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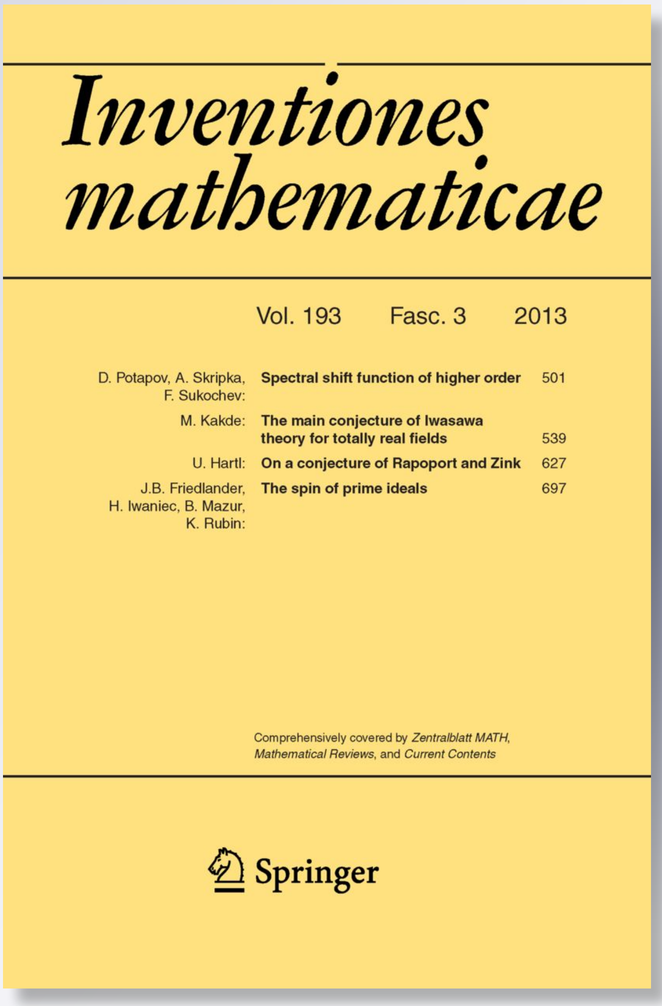

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The sphere covering inequality and its applications

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Abstract In this paper, we show that the total area of two *distinct* surfaces with Gaussian curvature equal to 1, which are also conformal to the Euclidean unit disk with the same conformal factor on the boundary, must be at least 4π. In other words, the areas of these surfaces must cover the whole unit sphere after a proper rearrangement. We refer to this lower bound of total area as the Sphere Covering Inequality. The inequality and its generalizations are applied to a number of open problems related to Moser–Trudinger type inequalities, mean field equations and Onsager vortices, etc, and yield optimal results. In particular, we prove a conjecture proposed by Chang and Yang (Acta Math 159(3–4):215–259, 1987) in the study of Nirenberg problem in conformal geometry.

1 Introduction

A large number of important second order nonlinear elliptic equations involve exponential nonlinearities. These equations arise, for example, in the study of Gaussian curvature of surfaces with metrics conformal to Euclidean metric

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([13,15,16,19,20], etc.), Moser–Trudinger type inequalities [1,2,7,23,24,26, 30,38,40,41,45], the mean field theory of statistical mechanics of classical vortices and thermodynamics [4,6,11,12,14,17,33,36], and self gravitating cosmic string configurations in the framework of Einstein's general relativity [18,42,46]. In this article, we shall prove a basic and important inequality which becomes a crucial tool for tackling several open problems in the above mentioned areas.

Let us consider the equation

$$\Delta v + e^{2v} = 0, \quad y \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a C^2 simply-connected region. It is well-known that for a solution $v \in C^2(\bar{\Omega})$ of (1.1), the two dimensional Riemannian manifold with boundary (Ω, g) with a conformal Euclidean metric $dg = e^{2v}dy$ has Gaussian curvature equal to 1 everywhere. The total area as well as the total curvature of such manifold is equal to $A = \int_\Omega e^{2v}dy$. The well-known Gauss–Bonnet Theorem states that

$$A = \int_\Omega e^{2v}dy = \int_\Omega dg = 2\pi - \int_{\partial\Omega} \kappa_g dl_g$$

where κ_g is the geodesic curvature and dl_g is the length parameter of $\partial\Omega$. From the equation, it is also easy to see that

