

Multiphase Distribution Locational Marginal Prices: Approximation and Decomposition

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Abstract—We propose a multiphase distribution locational marginal price (DLMP) model. Compared to existing DLMP models in the literature, the proposed model has three distinctive features: i) It provides linear approximation of relevant DLMP components which captures global behavior of nonlinear functions; ii) it decomposes into most general components, i.e., energy, loss, congestion, voltage violations; and iii) it incorporates both wye and delta grid connections along with unbalanced loadings. The developed model is tested on a benchmark IEEE 13-bus unbalanced distribution system with the inclusion of distributed generators (DGs).

Index Terms—Distribution Locational Marginal Prices (DLMPs), Approximation, Distributed Generators (DGs)

I. INTRODUCTION

Locational marginal prices (LMPs) at the transmission grid level provide the theoretical foundation to form deregulated energy markets [1]. However, as power systems increasingly face the integration of dispersed energy resources, such as distributed generators (DGs), at the distribution grid level, transmission level energy markets might not be solely sufficient for achieving economic operation in power systems. To this end, the idea of establishing distribution grid locational marginal prices (DLMPs) has started to gain attention in the power system community [2]. Essentially, DLMPs are the distribution grid variants of transmission grid (wholesale) LMPs, aimed at generating economically efficient price signals and facilitating the integration of contemporary active devices (e.g. DGs) in the distribution systems [2]–[4]. As in the past, efforts to standardize LMP models proved instrumental in realizing competitive wholesale markets, similar efforts are required for standardizing DLMP models. However as compared to transmission grids, distribution grids contain higher degree of nonlinear power flows along with multiphase and unbalance characteristics, preventing readily standardized LMP models to be deployed for calculating DLMPs.

Recently, there have been few works on three-phase DLMP models [5]–[8]. In [5], a full Alternating Current (AC) optimal power flow model and in [6] SDP-based (relaxed AC) optimal power flow were developed to calculate DLMPs. The works in [7], [8] considered a linearized distribution grid model to calculate DLMPs. All of the above-mentioned works have still not addressed the decomposition of DLMPs into the non-negligible components of 1) system active and reactive power losses, 2) line flows and 3) voltage binding limits. In addition, DLMP calculations under a mixture of wye and delta grid connections have also not been addressed in the literature. This paper addresses these issues by providing a DLMP model which not only decomposes into the above-stated components but also caters for multiphase nature of distribution grids.

In this paper, we present a multiphase DLMP model which accounts for non-linear power flows along with a mixture of delta and wye connections. To tackle multiphase nature of distribution grids, we build upon the recent works on approximate multiphase distribution grid models [9] and extend it to include key components, such as system losses and congestion. Coherent to the main results of [9], our above mentioned extensions also capture global behavior of their actual non-linear counterparts. Moreover, these extensions help in developing an approximate DLMP model, which is decomposable to the extent that it shares similarities with the existing standard LMP model. This feature might prove crucial for practical realization of DLMPs in future power systems.

Notations: Bold upper-case (lower-case) letters are used for matrices (column vectors) and non-bold lower-case for scalars, $(\cdot)^\top$ for transposition. For complex variable $c \in \mathbb{C}$, $\Re(c)$, $\Im(c)$, and $\bar{c}$ denote its real part, imaginary part and complex conjugate, respectively. For size n vector $\mathbf{x} \in \mathbb{C}^n$, $\text{diag}(\mathbf{x})$ returns a $n \times n$ matrix with entries of $\mathbf{x}$ at its diagonal. Vector $\mathbf{1}_n$ is a size n vector of all ones and letter $j := \sqrt{-1}$.

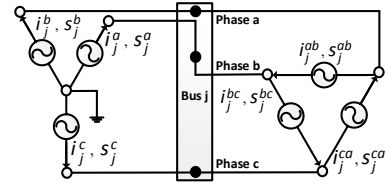


Fig. 1. Exemplary three-phase grounded wye-connected (left) and delta-connected (right) sources with net current and complex power injections. Similar connections can be made for loads. Procedure to translate mixture of wye-connected and delta-connected sources/loads for both primary and secondary distribution transformer is given in [9].

II. MULTIPHASE GRID & DLMP MODEL

For exposition simplicity, we present (in both Sec. II and III) the three-phase grid model. However, we show its extension to a generic multiphase system in Sec. II-E and demonstrate its deployment in Sec. IV. Also, we deploy constant PQ injection models, used both in the classic analysis of distribution systems [10] as well as in the recent voltage control problems [9]. However, the results can be naturally extended to a more general load model, as shown in [9].

A. Preliminaries

For bus j in Fig. 1 with three-phases $\{a, b, c\}$, we have: complex net current injections for each phase-ground $\mathbf{i}_j := (i_j^a, i_j^b, i_j^c)^\top$ and phase-phase $\mathbf{i}_j^\Delta := (i_j^{ab}, i_j^{bc}, i_j^{ca})^\top$ connections along with net complex power injections from wye-connected $\mathbf{s}_j^Y := (s_j^a, s_j^b, s_j^c)^\top$ and delta-connected $\mathbf{s}_j^\Delta :=$

$(s_j^{ab}, s_j^{bc}, s_j^{ca})^\top$ sources. Now for each phase-to-ground voltage represented as $\mathbf{v}_j := (v_j^a, v_j^b, v_j^c)^\top$, we have,

$$\begin{aligned} \mathbf{s}_j^\Delta &= \text{diag}(\mathbf{\Gamma} \mathbf{v}_j) \bar{\mathbf{i}}_j^\Delta, \\ \mathbf{s}_j^Y &= \text{diag}(\mathbf{v}_j) \bar{\mathbf{i}}_j - \text{diag}(\mathbf{v}_j) \mathbf{\Gamma}^\top \bar{\mathbf{i}}_j^\Delta, \end{aligned} \quad (1)$$

