \\
N_{25}^{(2)} & -N_{15}^{(2)} & -N_{25}^{(2)} & N_{15}^{(2)} & 0
\end{pmatrix}.
\tag{22}
$$

with

$$
N_{11}^{(2)} = \lambda^2 - \frac{1}{2}(pq + 2\alpha\beta), N_{12}^{(2)} = p\lambda + \frac{1}{2}p_x - hp, N_{13}^{(2)} = \lambda^2 - \left(ps + qr + rs + \frac{1}{2}pq - \alpha\beta\right),
$$

$$
N_{14}^{(2)} = (p + 2r)\lambda + \frac{1}{2}p_x + \frac{1}{2}r_x - hp - hr, N_{15}^{(2)} = \alpha\lambda + \frac{1}{2}\alpha_x - h\alpha, N_{21}^{(2)} = q\lambda - \frac{1}{2}q_x - hq,
$$

$$
N_{23}^{(2)} = (q + 2s)\lambda - \frac{1}{2}q_x - \frac{1}{2}s_x - hq - hs, N_{25}^{(2)} = \beta\lambda - \frac{1}{2}\beta_x - h\beta,
$$

$$
N_{33}^{(2)} = 2\lambda^2 - (ps + qr + rs + pq), N_{34}^{(2)} = 2(p + r)\lambda + p_x + \frac{1}{2}r_x - 2hp - hr,
$$

$$
N_{43}^{(2)} = 2(q + s)\lambda - q_x - \frac{1}{2}s_x - 2hq - hs.
$$

4 | SUPER BI-HAMILTONIAN STRUCTURES

In what follows, we shall find super bi-Hamiltonian structures of the nonlinear super integrable couplings of a generalized super AKNS hirearchy (20). To this end, we shall apply the super variational identities, which were discussed in Ma et al.[36]

$$\frac{\delta}{\delta u}\int Str\left(N\frac{\partial M}{\partial \lambda}\right)dx=\left(\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\right)Str\left(\frac{\partial M}{\partial u}N\right), \tag{23}$$

where *Str* denotes the super trace. It is not difficult to find that

$$\begin{aligned}
&Str\left(N\frac{\partial M}{\partial \lambda}\right)=4A+2E, Str\left(\frac{\partial M}{\partial p}N\right)=2C+G+2\mu s(2A+E),\\
&Str\left(\frac{\partial M}{\partial q}N\right)=2B+F+2\mu r(2A+E), Str\left(\frac{\partial M}{\partial \alpha}N\right)=2\delta+4\mu\beta(2A+E),\\
&Str\left(\frac{\partial M}{\partial \beta}N\right)=-2\rho-4\mu\alpha(2A+E), Str\left(\frac{\partial M}{\partial r}N\right)=C+G+2\mu(q+s)(2A+E),\\
&Str\left(\frac{\partial M}{\partial s}N\right)=B+F+2\mu(p+r)(2A+E).
\end{aligned} \tag{24}$$

Substituting Equation 24 into 23 and comparing the coefficient of λ^{-n-2} of both sides of Equation 23 yield

$$\begin{pmatrix}\frac{\delta}{\delta p}\\ \frac{\delta}{\delta q}\\ \frac{\delta}{\delta \alpha}\\ \frac{\delta}{\delta \beta}\\ \frac{\delta}{\delta r}\\ \frac{\delta}{\delta s}\end{pmatrix}\int(4a_{n+2}+2e_{n+2})dx=(\gamma-n-1)\begin{pmatrix}2c_{n+1}+g_{n+1}+2\mu s(2a_{n+1}+e_{n+1})\\ 2b_{n+1}+f_{n+1}+2\mu r(2a_{n+1}+e_{n+1})\\ 2\delta_{n+1}+4\mu\beta(2a_{n+1}+e_{n+1})\\ -2\rho_{n+1}-4\mu\alpha(2a_{n+1}+e_{n+1})\\ c_{n+1}+g_{n+1}+2\mu(q+s)(2a_{n+1}+e_{n+1})\\ b_{n+1}+f_{n+1}+2\mu(p+r)(2a_{n+1}+e_{n+1})\end{pmatrix}. \tag{25}$$

