

Modeling and sensitivity analysis methodology for hybrid dynamical system



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ABSTRACT

This paper provides a complete mathematical framework to compute the sensitivities with respect to system parameters for any second order hybrid Ordinary Differential Equation (ODE) and ranked 1 and 3 Differential Algebraic Equation (DAE) system. The hybrid system is characterized by discontinuities in the velocity state variables due to an impulsive forces at the time of event. Such system may also exhibit at the time of event a change in the equation of motions, or in the kinematic constraints.

The methodology and the tools developed in this study compute the sensitivities of the states of the model and of the general cost functionals with respect to model parameters for both, unconstrained and constrained, hybrid mechanical systems. The analytical methodology that solves this problem is structured based on jumping conditions for both, the velocity state variables and the sensitivities matrix. The proposed analytical approach is then benchmarked against a known numerical method.

Finally, this study emphasizes the penalty formulation for modeling constrained mechanical systems since this formalism has the advantage that it incorporates the kinematic constraints inside the equation of motion, thus easing the numerical integration, works well with redundant constraints, and avoids kinematic bifurcations.

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1. Introduction

Sensitivity analysis plays a key role in a wide range of computational engineering problems, such as design optimization, optimal control, and implicit time integration methods, by providing derivative information for gradient based algorithms and methods. Sensitivity analysis quantifies the effect of small changes in the system parameters onto the outputs of interest [1]. Specifically, in the design of mechanical systems, sensitivity analysis reveals the system parameters that affect the given performance criterion the most, thus providing directions for mechanical design improvements. Sensitivity analysis enables gradient-based optimization by providing the derivative of the cost function with respect to design variables. In adaptive control systems, sensitivity analysis allows assessing the stability of a system by accounting for the effects of system disturbances and system parameters inaccuracies.

The most widely used techniques for sensitivity analysis are the direct and the adjoint methods. These approaches are complementary, as the direct sensitivity provides information on how parametric uncertainties propagate through the system dynamics, while the adjoint method is suitable for inverse modeling, in the sense that it can be used to identify the origin of uncertainty in a given model output [2].

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Numerical approximations of sensitivities are often calculated by finite difference methods where the deviation of the state trajectories are evaluated after system parameters disturbances or variations in the initial conditions are added to the system. Because of the simplicity of this method, which does not require any additional inputs other than the provided model, this technique is broadly used. However, the accuracy of the results is severely limited by the perturbation size and by the roundoff and cancellation errors [3].

This study develops a general and unify formulation for *direct* sensitivity analysis for hybrid dynamical system. In the context of this study, the term hybrid refers to a continuous system that encounters a finite number of events where some of the state variables jump to different values; the dynamics of a hybrid system is piecewise-smooth in time. The formalism presented and developed in this study does not support mechanisms that deal with infinite number of events and the grazing phenomenon where the trajectories of the system would make a tangential contact with the event function. For methods that take into account such a phenomenon, the reader may refer to [4–6].

We treat unconstrained systems modeled by ODEs, as well as constrained multibody systems modeled by differential algebraic equations (DAEs) and ODE penalty formulation. The penalty formulation is a constraint violation stabilization technique that incorporates all the kinematic constraints (position, velocity, and the acceleration constraint equations) into the equation of motion. A first approach of the penalty formulation was presented by Park and Chiou (1988) [7]; the penalty formulation is mentioned by the authors to be more robust and easier to implement than the Baumgarte's method. The full formulation was presented in [8]. Kurdila (1993) [9] showed the convergence and stability of the method. De Jalon and Bayo [10] were in favor on this formulation as the penalty formulation works with redundant constraints and near kinematic singular configurations, whereas the Baumgarte's method fails for such configurations. The reader may refer to the book "Kinematic and Dynamic Simulation of Multibody Systems: The Real-Time Challenge" [10] for further details on the method.

We are especially interested in hybrid mechanical systems where sensitivity analysis involves the time, the position coordinates, the velocity coordinates, and the system parameters. The sensitivity of the time of event and the jumps in the sensitivities of the state variables at the time of event, are available in the literature [11–19].

In this study, a new graphical proof of the jumps in sensitivities at the time of an event is employed, which helps to better understand the conditions for the jump in the sensitivities. This paper provides a unified methodology for determining the system solutions, their sensitivities, and sensitivities of a cost function for different types of events. The first type of event is caused by an external impulse (e.g., a contact) leading to a sudden change of velocities. The second type of event is caused by a sudden change of the equations of motion. The third type of event is caused by a sudden change in the kinematic constraints.

The paper is organized as follows. A review of the direct sensitivity approach for smooth ODE systems, along with the quadrature variable of the running cost function, is introduced in Section 2. The extension of this approach to hybrid ODE systems is presented in Section 3. The sensitivity analysis for smooth constrained rigid multibody dynamics systems is reviewed in Section 4. Sensitivity analysis methodology for hybrid constrained systems is developed in Section 5. Sensitivity analysis methodology for constrained systems with a sudden change of the equation of motions is developed in Section 6. The proposed sensitivity analysis methodologies in Section 5 are applied to a five-bar mechanism in Section 7. Conclusions are drawn in Section 8.

2. Direct sensitivity analysis for smooth ODE systems

We start the discussion with a review of direct sensitivity analysis for dynamical systems governed by smooth ODEs.

2.1. Smooth ODE systems dynamics

In this study we consider second order systems of ordinary differential equations of the form:

$$\mathbf{M}(t, q, \rho) \cdot \ddot{q} = \mathbf{F}(t, q, \dot{q}, \rho), \quad t_0 \leq t \leq t_f, \quad q(t_0) = q_0(\rho), \quad \dot{q}(t_0) = \dot{q}_0(\rho), \quad (1)$$

or equivalently:

$$\ddot{q} = \mathbf{M}^{-1}(t, q, \rho) \cdot \mathbf{F}(t, q, \dot{q}, \rho) =: f^{\text{com}}(t, q, \dot{q}, \rho), \quad (2)$$

that arise from the description of the dynamics of mechanical systems. In (2) $t \in \mathbb{R}$ is time, $q \in \mathbb{R}^n$ is the generalized position vector and $\dot{q} \in \mathbb{R}^n$ is the generalized velocity vector, n is the dimension of generalized coordinates, and $\rho \in \mathbb{R}^p$ is the vector of system parameters, where p is the number of parameters. The dot notation ($\dot{\square}$ or $\ddot{\square}$) indicates the total (first or second order) derivative of a function or variable with respect to time. Subscripts indicate partial derivative with respect to a quantity, unless stated otherwise. The mass matrix $\mathbf{M} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^{n \times n}$ is assumed to be smooth with respect to all its arguments, invertible, and with an inverse $\mathbf{M}^{-1}$ that is also smooth with respect to all arguments. The right-hand side function $\mathbf{F} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ represents external and internal generalized forces and is assumed to be smooth with respect to all its arguments.

The state trajectories are obtained by integrating the equations of motion (2), which depend on the system parameters ρ . Consequently, the state trajectories (the solutions of the equations of motion) depend implicitly on time and on the parameters, $q = q(t, \rho)$ and $\dot{q} = \dot{q}(t, \rho)$. The state trajectories also depend implicitly on the initial conditions of (2). For clarity we denote the velocity state variables by $v = \dot{q} \in \mathbb{R}^n$.

Sensitivity analysis computes derivatives of the solutions of (2) with respect to the system parameters:

$$Q(t, \rho) := D_\rho q(t) := \frac{dq}{d\rho}(t, \rho) \in \mathbb{R}^{n \times p}, \quad V(t, \rho) := D_\rho v(t) = \frac{dv}{d\rho}(t, \rho) \equiv \frac{d\dot{q}}{d\rho}(t, \rho) = \dot{Q}(t, \rho) \in \mathbb{R}^{n \times p}. \quad (3)$$

The second order ODE (2) can be transformed into a first order reduced system as follows. With the velocity state variables $v := \dot{q} \in \mathbb{R}^n$ the system (2) can be written in the form:

$$\begin{bmatrix} 1 & 0 \\ 0 & M(t, q, \rho) \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ F(t, q, v, \rho) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ f^{\text{eom}}(t, q, v, \rho) \end{bmatrix}, \quad \begin{bmatrix} q(t_0) \\ v(t_0) \end{bmatrix} = \begin{bmatrix} q_0(\rho) \\ v_0(\rho) \end{bmatrix}. \quad (4)$$

Definition 1 (Cost Function). Consider a smooth ‘trajectory cost function’ $g : \mathbb{R}^{1+3n+p} \rightarrow \mathbb{R}$ and a smooth ‘terminal cost function’ $w : \mathbb{R} \times \mathbb{R}^{1+2n+p} \rightarrow \mathbb{R}$. A general cost function is defined as the sum of the costs along the trajectory plus the cost at the terminal point of the solution:

$$\psi = \int_{t_0}^{t_F} g(t, q, v, \dot{v}, \rho) dt + w(t_F, q(t_F, \rho), v(t_F, \rho), \rho). \quad (5)$$

Remark 1 (Accelerations in the Cost Function). Note that the trajectory cost function (5) includes accelerations via $\dot{v}$. Accelerations are not independent variables and they can be resolved in terms of positions and velocities:

$$g(t, q, v, \dot{v}, \rho) = g(t, q, v, f^{\text{eom}}(t, q, v, \rho), \rho) = \tilde{g}(t, q, v, \rho). \quad (6)$$

We prefer to keep accelerations as an explicit argument in the cost function (5) in order to give additional flexibility in practical applications. However, we will need to resolve the sensitivities of acceleration in terms of other sensitivities in subsequent calculations.

To further simplify the notation we define the ‘quadrature’ variable $z(t) \in \mathbb{R}$ as follows:

$$\begin{aligned} z(t, \rho) &:= \int_{t_0}^t \tilde{g}(t, q(t, \rho), v(t, \rho), \rho) dt \Leftrightarrow \\ \dot{z}(t, \rho) &= g(t, q, v, \dot{v}, \rho) = \tilde{g}(t, q(t, \rho), v(t, \rho), \rho), \quad t_0 \leq t \leq t_F, \quad z(t_0, \rho) = 0. \end{aligned} \quad (7)$$

The cost function (5) reads:

$$\psi = z(t_F, \rho) + w(t_F, q(t_F, \rho), v(t_F, \rho), \rho). \quad (8)$$

Definition 2 (The Canonical ODE System). The canonical system is obtained by combining the first order ODE dynamics (4) with Eq. (7) for the ‘quadrature’ variable:

$$x(t) := \begin{bmatrix} q(t) \\ v(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^{2n+1}; \quad \dot{x} = \begin{bmatrix} \dot{q} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v \\ f^{\text{eom}}(t, q(t, \rho), v(t, \rho), \rho) \\ \tilde{g}(t, q, v, \rho) \end{bmatrix} = F(t, x, \rho), \quad t_0 \leq t \leq t_F, \quad x(t_0, \rho) = \begin{bmatrix} q_0(\rho) \\ v_0(\rho) \\ 0 \end{bmatrix}. \quad (9a)$$

The canonical cost function (8) is purely a terminal cost function:

$$\psi = z(t_F, \rho) + w(t_F, q(t_F, \rho), v(t_F, \rho), \rho) = W(x(t_F, \rho), \rho). \quad (9b)$$

2.2. Direct sensitivity approach for smooth ODE systems

Definition 3 (The Sensitivity Analysis Problem). Our goal in this work is to perform a sensitivity analysis of the cost function, i.e., to compute the total derivative of the cost function (5) with respect to model parameters ρ :

$$D_\rho \psi = \frac{d\psi}{d\rho} \in \mathbb{R}^{1 \times p}. \quad (10)$$

Note that the cost function (5) depends on the system parameters ρ directly (through the direct dependency of g and w on ρ) as well as indirectly (through the dependency of q and v on ρ). The sensitivity (10) needs to account for all the direct and the indirect dependencies.

Remark 2 (The Direct Sensitivity Analysis Approach). In order to compute the sensitivity (10) we take a variational calculus approach [1,20,21]. Infinitesimal changes in parameters

$$\rho \rightarrow \rho + \delta\rho \in \mathbb{R}^p, \quad (11)$$

lead to a total change in the cost function as follows:

$$\psi \rightarrow \psi + \delta\psi, \quad \delta\psi = \sum_{i=1}^p \frac{d\psi}{d\rho_i} \cdot \delta\rho_i = D_\rho \psi \cdot \delta\rho. \quad (12)$$

The direct sensitivity analysis computes each element $(D_\rho \psi)_i = \partial\psi / \partial\rho_i$ of the derivative (10) by accounting for changes in the cost function that result from changing each individual parameter $\delta\rho_i, i = 1 \dots p$.

2.3. Direct sensitivity analysis with respect to system parameters solved analytically

Definition 4 (The Tangent Linear Model (TLM)). Consider the ‘position sensitivity’ matrix $Q(t, \rho)$ and the ‘velocity sensitivity’ matrix $V(t, \rho)$ defined in (3):

$$Q_i(t, \rho) := \frac{dq(t, \rho)}{d\rho_i} \in \mathbb{R}^n, \quad i = 1, \dots, p; \quad Q(t, \rho) := [Q_1(t, \rho) \cdots Q_p(t, \rho)] \in \mathbb{R}^{n \times p}, \quad (13a)$$

$$V_i(t, \rho) := \frac{dv(t, \rho)}{d\rho_i} \in \mathbb{R}^n, \quad i = 1, \dots, p; \quad V(t, \rho) := [V_1(t, \rho) \cdots V_p(t, \rho)] \in \mathbb{R}^{n \times p}. \quad (13b)$$

These sensitivities evolve in time according to the *tangent linear model* (TLM) equations [1,20,21], obtained by differentiating the equations of motion (2) with respect to the parameters:

$$\begin{cases} \dot{Q}_i = \frac{d\dot{q}}{d\rho_i} = \frac{dv}{d\rho_i} = V_i, \\ \dot{V}_i = \frac{d\dot{v}}{d\rho_i} = f_q^{\text{eom}}(t, q, v, \rho) \cdot \frac{dq}{d\rho_i} + f_v^{\text{eom}}(t, q, v, \rho) \cdot \frac{dv}{d\rho_i} + f_{\rho_i}^{\text{eom}}(t, q, v, \rho) \\ = f_q^{\text{eom}}(t, q, v, \rho) \cdot Q_i + f_v^{\text{eom}}(t, q, v, \rho) \cdot V_i + f_{\rho_i}^{\text{eom}}(t, q, v, \rho), \end{cases} \quad i = 1, \dots, p, \quad t_0 \leq t \leq t_F, \quad (14a)$$

with the initial conditions

$$Q_i(t_0, \rho) = \frac{dq_0}{d\rho_i}, \quad V_i(t_0, \rho) = \frac{dv_0}{d\rho_i}, \quad i = 1, \dots, p. \quad (14b)$$

The expressions $f_q^{\text{eom}}, f_v^{\text{eom}}$, and $f_{\rho_i}^{\text{eom}}$ denote the partial derivatives of f^{eom} with respect to the subscripted variables.

