

OPTIMAL CONSUMPTION OF MULTIPLE GOODS IN INCOMPLETE MARKETS

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ABSTRACT. We consider the problem of optimal consumption of multiple goods in incomplete semimartingale markets. We formulate the dual problem and identify conditions that allow for existence and uniqueness of the solution and give a characterization of the optimal consumption strategy in terms of the dual optimizer. We illustrate our results with examples in both complete and incomplete models. In particular, we construct closed-form solutions in some incomplete models.

1. INTRODUCTION

The problem of optimal consumption of multiple goods has been investigated in [5, 2]. For a single consumption good in continuous-time settings, it was first formulated in [17]. Since then, this problem was analyzed in a large number of papers in both complete and incomplete settings with a range of techniques based on Hamilton-Jacobi-Bellman equations, backward stochastic differential equations, and convex duality being used for its analysis.

In the present paper, we formulate a problem of optimal consumption of multiple goods in a general incomplete semimartingale model of a financial market. We construct the dual problem and characterize optimal consumption policies in terms of the solution to the dual problem. We also identify mathematical conditions, that allow for existence and uniqueness of the solution and a dual characterization. We illustrate our results by examples, where in particular we obtain closed-form solutions in incomplete markets.

Our proofs rely on certain results on weakly measurable correspondences for Carathéodory functions, multidimensional convex-analytic techniques, and some recent advances in stochastic analysis in mathematical finance, in particular, the characterization of the “no unbounded profit with bounded risk” condition in terms of non-emptiness of the set of equivalent local martingale deflators from [3, 9] and sharp conditions for solvability of the expected utility maximization problem in a single good setting from [19]. Measurability-wise the price processes of consumption

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goods are only needed to be optional. Strict positivity is also required, but no boundedness away from zero or infinity is supposed. Comparing to [3, 19], apart from the conditions on the price process of the consumption goods that are unequivocal in single-good settings, one of the leading challenges in the present work was handling the multidimensionality of the utility process in the spatial variable. The key step in here is an introduction of an auxiliary utility process (see (2.7) below) and its representation as a pointwise image function (in the convex-analytic sense) of the original multivalued utility process under the corresponding linear transformation, which is also identified. Further, a challenging point in the proofs that only emerges in the case of multiple goods, but not in single-good settings, is the measurability of the candidate optimizer. It is proven via establishing weak measurability of a certain correspondence, in the spirit of [1]. Here the model assumptions, in particular, the aforementioned optionality and strict positivity of the consumption goods processes, play an important role. Finally, the present paper provides a general framework for analyzing numerous questions related to stability and asymptotics in the multiple goods settings from both mathematical and economic viewpoints.

The remainder of this paper is organized as follows: in Section 2 we specify the model setting, formulate the problem, and state main results (in Theorem 2.4). In Section 3 we discuss various specific cases. In particular, we present there the structure of the solution in complete models and the additive utility case as well as closed-form solutions in some incomplete models (with and without an additive structure of the utility). We conclude the paper with Section 4, which contains proofs.

2. SETTING AND MAIN RESULTS

2.1. Setting. Let $\tilde{S} = (\tilde{S}_t)_{t \geq 0}$ an $\mathbb{R}^d$ -valued semimartingale, representing the discounted prices¹ of d risky assets on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$, with $\mathcal{F}_0$ being the trivial σ -algebra. We fix a *stochastic clock* $\kappa = (\kappa_t)_{t \geq 0}$, which is a nondecreasing, càdlàg, adapted process, such that

$$(2.1) \quad \kappa_0 = 0, \quad \mathbb{P}(\kappa_\infty > 0) > 0 \quad \text{and} \quad \kappa_\infty \leq \bar{A},$$

where $\bar{A}$ is a positive constant. The stochastic clock κ specifies times when consumption is assumed to occur. Various optimal investment-consumption problems can be recovered from the present general setting by suitably specifying the clock process κ . For example, the problem of maximizing expected utility of terminal wealth at some finite investment horizon $T < \infty$ can be recovered by simply letting $\kappa \triangleq \mathbb{I}_{[T, \infty[}$. Likewise, maximization of expected utility from consumption only up to a finite horizon $T < \infty$ can be obtained by letting $\kappa_t \triangleq \min(t, T)$, for $t \geq 0$. Other specifications include maximization of utility from lifetime consumption, from consumption at a finite set of stopping times, and from terminal wealth at a random horizon, see e.g., [19, Examples 2.5-2.9] for a description of possible standard choices of the clock process κ .

¹Since we allow preferences to be stochastic (see the definition below), there is no loss of generality in assuming that asset prices are discounted, see [19, Remark 2.2] for a more detailed explanation of this observation.

We suppose that there are m different consumption goods, where S_t^k denotes the discounted price of commodity k at time t . We assume that for each $k \in \{1, \dots, m\}$, $S^k = (S_t^k)_{t \geq 0}$ is a *strictly* positive optional processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$. We denote $S = (S^1, \dots, S^m)$.

A *portfolio* is defined by a triplet $\Pi = (x, H, c)$, where $x \in \mathbb{R}$ represents an initial capital, $H = (H_t)_{t \geq 0}$ is a d -dimensional $\tilde{S}$ -integrable process, H_t^j represents the holdings in the j -th risky asset at time t , $j \in \{1, \dots, d\}$, $t \geq 0$, c is an m -dimensional *consumption process*, whose every component $(c_t^k)_{t \geq 0}$ is a nonnegative optional process representing the consumption *rate* of commodity k , $k = \{1, \dots, m\}$. The wealth process $X = (X_t)_{t \geq 0}$ of a portfolio $\Pi = (x, H, c)$ is defined as

$$(2.2) \quad X_t \triangleq x + \int_0^t H_u \, d\tilde{S}_u - \int_0^t c_u \cdot S_u \, d\kappa_u, \quad t \geq 0,$$

where $\cdot$ denotes the dot product in $\mathbb{R}^m$.