The identity with $n=0$ tells $\gamma=0$. Thus, we have

$$\begin{pmatrix}2c_{n+1}+g_{n+1}+2\mu s(2a_{n+1}+e_{n+1})\\ 2b_{n+1}+f_{n+1}+2\mu r(2a_{n+1}+e_{n+1})\\ 2\delta_{n+1}+4\mu\beta(2a_{n+1}+e_{n+1})\\ -2\rho_{n+1}-4\mu\alpha(2a_{n+1}+e_{n+1})\\ c_{n+1}+g_{n+1}+2\mu(q+s)(2a_{n+1}+e_{n+1})\\ b_{n+1}+f_{n+1}+2\mu(p+r)(2a_{n+1}+e_{n+1})\end{pmatrix}=\frac{\delta\tilde{H}_{n+1}}{\delta u}, \tag{26}$$

where $\tilde{H}_{n+1}=-2\int\frac{2a_{n+2}+e_{n+2}}{n+1}dx$. Moreover, a direct calculation yields to the following recursive relationship

$$\begin{pmatrix}c_{n+1}\\ b_{n+1}\\ \delta_{n+1}\\ \rho_{n+1}\\ g_{n+1}\\ f_{n+1}\end{pmatrix}=R\begin{pmatrix}2c_{n+1}+g_{n+1}+2\mu s(2a_{n+1}+e_{n+1})\\ 2b_{n+1}+f_{n+1}+2\mu r(2a_{n+1}+e_{n+1})\\ 2\delta_{n+1}+4\mu\beta(2a_{n+1}+e_{n+1})\\ -2\rho_{n+1}-4\mu\alpha(2a_{n+1}+e_{n+1})\\ c_{n+1}+g_{n+1}+2\mu(q+s)(2a_{n+1}+e_{n+1})\\ b_{n+1}+f_{n+1}+2\mu(p+r)(2a_{n+1}+e_{n+1})\end{pmatrix}, \tag{27}$$

where R is given by

$$R=\begin{pmatrix}R_{11}&R_{12}&R_{13}\\ R_{21}&R_{22}&R_{23}\\ R_{31}&R_{32}&R_{33}\end{pmatrix}$$

with

$$R_{11}=\begin{pmatrix}1+2\mu q\partial^{-1}p&-2\mu q\partial^{-1}q\\ 2\mu p\partial^{-1}p&1-2\mu p\partial^{-1}q\end{pmatrix}, R_{12}=\begin{pmatrix}\mu q\partial^{-1}\alpha&-\mu q\partial^{-1}\beta\\ \mu p\partial^{-1}\alpha&-\mu p\partial^{-1}\beta\end{pmatrix},$$

$$R_{13}=\begin{pmatrix}-1+2\mu q\partial^{-1}r & -2\mu q\partial^{-1}s\\ 2\mu p\partial^{-1}r & -1-2\mu p\partial^{-1}s\end{pmatrix}, R_{21}=\begin{pmatrix}-2\mu\beta\partial^{-1}p & 2\mu\beta\partial^{-1}q\\ -2\mu\alpha\partial^{-1}p & 2\mu\alpha\partial^{-1}q\end{pmatrix},$$

$$R_{22}=\begin{pmatrix}\frac{1}{2}-\mu\beta\partial^{-1}\alpha & \mu\beta\partial^{-1}\beta\\ -\mu\alpha\partial^{-1}\alpha & -\frac{1}{2}+\mu\alpha\partial^{-1}\beta\end{pmatrix}, R_{23}=\begin{pmatrix}-2\mu\beta\partial^{-1}r & 2\mu\beta\partial^{-1}s\\ -2\mu\alpha\partial^{-1}r & 2\mu\alpha\partial^{-1}s\end{pmatrix}.$$