Remark 3. The partial derivatives $\partial f^{\text{eom}} / \partial \zeta$ are obtained by differentiating (2) with respect to $\zeta \in \{q, v, \rho\}$:

$$\frac{\partial f^{\text{eom}}}{\partial \zeta} = \frac{\partial (M^{-1} F)}{\partial \zeta} = -M^{-1} M_\zeta M^{-1} F + M^{-1} F_\zeta = M^{-1} (F_\zeta - M_\zeta f^{\text{eom}}) = M^{-1} (F_\zeta - M_\zeta \dot{v}). \quad (15)$$

Definition 5 (The Quadrature Sensitivity). Similarly, let the ‘quadrature sensitivity’ vector $Z(t, \rho)$ be the Jacobian of the ‘quadrature’ variable $z(t, \rho)$ (7) with respect to the parameters ρ :

$$Z_i(t, \rho) := \frac{\partial z(t, \rho)}{\partial \rho_i}, \quad i = 1, \dots, p; \quad Z(t, \rho) := \nabla_\rho z(t, \rho) = [Z_1(t, \rho) \cdots Z_p(t, \rho)] \in \mathbb{R}^{1 \times p}. \quad (16)$$

The time evolution equations of the quadrature sensitivities are given by the TLM obtained by differentiating (7) with respect to the parameters:

$$\begin{aligned} \dot{Z}_i &= \frac{dg(t, q, v, \dot{v}, \rho)}{d\rho_i} = g_q \cdot Q_i + g_v \cdot V_i + g_{\dot{v}} \cdot \dot{V}_i + g_{\rho_i} \\ &= (g_q + g_{\dot{v}} f_q^{\text{eom}}) \cdot Q_i + (g_v + g_{\dot{v}} f_v^{\text{eom}}) \cdot V_i + g_{\rho_i} + g_{\dot{v}} \cdot f_{\rho_i}^{\text{eom}}, \\ &t_0 \leq t \leq t_F, \quad Z_i(t_0, \rho) = 0, \quad i = 1, \dots, p. \end{aligned} \quad (17)$$

Definition 6 (Canonical Sensitivity ODE). The solutions given by (9a), the TLM given by (14), and the sensitivity quadrature equations (60) need to be solved together forward in time, leading to the canonical sensitivity ODE that computes the derivatives of the cost function with respect to the system parameters ρ for smooth systems:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \\ \dot{z} \\ [\dot{Q}_i]_{i=1, \dots, p} \\ [\dot{V}_i]_{i=1, \dots, p} \\ [\dot{Z}_i]_{i=1, \dots, p} \end{bmatrix} = \begin{bmatrix} v \\ f^{\text{eom}} \\ \tilde{g} \\ [V_i]_{i=1, \dots, p} \\ [f_q^{\text{eom}} Q_i + f_v^{\text{eom}} V_i + f_{\rho_i}^{\text{eom}}]_{i=1, \dots, p} \\ [(g_q + g_{\dot{v}} f_q^{\text{eom}}) \cdot Q_i + (g_v + g_{\dot{v}} f_v^{\text{eom}}) \cdot V_i + g_{\rho_i} + g_{\dot{v}} \cdot f_{\rho_i}^{\text{eom}}]_{i=1, \dots, p} \end{bmatrix}, \quad (18)$$

where the state vector of the canonical sensitivity ODE is :

$$X = [q^T, v^T, z, Q_1^T, \dots, Q_p^T, V_1^T, \dots, V_p^T, Z_1, \dots, Z_p]^T \in \mathbb{R}^{(n+1)(p+1)}. \quad (19)$$

Remark 4 (*The Sensitivities of the Cost Function*). Once the quadrature sensitivities (17) have been calculated, the sensitivities of the cost function with respect to each parameter are computed as follows:

$$\begin{aligned} \frac{d\psi}{d\rho_i} = & Z_i(t_F) + w_q(t_F, q(t_F, \rho), v(t_F, \rho), \rho) \cdot Q_i(t_F, \rho) + w_v(t_F, q(t_F, \rho), v(t_F, \rho), \rho) \cdot V_i(t_F, \rho) \\ & + w_{\rho_i}(t_F, q(t_F, \rho), v(t_F, \rho), \rho), \quad i = 1, \dots, p. \end{aligned} \quad (20)$$

2.4. Direct sensitivity analysis with respect to system parameters solved with the complex finite difference method

An accurate numerical method for sensitivity analysis of a smooth ODE system with respect to the system parameters ρ is the complex finite difference method [1,20,21]. Add a small *complex* perturbation to one parameter:

$$\tilde{\rho}_j = \begin{cases} \rho_j & \text{for } j \neq \ell, \\ \rho_\ell + i \Delta\rho & \text{for } j = \ell, \end{cases} \quad j = 1, \dots, p, \quad (21)$$

and solve the canonical ODE system (9a) for this perturbed values of the parameters to obtain:

$$q(t, \tilde{\rho}), \quad v(t, \tilde{\rho}), \quad z(t, \tilde{\rho}), \quad \psi(\tilde{\rho}) = z(t_F, \tilde{\rho}) + w(q(t_F, \tilde{\rho}), v(t_F, \tilde{\rho}), \tilde{\rho}). \quad (22)$$

The sensitivities are approximated numerically by the imaginary parts of the state variables:

$$Q_\ell(t, \rho) \approx -\frac{\text{imag}(q(t, \tilde{\rho}))}{\|\Delta\rho\|}, \quad V_\ell(t, \rho) \approx -\frac{\text{imag}(v(t, \tilde{\rho}))}{\|\Delta\rho\|}, \quad \frac{d\psi}{d\rho_\ell} \approx -\frac{\text{imag}(\psi(\tilde{\rho}))}{\|\Delta\rho\|}. \quad (23)$$

We next discuss an approach to sensitivity analysis that accounts for discontinuities in the state variables.

3. Direct sensitivity analysis for hybrid ODE system

3.1. Hybrid ODE system

Definition 7 (*Hybrid Dynamics*). A hybrid mechanical system is a piecewise-in-time continuous dynamic ODE described by (2) that exhibits discontinuous dynamic behavior in the generalized velocity state vector at a finite number of time moments (no zeno phenomenon [6]). Each such moment is a ‘time of event’ t_{eve} and corresponds to a triggering event described by the equation:

$$r(q|_{t_{\text{eve}}}) = 0, \quad (24)$$

where $r : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth ‘event function’.

Remark 5. In the context of this study, we assume that there are no grazing phenomenon where the system trajectory would make tangential contact with an event triggering hypersurface [4–6].

Definition 8 (*Characterization of an Event*). For hybrid systems variables can change values during the event. For this reason we distinguish between the value of a variable right before the event $\mathbf{x}|_{t_{\text{eve}}}^-$, and its value right after the event $\mathbf{x}|_{t_{\text{eve}}}^+$:

$$\mathbf{x}|_{t_{\text{eve}}}^- := \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} \mathbf{x}(t_{\text{eve}} - \varepsilon), \quad \mathbf{x}|_{t_{\text{eve}}}^+ := \lim_{\varepsilon > 0, \varepsilon \rightarrow 0} \mathbf{x}(t_{\text{eve}} + \varepsilon). \quad (25)$$

The limits exist since the evolution of the system is smooth in time both before and after the event.

We consider an event happening at time t_{eve} that applies a finite energy impulse force to the system. Such an impulse force does not change the generalized position state variables, and therefore:

$$q|_{t_{\text{eve}}}^+ = q|_{t_{\text{eve}}}^- = q|_{t_{\text{eve}}}. \quad (26)$$

However, the finite energy event can abruptly change the generalized velocity state vector $\dot{q}$ from its value $v|_{t_{\text{eve}}}^-$ right before the event to a new value $v|_{t_{\text{eve}}}^+$ right after the event. The ‘jump function’ at the time of event t_{eve} characterizes the change in the generalized velocity during the event:

$$v|_{t_{\text{eve}}}^+ = h(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \Leftrightarrow \dot{q}|_{t_{\text{eve}}}^+ = h(t_{\text{eve}}, q|_{t_{\text{eve}}}, \dot{q}|_{t_{\text{eve}}}^-, \rho). \quad (27)$$

Remark 6 (*Multiple Events*). In many cases the change can be triggered by one of multiple events. Each individual event is described by the event function $r_\ell : \mathbb{R}^n \rightarrow \mathbb{R}$, $\ell = 1, \dots, e$. The detection of the next event, which can be one of the possible e options, is described by:

$$r_1(q|_{t_{\text{eve}}}) \cdot r_2(q|_{t_{\text{eve}}}) \dots r_e(q|_{t_{\text{eve}}}) = 0, \quad (28)$$

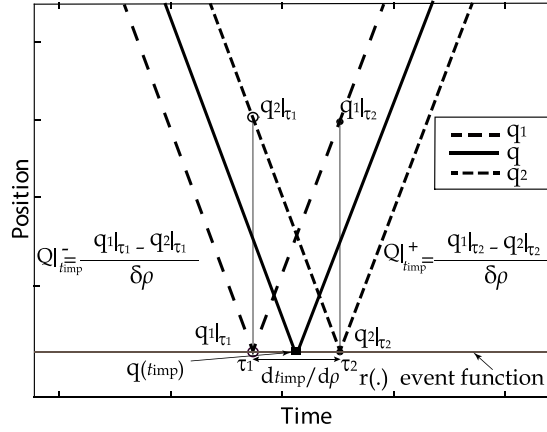


Fig. 1. Schematic visualization of the jump in the sensitivity of the position.

and if event ℓ takes place then $r_\ell = 0$ and the corresponding jump is:

$$v|_{t_{\text{eve}}}^+ = h_\ell(q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-). \quad (29)$$

Remark 7 (*Numerical Implementation of Events*). Numerical solutions of hybrid systems use an event detection mechanism. The event function (24) is implemented in the numerical time solver such that the integrator is stopped at the solution of (24). The jump function (27) is implemented as a callback function that is executed after the event is detected. The numerical integration resumes with new initial conditions after the jump.

Definition 9 (*Twin Perturbed System*). Consider two versions of the system (2) with identical dynamics and initial conditions, but with different parameters values ρ_1 and ρ_2 , respectively. Without loss of generality in this proof we consider the scalar parameter case $p = 1$; the general equation (35) can be proven element by element by considering sensitivities with respect to individual parameters. The two parameters are infinitesimally small perturbations $\delta\rho$ of the reference parameter value ρ :

$$\rho_1 = \rho - \frac{\delta\rho}{2}; \quad \rho_2 = \rho + \frac{\delta\rho}{2}. \quad (30)$$

We denote by $q_1(t) = q(t, \rho_1)$, $v_1(t) = v(t, \rho_1)$, and $z_1(t) = z(t, \rho_1)$ the position and velocity states, and the quadrature variable of the first system, respectively. We denote by $q_2(t) = q(t, \rho_2)$, $v_2(t) = v(t, \rho_2)$, and $z_2(t) = z(t, \rho_2)$ the position and velocity states, and the quadrature variable of the second system, respectively.

Assume that the sign of the perturbation $\delta\rho$ is such that $\tau_2 > t_{\text{eve}} > \tau_1$, and denote $\delta\tau = \tau_2 - \tau_1$. Since $\delta\rho$ is infinitesimally small, so is $\delta\tau$. The trajectories of the positions $q(t)$, $q_1(t)$, and $q_2(t)$, as well as the trajectories of the velocities $v(t)$, $v_1(t)$, and $v_2(t)$, are schematically illustrated in Figs. 1 and 2, respectively. As shown in Fig. 1 the first system meets the event described by the function (24) at the time of event $t_{\text{eve}}(\rho_1) = \tau_1$, when its position state is $q_1|_{\tau_1}$. The second system meets the event at time $t_{\text{eve}}(\rho_2) = \tau_2$, when its position state is $q_2|_{\tau_2}$. Note that in the limit of vanishing $\delta\rho$ we have:

$$\delta\rho \rightarrow 0 \Rightarrow v_1|_{\tau_1}^- \rightarrow v|_{t_{\text{eve}}}^-, \quad v_1|_{\tau_2} \rightarrow v|_{t_{\text{eve}}}^+, \quad v_1|_{\tau_1}^+ \rightarrow v|_{t_{\text{eve}}}^+; \quad v_2|_{\tau_1} \rightarrow v|_{t_{\text{eve}}}^-, \quad v_2|_{\tau_2}^- \rightarrow v|_{t_{\text{eve}}}^-, \quad v_2|_{\tau_2}^+ \rightarrow v|_{t_{\text{eve}}}^+. \quad (31)$$

Let $Q|_{t_{\text{eve}}}^+, Q|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized positions (13a) right before and right after the event, respectively. Let $V|_{t_{\text{eve}}}^+, V|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized velocities (13b) right before and right after the event, respectively. Our methodology to find these sensitivities is to first evaluate the states $q_1(t)$, $v_1(t)$ and $q_2(t)$, $v_2(t)$ of each of the twin perturbed systems at both τ_1 and τ_2 . The sensitivities are obtained from their definition by taking differences of states of the two systems, dividing them by the perturbation in parameters, and taking the limits, for example:

$$Q_i|_{t_{\text{eve}}}^- = \lim_{\delta\rho_i \rightarrow 0} \frac{q_2|_{\tau_1} - q_1|_{\tau_1}}{\delta\rho_i}, \quad V_i|_{t_{\text{eve}}}^+ = \lim_{\delta\rho_i \rightarrow 0} \frac{v_2|_{\tau_2}^+ - v_1|_{\tau_2}}{\delta\rho_i}, \quad i = 1, \dots, p. \quad (32)$$

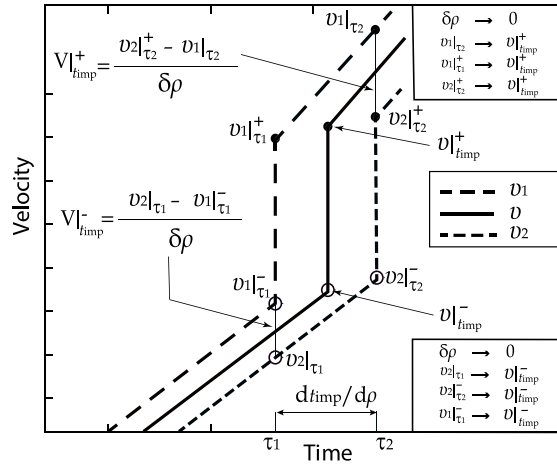


Fig. 2. Schematic visualization of the jump in the sensitivity of the velocity.