where, $\mathbf{\Gamma} := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

B. Grid Model

For a grid with one slack bus and n PQ buses, the relationship of (1) are easily extended to the whole grid except slack-bus as: $\mathbf{v} := (\mathbf{v}_1^\top, \dots, \mathbf{v}_n^\top)^\top$, $\mathbf{i} := (\mathbf{i}_1^\top, \dots, \mathbf{i}_n^\top)^\top$, $\mathbf{i}^\Delta := ((\mathbf{i}_1^\Delta)^\top, \dots, (\mathbf{i}_n^\Delta)^\top)^\top$, $\mathbf{s}^Y := ((\mathbf{s}_1^Y)^\top, \dots, (\mathbf{s}_n^Y)^\top)^\top$, $\mathbf{s}^\Delta := ((\mathbf{s}_1^\Delta)^\top, \dots, (\mathbf{s}_n^\Delta)^\top)^\top$ all having size $\mathbb{C}^{3n}$. The load-flow problem is then to satisfy the following set of equations [9]:

$$\text{diag}(\mathbf{H}^\top \bar{\mathbf{i}}^\Delta) \mathbf{v} + \mathbf{s}^Y = \text{diag}(\mathbf{v}) \bar{\mathbf{i}}, \quad (2a)$$

$$\mathbf{s}^\Delta = \text{diag}(\mathbf{H} \mathbf{v}) \bar{\mathbf{i}}^\Delta, \quad (2b)$$

$$\mathbf{i} = \mathbf{Y}_{L0} \mathbf{v}_0 + \mathbf{Y}_{LL} \mathbf{v}, \quad (2c)$$

where $\mathbf{v}_0 := (v_0^a, v_0^b, v_0^c)^\top$ is the voltage at the slack bus. Matrix $\mathbf{H}$ is a $3n \times 3n$ block diagonal version of $\mathbf{\Gamma}$. Matrix $\mathbf{Y}_{L0} := [\mathbf{Y}_{01}, \mathbf{0}_{3 \times (n-3)}]^\top \in \mathbb{C}^{3n \times 3}$, where $\mathbf{Y}_{01} \in \mathbb{C}^{3 \times 3}$ is the π -model admittance matrix of the slack bus connection to its neighboring buses. Matrix $\mathbf{Y}_{LL} \in \mathbb{C}^{3n \times 3n}$ is the grid admittance matrix.

To solve (2), in [9], it was shown that for given $(\mathbf{s}^\Delta, \mathbf{s}^Y, \mathbf{v}^0)$ for each iteration t , the following fixed point equation exists:

$$\mathbf{v}^{(t+1)} = \mathbf{w} + \mathbf{Y}_{LL}^{-1} (\text{diag}(\bar{\mathbf{v}}^t)^{-1} \bar{\mathbf{s}}^Y + \mathbf{H}^\top \text{diag}(\mathbf{H} \bar{\mathbf{v}}^t)^{-1} \bar{\mathbf{s}}^\Delta) \quad (3)$$

where $\mathbf{w} := \mathbf{Y}_{LL}^{-1} \mathbf{Y}_{L0} \mathbf{v}^0$ is the no-load voltage.

C. Grid Model Extension

Let $\hat{\mathbf{v}}$ be the solution upon convergence of (3). Now relevant for DLMP calculation, we extend the above grid model to account for line flows and system losses. Consider bus j and k connected by a three-phase distribution line element (j, k) , then the complex line flows “from” $\mathbf{s}_{jk}^f := (s_{jk}^{f,a}, s_{jk}^{f,b}, s_{jk}^{f,c})^\top$ and “to” $\mathbf{s}_{jk}^t := (s_{jk}^{t,a}, s_{jk}^{t,b}, s_{jk}^{t,c})^\top$ directions of line are:

$$\mathbf{s}_{jk}^f = \text{diag}(\hat{\mathbf{v}}_j) \left(\bar{\mathbf{Y}}_{jk}^{sh} \bar{\mathbf{v}}_j + \bar{\mathbf{Y}}_{jk}^{ph} (\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_k) \right), \quad (4)$$

$$\mathbf{s}_{jk}^t = \text{diag}(\hat{\mathbf{v}}_k) \left(\bar{\mathbf{Y}}_{jk}^{sh} \bar{\mathbf{v}}_k + \bar{\mathbf{Y}}_{jk}^{ph} (\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_k) \right), \quad (5)$$

where $\mathbf{Y}_{jk}^{sh} \in \mathbb{C}^{3 \times 3}$ and $\mathbf{Y}_{jk}^{ph} \in \mathbb{C}^{3 \times 3}$ are given by the standard distribution grid line's π -model shunt admittance and phase admittance matrices [10]. Similarly, complex losses s_{jk}^l for line (j, k) can be written as,

$$s_{jk}^l = (\hat{\mathbf{v}}_j - \hat{\mathbf{v}}_k)^\top \left(\bar{\mathbf{Y}}_{jk}^{sh} \bar{\mathbf{v}}_j + \bar{\mathbf{Y}}_{jk}^{ph} (\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_k) \right) \quad (6)$$

For extending these relationship to the whole grid, now consider l lines containing all three-phase connected (j, k) bus pairs, then the “from”/“to” line flows vector $\mathbf{s}^{f/t} :=$

$((\mathbf{s}_1^{f/t})^\top, \dots, (\mathbf{s}_l^{f/t})^\top)^\top \in \mathbb{C}^{3l}$ and complex total grid losses $\mathbf{s}^l := \mathbf{1}_l^\top \mathbf{s}_{j,k}^l$ are expressed as:

$$\mathbf{s}^{f/t} = \text{diag}(\mathbf{A}^{f/t} \hat{\mathbf{v}}) \bar{\mathbf{Y}} \mathbf{A} \hat{\mathbf{v}}, \quad (7)$$

$$\mathbf{s}^l = (\mathbf{A} \hat{\mathbf{v}})^\top \bar{\mathbf{Y}} \mathbf{A} \hat{\mathbf{v}}, \quad (8)$$

where matrix $\mathbf{A}^{f/t}$ is a $3l \times 3n$ incidence matrix, containing 1's at buses connected at the “from”/“to” ends of lines, zero elsewhere (e.g. as (4)–(5) is constructed for one line element between two buses). Matrix $\mathbf{A}$ is a $3l \times 3n$ containing 1 and -1 at appropriate buses connected to lines, zero elsewhere. Matrix $\bar{\mathbf{Y}} \in \mathbb{C}^{3l \times 3l}$ contains standard π models of all grid-connected devices (e.g. see (4)–(5) for one line element) [10]. Note that due to standard π -model deployment, the following relationship also exists: $\mathbf{Y}_{LL} := \mathbf{A}^\top \bar{\mathbf{Y}} \mathbf{A}$.