$$R_{31}=\begin{pmatrix}-1-2\mu(2q+s)\partial^{-1}p & 2\mu(2q+s)\partial^{-1}q\\ -2\mu(2p+r)\partial^{-1}p & -1+2\mu(2p+r)\partial^{-1}q\end{pmatrix},$$

$$R_{32}=\begin{pmatrix}-\mu(2q+s)\partial^{-1}\alpha & \mu(2q+s)\partial^{-1}\beta\\ -\mu(2p+r)\partial^{-1}\alpha & \mu(2p+r)\partial^{-1}\beta\end{pmatrix},$$

$$R_{33}=\begin{pmatrix}2-2\mu(2q+s)\partial^{-1}r & 2\mu(2q+s)\partial^{-1}s\\ -2\mu(2p+r)\partial^{-1}r & 2+2\mu(2p+r)\partial^{-1}s\end{pmatrix}.$$

Thus, the hierarchy (20) possesses the following super Hamiltonian structure

$$u_{t_n}=Q\begin{pmatrix}c_{n+1}\\ b_{n+1}\\ \delta_{n+1}\\ \rho_{n+1}\\ g_{n+1}\\ f_{n+1}\end{pmatrix}=QR\begin{pmatrix}2c_{n+1}+g_{n+1}+2\mu s(2a_{n+1}+e_{n+1})\\ 2b_{n+1}+f_{n+1}+2\mu r(2a_{n+1}+e_{n+1})\\ 2\delta_{n+1}+4\mu\beta(2a_{n+1}+e_{n+1})\\ -2\rho_{n+1}-4\mu\alpha(2a_{n+1}+e_{n+1})\\ c_{n+1}+g_{n+1}+2\mu(q+s)(2a_{n+1}+e_{n+1})\\ b_{n+1}+f_{n+1}+2\mu(p+r)(2a_{n+1}+e_{n+1})\end{pmatrix}=J\frac{\delta\tilde{H}_n}{\delta u}, \tag{28}$$

where

$$Q=\begin{pmatrix}Q_{11} & Q_{12} & Q_{13}\\ Q_{21} & Q_{22} & Q_{23}\\ Q_{31} & Q_{32} & Q_{33}\end{pmatrix}$$

with

$$Q_{11}=\begin{pmatrix}-4\mu p\partial^{-1}r & 2+4\mu p\partial^{-1}s\\ -2+4\mu q\partial^{-1}r & -4\mu q\partial^{-1}s\end{pmatrix}, Q_{12}=\begin{pmatrix}4\mu p\partial^{-1}\alpha & 4\mu p\partial^{-1}\beta\\ -4\mu q\partial^{-1}\alpha & -4\mu q\partial^{-1}\beta\end{pmatrix},$$

$$Q_{13}=\begin{pmatrix}-4\mu p\partial^{-1}(p+r) & 4\mu p\partial^{-1}(q+s)\\ 4\mu q\partial^{-1}(p+r) & -4\mu q\partial^{-1}(q+s)\end{pmatrix}, Q_{21}=\begin{pmatrix}-2\mu\alpha\partial^{-1}r & 2\mu\alpha\partial^{-1}s\\ 2\mu\beta\partial^{-1}r & -2\mu\beta\partial^{-1}s\end{pmatrix},$$

$$Q_{22}=\begin{pmatrix}2\mu\alpha\partial^{-1}\alpha & 1+2\mu\alpha\partial^{-1}\beta\\ -1-2\mu\beta\partial^{-1}\alpha & -2\mu\beta\partial^{-1}\beta\end{pmatrix}$$