3.2. The sensitivity of the time of event with respect to the system parameters

Theorem 1 (Sensitivity of the Time of Event [11,12,15,18,22,23]). Let $r(\cdot) \in \mathbb{R}$ be the scalar event function defined by (24), and $dr/dq \in \mathbb{R}^{1 \times n}$ be its Jacobian. The sensitivity of the time of event with respect to the system parameters is:

$$\frac{dt_{\text{eve}}}{d\rho} = -\frac{\frac{dr}{dq}(q|_{t_{\text{eve}}}) \cdot Q|_{t_{\text{eve}}}^-}{\frac{dr}{dq}(q|_{t_{\text{eve}}}) \cdot v|_{t_{\text{eve}}}^-} \in \mathbb{R}^{1 \times p}. \quad (33)$$

Proof. The time at which the event function becomes zero is indirectly dependent on the system parameters ρ . We evaluate the derivative of Eq. (24) with respect to the system parameters:

$$0 = r(q(t_{\text{eve}}, \rho)) \Rightarrow 0 = \frac{dr}{d\rho} = \frac{dr}{dq} \left(\frac{dq}{d\rho} + \dot{q} \frac{dt_{\text{eve}}}{d\rho} \right). \quad (34)$$

Rearrange the terms in (34) to obtain (33). $\square$

Remark 8. The sensitivity of the time of event with respect to the system parameters exists only for situations that do not involve grazing where $\frac{dr}{dq} = 0$ for such case.

3.3. The jump in the sensitivity of the position state vector due to the event

This section provides the jumps in the sensitivities of the position state vector $q(t)$ at the time of event [11,12,15,18,22,23]. Due to the nonzero inertia, the position state variable is continuous in time (26). However, its sensitivity can be discontinuous at the time of event, as established next.

Theorem 2 (Jump in Position Sensitivity [13,18,22,23]). Let $v|_{t_{\text{eve}}}^+, v|_{t_{\text{eve}}}^- \in \mathbb{R}^n$ be the generalized velocity state vectors after and before the event, respectively; the corresponding velocity jump function was introduced in (27). Let $Q|_{t_{\text{eve}}}^+$ and $Q|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized position state vectors after and before the event, respectively. The jump equation of the sensitivities of the generalized position state vector is:

$$Q|_{t_{\text{eve}}}^+ = Q|_{t_{\text{eve}}}^- - \left(v|_{t_{\text{eve}}}^+ - v|_{t_{\text{eve}}}^- \right) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (35)$$

Proof. Consider the twin perturbed systems from Definition 9. The evolution of positions is illustrated in Fig. 1, where the two different dashed line trajectories represent the position variables of the two perturbed systems. The jump in the velocity state variables occurs at time τ_1 only for the first system. The position variables at time τ_2 for both systems are:

$$\begin{aligned} q_1|_{\tau_2} &= q_1|_{\tau_1} + h(v|_{\tau_1}^-) \delta\tau, \\ q_2|_{\tau_2} &= q_2|_{\tau_1} + v_2|_{\tau_1} \delta\tau. \end{aligned} \quad (36)$$

Subtract the two equations and scale by the perturbation in the parameters:

$$\frac{q_2|_{\tau_2} - q_1|_{\tau_2}}{\delta\rho} = - (v_1|_{\tau_1}^+ - v_2|_{\tau_1}) \frac{\delta\tau}{\delta\rho} + \frac{q_2|_{\tau_1} - q_1|_{\tau_1}}{\delta\rho}. \quad (37)$$

Using (31) and taking the limit $\delta\rho \rightarrow 0$ in (37) we obtain (35). The trajectory state differences are illustrated by the vertical lines in Fig. 1.

3.4. The jump in the sensitivity of the velocity state vector due to the event

This section provides the jumps in the sensitivities of the velocity state vector $v(t)$ at the time of event [11,12,15,18] corresponding to the jump function (27).

Theorem 3 (Jump in Velocity Sensitivity. [11,12,15,18]). *Let $V|_{t_{\text{eve}}}^+, V|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized position state vectors after and before the event, respectively. Let $v|_{t_{\text{eve}}}^+$ and $v|_{t_{\text{eve}}}^- \in \mathbb{R}^n$ be the velocity state vectors after and before the event affected by the jump function (27), respectively. Let $\ddot{q}|_{t_{\text{eve}}}^+$ and $\ddot{q}|_{t_{\text{eve}}}^- \in \mathbb{R}^n$ be the generalized acceleration state vectors after and before the event, respectively. The jump equation of the sensitivities of the generalized velocity state vector is:*

$$V|_{t_{\text{eve}}}^+ = h_q|_{t_{\text{eve}}}^- \cdot Q|_{t_{\text{eve}}}^- + h_v|_{t_{\text{eve}}}^- \cdot V|_{t_{\text{eve}}}^- + (h_q|_{t_{\text{eve}}}^- \cdot v|_{t_{\text{eve}}}^- - \ddot{q}|_{t_{\text{eve}}}^+ + h_v|_{t_{\text{eve}}}^- \cdot \ddot{q}|_{t_{\text{eve}}}^- + h_t|_{t_{\text{eve}}}^-) \cdot \frac{dt_{\text{eve}}}{d\rho} + h_\rho|_{t_{\text{eve}}}^-, \quad (38)$$

where the Jacobians of the jump function are:

$$\begin{aligned} h_t|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial t}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^f, & h_q|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial q}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f \times n}, \\ h_v|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial v}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f \times f}, & h_\rho|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial \rho}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f \times p}. \end{aligned} \quad (39)$$

Proof. We consider again the twin perturbed systems from Definition 9. The jumps in velocities are illustrated in Fig. 2. The velocities for each system are determined as follows:

$$\begin{aligned} v_1|_{\tau_2} &= v_1|_{\tau_1}^+ + f^{\text{eom}}(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^+, \rho_1) \delta\tau, \\ &= h(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^-, \rho_1) + f^{\text{eom}}(\tau_1, q_1|_{\tau_1}, h(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^-, \rho_1), \rho_1) \delta\tau, \\ v_2|_{\tau_2}^+ &= h(\tau_2, q_2|_{\tau_2}, v_2|_{\tau_2}^-, \rho_2) \\ &= h(\tau_2, q_2|_{\tau_1} + v_2|_{\tau_1} \delta\tau, v_2|_{\tau_1} + f^{\text{eom}}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) \delta\tau, \rho_2) \\ &\approx h(\tau_2, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) + \frac{dh}{dq}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot v_2|_{\tau_1} \delta\tau \\ &\quad + \frac{dh}{dv}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot f^{\text{eom}}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) \delta\tau, \end{aligned} \quad (40)$$

where f^{eom} is the instantaneous acceleration of the system from (2). The last relation represents a linearization (first order Taylor expansion) that is infinitely accurate since $\delta\tau$ is infinitesimally small. The scaled difference between the velocity state vectors at the time of event is:

$$\begin{aligned} \frac{v_2|_{\tau_2}^+ - v_1|_{\tau_2}}{\delta\rho} &\approx \frac{h(\tau_2, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) - h(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^-, \rho_1)}{\delta\rho} \\ &\quad - f^{\text{eom}}(\tau_1, q_1|_{\tau_1}, h(q_1|_{\tau_1}, v_1|_{\tau_1}^-), \rho_1) \frac{\delta\tau}{\delta\rho} \\ &\quad + \frac{dh}{dq}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot v_2|_{\tau_1} \cdot \frac{\delta\tau}{\delta\rho} + \frac{dh}{dv}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot f^{\text{eom}}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) \cdot \frac{\delta\tau}{\delta\rho} \\ &\approx \frac{dh}{dt}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) \cdot \frac{\tau_2 - \tau_1}{\delta\rho} + \frac{dh}{dq}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) \cdot \frac{q_2|_{\tau_1} - q_1|_{\tau_1}}{\delta\rho} + \frac{dh}{dv}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) \cdot \frac{v_2|_{\tau_1} - v_1|_{\tau_1}^-}{\delta\rho} \\ &\quad + \frac{dh}{d\rho}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) - f^{\text{eom}}(\tau_1, q_1|_{\tau_1}, h(q_1|_{\tau_1}, v_1|_{\tau_1}^-), \rho_1) \frac{\delta\tau}{\delta\rho} \\ &\quad + \frac{dh}{dq}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot v_2|_{\tau_1} \cdot \frac{\delta\tau}{\delta\rho} + \frac{dh}{dv}(q_2|_{\tau_1}, v_2|_{\tau_1}) \cdot f^{\text{eom}}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) \cdot \frac{\delta\tau}{\delta\rho}. \end{aligned}$$

Taking the limit $\delta\rho \rightarrow 0$ and using (31) yields:

$$\begin{aligned} \frac{dh}{dq}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) &\rightarrow h_q|_{t_{\text{eve}}}, & \frac{dh}{dq}(q_2|_{\tau_1}, v_2|_{\tau_1}) &\rightarrow h_q|_{t_{\text{eve}}}, \\ \frac{dh}{dv}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) &\rightarrow h_v|_{t_{\text{eve}}}, & \frac{dh}{dv}(q_2|_{\tau_1}, v_2|_{\tau_1}) &\rightarrow h_v|_{t_{\text{eve}}}, \\ \frac{dh}{dt}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) &\rightarrow h_t|_{t_{\text{eve}}}, & \frac{dh}{d\rho}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) &\rightarrow h_\rho|_{t_{\text{eve}}}, \\ f^{\text{eom}}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) &\rightarrow f^{\text{eom}}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}, \rho) = \ddot{q}|_{t_{\text{eve}}}, \\ f^{\text{eom}}(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^+, \rho_1) &\rightarrow f^{\text{eom}}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}, \rho) = \ddot{q}|_{t_{\text{eve}}}^+. \end{aligned} \quad (41)$$

which leads to (38).

For simplicity we denote the derivatives of the jump function with respect to $\zeta \in \{t, q, v, \rho\}$ by:

$$\begin{aligned} \frac{dh}{d\zeta}(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^-, \rho_1) &= \frac{dh}{d\zeta}(q_1|_{\tau_1}, v_1|_{\tau_1}^-) \\ \frac{dh}{d\zeta}(\tau_1, q_2|_{\tau_1}, v_2|_{\tau_1}, \rho_2) &= \frac{dh}{d\zeta}(q_2|_{\tau_1}, v_2|_{\tau_1}) \quad \square \end{aligned} \quad (42)$$

Theorem 4 (Events That Only Change the Acceleration). *We now consider an event where the system undergoes a sudden change of the equation of motions (1) at t_{eve} . Let $\ddot{q}|_{t_{\text{eve}}}^+$ and $\ddot{q}|_{t_{\text{eve}}}^- \in \mathbb{R}^n$ be the generalized acceleration state vectors right after and right before the event, respectively:*

$$\ddot{q}|_{t_{\text{eve}}}^- = f^{\text{eom}-}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}, \rho) \xrightarrow{\text{event}} \ddot{q}|_{t_{\text{eve}}}^+ = f^{\text{eom}+}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}, \rho) =: f^{\text{eom}+}|_{t_{\text{eve}}}. \quad (43)$$

There is no abrupt jump in the system velocity, $v|_{t_{\text{eve}}}^+ = v|_{t_{\text{eve}}}^-$, and therefore the jump function (27) is identity. Let $V|_{t_{\text{eve}}}^+, V|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized position state vectors right after and before the event, respectively. The jump equation of the sensitivities of the generalized velocity state vector is:

$$V|_{t_{\text{eve}}}^+ = V|_{t_{\text{eve}}}^- - (\ddot{q}|_{t_{\text{eve}}}^+ - \ddot{q}|_{t_{\text{eve}}}^-) \cdot \frac{dt_{\text{eve}}}{d\rho} = V|_{t_{\text{eve}}}^- - (f^{\text{eom}+}|_{t_{\text{eve}}} - f^{\text{eom}-}|_{t_{\text{eve}}}) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (44)$$

Proof. For the type of events under consideration we have that:

$$\frac{dh}{dq} = 0, \quad \frac{dh}{dv} = 1. \quad (45)$$

Using this in (38) leads to (44). $\square$

3.5. The jump in the sensitivity of the cost functional due to the event

We now consider the sensitivity of the quadrature variable $z(t)$. Due to the integral form of (7) defining z , the quadrature variable is continuous in time:

$$z|_{t_{\text{eve}}}^+ = z|_{t_{\text{eve}}}^- = z|_{t_{\text{eve}}}. \quad (46)$$

However, its sensitivity can be discontinuous at the event time, as established next.

Theorem 5 (Jump in Quadrature Sensitivity). *Let $Z|_{t_{\text{eve}}}^+$ and $Z|_{t_{\text{eve}}}^-$, with $Z \in \mathbb{R}^p$, be the sensitivities of the quadrature variable $z(t)$ (Definition 5) right after and right before the event, respectively. Let*

$$g|_{t_{\text{eve}}}^+ := \tilde{g}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^+, \rho), \quad g|_{t_{\text{eve}}}^- := \tilde{g}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho), \quad (47)$$

be the running cost function evaluated right after and right before the event, respectively. The sensitivity of the cost functional changes during the event as follows:

$$Z|_{t_{\text{eve}}}^+ = Z|_{t_{\text{eve}}}^- - (g|_{t_{\text{eve}}}^+ - g|_{t_{\text{eve}}}^-) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (48)$$

Proof. Consider again the twin perturbed systems from Definition 9, and evaluate the associated quadrature variables (7) at the event:

$$\begin{aligned} z_1|_{\tau_2} &= z_1|_{\tau_1} + \int_{\tau_1}^{\tau_2} \tilde{g}(t, q_1(t), v_1(t), \rho_1) dt = z_1|_{\tau_1} + \tilde{g}(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^+, \rho_1) \delta\tau, \\ z_2|_{\tau_2} &= z_2|_{\tau_1} + \int_{\tau_1}^{\tau_2} \tilde{g}(t, q_2(t), v_2(t), \rho_2) dt = z_2|_{\tau_1} + \tilde{g}(\tau_2, q_2|_{\tau_2}, v_2|_{\tau_2}^-, \rho_2) \delta\tau. \end{aligned} \quad (49)$$

Subtract the two equations and scale by the parameter perturbation to obtain:

$$\frac{z_2|_{\tau_2} - z_1|_{\tau_2}}{\delta\rho} = \frac{z_2|_{\tau_1} - z_1|_{\tau_1}}{\delta\rho} + (\tilde{g}(\tau_2, q_2|_{\tau_2}, v_2|_{\tau_2}^-, \rho_2) - \tilde{g}(\tau_1, q_1|_{\tau_1}, v_1|_{\tau_1}^+, \rho_1)) \frac{\delta\tau}{\delta\rho}. \quad (50)$$

Taking the limit $\delta\rho \rightarrow 0$ leads to (48). $\square$

Remark 9. When there are multiple events along the trajectory jumps in sensitivity (48) will happen for each one. The jump of the quadrature variable Z is governed by the values of the cost function g before and after the event.