2.2. Absence of arbitrage. The main objective of this part is to specify the no-arbitrage type condition (NUPBR) below. As it is commonly done in the literature (see for example [16]), we begin defining $\mathcal{X}$ to be the collection of all nonnegative wealth processes associated to portfolios of the form $\Pi = (1, H, 0)$, i.e.,

$$\mathcal{X} \triangleq \left\{ X \geq 0 : X_t = 1 + \int_0^t H_u \, d\tilde{S}_u, \quad t \geq 0 \right\}.$$

In this paper, we suppose the following no-arbitrage-type condition:

$$(NUPBR) \quad \text{the set } \mathcal{X}_T \triangleq \{X_T : X \in \mathcal{X}\} \text{ is bounded in probability, for every } T \in \mathbb{R}_+,$$

where (NUPBR) stands for *no unbounded profit with bounded risk*. This condition was originally introduced in [10]. It is proven in [12, Proposition 1], that (NUPBR) is equivalent to another (weak) no-arbitrage condition, namely absence of *arbitrages of the first kind* on $[0, T]$, see [14, Definition 1].

A useful characterization of (NUPBR) is given via the set of *equivalent local martingale deflators* (ELMD) that is defined as follows:

$$(2.3) \quad \mathcal{Z} \triangleq \left\{ Z > 0 : \begin{array}{l} Z \text{ is a càdlàg local martingale such that } Z_0 = 1 \text{ and} \\ ZX = (Z_t X_t)_{t \geq 0} \text{ is a local martingale for every } X \in \mathcal{X} \end{array} \right\}.$$

It is proven in [3, Proposition 2.1] (see also [9]) that condition (NUPBR) holds if and only if $\mathcal{Z} \neq \emptyset$. This result was previously established in the one-dimensional case in the finite time horizon in [13, Theorem 2.1]. Also, [24, Theorem 2.6] contains a closely related result (in a finite time horizon) in terms of *strict σ -martingale densities*, see [24] for the corresponding definition and details.

Remark 2.1. Condition (NUPBR) is weaker than the existence of an equivalent martingale measure (see for example [4, p. 463] for the definition an equivalent martingale measure), another classical no-arbitrage type assumption, which in the infinite time horizon is even stronger than

$$(2.4) \quad \{Z \in \mathcal{Z} : Z \text{ is a martingale}\} \neq \emptyset.$$

Note that in the *finite time horizon* setting, (2.4) is equivalent to the existence of an equivalent martingale measure. Besides, (2.4) is apparently stronger than (NUPBR) (by comparison of (2.3) and (2.4) combined with [3, Proposition 2.1]). We also would like to point out that (2.4) holds in every original formulation of [17], where the problem of optimal consumption from investment (in a single consumption good setting) was introduced, including the infinite-time horizon case. In general, (2.4) can be stronger than (NUPBR). A classical example, where (NUPBR) holds but (2.4) fails, corresponds to the three-dimensional Bessel process driving the stock price, see e.g., [10, Example 4.6].

2.3. Admissible consumptions. For a given initial capital $x > 0$, an m -dimensional optional consumption process c is said to be x -admissible if there exists an $\mathbb{R}^d$ -valued predictable $\tilde{S}$ -integrable process H such that the wealth process X in (2.2), corresponding to the portfolio $\Pi = (x, H, c)$ is nonnegative; the set of x -admissible consumption processes corresponding to a stochastic clock κ is denoted by $\mathcal{A}(x)$. For brevity, we denote $\mathcal{A} \triangleq \mathcal{A}(1)$.

2.4. Preferences of a rational economic agent. Building from the formulation of [18], we assume that preferences of a rational economic agent are represented by a *optional utility-valued process* (or simply a *utility process*) $U = U(t, \omega, x) : [0, \infty) \times \Omega \times [0, \infty)^m \rightarrow \mathbb{R} \cup \{-\infty\}$, where for every $(t, \omega) \in [0, \infty) \times \Omega$, $U(t, \omega, \cdot)$ is an Inada-type utility function, i.e., $U(t, \omega, \cdot)$ satisfies the following (technical) assumption.

Assumption 2.2. For every $(t, \omega) \in [0, \infty) \times \Omega$, the function

$$\mathbb{R}_+^m \ni x \mapsto U(t, \omega, x) \in \mathbb{R} \cup \{-\infty\}$$

is strictly concave, strictly increasing in every component, finite-valued and continuously differentiable in the interior of the positive orthant, and satisfies the Inada conditions

$$\lim_{x_i \downarrow 0} \partial_{x_i} U(t, \omega, x) = \infty \quad \text{and} \quad \lim_{x_i \uparrow \infty} \partial_{x_i} U(t, \omega, x) = 0, \quad i = 1, \dots, m,$$

where $\partial_{x_i} U(t, \omega, \cdot) : \mathbb{R}_+^m \mapsto \mathbb{R}$ is the partial derivative of $U(t, \omega, \cdot)$ with respect to the i -th spatial variable². On the boundary of the first orthant, by upper semicontinuity, we suppose that $U(t, \omega, x) = \limsup_{x' \rightarrow x} U(t, \omega, x')$ (note that some of these values may be $-\infty$ and that $U(t, \omega, x) = \lim_{t \downarrow 0} U(t, \omega, x + t(x' - x))$, where x' is an arbitrary element in the interior of the first orthant, see [7, Proposition B.1.2.5]). Finally, for every $x \in \mathbb{R}_+^m$, we assume that the stochastic process $U(\cdot, \cdot, x)$ is optional.