$$Q_{23}=\begin{pmatrix}-2\mu\alpha\partial^{-1}(p+r) & 2\mu\alpha\partial^{-1}(q+s)\\ 2\mu\beta\partial^{-1}(p+r) & -2\mu\beta\partial^{-1}(q+s)\end{pmatrix}, Q_{31}=\begin{pmatrix}-4\mu r\partial^{-1}r & 4\mu r\partial^{-1}s\\ 4\mu s\partial^{-1}r & -4\mu s\partial^{-1}s\end{pmatrix},$$

$$Q_{32}=\begin{pmatrix}4\mu r\partial^{-1}\alpha & 4\mu r\partial^{-1}\beta\\ -4\mu s\partial^{-1}\alpha & -4\mu s\partial^{-1}\beta\end{pmatrix}, Q_{33}=\begin{pmatrix}-4\mu r\partial^{-1}(p+r) & 2+4\mu r\partial^{-1}(q+s)\\ -2+4\mu s\partial^{-1}(p+r) & -4\mu s\partial^{-1}(q+s)\end{pmatrix},$$

and

$$J=QR=\begin{pmatrix}J_1 & J_2 & -J_1\\ 0 & J_3 & J_4\\ -J_1 & -J_2 & J_5\end{pmatrix}$$

with

$$J_1=\begin{pmatrix}8\mu p\partial^{-1}p & 2-4\mu p\partial^{-1}q\\ -2-8\mu q\partial^{-1}p & 8\mu q\partial^{-1}q\end{pmatrix},J_2=\begin{pmatrix}4\mu p\partial^{-1}\alpha & -4\mu p\partial^{-1}\beta\\ -4\mu q\partial^{-1}\alpha & 4\mu q\partial^{-1}\beta\end{pmatrix},$$

$$J_3=\begin{pmatrix}0 & -\frac{1}{2}\\ -\frac{1}{2} & 0\end{pmatrix},J_4=\begin{pmatrix}-4\mu\alpha\partial^{-1}(p+r) & 4\mu\alpha\partial^{-1}(q+s)\\ 4\mu\beta\partial^{-1}(p+r) & -4\mu\beta\partial^{-1}(q+s)\end{pmatrix},$$

$$J_5=\begin{pmatrix}-8\mu r\partial^{-1}(p+r)-8\mu p\partial^{-1}r & 4+8\mu r\partial^{-1}(q+s)+8\mu p\partial^{-1}s\\ -4+8\mu s\partial^{-1}(p+r)+8\mu q\partial^{-1}r & -8\mu s\partial^{-1}(q+s)-8\mu q\partial^{-1}s\end{pmatrix}.$$

It could be proved that J is a super Hamiltonian operator.

Specially, by making use of the recursive relationship (14), the hierarchy (20) possesses the following super bi-Hamiltonian structure

$$u_{t_n}=QL\begin{pmatrix}c_n\\ b_n\\ \delta_n\\ \rho_n\\ g_n\\ f_n\end{pmatrix}=QLR\begin{pmatrix}2c_n+g_n+2\mu s(2a_n+e_n)\\ 2b_n+f_n+2\mu r(2a_n+e_n)\\ 2\delta_n+4\mu\beta(2a_n+e_n)\\ -2\rho_n-4\mu\alpha(2a_n+e_n)\\ c_n+g_n+2\mu(q+s)(2a_n+e_n)\\ b_n+f_n+2\mu(p+r)(2a_n+e_n)\end{pmatrix}=P\frac{\delta\tilde{H}_{n-1}}{\delta u},n\geq 2. \tag{29}$$

where the second compatible super Hamiltonian operator $P=QLR=(P_{ij})_{6\times 6}, i,j=1,2,\ \ldots\ ,6$ is given by