4. Direct sensitivity analysis for constrained multibody systems with smooth trajectories

This section reviews the direct sensitivity analysis for constrained systems governed by differential algebraic equations (DAEs). The presentation follows the authors' earlier work [1,20,21,24–26].

4.1. Representation of constrained multibody systems

Constrained multibody systems must satisfy the following kinematic constraints:

$$0 = \Phi, \quad (51a)$$

$$0 = \dot{\Phi} = \Phi_q \dot{q} + \Phi_t \Rightarrow \Phi_q v = -\Phi_t, \quad (51b)$$

$$0 = \ddot{\Phi} = \Phi_q \ddot{q} + \Phi_{q,q}(\dot{q}, \dot{q}) + \Phi_{t,q} \dot{q} + \Phi_{t,t} \Rightarrow \Phi_q \dot{v} = -(\Phi_q v)v - \Phi_{t,q} v - \Phi_{t,t} := C. \quad (51c)$$

Here (51a) is a holonomic position constraint equation $\Phi(t, q, \rho) = 0$, where $\Phi : \mathbb{R}^{1+n+p} \rightarrow \mathbb{R}^m$ is a smooth 'position constraint' function. The velocity (51b) and the acceleration (51c) kinematic constraints are found by differentiating the position constraint with respect to time.

There are two main approaches to solve such system, the DAE approach through direct inclusion of the algebraic constraints in the dynamics, and the ODE approach through either following locally the independent coordinates (Maggi) or through a penalty formulation.

4.2. Direct sensitivity analysis for smooth systems in the index-3 differential–algebraic formulation

Definition 10 (Constrained Multibody Dynamics: The Index-3 DAE Formulation). A constrained rigid multibody dynamics system is described by the following index-3 differential–algebraic equations (DAEs) [20]:

$$\begin{cases} \dot{q} &= v, \\ M(t, q, \rho) \cdot \dot{v} &= F(t, q, v, \rho) + \Phi_q^T(t, q, \rho) \cdot \lambda, \quad t_0 \leq t \leq t_f, \quad q(t_0) = q_0(\rho), \quad v(t_0) = v_0(\rho). \\ \Phi(t, q, \rho) &= 0, \end{cases} \quad (52)$$

Unlike the ODE formulation (2) the position vector of the system (52) is constrained by the Eq. (51a). The joint forces $\Phi_q^T \lambda$ ensure that the system solution obeys the constraints at all points along the trajectory, and $\lambda \in \mathbb{R}^m$ are Lagrange multipliers associated with the position constraint (51a).

Sensitivities of the position and velocity state variables are defined in (3). In addition, we need to consider the sensitivity of the Lagrange multipliers with respect to system parameters:

$$\Lambda(t, \rho) := D_\rho \lambda(t) := \frac{d\lambda}{d\rho}(t, \rho) \in \mathbb{R}^{m \times p}. \quad (53)$$

Definition 11 (TLM of the Index-3 DAE Formulation). Sensitivities of solutions (3) and multipliers (53) of the system (52) with respect to parameters evolve according to the tangent linear model derived in [1,20,21,24–26]:

$$\begin{cases} \dot{Q} = V, \\ M \cdot \dot{V} = F_v \cdot V - (M_q \dot{v} + \Phi_{q,q}^T \lambda - F_q) \cdot Q - \Phi_q^T \cdot \Lambda + F_\rho - M_\rho \dot{v} - \Phi_{q,\rho}^T \lambda, \\ \Phi_q \cdot Q = -\Phi_\rho, \end{cases}$$

with initial conditions given by (14b).

4.3. Direct sensitivity analysis for smooth systems in the index-1 differential–algebraic formulation

Definition 12 (Constrained Multibody Dynamics: The Index-1 DAE Formulation). The index-1 formulation of the equations of motion is obtained by replacing the position constraint (51a) in (52) with the acceleration constraint (51c):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & M(t, q, \rho) & \Phi_q^T(t, q, \rho) \\ 0 & \Phi_q(t, q, \rho) & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q} \\ \dot{v} \\ \lambda \end{bmatrix} = \begin{bmatrix} v \\ F(t, q, v, \rho) \\ C(t, q, v, \rho) \end{bmatrix}, \quad t_0 \leq t \leq t_f, \quad q(t_0) = q_0(\rho), \quad v(t_0) = v_0(\rho), \quad (54)$$

or equivalently,

$$\dot{q} = v, \quad \begin{bmatrix} \dot{v} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} F \\ C \end{bmatrix} = \begin{bmatrix} f^{\text{DAE}-\dot{v}} \\ f^{\text{DAE}-\lambda} \end{bmatrix} = f^{\text{DAE}}(t, q, v, \rho). \quad (55)$$

The algebraic equation has the form $f^{\text{DAE}-\lambda} - \lambda = 0$.

Definition 13 (TLM of the Index-1 DAE Formulation). Sensitivities of solutions (3) and multipliers (53) of the system (54) with respect to parameters evolve according to the tangent linear model derived in [1,20,21,24–26]:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M & \Phi_q^T \\ 0 & \Phi_q & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{Q} \\ \dot{V} \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} V \\ F_v \cdot V - (M_q \dot{v} + \Phi_{q,q}^T \lambda - F_q) \cdot Q + F_\rho - M_\rho \dot{v} - \Phi_{q,\rho}^T \lambda \\ C_v \cdot V - (\Phi_{q,q} \dot{v} - C_q) \cdot Q + C_\rho - \Phi_{q,\rho} \dot{v} \end{bmatrix}, \quad (56)$$

with initial conditions given by (14b).

Definition 14 (Cost Function). Following Definition 13, consider a smooth scalar “trajectory cost function” g and a smooth scalar “terminal cost function” w . A general cost function has the form:

$$\psi = \int_{t_0}^{t_f} g(t, q(t, \rho), v(t, \rho), \dot{v}(t, \rho), \lambda(t, \rho), \rho) dt + w(t_f, q(t_f, \rho), v(t_f, \rho), \rho).$$

Note that the trajectory cost function (5) depends on both accelerations $\dot{v}$ and on the Lagrange multipliers λ . These are not independent variables and they can be resolved in terms of positions and velocities using (54), to obtain an equivalent regular trajectory cost function:

$$g(t, q, v, \dot{v}(t, q, v, \rho), \lambda(t, q, v, \rho), \rho) = \tilde{g}(t, q, v, \rho). \quad (57)$$

We keep accelerations and Lagrange multipliers (constraint forces) as explicit parameters in the cost function (57) in order to give additional flexibility in practical applications. In addition, we define the ‘quadrature’ variable $z(t) \in \mathbb{R}$ as follows:

$$z(t, \rho) := \int_{t_0}^t \tilde{g}(t, q(t, \rho), v(t, \rho), \rho) dt \quad \Leftrightarrow \quad (58a)$$

$$\dot{z}(t, \rho) = g(t, q, v, \dot{v}, \lambda, \rho) = \tilde{g}(t, q(t, \rho), v(t, \rho), \rho), \quad t_0 \leq t \leq t_f, \quad z(t_0, \rho) = 0. \quad (58b)$$

Definition 15 (The DAE Quadrature Sensitivity). Similarly, let the ‘quadrature sensitivity’ vector $Z(t, \rho)$ be the Jacobian of the ‘quadrature’ variable $z(t, \rho)$ (58a) with respect to the parameters ρ :

$$Z_i(t, \rho) := \frac{\partial z(t, \rho)}{\partial \rho_i}, \quad i = 1, \dots, p; \quad Z(t, \rho) := \nabla_\rho z(t, \rho) = [Z_1(t, \rho) \cdots Z_p(t, \rho)] \in \mathbb{R}^{1 \times p}. \quad (59)$$

The time evolution equations of the quadrature sensitivities are given by the TLM obtained by differentiating (58b) with respect to the parameters:

$$\begin{aligned} \dot{Z}_i &= \frac{dg(t, q, v, \dot{v}, \lambda, \rho)}{d\rho_i} \\ &= g_q \cdot Q_i + g_v \cdot V_i + g_{\dot{v}} \cdot \frac{df^{\text{DAE}-\dot{v}}}{d\rho} + g_\lambda \cdot \frac{df^{\text{DAE}-\lambda}}{d\rho} + g_{\rho_i} \\ &= (g_q + g_{\dot{v}} \cdot f_q^{\text{DAE}-\dot{v}} + g_\lambda \cdot f_q^{\text{DAE}-\lambda}) \cdot Q_i + (g_v + g_{\dot{v}} \cdot f_v^{\text{DAE}-\dot{v}} + g_\lambda \cdot f_v^{\text{DAE}-\lambda}) \cdot V_i + (g_{\rho_i} + g_{\dot{v}} \cdot f_{\rho_i}^{\text{DAE}-\dot{v}} + g_\lambda \cdot f_{\rho_i}^{\text{DAE}-\lambda}), \\ i &= 1, \dots, p, \quad t_0 \leq t \leq t_f, \quad Z_i(t_0, \rho) = 0. \end{aligned} \quad (60)$$

Definition 16 (Canonical Index-1 Sensitivity DAE). The canonical DAE system for the solution given by (55), the DAE TLM given by (56), and the sensitivity quadrature equations given by (60) need to be solved together forward in time, leading to the canonical sensitivity DAE that computes the derivatives of the cost function with respect to the system parameters ρ for

smooth systems:

$$= \begin{bmatrix} \dot{q} \\ \dot{v} \\ \lambda \\ \dot{z} \\ [\dot{Q}_i]_{i=1,\dots,p} \\ [\dot{V}_i]_{i=1,\dots,p} \\ [\dot{\Lambda}_i]_{i=1,\dots,p} \\ [\dot{Z}_i]_{i=1,\dots,p} \\ \begin{bmatrix} v \\ f_q^{\text{DAE}-\dot{v}} \\ f_v^{\text{DAE}-\lambda} \\ \tilde{g} \\ [V_i]_{i=1,\dots,p} \\ [f_q^{\text{DAE}-\dot{v}} Q_i + f_v^{\text{DAE}-\dot{v}} V_i + f_{\rho_i}^{\text{DAE}-\dot{v}}]_{i=1,\dots,p} \\ [f_q^{\text{DAE}-\lambda} Q_i + f_v^{\text{DAE}-\lambda} V_i + f_{\rho_i}^{\text{DAE}-\lambda}]_{i=1,\dots,p} \\ [(g_q + g_{\dot{v}} f_q^{\text{DAE}-\dot{v}} + g_{\lambda} f_q^{\text{DAE}-\lambda}) \cdot Q_i + (g_v + g_{\dot{v}} f_v^{\text{DAE}-\dot{v}} + g_{\lambda} f_v^{\text{DAE}-\lambda}) \cdot V_i + (g_{\rho_i} + g_{\dot{v}} \cdot f_{\rho_i}^{\text{DAE}-\dot{v}} + g_{\lambda} \cdot f_{\rho_i}^{\text{DAE}-\lambda})]_{i=1,\dots,p} \end{bmatrix} \end{bmatrix}, \quad (61)$$

where the state vector of the canonical index-1 sensitivity DAE (61) is :

$$X = [q^T, v^T, \lambda^T, z, Q_1^T, \dots, Q_p^T, V_1^T, \dots, V_p^T, \Lambda_1^T, \dots, \Lambda_p^T, Z_1, \dots, Z_p]^T \in \mathbb{R}^{(n+1)(p+1)}, \quad (62)$$

and the derivatives of the DAE function are:

$$f_q^{\text{DAE}} = \begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_q - M_q \dot{v} - \Phi_{q,q}^T \lambda \\ C_q - \Phi_{q,q} \dot{v} \end{bmatrix}, \quad f_v^{\text{DAE}} = \begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_v \\ C_v \end{bmatrix}, \quad f_{\rho}^{\text{DAE}} = \begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}^{-1} \begin{bmatrix} F_{\rho} - M_{\rho} \dot{v} - \Phi_{q,\rho}^T \lambda \\ C_{\rho} - \Phi_{q,\rho} \dot{v} \end{bmatrix}. \quad (63)$$

4.4. Direct sensitivity analysis for smooth systems in the penalty ODE formulation

Definition 17 (Constrained Multibody Dynamics Systems: The Penalty ODE Formulation). Define the extended mass matrix $\bar{M} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^{n \times n}$ as:

$$\bar{M}(t, q, v, \rho) := M(t, q, v, \rho) + \Phi_q^T(t, q, v, \rho) \cdot \alpha \cdot \Phi_q(t, q, v, \rho), \quad (64a)$$

where $\alpha \in \mathbb{R}^{m \times m}$ is the penalty factor of the ODE penalty formulation. Define the extended right-hand side function $\bar{F} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ as:

$$\bar{F}(t, q, v, \rho) := F(t, q, v, \rho) - \Phi_q^T \cdot \alpha \cdot (\dot{\Phi}_q v + \dot{\Phi}_t + 2\xi \omega \dot{\Phi} + \omega^2 \Phi), \quad (64b)$$

where $\xi \in \mathbb{R}$ and $\omega \in \mathbb{R}$ are the natural frequency and damping ratio coefficients of the formulation, respectively, and $\dot{\Phi}$ is the total time derivative of the kinematic constraints. The algebraic position constraints (51a) are removed and an auxiliary spring-damper force is added in (64b) to prevent the system from deviating away from the constraints.

In the penalty formulation the EOM of a constrained rigid multibody system is the second order ODE:

$$\begin{cases} \dot{q} = v, \\ \dot{v} = f^{\text{eom}}(t, q, v, \rho) = \bar{M}^{-1}(t, q, v, \rho) \cdot \bar{F}(t, q, v, \rho). \end{cases} \quad (64c)$$

The Lagrange multipliers associated to the constraint forces are estimated as follows:

$$\lambda^* = \alpha (\ddot{\Phi} + 2\xi \omega \dot{\Phi} + \omega^2 \Phi). \quad (64d)$$

The cost function (57) is formulated using the Lagrange multiplier estimates (64d), i.e., using the trajectory cost function $g(t, q, v, \dot{v}, \lambda^*, \rho)$. Sensitivities (3) of the position and velocity state variables of the system (64) with respect to parameters evolve according to the tangent linear model derived in [1,20,21,24–26]:

$$\begin{cases} \dot{Q} = V, \\ \bar{M} \cdot \dot{V} = (\bar{F}_q - \bar{M}_q \dot{v}) \cdot Q + \bar{F}_v \cdot V + \bar{F}_{\rho} - \bar{M}_{\rho} \dot{v}, \end{cases} \quad t_0 \leq t \leq t_F, \quad (65)$$

with initial conditions given by (14b). The derivatives $\bar{F}_q$, $\bar{F}_v$, $\bar{F}_{\rho}$, $\bar{M}_{\rho}$, and $\bar{M}_q$ are given in Appendix.