$$A = -\int_{\partial\Omega} \frac{\partial v}{\partial r} dl_g.$$

These formulas, though very useful in general, do not impose any restriction on the area of the surface, as the uniformization theorem says that every simply-connected Riemann surface is conformally equivalent to one of the three domains: the open unit disk, the complex plane, or the Riemann sphere. However, if there is another surface $(\Omega, \tilde{g})$ with a distinct conformal metric $d\tilde{g} = e^{2\tilde{v}}dy$ in Ω, where $\tilde{v} \in C^2(\bar{\Omega})$ is a solution of (1.1) and $\tilde{g} = g$ on $\partial\Omega$, we shall show

$$\tilde{A} + A = \int_\Omega (e^{2\tilde{v}} + e^{2v})dy \geq 4\pi. \tag{1.2}$$

Since the standard sphere has Gaussian curvature 1 and area 4π, and these two surfaces have total area bigger than or equal to that of the standard sphere, one may think that these two surfaces could cover the standard sphere if they are properly arranged (this will be made more rigorous later in Sect. 2.1). The equality obviously hold when the two surfaces are isometric to two complementing spherical caps on the standard sphere. We therefore refer to (1.2) as the Sphere Covering Inequality.

We will prove inequality (1.2) in a more general setting as follows.

Theorem 1.1 (The sphere covering inequality) *Let* Ω *be a simply-connected subset of* $\mathbb{R}^2$ *and assume* $v_i \in C^2(\overline{\Omega})$, $i = 1, 2$ *satisfy*

$$\Delta v_i + e^{2v_i} = f_i(y), \quad \int_\Omega e^{2v_i} \le 4\pi \tag{1.3}$$

where $f_2 \ge f_1 \ge 0$ *in* Ω*. If* $v_2 \ge v_1$, $v_2 \not\equiv v_1$ *in* ω *and* $v_2 = v_1$ *on* $\partial\omega$ *for some piecewise Liptschitz subdomain* $\omega \subset \Omega$*, then*

$$\int_\omega (e^{2v_1} + e^{2v_2}) dy \ge 4\pi. \tag{1.4}$$

Moreover, the equality only holds when $f_2 \equiv f_1 \equiv 0$ *in* ω*, and* $(\omega, e^{2v_i} dy)$*,* $i = 1, 2$ *are isometric to two complementary spherical caps on the standard unit sphere.*

For the simplicity of the equation, we may replace $2v$ by $u - \ln 2$ and consider

$$\Delta u + e^u = 0, \quad y \in \Omega.$$

Geometrically, this is equivalent to multiplying the conformal factor by $\sqrt{2}$ so the sphere in comparison has radius $\sqrt{2}$ and total area 8π. Indeed Theorem 1.1 is equivalent to Theorem 3.1 in Sect. 3.

The Sphere Covering Inequality is closely related to the symmetry of solutions of elliptic equations with exponential nonlinearity in $\mathbb{R}^2$. To see the connection, consider the equation

$$\Delta w + e^w = f \ge 0 \quad \text{in } \mathbb{R}^2, \tag{1.5}$$

and let w be a classical solution with a critical point located at $P \in \mathbb{R}^2$. Assume that f is smooth and evenly symmetric about a line passing through P. It follows from the Sphere Covering Inequality that if $\int_{\mathbb{R}^2} e^w dy < 16\pi$, then w must be symmetric about the line. More precisely, suppose $P = (p, 0)$ and $f(y_1, y_2) = f(y_1, -y_2)$ in $\mathbb{R}^2$. Define $\bar{w}(y_1, y_2) = w(y_1, -y_2)$, and set