$$\begin{aligned}
P_{11}&=2p\partial^{-1}p-4\mu p\partial^{-1}p\left(\frac{1}{2}\partial+h\right)+2\mu(\partial-2h)p\partial^{-1}p-4\mu^2p\Delta\partial^{-1}p,\\
P_{12}&=-2p\partial^{-1}q-4\mu p\partial^{-1}q\left(\frac{1}{2}\partial-h\right)-2\mu(\partial-2h)p\partial^{-1}q+4\mu^2p\Delta\partial^{-1}q,\\
P_{13}&=p\partial^{-1}\alpha-2\mu p\partial^{-1}\alpha(\partial+h)+\mu(\partial-2h)p\partial^{-1}\alpha-2\mu^2p\Delta\partial^{-1}\alpha,\\
P_{14}&=-\alpha-p\partial^{-1}\beta-2\mu p\partial^{-1}\beta(\partial-h)-\mu(\partial-2h)p\partial^{-1}\beta+2\mu^2p\Delta\partial^{-1}\beta,\\
P_{15}&=-2p\partial^{-1}p+4\mu p\partial^{-1}(2p+r)\left(\frac{1}{2}\partial+h\right)+2\mu(\partial-2h)p\partial^{-1}r-4\mu^2p\Delta\partial^{-1}r,\\
P_{16}&=2p\partial^{-1}q+4\mu p\partial^{-1}(2q+s)\left(\frac{1}{2}\partial-h\right)-2\mu(\partial-2h)p\partial^{-1}s+4\mu^2p\Delta\partial^{-1}s,\\
P_{21}&=-2q\partial^{-1}p+4\mu q\partial^{-1}p\left(\frac{1}{2}\partial+h\right)+2\mu(\partial+2h)q\partial^{-1}p+4\mu^2q\Delta\partial^{-1}p,\\
P_{22}&=2q\partial^{-1}q+4\mu q\partial^{-1}q\left(\frac{1}{2}\partial-h\right)-2\mu(\partial+2h)q\partial^{-1}q-4\mu^2q\Delta\partial^{-1}q,\\
P_{23}&=-\beta-q\partial^{-1}\alpha+2\mu q\partial^{-1}\alpha(\partial+h)+\mu(\partial+2h)q\partial^{-1}\alpha+2\mu^2q\Delta\partial^{-1}\alpha,\\
P_{24}&=q\partial^{-1}\beta+2\mu q\partial^{-1}\beta(\partial-h)-\mu(\partial+2h)q\partial^{-1}\beta-2\mu^2q\Delta\partial^{-1}\beta,\\
P_{25}&=2q\partial^{-1}p-4\mu q\partial^{-1}(2p+r)\left(\frac{1}{2}\partial+h\right)+2\mu(\partial+2h)q\partial^{-1}r+4\mu^2q\Delta\partial^{-1}r,\\
P_{26}&=-2q\partial^{-1}q-4\mu q\partial^{-1}(2q+s)\left(\frac{1}{2}\partial-h\right)-2\mu(\partial+2h)q\partial^{-1}s-4\mu^2q\Delta\partial^{-1}s,\\
P_{31}&=\alpha\partial^{-1}p-2\mu\alpha\partial^{-1}p\left(\frac{1}{2}\partial+h\right)+4\mu\beta p\partial^{-1}p-2\mu(\partial-h)\alpha\partial^{-1}p-2\mu^2\alpha\Delta\partial^{-1}p,\\
P_{32}&=\beta-\alpha\partial^{-1}q-2\mu\alpha\partial^{-1}q\left(\frac{1}{2}\partial-h\right)-4\mu\beta p\partial^{-1}q+2\mu(\partial-h)\alpha\partial^{-1}q+2\mu^2\alpha\Delta\partial^{-1}q,\\
P_{33}&=-\frac{1}{2}p+\frac{1}{2}\alpha\partial^{-1}\alpha-\mu\alpha\partial^{-1}\alpha(\partial+h)+2\mu\beta p\partial^{-1}\alpha-\mu(\partial-h)\alpha\partial^{-1}\alpha-\mu^2\alpha\Delta\partial^{-1}\alpha,\\
P_{34}&=\frac{1}{2}h-\frac{1}{2}\partial-\frac{1}{2}\alpha\partial^{-1}\beta-\mu\alpha\partial^{-1}\beta(\partial-h)-2\mu\beta p\partial^{-1}\beta+\mu(\partial-h)\alpha\partial^{-1}\beta+\mu^2\alpha\Delta\partial^{-1}\beta,\\
P_{35}&=-\alpha\partial^{-1}p+2\mu\alpha\partial^{-1}(2p+r)\left(\frac{1}{2}\partial+h\right)+4\mu\beta p\partial^{-1}r-2\mu(\partial-h)\alpha\partial^{-1}r-2\mu^2\alpha\Delta\partial^{-1}r,\\
P_{36}&=-\beta+\alpha\partial^{-1}q+2\mu\alpha\partial^{-1}(2q+s)\left(\frac{1}{2}\partial-h\right)-4\mu\beta p\partial^{-1}s+2\mu(\partial-h)\alpha\partial^{-1}s+2\mu^2\alpha\Delta\partial^{-1}s,
\end{aligned}$$