Definition 18 (The Canonical ODE Sensitivity). The canonical sensitivity ODE that computes the derivatives of the cost function with respect to the system parameters ρ for the smooth ODE penalty system (64) is the same than the ODE canonical system presented in (18) and extended to the cost function (57) formulated using the Lagrange multiplier estimates.

$$\begin{bmatrix} \dot{q} \\ \dot{v} \\ \dot{z} \\ [\dot{Q}_i]_{i=1,\dots,p} \\ [\dot{V}_i]_{i=1,\dots,p} \\ [\dot{Z}_i]_{i=1,\dots,p} \end{bmatrix} = \begin{bmatrix} v \\ f_v^{\text{eom}} \\ \tilde{g} \\ [V_i]_{i=1,\dots,p} \\ [f_q^{\text{eom}} \cdot Q_i + f_v^{\text{eom}} \cdot V_i + f_{\rho_i}^{\text{eom}}]_{i=1,\dots,p} \\ [(g_q + g_v \cdot f_q^{\text{eom}} + g_{\lambda^*} \cdot \lambda_q^*) \cdot Q_i + (g_v + g_v \cdot f_v^{\text{eom}} + g_{\lambda^*} \cdot \lambda_v^*) \cdot V_i + (g_{\rho_i} + g_v \cdot f_{\rho_i}^{\text{eom}} + g_v \cdot \lambda_{\rho}^*)]_{i=1,\dots,p} \end{bmatrix}, \quad (66)$$

where

$$f_q^{\text{eom}} = \bar{M}^{-1} (\bar{F}_q - \bar{M}_q \dot{v}), \quad f_v^{\text{eom}} = \bar{M}^{-1} \bar{F}_v, \quad f_{\rho_i}^{\text{eom}} = \bar{M}^{-1} (\bar{F}_{\rho} - \bar{M}_{\rho} \dot{v}), \quad (67)$$

and with the initial conditions given by (14b). The derivatives λ_q^* , λ_v^* and λ_{ρ}^* are given in Appendix.

Remark 10. The sensitivity of the estimated Lagrange multipliers

$$\Lambda^*(t, \rho) := D_{\rho} \lambda^*(t) := \frac{d\lambda^*}{d\rho}(t, \rho) \in \mathbb{R}^{m \times p} \quad (68)$$

is calculated as:

$$\Lambda_i^* = \lambda_q^* Q_i + \lambda_v^* V_i + \lambda_{\rho_i}^*, \quad i = 1, \dots, p. \quad (69)$$

5. Direct sensitivity analysis for hybrid constrained multibody systems

We now discuss constrained multibody systems when the dynamics is piecewise continuous in time.

5.1. Coordinates partitioning for hybrid multibody systems

The direct sensitivity analysis for a constrained rigid hybrid multibody dynamic system requires to find the jump conditions at the time of event. For this we need to distinguish between dependent and independent state variables and their sensitivities.

Assume that the Jacobian of the position constraint (51a) has full row rank at a given configuration, $\text{rank}(\Phi_q) = m$. One can rearrange the columns and split the Jacobian in two submatrices:

$$\Phi_q \cdot P^T = [\Phi_{q_{\text{dep}}} \quad \Phi_{q_{\text{dof}}}], \quad \Phi_{q_{\text{dep}}} \in \mathbb{R}^{m \times m}, \quad \Phi_{q_{\text{dof}}} \in \mathbb{R}^{m \times f}, \quad f = n - m, \quad (70)$$

such that the first block $\Phi_{q_{\text{dep}}}$ is nonsingular. Here $P \in \mathbb{R}^{n \times n}$ is a permutation matrix, obtained by permuting rows of identity matrix; the multiplication $\Phi_q \cdot P$ performs a permutation of the columns of Φ_q .

By the implicit function theorem one can partition locally the position state variables into independent coordinates $q_{\text{dof}} \in \mathbb{R}^f$ (the local ‘degrees of freedom’ of the system) and dependent coordinates $q_{\text{dep}} \in \mathbb{R}^m$, and solve for the dependent ones in terms of the degrees of freedom:

$$\Phi(t, q) = 0 \quad \text{and} \quad \Phi_{q_{\text{dep}}}(t, q) \text{ nonsingular} \quad \Rightarrow \quad q_{\text{dep}} = \zeta(t, q_{\text{dof}}). \quad (71)$$

This induces a corresponding local partitioning of the state variables into independent components $q_{\text{dof}}, v_{\text{dof}} \in \mathbb{R}^f$ and dependent components $q_{\text{dep}}, v_{\text{dep}} \in \mathbb{R}^m$:

$$P \cdot q = \begin{bmatrix} P_{\text{dep}} \\ P_{\text{dof}} \end{bmatrix} \cdot q = \begin{bmatrix} q_{\text{dep}} \\ q_{\text{dof}} \end{bmatrix}, \quad P \cdot v = \begin{bmatrix} P_{\text{dep}} \\ P_{\text{dof}} \end{bmatrix} \cdot v = \begin{bmatrix} v_{\text{dep}} \\ v_{\text{dof}} \end{bmatrix}, \quad (72)$$

where $P_{\text{dep}} \in \mathbb{R}^{m \times n}$ and $P_{\text{dof}} \in \mathbb{R}^{f \times n}$ consist the first m and the last f rows of P , respectively. Let:

$$R := -\Phi_{q_{\text{dep}}}^{-1} \Phi_{q_{\text{dof}}} \in \mathbb{R}^{m \times f}. \quad (73)$$

The velocity constraint equation (51b) becomes:

$$\Phi_{q_{\text{dep}}} v_{\text{dep}} + \Phi_{q_{\text{dof}}} v_{\text{dof}} = -\Phi_t \Rightarrow v_{\text{dep}} = -\Phi_{q_{\text{dep}}}^{-1} (\Phi_{q_{\text{dof}}} v_{\text{dof}} + \Phi_t) = R v_{\text{dof}} - \Phi_{q_{\text{dep}}}^{-1} \Phi_t. \quad (74)$$

Similarly, the acceleration constraint equation (51c) becomes:

$$\Phi_{q_{\text{dep}}} \dot{v}_{\text{dep}} + \Phi_{q_{\text{dof}}} \dot{v}_{\text{dof}} = C \Rightarrow \dot{v}_{\text{dep}} = -\Phi_{q_{\text{dep}}}^{-1} (\Phi_{q_{\text{dof}}} \dot{v}_{\text{dof}} - C) = R \dot{v}_{\text{dof}} + \Phi_{q_{\text{dep}}}^{-1} C. \quad (75)$$

From (70), (72), and (73) we have that:

$$\Phi_{q_{\text{dof}}} = \Phi_q \cdot P_{\text{dof}}^T \in \mathbb{R}^{m \times f}, \quad \Phi_{q_{\text{dep}}} = \Phi_q \cdot P_{\text{dep}}^T \in \mathbb{R}^{m \times m}, \quad R = -(\Phi_q \cdot P_{\text{dep}}^T)^{-1} \cdot \Phi_q \cdot P_{\text{dof}}^T. \quad (76)$$

5.2. Representation of constrained hybrid multibody systems

The hybrid dynamics of a constrained mechanical system refers to the smooth system defined in Section 4 subjected to a finite number of events, as discussed in Definition 7. Each event (24) happening at the ‘time of event’ t_{eve} introduces a kink in the trajectory of the mechanical system. At each event the velocity state vector of an *unconstrained* system undergoes a jump (27) that can be arbitrary, i.e., can be described by any smooth function $h(\cdot)$. In case of a *constrained* system we need a more comprehensive understanding of the event.

Definition 19 (Characterization of an Event for Constrained Multibody Systems). During an event at time t_{eve} a constrained mechanical system undergoes a sudden change in state characterized as follows:

- The constraints may change at the time of event (e.g., when a walking humanoid robot changes its supporting foot at each step). Consequently, the position constraint function (51a) changes from $\Phi^- : \mathbb{R}^{1+n+p} \rightarrow \mathbb{R}^{m^-}$ before event to $\Phi^+ : \mathbb{R}^{1+n+p} \rightarrow \mathbb{R}^{m^+}$ after event:

$$\Phi^-(t, q, \rho) \xrightarrow{\text{event}} \Phi^+(t, q, \rho).$$

The two constraint functions are different, and in particular the number of constraints can differ, $m^+ \neq m^-$.

- The Jacobians of the position constraints before and after the event have full row ranks at the event configuration $q|_{t_{\text{eve}}}$:

$$\text{rank}(\Phi_q^-(t, q, \rho)) = m^-, \quad \text{rank}(\Phi_q^+(t, q, \rho)) = m^+. \quad (77)$$

- Since the constraints can be different after and before the event, the partitions of variables into independent and dependent can also differ. We denote by $\square_{\text{dof}^-}$, $\square_{\text{dep}^-}$ the independent and dependent components before the event, and by $\square_{\text{dof}^+}$, $\square_{\text{dep}^+}$ the independent and dependent components after the event:

$$P^- \cdot v = \begin{bmatrix} v_{\text{dep}^-} \\ v_{\text{dof}^-} \end{bmatrix} \in \mathbb{R}^n, \quad v_{\text{dof}^-} \in \mathbb{R}^{f^-}, \quad P^+ \cdot v = \begin{bmatrix} v_{\text{dep}^+} \\ v_{\text{dof}^+} \end{bmatrix} \in \mathbb{R}^n, \quad v_{\text{dof}^+} \in \mathbb{R}^{f^+}. \quad (78)$$

Here P^- and P^+ are the permutation matrices that select the dependent and independent coordinates before and after the event, respectively. The dimensions of the velocity degrees of freedom vectors are $f^- = n - m^-$ and $f^+ = n - m^+$ before and after the event, respectively.

- The generalized position state variables remain the same (26), i.e., $q|_{t_{\text{eve}}}^+ = q|_{t_{\text{eve}}}^- = q|_{t_{\text{eve}}}$. Consequently, the state at the time of event $q|_{t_{\text{eve}}}$ need to satisfy both constraint functions:

$$\Phi^-|_{t_{\text{eve}}}^- := \Phi^-(t_{\text{eve}}, q|_{t_{\text{eve}}}, \rho) = 0, \quad \Phi^+|_{t_{\text{eve}}}^+ := \Phi^+(t_{\text{eve}}, q|_{t_{\text{eve}}}, \rho) = 0. \quad (79)$$

At the event the system moves from one constraint manifold to another, and $q|_{t_{\text{eve}}}$ is on the intersection of the two manifolds.

- The jump in velocity from right before the event to right after the event is defined in terms of the independent components, i.e., in terms of the velocity degrees of freedom:

$$v_{\text{dof}^+}|_{t_{\text{eve}}}^+ = h(t_{\text{eve}}, q|_{t_{\text{eve}}}, v_{\text{dof}^-}|_{t_{\text{eve}}}^-, \rho), \quad h : \mathbb{R}^{1+n+f^-+p} \rightarrow \mathbb{R}^{f^+}. \quad (80)$$

The jump function (80) is assumed to be smooth. Note that its formulation is not unique, since it depends on the selections of the degrees of freedom that are not unique.

- The velocity state vectors satisfy the velocity kinematic constraints (74). Consequently, the jumps in velocity (27) cannot be arbitrary for the dependent components. The dependent components of velocity are obtained from solving the velocity constraints (74):

$$\begin{aligned} v_{\text{dep}^+}|_{t_{\text{eve}}}^+ &= -\left(\Phi_{q_{\text{dep}^+}}^+|_{t_{\text{eve}}}^+\right)^{-1} \cdot \left(\Phi_{q_{\text{dof}^+}}^+|_{t_{\text{eve}}}^+ v_{\text{dof}^+}|_{t_{\text{eve}}}^+ + \Phi_t^+|_{t_{\text{eve}}}^+\right) \\ &= R^+|_{t_{\text{eve}}}^+ v_{\text{dof}^+}|_{t_{\text{eve}}}^+ - \left(\Phi_{q_{\text{dep}^+}}^+|_{t_{\text{eve}}}^+\right)^{-1} \cdot \Phi_t^+|_{t_{\text{eve}}}^+. \end{aligned} \quad (81)$$

Here $R^\pm$ are the matrices corresponding to the constraints $\Phi^\pm$.

Remark 11 (Collision Events). The proposed formalism (80)–(81) covers the case of elastic contact/collision/impact without change in the set of constraint equations, $\Phi^+ \equiv \Phi^-$. The impulsive (external) contact forces act to change the independent components of the velocity state (80).

Remark 12 (Hybrid DAE Jump Formulation). The proposed formalism (80)–(81) also covers the case where the event consists solely of a change of constraints $\Phi^+ \neq \Phi^-$, without any external force to modify the independent velocities. This type of event appears mainly in the humanoid robotics field where general and relative coordinates are used and inelastic collisions are considered. A popular approach in robotics is to solve for the DAE involving impulsive forces in the constraints at the time of event [27]:

$$\begin{bmatrix} \mathbf{M}|_{t_{\text{eve}}} & (\Phi_q^+)^T|_{t_{\text{eve}}} \\ \Phi_q^+|_{t_{\text{eve}}} & 0 \end{bmatrix} \cdot \begin{bmatrix} v|_{t_{\text{eve}}}^+ \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M}|_{t_{\text{eve}}} \cdot v|_{t_{\text{eve}}}^- \\ -\Phi_t^+|_{t_{\text{eve}}} \end{bmatrix}, \quad (82a)$$

or equivalently,

$$\begin{bmatrix} v|_{t_{\text{eve}}}^+ \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M}|_{t_{\text{eve}}} & (\Phi_q^+)^T|_{t_{\text{eve}}} \\ \Phi_q^+|_{t_{\text{eve}}} & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{M}|_{t_{\text{eve}}} \cdot v|_{t_{\text{eve}}}^- \\ -\Phi_t^+|_{t_{\text{eve}}} \end{bmatrix} = \begin{bmatrix} f^{\text{DAE-imp-v}}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \\ f^{\text{DAE-imp-}\lambda}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) \end{bmatrix}. \quad (82b)$$

Here $v|_{t_{\text{eve}}}^+$ contains both the independent and dependent coordinates. We see that the second equation in (82a) automatically imposes the velocity constraint (51b).