D. DLMP model

Let wye/delta-connected active and reactive power injections as $\mathbf{p}^{Y/\Delta} := \Re(\mathbf{s}^{Y/\Delta})$ and $\mathbf{q}^{Y/\Delta} := \Im(\mathbf{s}^{Y/\Delta})$, where in particular, $\mathbf{p}^{Y/\Delta} := ((\mathbf{p}_{g,1}^{Y/\Delta} - \mathbf{p}_{d,1}^{Y/\Delta})^\top, \dots, (\mathbf{p}_{g,n}^{Y/\Delta} - \mathbf{p}_{d,n}^{Y/\Delta})^\top)^\top$, $\mathbf{q}^{Y/\Delta} := ((\mathbf{q}_{g,1}^{Y/\Delta} - \mathbf{q}_{d,1}^{Y/\Delta})^\top, \dots, (\mathbf{q}_{g,n}^{Y/\Delta} - \mathbf{q}_{d,n}^{Y/\Delta})^\top)^\top$ where $\mathbf{p}_g^{Y/\Delta}$, $\mathbf{p}_d^{Y/\Delta}$, $\mathbf{q}_g^{Y/\Delta}$ and $\mathbf{q}_d^{Y/\Delta}$ of size $\mathbb{R}^{3n}$ are active power generation, active power demand, reactive power generation and reactive power demand, respectively. We make $p^l/q^l := \Re(s^l)/\Im(s^l)$ as active/reactive power grid losses and $\mathbf{p}^0/\mathbf{q}^0 := \Re(\mathbf{s}^0)/\Im(\mathbf{s}^0) \in \mathbb{R}^3$ as active/reactive power injected from transmission grid to the slack bus. For dispatchable active and reactive power generation, (9) presents a multiphase variant of the classic global energy balance formulation for calculating DLMPs [1], [2].

$$\min \quad \mathbf{c}_{p_g^Y}^\top \mathbf{p}_g^Y + \mathbf{c}_{p_g^\Delta}^\top \mathbf{p}_g^\Delta + \mathbf{c}_{q_g^Y}^\top \mathbf{q}_g^Y + \mathbf{c}_{q_g^\Delta}^\top \mathbf{q}_g^\Delta + \mathbf{c}_{p^0}^\top \mathbf{p}^0 + \mathbf{c}_{q^0}^\top \mathbf{q}^0 \quad (9a)$$

s.t.

$$\mathbf{1}_3^\top \mathbf{p}^0 + \mathbf{1}_{3n}^\top (\mathbf{p}_g^Y + \mathbf{p}_g^\Delta - \mathbf{p}_d^Y - \mathbf{p}_d^\Delta) = p^l : \lambda_p \quad (9b)$$

$$\mathbf{1}_3^\top \mathbf{q}^0 + \mathbf{1}_{3n}^\top (\mathbf{q}_g^Y + \mathbf{q}_g^\Delta - \mathbf{q}_d^Y - \mathbf{q}_d^\Delta) = q^l : \lambda_q \quad (9c)$$

$$|\mathbf{s}^f|^2 \leq (\mathbf{s}^{f+})^2 : \mu_{s_f}^+ \quad (9d)$$

$$|\mathbf{s}^t|^2 \leq (\mathbf{s}^{t+})^2 : \mu_{s_t}^+ \quad (9e)$$

$$\mathbf{v}^- \leq |\mathbf{v}| \leq \mathbf{v}^+ : \mu_v^-, \mu_v^+ \quad (9f)$$

$$\mathbf{p}_g^{Y-} \leq \mathbf{p}_g^Y \leq \mathbf{p}_g^{Y+} : \mu_{p_g^Y}^-, \mu_{p_g^Y}^+ \quad (9g)$$

$$\mathbf{q}_g^{Y-} \leq \mathbf{q}_g^Y \leq \mathbf{q}_g^{Y+} : \mu_{q_g^Y}^-, \mu_{q_g^Y}^+ \quad (9h)$$

$$\mathbf{p}_g^{\Delta-} \leq \mathbf{p}_g^\Delta \leq \mathbf{p}_g^{\Delta+} : \mu_{p_g^\Delta}^-, \mu_{p_g^\Delta}^+ \quad (9i)$$

$$\mathbf{q}_g^{\Delta-} \leq \mathbf{q}_g^\Delta \leq \mathbf{q}_g^{\Delta+} : \mu_{q_g^\Delta}^-, \mu_{q_g^\Delta}^+ \quad (9j)$$

The vector $\mathbf{c}_{(\mathbf{p}/\mathbf{q})_g^{Y/\Delta}/\mathbf{c}_{(\mathbf{p}/\mathbf{q})^0}}$ represents the marginal cost of providing energy from DGs/slack-bus, i.e. $(\mathbf{p}/\mathbf{q})_g^{Y/\Delta}/(\mathbf{p}/\mathbf{q})^0$. Problem (9) minimizes the overall cost for dispatching generation resources, with respect to the following constraints: 1) constraints (9b) and (9c) balance grid's active and reactive powers; 2) constraint (9d)/(9e) considers the square of the apparent power flow line limits in from/to directions; 3) the voltage magnitude $|\mathbf{v}|$ is constrained in (9f), whereas active and reactive power of each wye-connected and delta-connected generators are constrained through (9g) – (9j). The variables

listed to the right of the colon are the Lagrange multipliers and variables with superscript $+/-$ representing the respective maximum/minimum limits.