$$
\begin{aligned}
P_{41} &= \alpha - \beta\partial^{-1}p + 2\mu\beta\partial^{-1}p\left(\frac{1}{2}\partial + h\right) + 4\mu\alpha q\partial^{-1}p - 2\mu(\partial + h)\beta\partial^{-1}p + 2\mu^2\beta\Delta\partial^{-1}p,\\
P_{42} &= \beta\partial^{-1}q + 2\mu\beta\partial^{-1}q\left(\frac{1}{2}\partial - h\right) - 4\mu\alpha q\partial^{-1}q + 2\mu(\partial + h)\beta\partial^{-1}q - 2\mu^2\beta\Delta\partial^{-1}q,\\
P_{43} &= \frac{1}{2}h + \frac{1}{2}\partial - \frac{1}{2}\beta\partial^{-1}\alpha + \mu\beta\partial^{-1}\alpha(\partial + h) + 2\mu\alpha q\partial^{-1}\alpha - \mu(\partial + h)\beta\partial^{-1}\alpha + \mu^2\beta\Delta\partial^{-1}\alpha,\\
P_{44} &= \frac{1}{2}q + \frac{1}{2}\beta\partial^{-1}\beta + \mu\beta\partial^{-1}\beta(\partial - h) - 2\mu\alpha q\partial^{-1}\beta + \mu(\partial + h)\beta\partial^{-1}\beta - \mu^2\beta\Delta\partial^{-1}\beta,\\
P_{45} &= -\alpha + \beta\partial^{-1}p - 2\mu\beta\partial^{-1}(2p + r)\left(\frac{1}{2}\partial + h\right) + 4\mu\alpha q\partial^{-1}r - 2\mu(\partial + h)\beta\partial^{-1}r + 2\mu^2\beta\Delta\partial^{-1}r,\\
P_{46} &= -\beta\partial^{-1}q - 2\mu\beta\partial^{-1}(2q + s)\left(\frac{1}{2}\partial - h\right) - 4\mu\alpha q\partial^{-1}s + 2\mu(\partial + h)\beta\partial^{-1}s - 2\mu^2\beta\Delta\partial^{-1}s,\\
P_{51} &= -2p\partial^{-1}p - 4\mu r\partial^{-1}p\left(\frac{1}{2}\partial + h\right) - 2\mu(\partial - 2h)[(2p + r)\partial^{-1}p] - 4\mu^2 r\Delta\partial^{-1}p,\\
P_{52} &= -\partial + 2h + 2p\partial^{-1}q - 4\mu r\partial^{-1}q\left(\frac{1}{2}\partial - h\right) + 2\mu(\partial - 2h)[(2p + r)\partial^{-1}q] + 4\mu^2 r\Delta\partial^{-1}q,\\
P_{53} &= -p\partial^{-1}\alpha - 2\mu r\partial^{-1}\alpha(\partial + h) - \mu(\partial - 2h)[(2p + r)\partial^{-1}\alpha] - 2\mu^2 r\Delta\partial^{-1}\alpha,\\
P_{54} &= p\partial^{-1}\beta - 2\mu r\partial^{-1}\beta(\partial - h) + \mu(\partial - 2h)[(2p + r)\partial^{-1}\beta] + 2\mu^2 r\Delta\partial^{-1}\beta,\\
P_{55} &= 2(2p + r)\partial^{-1}p + 2(p + r)\partial^{-1}r + 4\mu r\partial^{-1}(2p + r)\left(\frac{1}{2}\partial + h\right)\\
&\quad - 2\mu(\partial - 2h)[(2p + r)\partial^{-1}r] - 4\mu^2 r\Delta\partial^{-1}r,\\