Our formalism covers this approach by defining the jump function given by (80) as:

$$v_{\text{dof}+}|_{t_{\text{eve}}}^+ = \mathbf{P}_{\text{dof}+} f^{\text{DAE-imp-v}}(t_{\text{eve}}, q|_{t_{\text{eve}}}, v|_{t_{\text{eve}}}^-, \rho) =: h(t_{\text{eve}}, q|_{t_{\text{eve}}}, v_{\text{dof}-}|_{t_{\text{eve}}}^-, \rho). \quad (83)$$

5.3. The jump in the sensitivity of the position state vector

The jump conditions at the time of event in the sensitivity state vector for a constrained rigid multibody involve finding the sudden change in values of the sensitivity with respect to the system parameters ρ of the position and the dependent and independent velocity state variables due the impulsive jump of the independent velocity state variables.

Remark 13 (Partitioning of Sensitivity Matrices). The partitioning of state variables into dependent and independent (78) induces a similar partitioning of the sensitivity matrices (3):

$$\mathbf{P} \cdot \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{\text{dep}} \\ \mathbf{Q}_{\text{dof}} \end{bmatrix} \in \mathbb{R}^{n \times p}, \quad \mathbf{Q}_{\text{dof}} \in \mathbb{R}^{f \times p}, \quad \mathbf{P} \cdot \mathbf{V} = \begin{bmatrix} \mathbf{V}_{\text{dep}} \\ \mathbf{V}_{\text{dof}} \end{bmatrix} \in \mathbb{R}^{n \times p}, \quad \mathbf{V}_{\text{dof}} \in \mathbb{R}^{f \times p}. \quad (84)$$

Differentiation of the position constraint equation (51a) with respect to the system parameters ρ gives:

$$0 = \frac{d\Phi(t, q(t, \rho), \rho)}{d\rho} = \Phi_q \cdot \mathbf{Q} + \Phi_\rho = \Phi_{q_{\text{dep}}} \cdot \mathbf{Q}_{\text{dep}} + \Phi_{q_{\text{dof}}} \cdot \mathbf{Q}_{\text{dof}} + \Phi_\rho, \quad (85)$$

and therefore:

$$\mathbf{Q}_{\text{dep}} = -\Phi_{q_{\text{dep}}}^{-1} (\Phi_{q_{\text{dof}}} \cdot \mathbf{Q}_{\text{dof}} + \Phi_\rho) = \mathbf{R} \cdot \mathbf{Q}_{\text{dof}} - \Phi_{q_{\text{dep}}}^{-1} \Phi_\rho. \quad (86)$$

Similarly, differentiation of the velocity constraint equation (51b) with respect to the system parameters gives:

$$\begin{aligned} 0 &= \frac{d}{d\rho} (\Phi_q(t, q(t, \rho), \rho) v(t, \rho) + \Phi_t(t, q(t, \rho), \rho)) \\ &= \Phi_q \cdot \mathbf{V} + (\Phi_{q,q} v + \Phi_{q,t}) \cdot \mathbf{Q} + \Phi_{\rho,q} v + \Phi_{\rho,t} \\ &= \Phi_{q_{\text{dep}}} \cdot \mathbf{V}_{\text{dep}} + \Phi_{q_{\text{dof}}} \cdot \mathbf{V}_{\text{dof}} + (\Phi_{q,q} v + \Phi_{q,t}) \cdot \mathbf{Q} + \Phi_{\rho,q} v + \Phi_{\rho,t}, \end{aligned}$$

and therefore:

$$\mathbf{V}_{\text{dep}} = \mathbf{R} \cdot \mathbf{V}_{\text{dof}} - \Phi_{q_{\text{dep}}}^{-1} ((\Phi_{q,q} v + \Phi_{q,t}) \cdot \mathbf{Q} + \Phi_{\rho,q} v + \Phi_{\rho,t}). \quad (87)$$

Remark 14 (Sensitivity of the Time of Event for Constrained Systems). The time of event depends only on the position state and on the event function (24). Consequently, the sensitivity of the time of event for constrained systems is the same as for unconstrained systems, and is given by (33) in Theorem 1.

Theorem 6 (Jump in Position Sensitivity for Constrained Systems). Let $\mathbf{Q}|_{t_{\text{eve}}}^+$ and $\mathbf{Q}|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivities of the generalized position state vectors right after and right before the event, respectively. The independent components of the sensitivity of the generalized positions right after the event are:

$$\mathbf{Q}_{\text{dof}+}|_{t_{\text{eve}}}^+ = \mathbf{Q}_{\text{dof}+}|_{t_{\text{eve}}}^- - \left(v_{\text{dof}+}|_{t_{\text{eve}}}^+ - v_{\text{dof}+}|_{t_{\text{eve}}}^- \right) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (88a)$$

The dependent components of the sensitivity of the generalized positions right after the event are given by Eq. (86), using the after-event constraints:

$$Q_{\text{dep}+}|_{t_{\text{eve}}}^+ = R^+|_{t_{\text{eve}}}^+ \cdot Q_{\text{dof}+}|_{t_{\text{eve}}}^+ - \left(\Phi_{\text{qdep}+}^+|_{t_{\text{eve}}}^+ \right)^{-1} \Phi_{\rho}^+|_{t_{\text{eve}}}^+. \quad (88b)$$

Proof. The proof of the jump in the independent coordinates (88a) follows closely the proof of Theorem 2. The equation for dependent coordinates (88b) follows from the linearized position constraint equation (86). $\square$

Remark 15. From (70) and (72) we can rewrite (88a) as:

$$P_{\text{dof}+}^+ \cdot \left(Q|_{t_{\text{eve}}}^+ - Q|_{t_{\text{eve}}}^- \right) = -P_{\text{dof}+}^+ \cdot \left(v|_{t_{\text{eve}}}^+ - v|_{t_{\text{eve}}}^- \right) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (89)$$

5.4. The jump in the sensitivity of the velocity state vector

Theorem 7 (Jump in Velocity Sensitivity for Constrained System). Let $V|_{t_{\text{eve}}}^+, V|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivity matrices of the generalized velocity state vectors after and before the event, respectively. The independent coordinates of the velocity sensitivities right after the event are given by:

$$V_{\text{dof}+}|_{t_{\text{eve}}}^+ = h_q|_{t_{\text{eve}}}^- \cdot Q|_{t_{\text{eve}}}^- + h_{v_{\text{dof}-}}|_{t_{\text{eve}}}^- \cdot V_{\text{dof}-}|_{t_{\text{eve}}}^- + \left(h_q|_{t_{\text{eve}}}^- \cdot v|_{t_{\text{eve}}}^- - \ddot{q}_{\text{dof}+}|_{t_{\text{eve}}}^+ + h_{v_{\text{dof}-}}|_{t_{\text{eve}}}^- \cdot \ddot{q}_{\text{dof}-}|_{t_{\text{eve}}}^- + h_t|_{t_{\text{eve}}}^- \right) \cdot \frac{dt_{\text{eve}}}{d\rho} + h_{\rho}|_{t_{\text{eve}}}^-, \quad (90a)$$

where the Jacobians of the jump function are:

$$\begin{aligned} h_q|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial q}(q|_{t_{\text{eve}}}, v_{\text{dof}-}|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f^+ \times n}, & h_{v_{\text{dof}-}}|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial v_{\text{dof}-}}(q|_{t_{\text{eve}}}, v_{\text{dof}-}|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f^+ \times f^-}, \\ h_t|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial t}(q|_{t_{\text{eve}}}, v_{\text{dof}-}|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^f, & h_{\rho}|_{t_{\text{eve}}}^- &:= \frac{\partial h}{\partial \rho}(q|_{t_{\text{eve}}}, v_{\text{dof}-}|_{t_{\text{eve}}}^-, \rho) \in \mathbb{R}^{f \times p}. \end{aligned} \quad (90b)$$

The dependent components of the velocity sensitivities right after the event are calculated via (87), using the after-event constraints:

$$V_{\text{dep}+}|_{t_{\text{eve}}}^+ = -\left(\Phi_{\text{qdep}+}^+|_{t_{\text{eve}}}^+ \right)^{-1} \left(\Phi_{\text{qdof}+}^+ \cdot V_{\text{dof}+}|_{t_{\text{eve}}}^+ + \left(\Phi_{q,q}^+ v + \Phi_{t,q}^+ \right) \cdot Q + \Phi_{q,\rho}^+ v + \Phi_{t,\rho}^+ \right)|_{t_{\text{eve}}}^+. \quad (90c)$$

Proof. The proof of the jump in the independent coordinates (90a) follows closely the proof of Theorem 3. The equation for dependent coordinates (90c) follows from the linearized velocity constraint equation (87). $\square$

5.5. The jump in the sensitivity of the velocity state vector using the hybrid DAE jump formulation

Consider the case of a sudden change in constraints discussed in Remark 12. The jump in the velocity sensitivity for constrained systems due to impulsive forces is determined as follows:

$$\begin{aligned} \begin{bmatrix} M|_{t_{\text{eve}}}^+ & \Phi_q^{+T}|_{t_{\text{eve}}}^+ \\ \Phi_q^+|_{t_{\text{eve}}}^+ & 0 \end{bmatrix} \cdot \begin{bmatrix} V|_{t_{\text{eve}}}^+ \\ \delta\lambda \end{bmatrix} &= - \begin{bmatrix} M|_{t_{\text{eve}}}^+ q|_{t_{\text{eve}}}^+ \cdot (v|_{t_{\text{eve}}}^+ - v|_{t_{\text{eve}}}^-) + \Phi_{q,q}^{+T}|_{t_{\text{eve}}}^+ \cdot \delta\lambda \\ \Phi_{q,q}^+|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^+ \end{bmatrix} \cdot Q|_{t_{\text{eve}}}^+ + \begin{bmatrix} M|_{t_{\text{eve}}}^+ \\ 0 \end{bmatrix} \cdot V|_{t_{\text{eve}}}^- \\ &\quad - \begin{bmatrix} M_{\rho}|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^+ + \Phi_{q,\rho}^{+T}|_{t_{\text{eve}}}^+ \cdot \delta\lambda \\ \Phi_{q,\rho}^+|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^+ + \Phi_{t,q}^+|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^- + \Phi_{t,v}^+|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^- + \Phi_{t,\rho}^+|_{t_{\text{eve}}}^+ \cdot v|_{t_{\text{eve}}}^- \end{bmatrix}, \end{aligned} \quad (91)$$

or equivalently:

$$\begin{aligned} \begin{bmatrix} V|_{t_{\text{eve}}}^+ \\ \delta\lambda \end{bmatrix} &= - \begin{bmatrix} M & \Phi_q^{+T} \\ \Phi_q^+ & 0 \end{bmatrix}^{-1} \begin{bmatrix} M_q \cdot (v|_{t_{\text{eve}}}^+ - v|_{t_{\text{eve}}}^-) + \Phi_{q,q}^{+T} \cdot \delta\lambda \\ \Phi_{q,q}^+ \cdot v|_{t_{\text{eve}}}^+ \end{bmatrix} \cdot Q|_{t_{\text{eve}}}^+ + \begin{bmatrix} M & \Phi_q^{+T} \\ \Phi_q^+ & 0 \end{bmatrix}^{-1} \begin{bmatrix} M \\ 0 \end{bmatrix} \cdot V|_{t_{\text{eve}}}^- \\ &\quad - \begin{bmatrix} M & \Phi_q^{+T} \\ \Phi_q^+ & 0 \end{bmatrix}^{-1} \begin{bmatrix} M_{\rho} \cdot v|_{t_{\text{eve}}}^+ + \Phi_{q,\rho}^{+T} \cdot \delta\lambda \\ \Phi_{q,\rho}^+ \cdot v|_{t_{\text{eve}}}^+ + \Phi_{t,\rho}^+ \cdot v|_{t_{\text{eve}}}^- \end{bmatrix} - \begin{bmatrix} M & \Phi_q^{+T} \\ \Phi_q^+ & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \Phi_{t,q}^+ \cdot v|_{t_{\text{eve}}}^- + \Phi_{t,v}^+ \cdot v|_{t_{\text{eve}}}^- \end{bmatrix}, \end{aligned} \quad (92)$$

which simplifies to:

$$\begin{bmatrix} V|_{t_{\text{eve}}}^+ \\ \delta\lambda \end{bmatrix} = f_q^{\text{DAE-imp}} \cdot Q|_{t_{\text{eve}}}^+ + f_{v|_{t_{\text{eve}}}^-}^{\text{DAE-imp}} \cdot V|_{t_{\text{eve}}}^- + f_{\rho}^{\text{DAE-imp}} + f_t^{\text{DAE-imp}} \quad (93)$$

5.6. The jump in the sensitivity of the Lagrange multipliers

Remark 16. When the DAE formalism is selected to model the smooth dynamics of a constrained mechanical system, the jump in the sensitivity of the Lagrange multipliers (53) from $\Lambda|_{t_{\text{eve}}}^- \rightarrow \Lambda|_{t_{\text{eve}}}^+$ at the time of event is:

$$\Lambda_i|_{t_{\text{eve}}}^+ = \Lambda_i|_{t_{\text{eve}}}^- + f_q^{\text{DAE}-\lambda}|_{t_{\text{eve}}}^+ Q_i|_{t_{\text{eve}}}^+ + f_v^{\text{DAE}-\lambda}|_{t_{\text{eve}}}^+ V_i|_{t_{\text{eve}}}^+ + f_{\rho_i}^{\text{DAE}-\lambda}|_{t_{\text{eve}}}^+, \quad i = 1, \dots, p \quad (94)$$

Remark 17. When the ODE penalty formalism is selected to model the smooth dynamics of a constrained mechanical system, the jump in the sensitivity of the estimated Lagrange multipliers (68) from $\Lambda^*|_{t_{\text{eve}}}^- \rightarrow \Lambda^*|_{t_{\text{eve}}}^+$ at the time of event is:

$$\Lambda_i^*|_{t_{\text{eve}}}^+ = \Lambda_i^*|_{t_{\text{eve}}}^- + \lambda_q^*|_{t_{\text{eve}}}^+ Q_i|_{t_{\text{eve}}}^+ + \lambda_v^*|_{t_{\text{eve}}}^+ V_i|_{t_{\text{eve}}}^+ + \lambda_{\rho_i}^*|_{t_{\text{eve}}}^+, \quad i = 1, \dots, p \quad (95)$$

5.7. The sensitivity of the cost function for hybrid systems

Remark 18. The formalism that computes the sensitivities of the cost function with respect to parameters for hybrid systems does not change from the formalism presented for smooth systems illustrated in Remark 4. Indeed, the sensitivities of the cost function sum all the sensitivities of the trajectories and the quadrature variables evaluated at the final time. Any jump in the sensitivities of the trajectories and quadrature variables were anteriorly computed. The jump in the sensitivities of the quadrature variables are given by (48).