$$\tilde{v} := w - \bar{w}.$$

Then $\tilde{v}$ satisfies

$$\Delta \tilde{v} + c\tilde{v} = 0,$$

where

$$c = \frac{e^w - e^{\bar{w}}}{w - \bar{w}} \geq 0$$

is a smooth function. Suppose $\tilde{v} \not\equiv 0$. Classic results imply that the nodal set of $\tilde{v}$ consists of finitely many immersed smooth curves which have only finitely many self-intersection points in any compact domain (see, e.g. Theorem 2.5 of [22]). These intersecting points are sometime called nodal points, they are also the critical points of $\tilde{v}$. Note that the nodal set of $\tilde{v}$ contains y_1-axis and it follows from the classic results that the nodal curve of $\tilde{v}$ divides a neighborhood of P into at least four regions (see, e.g. Theorem 2.2 of [22], or [8,31]). It also follows easily from Hopf's lemma that $\tilde{v}$ changes sign in $B_\epsilon(P) \cap \{y_2 > 0\}$ for any $\epsilon > 0$ since it vanishes on y_1-axis. Consequently there exist at least two simply-connected regions $\Omega_1, \Omega_2 \subset \mathbb{R}^2_+$ such that $\tilde{v} > 0$ in Ω_1, $\tilde{v} < 0$ in Ω_2, and $\tilde{v} = 0$ on $\partial\Omega_1 \cup \partial\Omega_2$. Therefore on each Ω_i, $i = 1, 2$, the Eq. (1.5) has two distinct solutions, w and $\bar{w}$, satisfying the assumptions of Theorem 3.1, which is an equivalent form of Theorem 1.1 for (1.5). Thus

$$\int_{\mathbb{R}^2} e^w dy \geq \int_{\Omega_1} (e^w + e^{\bar{w}}) dy + \int_{\Omega_2} (e^w + e^{\bar{w}}) dy \geq 16\pi,$$

which is a contradiction and leads to the symmetry of w.

The above argument is at the core of the proof of the symmetry results in this paper, and consists of two main ingredients: the Sphere Covering Inequality and the nodal set analysis. The idea of using nodal sets to prove symmetry results for elliptic equations with exponential nonlinearity was used by Lin and others to obtain symmetry results for mean field equations in $\mathbb{R}^2$ and on S^2 and flat tori (see, e.g., [6,26,35,37], etc). The key in their arguments is Proposition 3.2 in Sect. 3, which has a geometrical interpretation in terms of the extremal first eigenvalue (see Remark 3.4). Note that Proposition 3.2 may be regarded as a special limiting case of the Sphere Covering Inequality, although the geometric meaning of the Sphere Covering Inequality itself still remains unclear to us and worth further exploring. In this paper, the Sphere Covering Inequality will be used to solve several important open problems. Below we introduce some of the problems.

1.1 Best constant in a Moser–Trudinger type inequality

Let S^2 be the unit sphere and for $u \in H^1(S^2)$ define

$$J_\alpha(u) = \frac{\alpha}{4}\int_{S^2} |\nabla u|^2 d\omega_1 + \int_{S^2} u d\omega_1 - \log \int_{S^2} e^u d\omega_1, \tag{1.6}$$

where the volume form $d\omega_1$ is normalized so that $\int_{S^2} d\omega_1 = 1$. The well-known Moser–Trudinger inequality [38] says that J_α is bounded below if and only if $\alpha \geq 1$. Onofri [40] showed that for $\alpha \geq 1$ the best lower bound is equal to zero. On the other hand, Aubin [2] proved that if J_α is restricted to

$$\mathcal{M} := \left\{ u \in H^1(S^2) : \int_{S^2} e^u x_i = 0, \quad i = 1, 2, 3 \right\},$$

then for $\alpha \geq \frac{1}{2}$, J_α is bounded below and the infimum is attained in $\mathcal{M}$. In 1987 Chang and Yang [15], in their work on prescribing Gaussian curvature on S^2 (see also [16]), showed that for α close to 1 the best constant again is equal to zero, and this led to the following conjecture.

Conjecture A For $\alpha \geq \frac{1}{2}$

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$

In 1998, Feldman et al. [25] proved that this conjecture is true for axially symmetric functions when $\alpha > \frac{16}{25} - \epsilon$. Later the first author and Wei [30], and independently Lin [36] proved Conjecture A for radially symmetric function, but the problem remained open for non-axially symmetric functions.

In [26] Ghoussoub and Lin, showed that Conjecture A holds true for $\alpha \geq \frac{2}{3} - \epsilon$, for some $\epsilon > 0$. See Chapter 19 in [27] for a complete history of the problem. In this paper, among other results, we will prove that Conjecture A is true.