E. Generic Multiphase Grid & DLMP Model

Equations in (2) can be easily modified to represent generic number of phases in the grid. For buses, the only required modification would be to appropriately collect vectors $\mathbf{s}^\Delta$, $\mathbf{s}^Y$ and $\mathbf{v}$ along with the reconstruction of matrix $\mathbf{H}$ as a $n^\Delta \times n^\phi$ matrix, where n^ϕ is the available nodal connection and n^Δ is the available phase-phase connections. For lines, reconstruction of matrices $\mathbf{A}$ and $\mathbf{A}^f$ as $l^\phi \times n^\phi$ can be performed, where l^ϕ is the total number of lines inclusive of available phases per line. Similarly, (9) is generalized by modifying power generation/load vectors to size n_g/n_l , where n_g/n_l are the total number of generation/loads inclusive of their phases.

III. APPROXIMATE MULTIPHASE DLMP MODEL

From (2), it can be seen that $\mathbf{v}$ (also $|\mathbf{v}|$) is nonconvex in terms of injections. This renders the problem (9) highly nonlinear and nonconvex, making it difficult to be solved using off-the-shelf optimization solvers [2]. To this end, we present an approximate multiphase grid model (Sec. III-A), easily translatable to obtain an approximate DLMP model (Sec. III-B).

A. Approximate Grid Model

Let augmented injection vector be $\mathbf{x}^{Y/\Delta} := (\mathbf{p}^{Y/\Delta})^\top, (\mathbf{q}^{Y/\Delta})^\top)^\top$, then as required in (9), our goal is to obtain the following linearized relationship in $\mathbf{x}^{Y/\Delta}$:

$$|\tilde{\mathbf{v}}| = \mathbf{M}_{|\mathbf{v}|}^Y \mathbf{x}^Y + \mathbf{M}_{|\mathbf{v}|}^\Delta \mathbf{x}^\Delta + \mathbf{a}, \quad (10a)$$

$$|\tilde{\mathbf{s}}^f|^2 = \mathbf{M}_{\mathbf{s}^f}^Y \mathbf{x}^Y + \mathbf{M}_{\mathbf{s}^f}^\Delta \mathbf{x}^\Delta + \mathbf{b}, \quad (10b)$$

$$|\tilde{\mathbf{s}}^t|^2 = \mathbf{M}_{\mathbf{s}^t}^Y \mathbf{x}^Y + \mathbf{M}_{\mathbf{s}^t}^\Delta \mathbf{x}^\Delta + \mathbf{c}, \quad (10c)$$

$$\tilde{p}^l = \mathbf{M}_{p^l}^Y \mathbf{x}^Y + \mathbf{M}_{p^l}^\Delta \mathbf{x}^\Delta + \mathbf{d}, \quad (10d)$$

$$\tilde{q}^l = \mathbf{M}_{q^l}^Y \mathbf{x}^Y + \mathbf{M}_{q^l}^\Delta \mathbf{x}^\Delta + \mathbf{e}. \quad (10e)$$

1) *Voltage Magnitude*: The explicit relationship for a linearized voltage expression $\tilde{\mathbf{v}}$ is obtained by considering the first iteration of (3), initialized at $\hat{\mathbf{v}}$ as:

$$\tilde{\mathbf{v}} = \mathbf{M}^Y \mathbf{x}^Y + \mathbf{M}^\Delta \mathbf{x}^\Delta + \mathbf{w} \quad (11)$$

where,

$$\mathbf{M}^Y := (\mathbf{Y}_{LL}^{-1} \text{diag}(\hat{\mathbf{v}})^{-1}, -j\mathbf{Y}_{LL}^{-1} \text{diag}(\hat{\mathbf{v}})^{-1})$$

$$\mathbf{M}^\Delta := (\mathbf{Y}_{LL}^{-1} \mathbf{H}^\top \text{diag}(\mathbf{H}\hat{\mathbf{v}})^{-1}, -j\mathbf{Y}_{LL}^{-1} \mathbf{H}^\top \text{diag}(\mathbf{H}\hat{\mathbf{v}})^{-1})$$

The approximation $|\tilde{\mathbf{v}}|$ in (10a) then follows [9]:

$$\mathbf{M}_{|\mathbf{v}|}^Y := \frac{\partial |\mathbf{v}|}{\partial \mathbf{x}^Y} = \text{diag}(|\hat{\mathbf{v}}|)^{-1} \Re(\text{diag}(\hat{\mathbf{v}}) \mathbf{M}^Y), \quad (12a)$$

$$\mathbf{M}_{|\mathbf{v}|}^\Delta := \frac{\partial |\mathbf{v}|}{\partial \mathbf{x}^\Delta} = \text{diag}(|\hat{\mathbf{v}}|)^{-1} \Re(\text{diag}(\hat{\mathbf{v}}) \mathbf{M}^\Delta), \quad (12b)$$

$$\mathbf{a} := |\hat{\mathbf{v}}| - \mathbf{M}_{|\mathbf{v}|}^Y \hat{\mathbf{x}}^Y - \mathbf{M}_{|\mathbf{v}|}^\Delta \hat{\mathbf{x}}^\Delta \quad (12c)$$

Note that the above approximation is different than a local (tangent) approximation, and gives a more global behavior. In principle, the approximation in (12) gives an interpolation between a specified loading $(\hat{\mathbf{v}}, \hat{\mathbf{s}})$ and at no load $(\mathbf{w}, \mathbf{0})$ [9]. For more information on approximation error bounds and computational efficiency, refer to [9].