Remark 2.3. The Inada conditions in Assumption 2.2 were introduced in [8]. These are technical assumptions that have natural economic interpretations and that allow for a deeper tractability of the problem (as e.g., in [16]). Likewise, the semicontinuity of U is imposed for regularity purposes. It also used in e.g., [22, 23].

²For the results below, we only need to specify the gradient of $U(t, \omega, \cdot)$ in the *interior* of the first orthant, i.e., at the points $x \in \mathbb{R}^m$, where $U(t, \omega, x)$ is (finite-valued and) differentiable.

In particular, modeling preferences via utility process allows to take into account utility maximization problems under a change of numéraire (see e.g., [20, Example 4.2]). This is the primary reason why we suppose that the prices of the traded stocks are discounted, as this allows to simplify notations without any loss of generality. Note also that Assumption 2.2 does not make any requirement on the *asymptotic elasticity* of U , introduced in [16].

To a utility process U satisfying Assumption 2.2, we associate the *primal value function*, defined as

$$(2.5) \quad u(x) \triangleq \sup_{\mathbf{c}=(c^1, \dots, c^m) \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty U(t, \omega, \mathbf{c}_t(\omega)) d\kappa_t \right], \quad x > 0.$$

To ensure that the integral above is well-defined, we use the convention

$$(2.6) \quad \mathbb{E} \left[\int_0^\infty U(t, \omega, \mathbf{c}_t(\omega)) d\kappa_t \right] \triangleq -\infty \quad \text{if} \quad \mathbb{E} \left[\int_0^\infty U^-(t, \omega, \mathbf{c}_t(\omega)) d\kappa_t \right] = \infty,$$

where $U^-(t, \omega, \cdot)$ is the negative part of $U(t, \omega, \cdot)$. Note that formulation (2.5) is a generalization of the formulation in [18, p. 205]. In the form (2.5), we allow for stochastic preferences and include several standard formulations as particular cases.

2.5. Dual problem. In order to specify model assumptions that ensure existence and uniqueness of solutions to (2.5) and to give a characterization of this solution, we need to formulate an appropriate dual problem. Let us define

$$(2.7) \quad U^*(t, \omega, x) \triangleq \sup_{\substack{c \in \mathbb{R}_+^m: \\ c \cdot S_t(\omega) \leq x}} U(t, \omega, c), \quad (t, \omega, x) \in [0, \infty) \times \Omega \times [0, \infty).$$

Let us set a family of transformations $A : [0, \infty) \times \Omega \times \mathbb{R}^m \mapsto \mathbb{R}$, as

$$A(t, \omega, c) \triangleq c \cdot S_t(\omega), \quad (t, \omega, c) \in [0, \infty) \times \Omega \times [0, \infty)^m.$$

Note that for every $(t, \omega) \in [0, \infty) \times \Omega$, $A(t, \omega, \cdot)$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}$ and $U^*(t, \omega, \cdot)$ is the image of $U(t, \omega, \cdot)$ under $A(t, \omega, \cdot)$ (see e.g., [7, p. 96] for the definition and properties of the *image of a function under a linear mapping*³). We define a stochastic field V^* as the pointwise conjugate of U^* (equivalently, as the pointwise conjugate of the image function of U under A) in the sense that

$$V^*(t, \omega, y) \triangleq \sup_{x > 0} (U^*(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, \infty) \times \Omega \times [0, \infty),$$

where $\sup_{x > 0}$ and $\sup_{x \geq 0}$ coincide thanks to continuity of U^* established in Lemma 4.1. We also introduce the following set of dual processes:

$$\mathcal{Y}(y) \triangleq \text{cl}\{Y : Y \text{ is càdlàg adapted and} \\ 0 \leq Y \leq yZ \text{ (d}\kappa \times \mathbb{P}\text{)-a.e. for some } Z \in \mathcal{Z}\},$$

³Equivalently, see [21, Theorem 5.2], where $U^*(t, \omega, \cdot)$ is named the image of $U(t, \omega, \cdot)$ under the linear transformation $A(t, \omega, \cdot)$, $(t, \omega) \in [0, \infty) \times \Omega$.

where the closure is taken in the topology of convergence in measure ($d\kappa \times \mathbb{P}$) on the measure space of real-valued optional processes $([0, \infty) \times \Omega, \mathcal{O}, d\kappa \times \mathbb{P})$, where $\mathcal{O}$ is the optional sigma-field. We write $\mathcal{Y} \triangleq \mathcal{Y}(1)$ for brevity. Note that $\mathcal{Y}$ is closely related to - but different from - the set with the same name in [16]. The value function of the dual optimization problem, or equivalently, the *dual value function*, is then defined as

$$(2.8) \quad v(y) \triangleq \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V^*(t, \omega, Y_t(\omega)) d\kappa_t \right], \quad y > 0,$$

with the convention $\mathbb{E}[\int_0^\infty V^*(t, \omega, Y_t(\omega)) d\kappa_t] \triangleq \infty$ if $\mathbb{E}[\int_0^\infty V^{*+}(t, \omega, Y_t(\omega)) d\kappa_t] = \infty$, where $V^{*+}(t, \omega, \cdot)$ is the positive part of $V^*(t, \omega, \cdot)$. Note that, in the single-good, but otherwise similar, settings, properties of the dual value function are investigated in [19, 3]. We are now in a position to state the following theorem, which is the main result of this paper.