Theorem 1.2 *For* $\alpha \geq \frac{1}{2}$

$$\inf_{u \in \mathcal{M}} J_\alpha(u) = 0.$$

Indeed we shall apply Theorem 3.1 to show that the corresponding Euler-Lagrange equation

$$\frac{\alpha}{2}\Delta u + \frac{e^u}{\int_{S^2} e^u d\omega_1} - 1 = 0 \text{ on } S^2 \tag{1.7}$$

has only constant solutions for $\alpha \geq 1/2$ except $\alpha = 1$. In the latter case $\alpha = 1$, all solutions of (1.7) are classified and has a family of axially symmetric solutions with a scaling parameter which blows up when the parameter goes to infinity and is usually called a standard bubble. See (2.3) for the explicit formula of this family of solutions after stereographic projection.

1.2 A mean field equation with singularity on S^2

Consider the mean field equation

$$\Delta_g u + \lambda \left(\frac{e^u}{\int_{S^2} e^u d\omega_2} - \frac{1}{4\pi} \right) = 4\pi\bar{\alpha}\big(\delta(P) - \frac{1}{4\pi}\big) \text{ on } S^2, \tag{1.8}$$

where g is the standard metric on S^2 with the corresponding volume form $d\omega_2$ such that $\int_{S^2} d\omega_2 = 4\pi$, $\bar{\alpha} > -1$, $\lambda > 0$, and $P \in S^2$. The problems is motivated by the study of vortices in self-dual gauge field theory. It is known that when $\lambda \in (0, 8\pi(1 + \alpha_-))$, where $\alpha_- = \min\{\bar{\alpha}, 0\}$, there exists a unique solution to (1.8) and the solution is axially symmetric; while when $\lambda \in [8\pi(1+\alpha_-), 8\pi(1+\alpha_+)]$ and $\bar{\alpha} \neq 0$, where $\alpha_+ = \max\{\bar{\alpha}, 0\}$, there exists no solution to (1.8). It is also shown that when $\lambda > 8\pi(2 + \bar{\alpha})$, there exist multiple solutions (see [44] or [6]).

On the other hand, it is shown in [36] that when $\lambda \in (8\pi(1+\alpha_+), 8\pi(2+\bar{\alpha}))$, there exists a unique solution in the class of axially symmetric functions. In particular, in [6] Bartolucci et al. studied symmetry of solutions of (1.8) under the assumption

$$\lambda = 4\pi(3 + \bar{\alpha}) \tag{1.9}$$

and showed that (1.8) admits a solution if and only if $\bar{\alpha} \in (-1, 1)$. Then, via a new bubbling phenomenon, they proved that there exists $\delta > 0$ such that for $\bar{\alpha} \in (1 - \delta, 1)$ the equation (1.8) admits a unique solution that in addition is axially symmetric about the direction $\overrightarrow{OP}$ where O is the origin in $\mathbb{R}^3$, and proposed the following

Conjecture B All solutions of (1.8)–(1.9) are axially symmetric about $\overrightarrow{OP}$ for every $\bar{\alpha} \in (-1, 1)$.

In Sect. 5, we shall use the Sphere Covering Inequality to provide an affirmative answer to the above question. Indeed we will prove the following result for general λ.

Theorem 1.3 *For every $\lambda \in (0, 16\pi]$ and $\bar{\alpha} > -1$, any solution of* (1.8) *must be axially symmetric.*

We note that if $\bar{\alpha} = 0$, (1.8) is equivalent to (1.7) with $\lambda = 8\pi/\alpha$, and therefore the above comments on (1.7) apply here. If $\bar{\alpha} \neq 0$, the axis of symmetry must be $\overrightarrow{OP}$.

As a consequence, we have

Theorem 1.4 *For every* $\bar{\alpha} \in (-1, 1)$*, there exists a unique solution (upto a constant) to* (1.8)–(1.9)*. Moreover, the solution must be axially symmetric about* $\overrightarrow{OP}$*.*

1.3 A mean field equation for the spherical Onsager vortex

Consider the following equation

$$\Delta_g u(x) + \frac{\exp(\tilde{\alpha} u(x) - \gamma\langle n, x\rangle)}{\int_{S^2} \exp(\tilde{\alpha} u(x) - \gamma\langle n, x\rangle) d\omega_2} - \frac{1}{4\pi} = 0 \text{ on } S^2, \quad (1.10)$$

where g is the standard metric on S^2 with corresponding volume form $d\omega_2$, $\mathbf{n}$ is a unit vector in R^3, $\tilde{\alpha} \geq 0$, and $\gamma \in R$. Since $\gamma < 0$ can be changed to $-\gamma$ by replacing the north pole with the south pole, we only need to consider the case $\gamma \geq 0$. This equation is invariant up to adding a constant and we seek a normalized solution with

$$\int_{S^2} u d\omega_2 = 0. \quad (1.11)$$

In [36], Lin showed that if $\tilde{\alpha} < 8\pi$, then for $\gamma \geq 0$ the equation (1.10) has a unique solution that in addition is axially symmetric with respect to $\mathbf{n}$. In this case the coefficient in the equation is radially decreasing and therefore the moving plane method applies [see (5.19)]. He also conjectured the following