2) *Approximate Flows*: From the solution $\hat{\mathbf{v}}$, let squared line flow be obtained as: $|\mathbf{s}^f|^2 := \text{diag}(\bar{\mathbf{s}}^f) \mathbf{s}^f$, then its sensitivity with respect to $\mathbf{x}^Y$ is:

$$\begin{aligned} \frac{\partial |\mathbf{s}^f|^2}{\partial \mathbf{x}^Y} &= \text{diag}(\bar{\mathbf{s}}^f) \frac{\partial \mathbf{s}^f}{\partial \mathbf{x}^Y} + \text{diag}(\mathbf{s}^f) \frac{\partial \bar{\mathbf{s}}^f}{\partial \mathbf{x}^Y} \\ &= \text{diag} \left(\Re(\mathbf{s}^f) - j\Im(\mathbf{s}^f) \right) \left(\frac{\partial \Re(\mathbf{s}^f)}{\partial \mathbf{x}^Y} + j \frac{\partial \Im(\mathbf{s}^f)}{\partial \mathbf{x}^Y} \right) \\ &\quad + \text{diag} \left(\Re(\mathbf{s}^f) + j\Im(\mathbf{s}^f) \right) \left(\frac{\partial \Re(\mathbf{s}^f)}{\partial \mathbf{x}^Y} - j \frac{\partial \Im(\mathbf{s}^f)}{\partial \mathbf{x}^Y} \right) \\ &= 2 \left(\text{diag}(\Re(\mathbf{s}^f)) \frac{\partial \Re(\mathbf{s}^f)}{\partial \mathbf{x}^Y} + \text{diag}(\Im(\mathbf{s}^f)) \frac{\partial \Im(\mathbf{s}^f)}{\partial \mathbf{x}^Y} \right) \\ &= 2 \underbrace{\left(\text{diag}(\Re(\mathbf{s}^f)), \text{diag}(\Im(\mathbf{s}^f)) \right)}_{\mathbf{M}_{\mathbf{s}^f}^Y} \left(\mathbf{X}_{\mathbf{s}^f}^Y + \mathbf{Y}_{\mathbf{s}^f}^Y \right) \begin{bmatrix} \Re(\mathbf{M}^Y) \\ \Im(\mathbf{M}^Y) \end{bmatrix} \end{aligned} \quad (13)$$

where $\mathbf{X}_{\mathbf{s}^f}^Y := \langle \text{diag}(\bar{\mathbf{Y}}^l \mathbf{A} \hat{\mathbf{v}}) \mathbf{A}^f \rangle$ and $\mathbf{Y}_{\mathbf{s}^f}^Y := \langle \text{diag}(\mathbf{A}^f \hat{\mathbf{v}}) \bar{\mathbf{Y}}^l \mathbf{A} \rangle \mathbf{N}_1$ are constructed by combining $\mathbf{N}_1 := \begin{bmatrix} \mathbf{1}_{3l \times 3n} & \mathbf{0}_{3l \times 3n} \\ \mathbf{0}_{3l \times 3n} & -\mathbf{1}_{3l \times 3n} \end{bmatrix}$ and $\langle \cdot \rangle := \begin{bmatrix} \Re(\cdot) & -\Im(\cdot) \\ \Im(\cdot) & \Re(\cdot) \end{bmatrix}$ with (7). Now by replacing $\mathbf{x}^Y$ with $\mathbf{x}^\Delta$ in (13), we simply obtain $\mathbf{M}_{\mathbf{s}^f}^\Delta$. Finally, at $(\mathbf{w}, \mathbf{0})$ no line flows exist, making $\mathbf{b} := \mathbf{0}$ to complete the approximation in (10b). Exactly similar procedure exists for obtaining (10c).

3) *Approximate Losses*: Similarly, for calculating (10d) and (10e), we first present sensitivity of complex power loss of (8) at $\hat{\mathbf{v}}$, with respect to $\mathbf{x}^Y$ as:

$$\begin{bmatrix} \mathbf{M}_{p^l}^Y \\ \mathbf{M}_{q^l}^Y \end{bmatrix} = \begin{bmatrix} \frac{\partial \Re(\mathbf{s}^l)}{\partial \mathbf{x}^Y} \\ \frac{\partial \Im(\mathbf{s}^l)}{\partial \mathbf{x}^Y} \end{bmatrix} = (\mathbf{X}_{\mathbf{s}^l}^Y + \mathbf{Y}_{\mathbf{s}^l}^Y) \begin{bmatrix} \Re(\mathbf{M}^Y) \\ \Im(\mathbf{M}^Y) \end{bmatrix} \quad (14)$$

where $\mathbf{X}_{\mathbf{s}^l}^Y := \langle (\bar{\mathbf{Y}}^l \mathbf{A} \hat{\mathbf{v}})^\top \mathbf{A} \rangle \mathbf{N}_2$ and $\mathbf{Y}_{\mathbf{s}^l}^Y := \langle (\mathbf{A} \hat{\mathbf{v}})^\top \bar{\mathbf{Y}}^l \mathbf{A} \rangle \mathbf{N}_2$. Here $\mathbf{N}_2 := \begin{bmatrix} \mathbf{1}_{1 \times 3n} & \mathbf{0}_{1 \times 3n} \\ \mathbf{0}_{1 \times 3n} & -\mathbf{1}_{1 \times 3n} \end{bmatrix}$ and $\langle \cdot \rangle$ are used with (8). Now by replacing $\mathbf{x}^Y$ with $\mathbf{x}^\Delta$ in (14), we obtain $\begin{bmatrix} \mathbf{M}_{p^l}^\Delta \\ \mathbf{M}_{q^l}^\Delta \end{bmatrix}$. Finally, as zero losses occur at $(\mathbf{w}, \mathbf{0})$, we set $(\mathbf{d}, \mathbf{e}) = (\mathbf{0}, \mathbf{0})$ and recover upper and lower half of (14) to complete approximations (10d) and (10e), respectively.

B. Approximate DLMP Model

Finally, the approximate DLMP model is constructed by replacing the non-convex constraints (9b)–(9f) in (9) with their approximate counterparts in (10). Note that as the cost functions (9a) are linear, the resultant approximate DLMP problem is a linear program (LP). This property is highly favorable in terms of solution robustness and computational efficiency, as numerous off-the-shelf solvers can readily solve large-scale LPs. With the above-mentioned modification, the Lagrangian of the approximate DLMP model follows:

$$\begin{aligned} \phi &= \mathbf{c}_{p_g}^\top \mathbf{p}_g^Y + \mathbf{c}_{p_g}^\top \mathbf{p}_g^\Delta + \mathbf{c}_{q_g}^\top \mathbf{q}_g^Y + \mathbf{c}_{q_g}^\top \mathbf{q}_g^\Delta + \mathbf{c}_{p_0}^\top \mathbf{p}_0 + \mathbf{c}_{q_0}^\top \mathbf{q}_0 \\ &\quad - \lambda_p \left(\mathbf{1}_3^\top \mathbf{p}_0 + \mathbf{1}_{3n}^\top (\mathbf{p}_g^Y + \mathbf{p}_g^\Delta - \mathbf{p}_d^Y - \mathbf{p}_d^\Delta) - \tilde{p}^d \right) \\ &\quad - \lambda_q \left(\mathbf{1}_3^\top \mathbf{q}_0 + \mathbf{1}_{3n}^\top (\mathbf{q}_g^Y + \mathbf{q}_g^\Delta - \mathbf{q}_d^Y - \mathbf{q}_d^\Delta) - \tilde{q}^d \right) \\ &\quad - (\boldsymbol{\mu}_{s^f}^+)^\top \left(|\tilde{\mathbf{s}}^f|^2 - (\mathbf{s}^f)^2 \right) - (\boldsymbol{\mu}_{s^t}^+)^\top \left(|\tilde{\mathbf{s}}^t|^2 - (\mathbf{s}^t)^2 \right) \\ &\quad - (\boldsymbol{\mu}_v^+)^\top \left(|\tilde{\mathbf{v}}| - \mathbf{v}^+ \right) + (\boldsymbol{\mu}_v^-)^\top \left(|\tilde{\mathbf{v}}| - \mathbf{v}^- \right) \end{aligned}$$

$$\begin{aligned}
& -(\mu_{p_g^+}^+)^T (\mathbf{p}_g^Y - \mathbf{p}_g^{Y+}) + (\mu_{p_g^-}^-)^T (\mathbf{p}_g^Y - \mathbf{p}_g^{Y-}) \\
& -(\mu_{q_g^+}^+)^T (\mathbf{q}_g^Y - \mathbf{q}_g^{Y+}) + (\mu_{q_g^-}^-)^T (\mathbf{q}_g^Y - \mathbf{q}_g^{Y-}) \\
& -(\mu_{p_g^\Delta}^+)^T (\mathbf{p}_g^\Delta - \mathbf{p}_g^{\Delta+}) + (\mu_{p_g^\Delta}^-)^T (\mathbf{p}_g^\Delta - \mathbf{p}_g^{\Delta-}) \\
& -(\mu_{q_g^\Delta}^+)^T (\mathbf{q}_g^\Delta - \mathbf{q}_g^{\Delta+}) + (\mu_{q_g^\Delta}^-)^T (\mathbf{q}_g^\Delta - \mathbf{q}_g^{\Delta-}). \quad (15)
\end{aligned}$$

DLMPs are defined as the marginal value of providing incremental demand at a specified bus. Using this definition, we can obtain for wye connections, active power DLMPs Π_p^Y as:

$$\begin{aligned}
\Pi_p^Y &:= \frac{\partial \phi}{\partial \mathbf{p}_d^Y} = \underbrace{\Pi_p^{Y(e)}}_{\lambda_p \mathbf{1}_{3n}} - \underbrace{\Pi_p^{Y(l)}}_{\lambda_p \mathbf{M}_{p^l}^{Y(p)} - \lambda_q \mathbf{M}_{p^l}^{Y(q)}} \\
&\quad + \underbrace{\Pi_p^{Y(c)}}_{(\mathbf{M}_{sf}^{Y(p)})^T \mu_{sf}^+ + (\mathbf{M}_{st}^{Y(p)})^T \mu_{st}^+ + (\mathbf{M}_{|v|}^{Y(p)})^T (\mu_v^+ + \mu_v^-)} + \underbrace{\Pi_p^{Y(v)}}_{(\mathbf{M}_{sf}^{Y(q)})^T \mu_{sf}^+ + (\mathbf{M}_{st}^{Y(q)})^T \mu_{st}^+ + (\mathbf{M}_{|v|}^{Y(q)})^T (\mu_v^+ - \mu_v^-)}, \quad (16)
\end{aligned}$$

and reactive power DLMPs Π_q^Y as:

$$\begin{aligned}
\Pi_q^Y &:= \frac{\partial \phi}{\partial \mathbf{q}_d^Y} = \underbrace{\Pi_q^{Y(e)}}_{\lambda_q \mathbf{1}_{3n}} - \underbrace{\Pi_q^{Y(l)}}_{\lambda_p \mathbf{M}_{q^l}^{Y(p)} - \lambda_q \mathbf{M}_{q^l}^{Y(q)}} \\
&\quad + \underbrace{\Pi_q^{Y(c)}}_{(\mathbf{M}_{sf}^{Y(q)})^T \mu_{sf}^+ + (\mathbf{M}_{st}^{Y(q)})^T \mu_{st}^+ + (\mathbf{M}_{|v|}^{Y(q)})^T (\mu_v^+ - \mu_v^-)} + \underbrace{\Pi_q^{Y(v)}}_{(\mathbf{M}_{sf}^{Y(p)})^T \mu_{sf}^+ + (\mathbf{M}_{st}^{Y(p)})^T \mu_{st}^+ + (\mathbf{M}_{|v|}^{Y(p)})^T (\mu_v^+ + \mu_v^-)}, \quad (17)
\end{aligned}$$

with the above deployed sub-matrices obtained from matrices in (10) as follows:

$$\begin{aligned}
\mathbf{M}_{|v|}^Y &:= (\mathbf{M}_{|v|}^{Y(p)}, \mathbf{M}_{|v|}^{Y(q)}), \\
\mathbf{M}_{sf}^Y &:= (\mathbf{M}_{sf}^{Y(p)}, \mathbf{M}_{sf}^{Y(q)}), \mathbf{M}_{st}^Y := (\mathbf{M}_{st}^{Y(p)}, \mathbf{M}_{st}^{Y(q)}) \\
\mathbf{M}_{p^l}^Y &:= (\mathbf{M}_{p^l}^{Y(p)}, \mathbf{M}_{p^l}^{Y(q)}), \mathbf{M}_{q^l}^Y := (\mathbf{M}_{q^l}^{Y(p)}, \mathbf{M}_{q^l}^{Y(q)}) \quad (18)
\end{aligned}$$

For delta connections DLMPs, similar expressions exist after simply evaluating for active power: $\Pi_p^\Delta := \frac{\partial \phi}{\partial \mathbf{p}_d^\Delta}$ and for reactive power: $\Pi_q^\Delta := \frac{\partial \phi}{\partial \mathbf{q}_d^\Delta}$.