Theorem 2.4. *Assume that conditions (2.1) and (NUPBR) hold true and let U satisfies Assumption 2.2. Let us also suppose that*

$$(2.9) \quad v(y) < \infty \quad \text{for every } y > 0 \quad \text{and} \quad u(x) > -\infty \quad \text{for every } x > 0.$$

Then we have

- (i) $u(x) < \infty$, for every $x > 0$, and $v(y) > -\infty$, for every $y > 0$, i.e., the value functions are finite-valued.
- (ii) The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly concave, strictly increasing and satisfy the Inada conditions

$$(2.10) \quad \begin{aligned} \lim_{x \downarrow 0} u'(x) &= \infty, & \lim_{y \downarrow 0} -v'(y) &= \infty, \\ \lim_{x \rightarrow \infty} u'(x) &= 0, & \lim_{y \rightarrow \infty} -v'(y) &= 0. \end{aligned}$$

- (iii) For every $x > 0$ and $y > 0$, the solutions $\hat{c}(x)$ to (2.5) and $\hat{Y}(y)$ to (2.8) exist and are unique and, if $y = u'(x)$, we have the optimality characterizations

$$(2.11) \quad \hat{Y}_t(y)(\omega) = \frac{\partial_{x_i} U(t, \omega, \hat{c}_t(x)(\omega))}{S_t^i(x)(\omega)}, \quad (d\kappa \times \mathbb{P})\text{-a.e.}, \quad i = 1, \dots, m.$$

and

$$(2.12) \quad \hat{Y}_t(y)(\omega) = U_x^*(t, \omega, \hat{c}_t(x)(\omega) \cdot S_t(\omega)), \quad (d\kappa \times \mathbb{P})\text{-a.e.},$$

with U_x^* denoting the partial derivative of U^* with respect to its third argument.

- (iv) For every $x > 0$, the constraint x is binding in the sense that

$$(2.13) \quad \mathbb{E} \left[\int_0^\infty \hat{c}_t(x) \cdot S_t \frac{\hat{Y}_t(y)}{y} d\kappa_t \right] = x, \quad \text{where } y = u'(x).$$

- (v) The functions u and v are Legendre conjugate, i.e.,

$$(2.14) \quad v(y) = \sup_{x > 0} (u(x) - xy), \quad y > 0, \quad u(x) = \inf_{y > 0} (v(y) + xy), \quad x > 0.$$

(vi) The dual value function v can be represented as

$$(2.15) \quad v(y) = \inf_{Z \in \mathcal{Z}} \mathbb{E} \left[\int_0^\infty V(t, \omega, yZ_t(\omega)) d\kappa_t(\omega) \right], \quad y > 0.$$

Remark 2.5 (On sufficient conditions for the validity of (2.9)). Condition (2.9) holds if there exists one primal element $c \in \mathcal{A}$ and one dual element $Y \in \mathcal{Y}$ such that

$$\mathbb{E} \left[\int_0^\infty U(t, \omega, zc_t(\omega)) d\kappa_t \right] > -\infty \quad \text{and} \quad \mathbb{E} \left[\int_0^\infty V^*(t, \omega, zY_t(\omega)) d\kappa_t \right] < \infty, \quad z > 0.$$

In particular, for every $x > 0$, as an m -dimensional optional process with constant values $(\frac{x}{Am}, \dots, \frac{x}{Am})$ belongs to $\mathcal{A}(x)$, a sufficient condition in (2.9) for the finiteness of u is

$$\mathbb{E} \left[\int_0^\infty U(t, \omega, \frac{x}{Am}, \dots, \frac{x}{Am}) d\kappa_t \right] > -\infty, \quad x > 0,$$

which typically holds if U is nonrandom. Likewise, as $\mathcal{Z} \neq \emptyset$ (by (NUPBR) and [3, Proposition 2.1]), finiteness of v holds if for one equivalent local martingale deflator Z , we have

$$\mathbb{E} \left[\int_0^\infty V^*(t, \omega, yZ_t(\omega)) d\kappa_t \right] < \infty, \quad y > 0.$$

3. EXAMPLES

COMPLETE MARKET SOLUTION AND DUAL CHARACTERIZATION

If the model is complete, the dual characterization of the optimal consumption policies has a particularly nice form, as $\mathcal{Z}$ contains a unique element, Z . The solutions corresponding to different y 's in the dual problem (2.8) are yZ , $y > 0$. Therefore, in (2.12) and (2.11) we have $\hat{Y}(y) = yZ$, $y > 0$.

SPECIAL CASE: ADDITIVE UTILITY

An important example of U^* corresponds to U having an additive form with respect to its spatial components, i.e., when

$$U(t, \omega, c_1, \dots, c_m) = U^1(t, \omega, c_1) + \dots + U^m(t, \omega, c_m), \quad (t, \omega) \in [0, \infty) \times \Omega,$$

where for every $k = 1, \dots, m$, U^k is a utility process in the sense of [19, Assumption 2.1] and a utility process in sense of the Assumption 2.2 with $m = 1$. In this case, for every $(t, \omega) \in [0, \infty) \times \Omega$, $U^*(t, \omega, \cdot)$ is given by the *infimal convolution* of $U^k(t, \omega, \cdot)$'s, see the definition in e.g., [21, p. 34]. Let $V^i(t, \omega, \cdot)$ denote the convex conjugate of $U^i(t, \omega, \cdot)$, $i = 1, \dots, m$. Then the convex conjugate of $U^*(t, \omega, \cdot)$ is $V^*(t, \omega, \cdot)$ given by

$$V^*(t, \omega, \cdot) = V^1(t, \omega, \cdot) + \dots + V^m(t, \omega, \cdot).$$

This result was established e.g., in [21, Theorem 16.4, p. 145]. In this case, the optimal $\hat{c}(x) = (\hat{c}^1(x), \dots, \hat{c}^m(x))$ has a more explicit characterization via $I_i(t, \omega, \cdot) \triangleq (U_x^i)^{-1}(t, \omega, \cdot)$, the pointwise inverse of the partial derivative of $U^i(t, \omega, \cdot)$ with respect to the third argument, as (2.11) can be solved for $\hat{c}^i(x)$, $i = 1, \dots, m$, as follows

$$(3.1) \quad \hat{c}_t^i(x)(\omega) = I_i \left(t, \omega, \hat{Y}_t(y)(\omega) S_t^i(\omega) \right), \quad (d\kappa \times \mathbb{P})\text{-a.e.}, \quad i = 1, \dots, m.$$