P_{56} &= 2\partial - 4h + 2(2q + s)\partial^{-1}q + 2(q + s)\partial^{-1}s + 4\mu r\partial^{-1}(2q + s)\left(\frac{1}{2}\partial - h\right)\\
&\quad + 2\mu(\partial - 2h)[(2p + r)\partial^{-1}s] + 4\mu^2 r\Delta\partial^{-1}s,\\
P_{61} &= -\partial - 2h + 2q\partial^{-1}p + 4\mu s\partial^{-1}p\left(\frac{1}{2}\partial + h\right) - 2\mu(\partial + 2h)[(2q + s)\partial^{-1}p] + 4\mu^2 s\Delta\partial^{-1}p,\\
P_{62} &= -2q\partial^{-1}q - 4\mu s\partial^{-1}q\left(\frac{1}{2}\partial - h\right) + 2\mu(\partial + 2h)[(2q + s)\partial^{-1}q] - 4\mu^2 s\Delta\partial^{-1}q,\\
P_{63} &= -p\partial^{-1}\alpha - 2\mu s\partial^{-1}\alpha(\partial + h) - \mu(\partial + 2h)[(2q + s)\partial^{-1}\alpha] + 2\mu^2 s\Delta\partial^{-1}\alpha,\\
P_{64} &= p\partial^{-1}\beta - 2\mu s\partial^{-1}\beta(\partial - h) + \mu(\partial + 2h)[(2q + s)\partial^{-1}\beta] - 2\mu^2 s\Delta\partial^{-1}\beta,\\
P_{65} &= 2\partial + 4h + 2(2p + r)\partial^{-1}p + 2(p + r)\partial^{-1}r + 4\mu s\partial^{-1}(2q + s)\left(\frac{1}{2}\partial + h\right)\\
&\quad - 2\mu(\partial + 2h)[(2q + s)\partial^{-1}r] + 4\mu^2 s\Delta\partial^{-1}r,\\
P_{66} &= 2\partial - 4h + 2(2q + s)\partial^{-1}q + 2(q + s)\partial^{-1}s + 4\mu s\partial^{-1}(2q + s)\left(\frac{1}{2}\partial - h\right)\\
&\quad + 2\mu(\partial + 2h)[(2q + s)\partial^{-1}s] - 4\mu^2 s\Delta\partial^{-1}s,
\end{aligned}
$$

with

$$
\Delta = \partial^{-1}(2q + s)\partial p + \partial^{-1}(2p + r)\partial q + \partial^{-1}(q + s)\partial r + \partial^{-1}(p + r)\partial s + 2\partial^{-1}\beta\partial\alpha - 2\partial^{-1}\alpha\partial\beta.
$$

5 | CONCLUSION AND DISCUSSIONS

In this paper, we presented an approach for constructing nonlinear super integrable couplings of super soliton equations through enlarging matrix Lie superalgebras. We took the Lie algebra $sl(2,1)$ as an example to illustrate the introduced idea to extend Lie superalgebras. Based on the enlarged Lie superalgebra $sl(4,1)$, we worked out nonlinear integrable couplings for a generalized super AKNS soliton hierarchy. The presented method in this paper can be applied to other generalized super integrable hierarchies, which will be our future problems to construct super integrable couplings.

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