C. DLMP Features

Note that in (16)/(17), the final DLMP Π_p^Y/Π_q^Y decomposes nicely into its respective 1) energy $\Pi_p^{Y(e)}/\Pi_q^{Y(e)}$, 2) system losses $\Pi_p^{Y(l)}/\Pi_q^{Y(l)}$, 3) line congestion $\Pi_p^{Y(c)}/\Pi_q^{Y(c)}$ and 4) voltage violation $\Pi_p^{Y(v)}/\Pi_q^{Y(v)}$ components. This decomposition not only represents various important grid conditions into DLMP modeling but may also help in standardizing DLMP models for their future practical realization. As an example, one can observe that matrices $\mathbf{M}_{p^l}^Y$ and $\mathbf{M}_{sf}^Y$ are nothing else but multiphase variants of famous Loss Factor (LF) and Power Transfer Distribution Factor (PTDF) matrices, deployed heavily in developing LMP models [1].

IV. SIMULATION SETUP & RESULTS

The proposed method is tested on the multiphase and unbalanced IEEE 13-bus radial grid [11], with addition of DGs, as shown in Fig. 2. The DGs are operated as follows: 1) all DGs can control their active and reactive power independently; 2) per-phase maximum and minimum active and reactive

power dispatch for all DGs is constrained within $[0, 0.5]$ MW and $[-0.25, 0.25]$ MVar and; 3) marginal cost of active and reactive power for all DGs as well as at the slack-bus is set at 100 \$/MWh and 50 \$/MVarh. Relevant grid approximation is given in Sec. IV-A, whereas DLMPs for selected grid points are presented in Sec. IV-B.

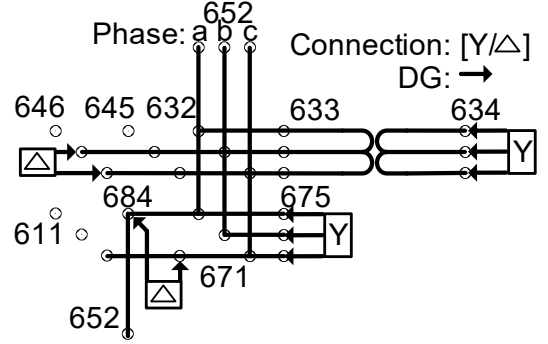


Fig. 2. The modified IEEE 13-bus radial distribution grid, with total rated load of 3.577 MW and 1.78 MVar [11]. For phase a, b and c, the total active (MW)/reactive (MVar) loading is 1.26/0.68, 0.97/0.37 and 1.35/0.67, respectively. For this case, the delta connected DGs (bus 646 and 684) are simply phase-phase connections. The grounded wye connected DGs (bus 675 and 634) have adjustable per-phase dispatch.

A. Multiphase Approximation Model

We analyze the presented approximation procedure of Sec. III-A by varying grid loading as: $\kappa \mathbf{s}^{\text{ref}}$, with $\kappa \in [-0.5, 1.5]$ and $\mathbf{s}^{\text{ref}}$ as the rated loading level of the grid [11]. We set DG outputs to zero in this analysis. Voltage magnitude approximations follow exactly similar results as given in [9], and hence are not presented here. In Fig. 3, approximation of system active $\tilde{p}^l$ and reactive $\tilde{q}^l$ power losses along with squared “from” line flow for the heaviest loaded phases of line $l \in \{632, 671\}$ i.e., $|\tilde{s}^{f, \{a, c\}}|^2_l$, are shown. Fig. 3 verifies that the approximation model captures global behavior, i.e., it makes an interpolation between the no-load ($\mathbf{w}, \mathbf{0}$) and load-flow solution ($\hat{\mathbf{v}}, \mathbf{s}^{\text{ref}}$). Load-flow solutions are obtained from fixed-point method (2), which has been validated in [9].

B. Multiphase DLMP Model

We obtain DLMPs using the approximated DLMP model developed in Sec. III-B. To demonstrate the full capability of the developed model, we enforce the grid voltage limit of 1.06 p.u. and squared “from”/“to” line flow limit of 0.4 MVA². The active and reactive power outputs of DGs are presented in Table I. Note that DGs are not dispatched equally, even though they share the exact same marginal cost. The most interesting case is for DG located at 675b, having zero output for both active and reactive power injections. This is because bus 675b has very small active power demand and a large reactive injection (capacitor-bank). Hence, any marginal injection at bus 675b is going to increase the overall grid dispatch cost. This makes DG at bus 675b to dispatch zero active and reactive power.

For selected buses with wye and delta connection, Table II and III respectively present active and reactive power DLMPs (in bold) along with their breakdown in terms of their energy, congestion, losses, and voltage components. Moreover,

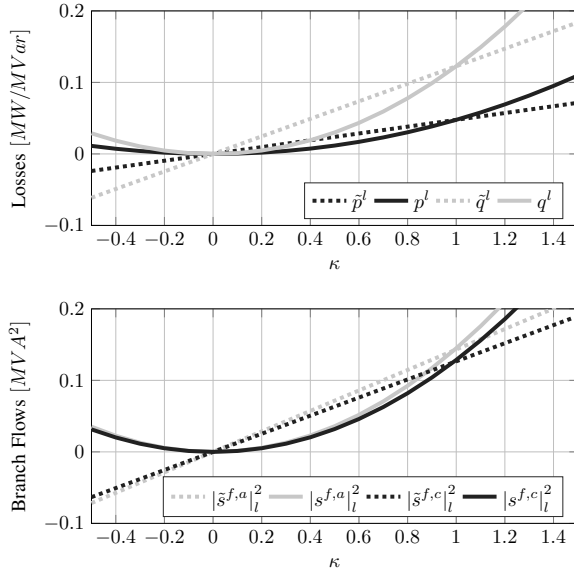


Fig. 3. Approximation of active and reactive power losses of the grid (top) and squared “from” line flow (bottom).