Conjecture C Let $\gamma > 0$ and $\tilde{\alpha} \leq 16\pi$. Then every solution u of (1.10) is axially symmetric with respect to $\mathbf{n}$.

In an attempt to prove this conjecture, in [35], C.S. Lin proved the following theorems for $\tilde{\alpha} > 8\pi$.

Theorem A ([35]) *For every* $\gamma > 0$*, there exists* $\alpha_0 = \alpha_0(\gamma) > 8\pi$ *such that, for* $8\pi < \tilde{\alpha} \leq \alpha_0$*, any solution* u *of* (1.10) *is axially symmetric.*

Theorem B ([35]) *Let* u_i *be a solution of* (1.10) *with* $\gamma = 0$ *and* $\tilde{\alpha}_i \to 16\pi$*. Suppose* $\lim_{i\to\infty} \sup u_i(x) = +\infty$*. Then* u_i *is axially symmetric with respect to some direction* $\mathbf{n}_i$ *in* R^3 *for* i *large enough.*

In Sect. 5, we shall apply the Sphere Covering Theorem to prove the following result.

Theorem 1.5 *Suppose* $8\pi < \tilde{\alpha} \leq 16\pi$ *and*

$$0 \leq \gamma \leq \frac{\tilde{\alpha}}{8\pi} - 1. \tag{1.12}$$

Then every solution of (1.10) *is axially symmetric with respect to* **n**. *In particular if* $\gamma = 0$ *and* $8\pi < \tilde{\alpha} \leq 16\pi$, *then the trivial solution* $u \equiv 0$ *is the only solution of* (1.10) *and* (1.11).

We note that (1.10) with $\tilde{\alpha} = 8\pi, \gamma = 0$ is equivalent to (1.8) with $\lambda = 8\pi, \bar{\alpha} = 0$ and (1.7) with $\alpha = 1$.

In all problems above and many others, there exists a critical number 8π for a quantity which may be interpreted as total area. In (1.6), the quantity is $4\pi/\alpha$, with α being a parameter; In (1.8), the quantity is λ; while in (1.10), the quantity is $\tilde{\alpha}$. The work of Brezis and Merle [10] and Li [34] as well as others showed that $8\pi m, m \in N$ are values where solutions of these type of equations may lose compactness and blow-up phenomena may happen. The critical level 8π also separates two significantly different cases in terms of the coerciveness of the associated functionals and the positivity of the linearized operators. A crucial tool is required to deal with the supercritical cases of many important problems in related research. The Sphere Covering Inequality provides exactly such a much needed tool. Besides the applications in this paper, other applications of the Sphere Covering Inequality will also be discussed in forthcoming papers [28,29].

The paper is organized as follows. In Sect. 2, we shall discuss some preliminary results about the classical Bol's inequality and prove a counterpart of Bol's inequality for radially symmetric functions which is needed for the proof of the Sphere Covering Inequality. In Sect. 3, we will prove the Sphere Covering Inequality. In Sect. 4, the Sphere Covering Inequality shall be applied to (1.6) to show the best constant. Finally, we will present a general symmetry result regarding Gaussian curvature equations on $\mathbb{R}^2$ which leads to optimal results for (1.8), (1.10) and others.