6. Direct sensitivity analysis for constrained mechanical systems with transition functions

The transition function refers to a sudden change of the governing function or vector field. In this section we discuss about direct sensitivity analysis for constrained mechanical with jump-discontinuity in the acceleration caused by a sudden change of the equation of motions at the time of event.

Definition 20 (*Change of EOM at the Time of Event*). Unlike previous methodology, the ODE penalty formulation (64) incorporates the kinematic constraints (position, velocity, and acceleration constraint equations) into the equation of motions and stabilize them over time. Therefore, any change in the set of kinematic constraints involves a change in the equation of motions, thus, a change in the acceleration vector (or right-hand side function). Because the ODE penalty formulation is a control based constraint stabilization method, the position constraint is not satisfied exactly right after the sudden change in the set of kinematic constraints:

$$\Phi^-|_{t_{\text{eve}}} = 0, \quad \Phi^+|_{t_{\text{eve}}} \neq 0. \quad (96)$$

This differs from (82) as there are no instantaneous kinematic jump in the velocity state variable.

Theorem 8 (*Jump in the Velocity Sensitivity for Constrained Systems Due to the Change of Equation of Motions*). Let $V|_{t_{\text{eve}}}^+, V|_{t_{\text{eve}}}^- \in \mathbb{R}^{n \times p}$ be the sensitivity matrices of the generalized velocity state vectors after and before the event, respectively. Let the event characterized as a change of equation of motions due to the change of constraints including in the equation of motions. The sensitivities of the independent velocities right after the event are given by

$$V_{\text{dof-}}|_{t_{\text{eve}}}^+ = V_{\text{dof-}}|_{t_{\text{eve}}}^- + (\ddot{q}_{\text{dof-}}|_{t_{\text{eve}}}^- - \ddot{q}_{\text{dof-}}|_{t_{\text{eve}}}^+) \cdot \frac{dt_{\text{eve}}}{d\rho}. \quad (97)$$

Proof. The proof of the jump in the independent coordinates (58a) follows closely the proof of Theorem 4. $\square$

Remark 19. The sensitivities of the dependent velocities right after the event are given by (90c). As well, the sensitivities of the position right after the event for the independent and dependent variables are given by (88a) and (88b), respectively.

Remark 20. The sudden change in the equation of motions at the event time is caused by a sudden change of forces acting on the system, such as constraint forces, friction forces, or a change of masses. The proposed formalism to calculate the jump conditions for systems with discontinuous right-hand sides remains valid for any type of change of the equation of motions.

Remark 21. The proposed formalism in calculating the jump conditions for systems with jerk discontinuity incorporates Remarks 17 and 18.

7. Case study: sensitivity analysis of a five-bar mechanism

The five-bar mechanism with two degrees of freedom, shown in Fig. 3a, is used as a case study to illustrate the sensitivity analysis approach for hybrid constrained multibody systems developed herein. The mechanism has five revolute joints located at points A, 1, 2, 3, and B where A and B are pinned. The fixed link between the two pinned points represent the fifth bar of the system. The state vector includes the natural coordinates of the point 1, 2, and 3 of the mechanism with

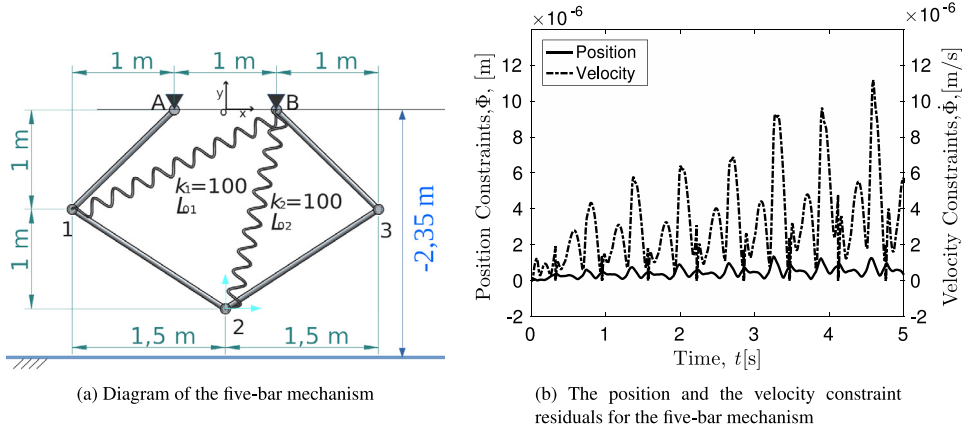


Fig. 3. Structure of the five-bar mechanism.

$q = [q_1^T q_2^T q_3^T]^T$, where $q_2 = [x_2 y_2]^T$ represents the two degrees of freedom, and $q_1 = [x_1 y_1]^T$ with $q_3 = [x_3 y_3]^T$ represent the dependent coordinates. The dependent coordinates are solved using the constraints of the system. The constraints are defined according to the fixed lengths between each set of points, as follows:

$$\Phi = \begin{bmatrix} \|q_A - q_1\|^2 - L_{A1}^2 \\ \|q_2 - q_1\|^2 - L_{21}^2 \\ \|q_3 - q_2\|^2 - L_{32}^2 \\ \|q_B - q_3\|^2 - L_{B3}^2 \end{bmatrix} = 0, \quad (98)$$

where the lengths $L_{A1} = L_{B3} = 1.4142$ m; $L_{A1} = L_{B3} = 1.8027$ m. The pinned points are located at $q_A = [-0.5 \ 0]^T$ and $q_B = [0.5 \ 0]^T$. The masses of the bars are $m_1 = 1$ kg, $m_2 = 1.5$ kg, $m_3 = 1.5$ kg, $m_4 = 1$ kg; the polar moments of inertia are assumed to be ideal, with uniform distribution of mass; the two springs have stiffness coefficients of $k_1 = k_2 = 100$ N/m and natural lengths of $L_{01} = 2.2360$ m and $L_{02} = 2.0615$ m.

A flat ground surface is located at -2.35 m and we define the event function $r(q|_{t_{\text{eve}}}) = y_2 - y_{\text{event}} = 0$ (24), where the bottom point of the five-bar mechanism (point 2) hits the ground at $y_{\text{event}} = -2.35$ m. Once the event is detected, the vertical velocity of point 2 jumps to its opposite value, while its horizontal velocity remains unchanged. This study computes the sensitivity of the bottom point of the five-bar mechanism (point 2) with respect to a small change in the initial lengths of the springs.

The canonical ODE for multibody systems (66) that computes the penalty ODE formulation for constrained multibody dynamics systems (64) and the sensitivities of such formulation is simulated for a time span of five seconds. The initial conditions for the two degrees of freedom were set randomly at $q_{2\text{init}} = [0.53029 \ -2.10283]^T$.

Fig. 3b shows the residuals of the constraint equations. The position constraints are satisfied within an error of 10^{-6} , while the velocity constraints are within an error of 10^{-5} , which is acceptable.

The trajectories of the position and velocity of point 2 of the five-bar mechanism along the vertical y axis are shown in Fig. 4a and 4b, respectively. These results show that point 2's vertical position bounces at -2.35 m, as expected, and its vertical velocity jumps to its opposite value at each time of event.

The trajectories of the sensitivity of the position and velocity of point 2 of the five-bar mechanism along the vertical y axis are shown in Fig. 5a and 5b, respectively. The jump equations of the sensitivity of the position and velocity at each time of event were determined from (88) and (7), respectively. The results of the analytical sensitivity presented in this paper is benchmarked against the central finite difference, which consists on differentiating the trajectories of the position and velocity of point 2 for a perturbed system on the parameters $\delta\rho$ with the nominal trajectories computed from the initial system. The difference is then divided by $\delta\rho$. We refer the central finite difference methodology as a numerical sensitivity analysis.

The analytical sensitivity is represented by the continuous line, while the central finite difference sensitivity is represented by the dashed line. There is an excellent correlation between the numerical and the analytical sensitivities, with a difference between the two trajectories of less than 0.1%. Note that the numerical sensitivity of the velocity of point 2 along the vertical axis tends to be really large in magnitude, $1/\delta\rho$ at each time of event. This is shown by the vertical dashed lines and it is due to the fact that the difference between the trajectories $v(\rho + \delta\rho)$ and $v(\rho - \delta\rho)$ increases considerably during the Δt period, as shown in Fig. 2. Indeed, the magnitude of the difference of the trajectories $|v(\rho + \delta\rho)|_{t_{\text{eve}}}^+ - v(\rho - \delta\rho)|_{t_{\text{eve}}}^-| = |h(v(\rho + \delta\rho)|_{t_{\text{eve}}}^-) - v(\rho - \delta\rho)|_{t_{\text{eve}}}^-| = |-v(\rho + \delta\rho)|_{t_{\text{eve}}}^- - v(\rho - \delta\rho)|_{t_{\text{eve}}}^-| \approx |-2v(\rho - \delta\rho)|_{t_{\text{eve}}}^-|$. This difference divided by $\delta\rho$ of magnitude 10^{-4} leads the jump in the numerical sensitivity of the velocity of point 2 to 10^4 of magnitude. Therefore,

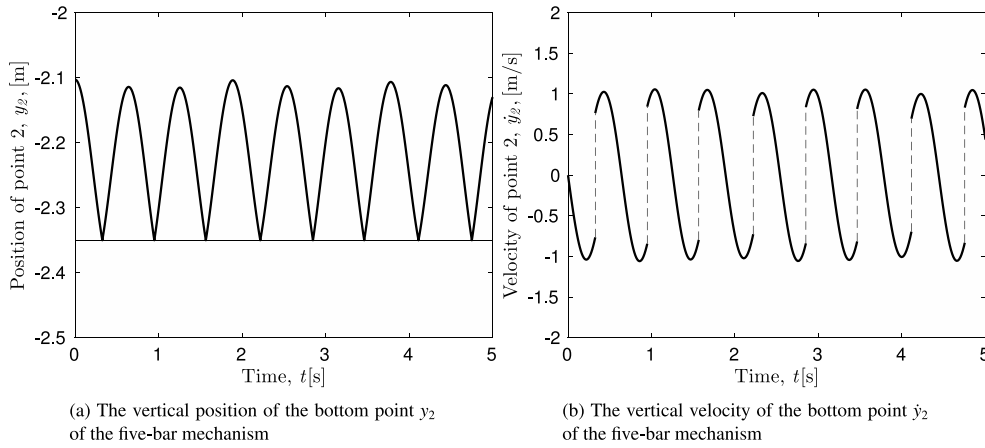


Fig. 4. Trajectories of the position and velocity of the bottom point the five-bar mechanism.

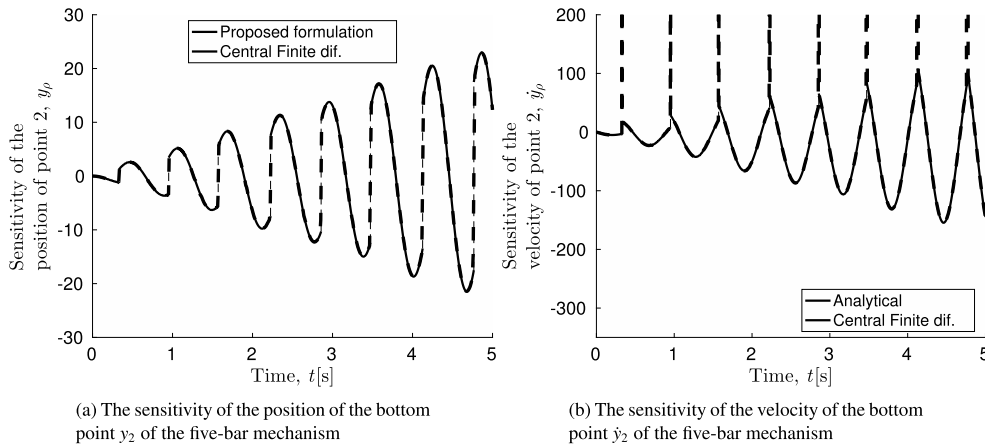


Fig. 5. Sensitivity analysis of the position and velocity of the bottom point of the five-bar mechanism.

this result shows that the novel analytical sensitivity method presented in this paper is considerably more robust than the numerical method, as it correctly calculates the sensitivity jumps and accurately determines the sensitivity trajectories after each event without any delta-like jumps.

The trajectories of the quadrature variables $z = \int_{t_0}^{t_F} \dot{y}_2 dt$ and $z = \int_{t_0}^{t_F} \ddot{y}_2 dt$ are shown in Fig. 6a and 6b, respectively. Note that $z = \int_{t_0}^{t_F} \dot{y}_2 dt$ matches the trajectory of the position of point 2 along the vertical axis in Fig. 4a, while $z = \int_{t_0}^{t_F} \ddot{y}_2 dt$ does not completely match the trajectory of the velocity of point 2 in Fig. 4b. This is due to the fact that the velocity variable is affected by the impulse function at the time of event, while the quadrature variable is not. The trajectories after each event differ by a constant since the quadrature variable evaluates the integral of the acceleration of point 2.

The trajectories of the sensitivities of the quadrature variables $z = \int_{t_0}^{t_F} \dot{y}_2 dt$ and $z = \int_{t_0}^{t_F} \ddot{y}_2 dt$ are presented in Fig. 7a and 7b, respectively. The sensitivities are with respect to the system parameters $\rho = [L_{01} \ L_{02}]$. The results of the analytical and numerical sensitivity analysis of the five-bar mechanism highlight the quasi-perfect correlation between the numerical and analytical sensitivities with a difference between the two trajectories of less than 0.1%. Note that a similar observation with the one previously made is valid here: the sensitivity of $z = \int_{t_0}^{t_F} \dot{y}_2 dt$ shown in Fig. 7a matches the trajectory of the sensitivity of the position illustrated in Fig. 5a, while the sensitivity of $z = \int_{t_0}^{t_F} \ddot{y}_2 dt$ shown in Fig. 7b does not completely match the trajectory of the sensitivity of the velocity from Fig. 5b because of the impulse function.