the respective DG’s marginal supply, i.e. $\frac{\partial \phi}{\partial \mathbf{p}_g^{Y/\Delta}} = \mathbf{c}_{\mathbf{p}_g^{Y/\Delta}} + \mu_{\mathbf{p}_g^{Y/\Delta}}^+ - \mu_{\mathbf{p}_g^{Y/\Delta}}^-$ for active power and $\frac{\partial \phi}{\partial \mathbf{q}_g^{Y/\Delta}} = \mathbf{c}_{\mathbf{q}_g^{Y/\Delta}} + \mu_{\mathbf{q}_g^{Y/\Delta}}^+ - \mu_{\mathbf{q}_g^{Y/\Delta}}^-$ for reactive power is also shown in Table II and III.

Table I. Active (MW) and Reactive Power (MVar) Dispatch from DGs.

	675a	675b	675c	634a	634b	634c	646bc	684ca
$\mathbf{p}_g^Y/\mathbf{p}_g^\Delta$	0.500	0	0.500	0.500	0.500	0.500	0.465	0.500
$\mathbf{q}_g^Y/\mathbf{q}_g^\Delta$	0.250	0	0.250	0.205	0.250	0.123	0.870	0.237

Table II. Active Power DLMPs (\$/MWh) for wye- and delta-connection.

Bus	$\Pi_{\mathbf{p}}^{Y(e)}$	$\Pi_{\mathbf{p}}^{Y(l)}$	$\Pi_{\mathbf{p}}^{Y(c)}$	$\Pi_{\mathbf{p}}^{Y(v)}$	$\Pi_{\mathbf{p}}^Y$	$\mathbf{c}_{\mathbf{p}_g^Y} \mu_{\mathbf{p}_g^Y}^+ - \mu_{\mathbf{p}_g^Y}^-$
675a		0.717	1.530	-0.040	102.13	2.133
675b		0.071	-0.014	-0.005	99.978	-0.021
675c		0.282	0.017	0.015	100.24	0.240
634a	99.92	0.253	-0.014	-0.129	100.03	0.030
634b		0.177	-0.014	-0.006	100.08	0.080
634c		0.175	0.017	-0.034	100.08	0.080
Bus	$\Pi_{\mathbf{p}}^{\Delta(e)}$	$\Pi_{\mathbf{p}}^{\Delta(l)}$	$\Pi_{\mathbf{p}}^{\Delta(c)}$	$\Pi_{\mathbf{p}}^{\Delta(v)}$	$\Pi_{\mathbf{p}}^\Delta$	$\mathbf{c}_{\mathbf{p}_g^\Delta} \mu_{\mathbf{p}_g^\Delta}^+ - \mu_{\mathbf{p}_g^\Delta}^-$
646bc		0.096	-0.002	-0.020	100.00	0
684ca	99.92	0.410	0.943	-0.015	101.26	1.260

As sum of both left and right side components to the final DLMPs (in bold) are equal in Table II & III, this validates that the obtained DLMPs represent true marginal value of the delivered power. Also, one can observe that each DLMP component has a physical interpretation. For example, congestion components $\Pi_{\mathbf{q}}^{Y(c)}$ and $\Pi_{\mathbf{q}}^{Y(v)}$ at locations 675a and 684ca are high, as they are the closest to the congested line {632a-671a}. Similarly, the voltage magnitude is binding at location 634a and 634c, which is immediately reflected in their respective $\Pi_{\mathbf{q}}^{Y(v)}$. Note that the negative $\Pi_{\mathbf{q}}^{Y(v)}$ shows that the marginal increase in reactive power demand (decrease in reactive power injection) improves the overall grid dispatch

Table III. Reactive Power DLMPs (\$/MVarh) for wye- and delta-connection.

Bus	$\Pi_{\mathbf{q}}^{Y(e)}$	$\Pi_{\mathbf{q}}^{Y(l)}$	$\Pi_{\mathbf{q}}^{Y(c)}$	$\Pi_{\mathbf{q}}^{Y(v)}$	$\Pi_{\mathbf{q}}^Y$	$\mathbf{c}_{\mathbf{q}_g^Y} \mu_{\mathbf{q}_g^Y}^+ - \mu_{\mathbf{q}_g^Y}^-$
675a		0.436	0.581	-0.056	50.741	0.741
675b		-0.128	0.014	0.030	49.704	-0.296
675c		0.567	0.003	-0.046	50.311	0.311
634a	49.79	0.456	-0.039	-0.204	50	0
634b		0.345	0.014	0.037	50.184	0.184
634c		0.360	0.002	-0.148	50	0
Bus	$\Pi_{\mathbf{q}}^{\Delta(e)}$	$\Pi_{\mathbf{q}}^{\Delta(l)}$	$\Pi_{\mathbf{q}}^{\Delta(c)}$	$\Pi_{\mathbf{q}}^{\Delta(v)}$	$\Pi_{\mathbf{q}}^\Delta$	$\mathbf{c}_{\mathbf{q}_g^\Delta} \mu_{\mathbf{q}_g^\Delta}^+ - \mu_{\mathbf{q}_g^\Delta}^-$
646bc		0.226	0	-0.013	50	0
684ca	49.79	0.388	-0.140	-0.035	50	0

cost. This also explains the reason for DGs at location 634a, 634c, 646bc, 684c to dispatch reactive power below their maximum limits.

V. CONCLUSION & FUTURE WORKS

This paper proposed a multiphase DLMP model, which utilized approximated losses, line flows and voltages, for fully decomposing into its respective components. Moreover, the developed model provided both active and reactive power DLMPs, along with the consideration of wye/delta connections and unbalanced loadings. With these features, we believe that the developed model is able to comprehensively represent distribution grid conditions in DLMPs.

In future, the developed model can be effectively deployed in designing local distribution grid markets, aimed for handling both instantaneous (real-time) and planned scheduling horizon (day-ahead) along with energy storage systems.

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