2 Bol's isoperimetric inequality

Bol's isoperimetric inequality plays a crucial role in the proof of our main results. In this section we present some preliminary results on Bol's inequality that will be used in subsequent sections. First we recall the classical Bol's isoperimetric inequality [3,9,43]:

Proposition 2.1 *Let* $\Omega \subset \mathbb{R}^2$ *be a simply-connected domain and assume* $u \in C^2(\Omega)$ *satisfies*

$$\Delta u + e^u \geq 0 \text{ and } \int_\Omega e^u \leq 8\pi. \tag{2.1}$$

Then for every $\omega \Subset \Omega$ *with a piecewise* C^1 *boundary (which may not be necessariely bounded), the following inequality holds*

$$\left(\int_{\partial\omega} e^{\frac{u}{2}}\right)^2 \geq \frac{1}{2}\left(\int_\omega e^u\right)\left(8\pi - \int_\omega e^u\right). \tag{2.2}$$

We first show an example for which the equality in Bol's inequality holds. For $\lambda > 0$, let us define U_λ by

$$U_\lambda := -2\ln\left(1 + \frac{\lambda^2|y|^2}{8}\right) + 2\ln(\lambda). \tag{2.3}$$

Then

$$\Delta U_\lambda + e^{U_\lambda} = 0,$$

and

$$\int_{B_r} e^{U_\lambda} dy = \frac{8\pi\lambda^2 r^2}{8 + \lambda^2 r^2},$$

for all $r > 0$, where B_r denotes the ball of radius r centered at the origin in $\mathbb{R}^2$. One can check that

$$\left(\int_{\partial B_r} e^{\frac{U_\lambda}{2}}\right)^2 = \frac{1}{2}\left(\int_{B_r} e^{U_\lambda}\right)\left(8\pi - \int_{B_r} e^{U_\lambda}\right),$$

for all $r > 0$ and $\lambda > 0$. Indeed, $e^{U_\lambda(y)}dy$ corresponds to the metric in a standard sphere with radius $\sqrt{2}$.

By examining the proof of Bol's inequality (see, e.g., [43]), it can be seen that if the equality holds for some ω in (2.2), then $\Delta u + e^u = 0$ in ω, and $e^{u(z)}dz = e^{U_\lambda(\xi)}d\xi$, where $z = g^{-1}(\xi)$ for some analytic function $g : \omega \to B_R$, and $\lambda > 0$. More precisely, let us consider the case when $\omega = \Omega$ is simply-connected, and follow the arguments in [43] by considering the harmonic lifting h of boundary value of u in Ω, i.e.,

$$\Delta h = 0 \text{ in } \Omega; \quad h = u \text{ on } \partial\Omega.$$

It is known that (see [39]) there is an analytic function $\xi = g(z)$ such that $e^h = |g'(z)|^2$. The equality in (2.2) implies that $g(\Omega) = B_R$ and $g(\partial\Omega) = g(\partial B_R)$ for some $R > 0$. Furthermore, letting

$$q(\xi) := e^{u\left(g^{-1}(\xi)\right)}|g'\left(g^{-1}(\xi)\right)|^{-2},$$

we have $e^u(z)dz = q(\xi)d\xi$ and $q(\xi)$ is radially symmetric. Therefore $q(\xi) = e^{U_\lambda(\xi)}$ for some $\lambda > 0$. In general, if ω is not simply-connected , Bartolucci and Lin showed in [5] that strict inequality holds in (2.2). So the equality may hold only for simply-connected ω. Note that if Ω is not simply-connected, Proposition 2.1 is not valid in general. Indeed, (2.2) does not hold for certain annular regions as shown in [5].

For the proof of our main results, we shall need the following counterpart of the Bol's inequality for radial functions. The proof is a modification of an argument by Suzuki [43] and we present it here for the sake of completeness.

Proposition 2.2 *Let B_R be the ball of radius R in $\mathbb{R}^2$ $\psi \in C^{0,1}(\overline{B_R})$ be a strictly decreasing, radial, Lipschitz function satisfying*

$$\int_{\partial B_r} |\nabla\psi| ds \le \int_{B_r} e^{\psi} dy \ \ a.e.\ r \in (0, R), \ \ and \ \ \int_{B_R} e^{\psi} \le 8\pi. \quad (2.4)$$

Then the following inequality holds

$$\left(\int_{\partial B_R} e^{\frac{\psi}{2}}\right)^2 \ge \frac{1}{2}\left(\int_{B_R} e^{\psi}\right)\left(8\pi - \int_{B_R} e^{\psi}\right). \quad (2.5)$$

Moreover if $\int_{\partial B_r} |\nabla\psi| ds \not\equiv \int_{B_r} e^{\psi} dy$ in $(0, R)$, then the inequality in (2.5) is strict.