Using (2.12), we can restate (3.1) as

$$\widehat{c}_t^i(x)(\omega) = I_i(t, \omega, U_x^*(t, \omega, \widehat{c}_t^*(x)(\omega))S_t^i(\omega)), \quad (\mathrm{d}\kappa \times \mathbb{P})\text{-a.e.}, \quad i = 1, \dots, m,$$

where $\widehat{c}^*(x)$ is the optimizer to the auxiliary problem (4.2) corresponding to the initial wealth $x > 0$.

Remark 3.1. In the following three examples we consider some incomplete models that admit closed-form solutions for one good and show how these results apply to multiple good settings.

EXAMPLE OF A CLOSED FORM SOLUTION IN AN INCOMPLETE MODEL WITH ADDITIVE LOGARITHMIC UTILITY

Let us suppose that d traded discounted assets are modeled with Ito processes of the form

$$(3.2) \quad d\widetilde{S}_t^i = \widetilde{S}_t^i b_t^i dt + \widetilde{S}_t^i \sum_{j=1}^n \sigma_t^{ij} dW_t^j, \quad i = 1, \dots, d, \quad \widetilde{S}_0 \in \mathbb{R}^d,$$

where W is an $\mathbb{R}^n$ -valued standard Brownian motion and b^i , σ^{ij} , $i = 1, \dots, d$, $j = 1, \dots, n$, are predictable processes, such that the unique strong solution to (3.2) exists, see e.g., [11]. Let us suppose that there are m consumption goods and that the value function of a rational economic agent is given by

$$\sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^T e^{-\nu t} \log(c_1, \dots, c_m) dt \right], \quad x > 0,$$

(with the same convention as the one specified after (2.5)), where an impatience rate ν and a time horizon T are positive constants. Note that in this case $\kappa_t = \frac{1-e^{-\nu t}}{\nu}$, $t \in [0, T]$, i.e., κ is deterministic. Let us also suppose that there exists an $\mathbb{R}^d$ -valued process γ , such that

$$b_t - \sigma_t \sigma_t^T \gamma_t = 0 \quad (\mathrm{d}\kappa \times \mathbb{P}) - a.e.$$

Let $\mathcal{E}$ denotes the Doleans-Dade exponent. Then, using [6, Theorem 3.1 and Example 4.2] and Theorem 2.4, we get

$$\begin{aligned} \widehat{c}_t^*(x) &= \frac{x\nu}{1 - e^{-\nu T}} \mathcal{E} \left(\int_0^T \gamma_s^T d\widetilde{S}_s \right)_t, \quad x > 0, \\ \widehat{c}_t^i(x) &= \frac{\widehat{c}_t^*(x)}{S_t^i M}, \quad i = 1, \dots, m, \quad x > 0, \\ \widehat{Y}_t(y) &= \frac{y}{\mathcal{E} \left(\int_0^T \gamma_s^T d\widetilde{S}_s \right)_t}, \quad y > 0, \quad t \in [0, T]. \end{aligned}$$

EXAMPLE OF A CLOSED-FORM SOLUTION AND DUAL CHARACTERIZATION IN AN INCOMPLETE ADDITIVE CASE

Let us fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the augmentation of the filtration generated by a two-dimensional Brownian motion (W^1, W^2) . Let us suppose that there are two traded securities: a risk-free asset B , such that

$$B_t = e^{rt}, \quad t > 0,$$

where r is a nonnegative constant, and a risky stock $\tilde{S}$ with the dynamics

$$d\tilde{S}_t = \tilde{S}_t \mu_t dt + \tilde{S}_t \sigma_t dW_t^1, \quad t \geq 0, \quad \tilde{S}_0 \in \mathbb{R}_+,$$

where processes μ and σ are such that $\theta_t = \frac{\mu_t - r}{\sigma_t}$, $t \geq 0$, the market price of risk process, follows the Ornstein-Uhlenbeck process

$$d\theta_t = -\lambda_\theta(\theta_t - \bar{\theta})dt + \sigma_\theta \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \quad t \geq 0, \quad \theta_0 \in \mathbb{R}_+,$$

where λ_θ , σ_θ , and $\bar{\theta}$ are positive constants, $\rho \in (-1, 1)$. Let us also assume that κ corresponds to the expected utility maximization from terminal wealth, i.e., $\kappa = \mathbb{I}_{[T, \infty[}$, $T \in \mathbb{R}_+$, that there are m consumption goods, where S^i , $i = 1, \dots, m$, are *deterministic*, and

$$U(T, \omega, c_1, \dots, c_m) = \frac{c_1^p}{p} + \dots + \frac{c_m^p}{p}, \quad (c_1, \dots, c_m) \in \mathbb{R}_+^m, \quad \omega \in \Omega,$$

where $p < 0$. Let us set

$$q \triangleq \frac{p}{1-p}, \quad A \triangleq \sum_{i=1}^m (S_T^i)^{-q}, \quad \text{and} \quad B \triangleq A^{1-p}.$$