8. Conclusions

In summary, this study provides a clear methodology for modeling constrained mechanical systems by using the penalty formulation. This formulation has the advantage that it incorporates the kinematic constraints inside the equation of motion. Thus, it avoids any discontinuities in the velocity trajectories due to change in the set of constraints at the time of event. This

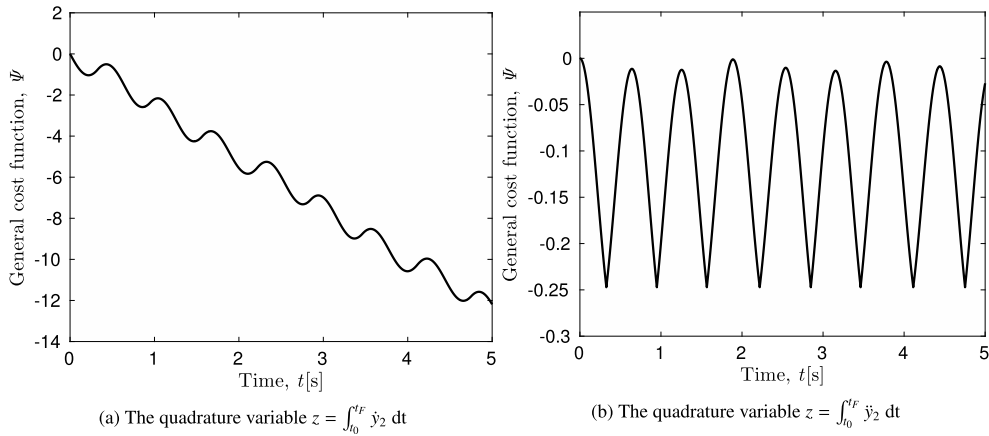


Fig. 6. The quadrature variables of the five-bar mechanism.

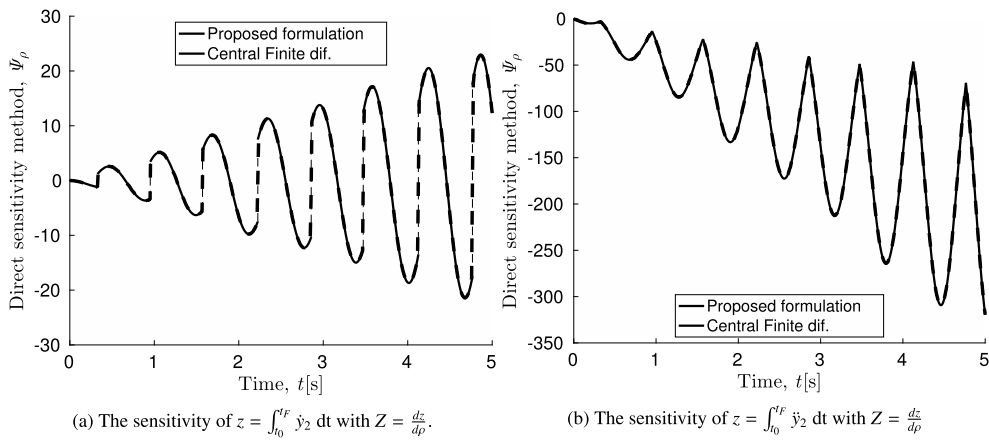


Fig. 7. Sensitivity analysis of the quadrature variables of the five-bar mechanism.

formalism is in the form of an ODE which considerably eases the numerical integration by avoiding algebraic variables. Also, this formalism works well for systems with redundant constraints and for systems that encounter a kinematic bifurcation. Although there exists a multitude of formalisms and methodology approaches for modeling a constrained mechanical system, most of them are ODE or DAE based formalisms. For this reason, this study also provides the DAE-ranked 1 and 3 for modeling constrained mechanical systems.

In addition, this study targets the needs of a large audience as it provides the tools for numerical implementation of direct sensitivity analysis for any kind of ODE and DAE formalisms. Direct and adjoint sensitivity analyses for mechanical systems with smooth trajectories have been studied in the literature [1,20,21,24–26]. The methodology and the tools developed in this study advance the state-of-the-art in the sensitivity analysis field as it provides a complete mathematical framework needed to compute the sensitivities of model states, and of general cost functionals, with respect to model parameters for both unconstrained and constrained hybrid mechanical systems. Jump conditions for sensitivity variables and general cost functionals are established for both the ordinary and the differential algebraic cases. These jump conditions specify the values of the sensitivities right after the event, given their value right before the event, and the characteristics of the impact. The impacts can be characterized as a sudden jump in the velocity state variables caused by impulsive forces, change in the equation of motion, or change in the kinematic constraints.

A five-bar mechanism with non-smooth contacts is used as a case study. The analytical sensitivities obtained by the proposed methodology are validated against numerical sensitivities computed by central finite differences. The results of the case study show that the analytical sensitivity is more robust than the numerical method, as it correctly calculates the sensitivities of piecewise trajectories without any delta-like jumps.

Current efforts focus on applying the hybrid system sensitivity analysis methodology to robotics, where sensitivities of the performance of a robotic system with respect to changes in the system configuration, under non-smooth impact conditions

are a topic of great interest. Ongoing work extends the current framework to perform adjoint sensitivity analysis of hybrid mechanical systems. Future work may explore other conditions, such as grazing and infinite events.

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Appendix. Terminology used in Section 4

In Eqs. (65) the terms $\bar{F}_q$, $\bar{F}_{\dot{q}}$, $\bar{F}_\rho$, $\bar{M}_q\ddot{q}$, and $\bar{M}_\rho\ddot{q}$ are given by the following expressions:

$$\bar{F}_q = F_q - \Phi_{qq}^T \alpha (\dot{\Phi}_q \dot{q} \dot{\Phi}_t + 2\xi \omega \dot{\Phi} + \omega^2 \Phi) - \Phi_q^T \alpha \left((\dot{\Phi}_q \dot{q})_q + (\dot{\Phi}_t)_q + 2\xi \omega (\Phi_{qq} \dot{q} + \Phi_{tq}) + \omega^2 \Phi_q \right), \quad (\text{A.1})$$

$$\bar{F}_{\dot{q}} = F_{\dot{q}} - \Phi_q^T \alpha (\Phi_{qq} \dot{q} + \dot{\Phi}_q + \Phi_{tq} + 2\xi \omega \Phi_q), \quad (\text{A.2})$$

$$\bar{F}_\rho = F_\rho - \Phi_{q\rho}^T \alpha (\dot{\Phi}_q \dot{q} + \dot{\Phi}_t + 2\xi \omega \dot{\Phi} + \omega^2 \Phi) - \Phi_q^T \alpha \left((\dot{\Phi}_q \dot{q})_\rho + \dot{\Phi}_{t\rho} + 2\xi \omega \dot{\Phi}_\rho + \omega^2 \Phi_\rho \right), \quad (\text{A.3})$$

$$\bar{M}_q\ddot{q} = M_q\ddot{q} + \Phi_{qq}^T (\alpha \Phi_q \ddot{q}) + \Phi_q^T \alpha (\Phi_{qq} \ddot{q}), \quad (\text{A.4})$$

$$\bar{M}_\rho\ddot{q} = M_\rho\ddot{q} + \Phi_{q\rho}^T (\alpha \Phi_q \ddot{q}) + \Phi_q^T \alpha (\Phi_{q\rho} \ddot{q}). \quad (\text{A.5})$$

In (66) the terms λ_y^* , $\lambda_{\dot{y}}^*$, λ_v^* , λ_q^* , and λ_ρ^* are given by the following expressions:

$$\lambda_y^* = [\lambda_q^* \quad \lambda_v^*] \quad (\text{A.6})$$

$$\lambda_{\dot{y}}^* = [0 \quad \lambda_v^*] \quad (\text{A.7})$$

$$\lambda_v^* = \alpha \Phi_q \quad (\text{A.8})$$

$$\lambda_v^* = \alpha [\Phi_{q,q} v + \dot{\Phi}_q + \Phi_{tq} + 2\xi \omega \Phi_q] \quad (\text{A.9})$$

$$\lambda_q^* = \alpha \left[\Phi_{q,q} \dot{v} + (\dot{\Phi}_q)_q v + (\Phi_t)_q + 2\xi \omega (\Phi_{q,q} v + \Phi_{tq}) + \omega^2 \Phi_q \right] \quad (\text{A.10})$$

$$\lambda_\rho^* = \alpha \left[\Phi_{q\rho} \dot{v} + (\dot{\Phi}_q)_\rho v + (\dot{\Phi}_t)_\rho + 2\xi \omega (\Phi_{q\rho} v + \Phi_{t\rho}) + \omega^2 \Phi_\rho \right] \quad (\text{A.11})$$

References

- [1] Y. Zhu, D. Dopico, C. Sandu, A. Sandu, Dynamic response optimization of complex multibody systems in a penalty formulation using adjoint sensitivity, *J. Comput. Nonlinear Dynam.* 10 (3) (2015) 031009. <http://dx.doi.org/10.1115/1.4029601>, paper CND-14-1268.
- [2] A. Sandu, D.N. Daescu, G.R. Carmichael, Direct and adjoint sensitivity analysis of chemical kinetic systems with KPP: Part I—theory and software tools, *Atmos. Environ.* 37 (36) (2003) 5083–5096. <http://dx.doi.org/10.1016/j.atmosenv.2003.08.019>.
- [3] K.-H. Chang, Chapter 18 - Structural Design Sensitivity Analysis, Academic Press, Boston, 2015, pp. 1001–1103. <http://dx.doi.org/10.1016/B978-0-12-382038-9.00018-1>.
- [4] M. Bernardo, C. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, vol. 163, Springer-Verlag, London, 2008.
- [5] P. Ballard, The dynamics of discrete mechanical systems with perfect unilateral constraints, *Arch. Ration. Mech. Anal.* 154 (3) (2000) 199–274. <http://dx.doi.org/10.1007/s002050000105>.
- [6] A.M. Pace, S.A. Burden, Piecewise - differentiable trajectory outcomes in mechanical systems subject to unilateral constraints, in: *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control, HSCC '17*, ACM, New York, NY, USA, 2017, pp. 243–252. <http://dx.doi.org/10.1145/3049797.3049807>.
- [7] K. Park, J. Chiou, Stabilization of computational procedures for constrained dynamical systems, *J. Guid. Control Dyn.* 11 (4) (1988) 365–370.
- [8] E. Bayo, J.G. De Jalon, M.A. Serna, A modified lagrangian formulation for the dynamic analysis of constrained mechanical systems, *Comput. Methods Appl. Mech. Engrg.* 71 (2) (1988) 183–195.
- [9] A.J. Kurdila, F.J. Narcowich, Sufficient conditions for penalty formulation methods in analytical dynamics, *Comput. Mech.* 12 (1) (1993) 81–96. <http://dx.doi.org/10.1007/BF00370488>.
- [10] J.G.d. Jalon, E. Bayo, *Kinematic and Dynamic Simulation of Multibody Systems: The Real Time Challenge*, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1994.
- [11] P.I. Barton, R.J. Allgor, W.F. Feehery, S. Galán, Dynamic optimization in a discontinuous world, *Ind. Eng. Chem. Res.* 37 (3) (1998) 966–981. <http://dx.doi.org/10.1021/ie970738y>.
- [12] S. Galán, W.F. Feehery, P.I. Barton, Parametric sensitivity functions for hybrid discrete/continuous systems, *Appl. Numer. Math.* 31 (1) (1999) 17–47. [http://dx.doi.org/10.1016/S0168-9274\(98\)00125-1](http://dx.doi.org/10.1016/S0168-9274(98)00125-1).

- [13] P.I. Barton, C.K. Lee, Modeling, simulation, sensitivity analysis, and optimization of hybrid systems, *ACM Trans. Model. Comput. Simul.* 12 (4) (2002) 256–289. <http://dx.doi.org/10.1145/643120.643122>.
- [14] J.E. Tolsma, P.I. Barton, Hidden discontinuities and parametric sensitivity calculations, *SIAM J. Sci. Comput.* 23 (6) (2002) 1861–1874. <http://dx.doi.org/10.1137/S106482750037281X>.
- [15] E. Rozenvasser, General sensitivity equations of discontinuous systems, *Autom. i. Telemekh.* (3) (1967) 52–56.
- [16] A. Saccon, N. van de Wouw, H. Nijmeijer, Sensitivity analysis of hybrid systems with state jumps with application to trajectory tracking, in: 53rd IEEE Conference on Decision and Control, 2014, pp. 3065–3070. <http://dx.doi.org/10.1109/CDC.2014.7039861>.
- [17] I.A. Hiskens, J. Alseddiqui, Sensitivity, approximation, and uncertainty in power system dynamic simulation, *IEEE Trans. Power Syst.* 21 (4) (2006) 1808–1820. <http://dx.doi.org/10.1109/TPWRS.2006.882460>.
- [18] I.A. Hiskens, M.A. Pai, Trajectory sensitivity analysis of hybrid systems, *IEEE Trans. Circuits Syst. I* 47 (2) (2000) 204–220. <http://dx.doi.org/10.1109/81.828574>.
- [19] F. Taringoo, P. Caines, On the geometry of switching manifolds for autonomous hybrid systems, *IFAC Proc. Volumes* 43 (12) (2010) 35–40. <http://dx.doi.org/10.3182/20100830-3-DE-4013.00008>.
- [20] D. Dopico, A. Sandu, Y. Zhu, C. Sandu, Direct and adjoint sensitivity analysis of ODE multibody formulations, *J. Comput. Nonlinear Dyn.* 10 (1) (2014) 011012. <http://dx.doi.org/10.1115/1.4026492>, paper CND-13-1184..
- [21] Y. Zhu, D. Dopico, C. Sandu, A. Sandu, Benchmarking of adjoint sensitivity-based optimization techniques using a vehicle ride case study, *Mech. Based Design Structures Mach.* (2017) <http://dx.doi.org/10.1080/15397734.2017.1338576>, in print, paper LMBD-2016-0203.
- [22] A. Donzé, O. Maler, Systematic Simulation Using Sensitivity Analysis, 2007, pp. 174–189.
- [23] W. Backer, Jump conditions for sensitivity coefficients, in: L. Radanović (Ed.), *Sensitivity Methods in Control Theory (Symp. Dubrovnik 1964)*, Pergamon Press, Oxford, 1964, pp. 168–175.
- [24] Y. Zhu, D. Dopico, A. Sandu, C. Sandu, Computing sensitivity analysis of vehicle dynamics based on multibody models, DETC2013/AVT-13212, in: *Proceedings of the ASME 2013 International Design Engineering Technical Conferences, Computers and Information in Engineering Conference*, ASME, Portland, USA, 2013.
- [25] Y. Zhu, D. Dopico, A. Sandu, C. Sandu, MBSVT. A library for the simulation and optimization of multibody systems, 2014, [online cited January 2015].
- [26] Y. Zhu, *Sensitivity Analysis and Optimization of Multibody Systems (Ph.D. thesis)*, Virginia Tech, 2014.
- [27] S. Kolathaya, A.D. Ames, Parameter to state stability of control Lyapunov functions for hybrid system models of robots, *Nonlinear Anal. Hybrid Syst.* (2016) <http://dx.doi.org/10.1016/j.nahs.2016.09.003>.