Then, by direct computations, we get

$$U^*(T, \omega, x) = \frac{x^p}{p} B, \quad x > 0.$$

Using the argument in [15], one can express the optimal trading strategy $\hat{H}(x)$ in a closed form in terms of a solution to a system of (nonlinear) ordinary differential equations (see [15, p. 147]), where $\hat{H}_t(x)$ is the number of shares of the risky asset in the portfolio at time t , $t \in [0, T]$. With $\hat{X}(x)$ such that

$$d\hat{X}_t(x) = \hat{H}_t(x) d\tilde{S}_t + (\hat{X}_t(x) - \hat{H}_t(x) \tilde{S}_t) r dt, \quad \hat{X}_0(x) = x,$$

using Theorem 2.4, we get

$$\begin{aligned} \hat{c}_T^*(x) &= \hat{X}_T(x), \quad x > 0, \\ \hat{Y}_T(y) &= \frac{y}{\mathbb{E}[(\hat{c}_T^*(1))^p]} (\hat{c}_T^*(1))^{p-1}, \quad y > 0, \\ \hat{c}_T^i(x) &= \frac{\hat{c}_T^*(x)}{A} (S_T^i)^{-(1+q)}, \quad x > 0. \end{aligned}$$

EXAMPLE OF A CLOSED-FORM SOLUTION AND DUAL CHARACTERIZATION IN AN INCOMPLETE NON-ADDITIVE CASE

Here we will suppose that $\kappa = \mathbb{I}_{[T, \infty[}$, where $T \in \mathbb{R}_+$, and let

$$U(t, \omega, c_1, c_2) = -\frac{c_1^{p_1}}{p_1} \frac{c_2^{p_2}}{p_2}, \quad p_1 < 0, p_2 < 0,$$

i.e., there are two consumption goods. One can see that $U(t, \omega, \cdot)$ is jointly concave, since the Hessian of $-U(t, \omega, \cdot)$ is positive definite on $\mathbb{R}_{++}^2$. We also extend $U(t, \omega, \cdot)$ to the boundary of $\mathbb{R}_+^2$ by $-\infty$. Then, with $p \triangleq p_1 + p_2 < 0$, U^* is given by

$$U^*(t, \omega, x) = \frac{x^p (-p_1)^{p_1-1} (-p_2)^{p_2-1}}{p (-p)^{p-1}} (S_t^1)^{-p_1} (S_t^2)^{-p_2}, \quad x > 0.$$

Let us define $G \triangleq \frac{(-p_1)^{p_1-1}(-p_2)^{p_2-1}}{(-p)^{p-1}}(S_T^1)^{-p_1}(S_T^2)^{-p_2}$. Then $U(T, \omega, x) = \frac{x^p}{p}G(\omega)$, $x > 0$. Let us suppose that W^1 and W^2 are two Brownian motions with a fixed correlation ρ such that $0 < |\rho| < 1$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the usual augmentation of the filtration generated by W^1 and W^2 and $(\mathcal{G}_t)_{t \geq 0}$ be the usual augmentation of the filtration generated by W^2 . We also assume that there is a bond B and a stock $\tilde{S}$ on the market. Their dynamics are given by

$$d\tilde{S}_t = \tilde{S}_t(\mu_t dt + \sigma_t dW_t^1), \quad \tilde{S}_0 \in \mathbb{R},$$

$$dB_t = B_t r_t dt, \quad B_0 = 1,$$

where the drift μ , volatility σ , and sport interest rate r are bounded, progressively measurable processes with respect to $(\mathcal{G}_t)_t \in [0, T]$, and σ is strictly positive.

Let us suppose that S_T^1 and S_T^2 are $\mathcal{G}_T$ -measurable random variables with moments of all orders. Then G is also $\mathcal{G}_T$ -measurable random variable with moments of all orders (by Hölder's inequality) and the auxiliary value function u^* defined in (4.2) satisfies the settings of [25]. Also, as $u^*(x) \geq \frac{x^p}{p}\mathbb{E}[G] > -\infty$ and since $V(T, \omega, \cdot)$ is negative-valued (thus, $v(y) \leq 0$), the assumption (2.9) holds.

Let us set

$$\lambda_t \triangleq \frac{\mu_t - r_t}{\sigma_t}, \quad \delta \triangleq \frac{1-p}{1-p+\rho^2 p}, \quad \frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \exp\left(-\frac{\rho^2 p^2}{2(1-p)^2} \int_0^T \lambda_s^2 ds + \frac{\rho p}{1-p} \int_0^T \lambda_s dW_s^2\right),$$

$$K_t \triangleq \frac{p}{(1-p)} \left(\lambda_t + \rho \delta \frac{\beta_t}{\mathbb{E}\mathbb{Q}[\exp(\int_0^T (r_s/\delta) ds) | \mathcal{F}_t]} \right), \quad t \in [0, T].$$

Then, using [25, Proposition 3.4] and Theorem 2.4, we deduce that

$$\hat{c}_T^*(x) = x \exp\left(\int_0^T (r + K_s \lambda_s - \frac{1}{2} K_s^2) ds + \int_0^T K_s dW_s^1\right), \quad x > 0,$$

$$\hat{Y}_T(y) = \frac{y}{\mathbb{E}[(\hat{c}_T^*(1))^p]} \exp\left(\int_0^T (p-1)(r + K_s \lambda_s - \frac{1}{2} K_s^2) ds + \int_0^T (p-1) K_s dW_s^1\right), \quad y > 0,$$

$$\hat{c}_T^i = \frac{\hat{c}_T^*(x) p_i}{p S_T^i}, \quad i = 1, 2, \quad x > 0,$$

are the optimizers to (2.8), (4.2), and (2.5), respectively. From Theorem 2.4, we conclude that for every $x > 0$, $\hat{c}_T^i(x)$, $i = 1, 2$, and $\hat{Y}_T(u'(x))$ are related via (2.11) and (2.12).

4. PROOFS

We begin from a characterization of the utility process U^* defined in (2.7).

Lemma 4.1. *Let U satisfies Assumption 2.2 and U^* be defined in (2.7). Then, U^* is an Inada-type utility process for $m = 1$ in the sense of Assumption 2.2.*

Proof. For every $(t, \omega) \in [0, \infty) \times \Omega$, as $U^*(t, \omega, \cdot)$ is an image function under an appropriate linear transformation of a concave function $U(t, \omega, \cdot)$, therefore using e.g., [7, Theorem B.2.4.2],

one can show that $U^*(t, \omega, \cdot)$ is concave. In order to show strict concavity of $U^*(t, \omega, \cdot)$, one can proceed as follows. First, for some positive numbers $x_1 \neq x_2$, let $\mathbf{c}^i \in \mathbb{R}_+^m$ be such that

$$(4.1) \quad \begin{aligned} \mathbf{c}^i \cdot S_t(\omega) &\leq x_i, \quad \text{and} \\ U^*(t, \omega, x_i) &= U(t, \omega, \mathbf{c}^i), \quad i = 1, 2. \end{aligned}$$

The existence of such $\mathbf{c}^i$'s follows from compactness of the domain of the optimization problem in the definition of $U^*(t, \omega, x)$ (for every $x > 0$) and upper semicontinuity of $U(t, \omega, \cdot)$. Since in (4.1), $\mathbf{c}^i$ necessarily satisfies inequality $\mathbf{c}^i \cdot S_t(\omega) \leq x_i$ with equality, $i = 1, 2$, from the strict monotonicity of $U(t, \omega, \cdot)$ in every spatial component and $x_1 \neq x_2$, we deduce that $\mathbf{c}^1 \neq \mathbf{c}^2$. Consequently, from *strict* concavity of $U(t, \omega, \cdot)$, we get

$$\begin{aligned} U^*\left(t, \omega, \frac{x_1 + x_2}{2}\right) &= \sup_{\substack{c \in \mathbb{R}_+^m: \\ c \cdot S_t(\omega) \leq \frac{x_1 + x_2}{2}}} U(t, \omega, c) \\ &\geq U\left(t, \omega, \frac{\mathbf{c}^1 + \mathbf{c}^2}{2}\right) \\ &> \frac{1}{2}U(t, \omega, \mathbf{c}^1) + \frac{1}{2}U(t, \omega, \mathbf{c}^2) \\ &= \frac{1}{2}U^*(t, \omega, x_1) + \frac{1}{2}U^*(t, \omega, x_2). \end{aligned}$$

Therefore, $U^*(t, \omega, \cdot)$ is strictly concave. As $U^*(t, \omega, \cdot)$ is increasing and strictly concave, it is *strictly* increasing.

For every $(t, \omega) \in [0, \infty) \times \Omega$ and $x > 0$, using the Inada conditions for $U(t, \omega, \cdot)$ one can show that there exists c in the *interior* of the first orthant, such that $c \cdot S_t(\omega) = x$ and $U^*(t, \omega, x) = U(t, \omega, c)$. As a result, differentiability of $U^*(t, \omega, \cdot)$ (in the third argument) follows from differentiability of $U(t, \omega, \cdot)$ and general properties of the subgradient of the image function, see e.g., [7, Corollary D.4.5.2]. As $U^*(t, \omega, \cdot)$ is concave and differentiable, we deduce that $U^*(t, \omega, \cdot)$ is *continuously* differentiable in the interior of its domain, see [7, Theorem D.6.2.4]. The Inada conditions for $U^*(t, \omega, \cdot)$ follow from the (version of the) Inada conditions for $U(t, \omega, \cdot)$ and [7, Theorem D.4.5.1, p.192].

For every $(t, \omega) \in [0, \infty) \times \Omega$, as $U(t, \omega, \cdot)$ is a closed concave function, using e.g., [21, Theorem 9.2, p. 75], we deduce that $U^*(t, \omega, \cdot)$ is also a *closed* concave function⁴. In particular, we get

$$U^*(t, \omega, 0) = \lim_{z \downarrow 0} U^*(t, \omega, z), \quad (t, \omega) \in [0, \infty) \times \Omega.$$

Finally, for every $x \geq 0$, $U^*(\cdot, \cdot, x)$ is optional as a supremum of countably many optional processes (where from continuity of $U(t, \omega, \cdot)$ in the relative interior of its effective domain, it is enough to take the supremum (in the definition of $U^*(t, \omega, \cdot)$) over the m -dimensional vectors, whose components take only *rational* values).

□

Remark 4.2. Lemma 4.1 asserts that U^* satisfies Assumption 2.1 in [19].

⁴Note that in general, the image of a closed convex or concave function under a linear transformation is not necessarily closed, see a discussion in [7, p.97].

For every $x > 0$, we denote by $\mathcal{A}^*(x)$ the set of 1-dimensional optional processes c^* , for which there exists an $\mathbb{R}^d$ -valued predictable $\tilde{S}$ -integrable process H , such that

$$X_t \triangleq x + \int_0^t H_u d\tilde{S}_u - \int_0^t c_u^* d\kappa_u, \quad t \geq 0,$$

is nonnegative, $\mathbb{P}$ -a.s. We also define

$$(4.2) \quad u^*(x) \triangleq \sup_{c^* \in \mathcal{A}^*(x)} \mathbb{E} \left[\int_0^\infty U(t, \omega, c_t^*(\omega)) d\kappa_t(\omega) \right], \quad x > 0.$$

with the convention analogous to (2.6):

$$\mathbb{E} \left[\int_0^\infty U^*(t, \omega, c_t^*(\omega)) d\kappa_t(\omega) \right] \triangleq -\infty, \quad \text{if} \quad \mathbb{E} \left[\int_0^\infty U^{*-}(t, \omega, c_t^*(\omega)) d\kappa_t(\omega) \right] = \infty.$$

Proof of Theorem 2.4. Let $x > 0$ be fixed and $c \in \mathcal{A}(x)$. Then $c_t^* \triangleq c_t \cdot S_t$, $t \geq 0$, is an optional process such that $c^* \in \mathcal{A}^*(x)$. Therefore,

$$(4.3) \quad u^*(x) \geq u(x) > -\infty, \quad x > 0.$$

Since U^* satisfies the assertions of Lemma 4.1, standard techniques in convex analysis show that $-V^*$ has the same properties as U^* . Therefore, optimization problems (4.2) and (2.8) satisfy the assumptions of [19, Theorem 3.2]. Consequently, [19, Theorem 3.2] applies, which in particular asserts that u^* and v are finite-valued and that for every $x > 0$, there exists a strictly positive optional process $\widehat{c}^*(x)$, the unique maximizer to (4.2).

Let us consider

$$(4.4) \quad \sup_{\substack{x \in \mathbb{R}_+^m: \\ x \cdot S_t(\omega) \leq \widehat{c}_t^*(x)(\omega)}} U(t, \omega, x), \quad (t, \omega) \in [0, \infty) \times \Omega,$$

and define a correspondence $\varphi : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m$ as follows

$$\varphi(t, \omega) \triangleq \left\{ x \in \mathbb{R}_+^m : x \cdot S_t(\omega) \leq \widehat{c}_t^*(x)(\omega) \right\}.$$

From *strict* positivity of the S^k 's and positivity and $(d\kappa \times \mathbb{P})$ -a.e. finiteness of $\widehat{c}^*(x)$ (by [19, Theorem 3.2]), we deduce that φ has nonempty⁵ compact values $(d\kappa \times \mathbb{P})$ -a.e. Let us consider the lower inverse of φ^l defined by

$$\varphi^l(G) \triangleq \{(t, \omega) \in [0, \infty) \times \Omega : \varphi(t, \omega) \cap G \neq \emptyset\}, \quad G \subset \mathbb{R}^m.$$

Let us also consider a subset of $\mathbb{R}^m$ of the form $A \triangleq [a_1, b_1] \times \cdots \times [a_m, b_m]$, where a_i 's and b_i 's are real numbers. In view of the weak measurability of φ (see [1, Definition 18.1, p. 592]) that we are planning to show, it is enough to consider $b_i \geq 0$, $i = 1, \dots, m$. In addition, let us set $\bar{a}_i = \max(0, a_i)$. One can see that for such a set A , as

$$\varphi^l(A) = \varphi^l([\bar{a}_1, b_1] \times \cdots \times [\bar{a}_m, b_m]),$$

with $\bar{a} \triangleq (\bar{a}_1, \dots, \bar{a}_m)$, we have

$$\varphi^l(A) = \left\{ (t, \omega) : \bar{a} \cdot S_t(\omega) \leq \widehat{c}_t^*(x)(\omega) \right\}.$$

⁵Note that the origin in $\mathbb{R}^m$ is in $\varphi(t, \omega)$ for every $(t, \omega) \in [0, \infty) \times \Omega$.

As $\widehat{c}^*(x)$ and S^i 's are optional processes and since $\varphi^l \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \bigcup_{n \in \mathbb{N}} \varphi^l(A_n)$ (see [1, Section 17.1], where A_n 's are subsets of $\mathbb{R}^m$), we deduce that $\varphi^l(G) \in \mathcal{O}$ for every open subset G of $\mathbb{R}^m$, i.e., φ is weakly measurable. As U is a Carathéodory function (see [1, Definition 4.50, p. 153]), we conclude from [1, Theorem 18.19, p. 605] that there exists an *optional* $\mathbb{R}^m$ -valued process $\widehat{c}_t(x)$, $t \in [0, T]$, the maximizer of (4.4) for $(d\kappa \times \mathbb{P})$ -a.e. $(t, \omega) \in [0, \infty) \times \Omega$. The uniqueness of such a maximizer follows from *strict* concavity of $U(t, \omega, \cdot)$ (for every $(t, \omega) \in [0, \infty) \times \Omega$)⁶. As $\widehat{c}^*(x) \in \mathcal{A}^*(x)$, we deduce that $\widehat{c}(x) \in \mathcal{A}(x)$. Combining this with (4.3), we conclude that $\widehat{c}(x)$ is the unique (up to an equivalence class) maximizer to (2.5).

For $x > 0$, let $\widehat{c}_t^i(x)$, $i = 1, \dots, m$, denote the components of $\widehat{c}_t(x)$. As $\widehat{c}_t(x)(\omega) \cdot S_t(\omega) = \widehat{c}_t^*(\omega)$, $(d\kappa \times \mathbb{P})$ -a.e., (where the argument here is similar to the discussion after (4.1)) relations (2.10), (2.12), (2.13), and (2.14) follow from [19, Theorem 3.2], whereas (2.15) results from [19, Theorem 3.3] (equivalently, from [3, Theorem 2.4]). In turn, combining (2.12) with [7, Theorem D.4.5.1], we get

$$\begin{aligned} \widehat{Y}_t(\omega) &= U_x^* \left(t, \omega, \widehat{c}_t^*(x)(\omega) \right) \\ &= \left\{ s(t, \omega) \in \mathbb{R} : S_t^i(\omega) s(t, \omega) = \partial_{x_i} U(t, \omega, \widehat{c}_t(x)(\omega)), \ i = 1, \dots, m \right\} \end{aligned} \quad (d\kappa \times \mathbb{P})\text{-a.e.},$$

i.e., (2.11) holds. $\square$

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⁶[1, Theorem 18.19, p. 605] gives a maximizer, which is a measurable multifunction, and from the uniqueness of the maximizer it is a single valued multifunction, for which the concept of measurability coincides with measurability for functions.

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