Proof Let $\beta{:=}\psi(R)$ and define

$$k(t) = \int_{\{\psi>t\}} e^{\psi} dy, \ \text{ and } \ \mu(t) = \int_{\{\psi>t\}} dy,$$

for $t > \beta$. Then

$$-k'(t) = \int_{\{\psi=t\}} \frac{e^{\psi}}{|\nabla\psi|} = -e^t\mu'(t), \ \text{ for } \ a.e.\ t > \beta.$$

Hence

$$-k(t)k'(t) \geq \int_{\{\psi=t\}} |\nabla\psi| \cdot \int_{\{\psi=t\}} \frac{e^{\psi}}{|\nabla\psi|} \tag{2.6}$$
$$= \left(\int_{\{\psi=t\}} e^{\psi/2}\right)^2 = e^t \left(\int_{\{\psi=t\}} ds\right)^2$$
$$= 4\pi e^t \int_{\{\psi>t\}} dy = 4\pi e^t \mu(t), \quad \text{for } a.e. \ t > \beta.$$

Therefore

$$\frac{d}{dt}\left[e^t\mu(t) - k(t) + \frac{1}{8\pi}k^2(t)\right] = e^t\mu(t) + \frac{1}{4\pi}k'(t)k(t) \leq 0, \quad \text{for} a.e. \ t > \beta.$$

Integrating on (β, ∞) we get

$$\left[e^t\mu(t) - k(t) + \frac{1}{8\pi}k^2(t)\right]_{\beta}^{\infty} = -\left(e^{\beta}\mu(\beta) - k(\beta) + \frac{1}{8\pi}k^2(\beta)\right) \leq 0. \tag{2.7}$$

Now notice that

$$k(\beta) = \int_{B_R} e^{\psi} dy$$

and

$$e^{\beta}\mu(\beta) = e^{\beta}\int_{B_R} dy = \frac{1}{4\pi}e^{\beta}\left(\int_{\partial B_R} ds\right)^2 = \frac{1}{4\pi}\left(\int_{\partial B_R} e^{\frac{\psi}{2}} ds\right)^2.$$

Thus (2.5) follows from the inequality (2.7). Finally if $\int_{\partial B_r} |\nabla\psi| ds \not\equiv \int_{B_r} e^{\psi} \in (0, R)$, then the inequality (2.6) will be strict in a set with a positive measure in $\{t > \beta\}$, and consequently (2.7) and (2.5) will also be strict. □

2.1 Rearrangement with respect to two measures

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and $\lambda > 0$, and suppose that $u \in C^2(\overline{\Omega})$ satisfies

$$\Delta u + e^u \geq 0, \quad \int_{\Omega} e^u dy < 8\pi. \tag{2.8}$$

Let $\phi \in C^2(\overline{\Omega})$ be constant on $\partial\Omega$. Then ϕ can be equimeasurably rearranged with respect to the measures $e^u dy$ and $e^{U_\lambda} dy$ (see [3,37,43]), where U_λ is defined in (2.3). More precisely, for $t > \min_{y\in\overline{\Omega}} \phi$ define

$$\Omega_t := \{\phi > t\} \subset \Omega,$$

and define Ω_t^* to be the ball centered at the origin in $\mathbb{R}^2$ such that

$$\int_{\Omega_t^*} e^{U_\lambda} dy = \int_{\Omega_t} e^u dy := a(t).$$

Then $a(t)$ is a right-continuous function, and $\phi^* : \Omega^* \to \mathbb{R}$ defined by $\phi^*(y) := \sup\{t \in \mathbb{R} : y \in \Omega_t^*\}$ provides an equimeasurable rearrangement of ϕ with respect to the measure $e^u dy$ and $e^{U_\lambda} dy$, i.e.

$$\int_{\{\phi^* > t\}} e^{U_\lambda} dy = \int_{\{\phi > t\}} e^u dy, \quad \forall t > \min_{y\in\overline{\Omega}} \phi, \tag{2.9}$$

where Ω^* a ball of radius $R < \infty$ and centered at the origin with $\int_{\Omega^*} e^{U_\lambda} dy = \int_\Omega e^u dy$. Note that $\phi^*(y)$ is a radial function and we will identify $\phi^*(y)$ with $\phi^*(|y|)$. We shall need the following lemma.

Proposition 2.3 *Assume that $u \in C^2(\overline{\Omega})$ satisfies (2.8) and $\phi \in C^1(\bar{\Omega})$ satisfies $\phi = c$ for some constant c on $\partial\Omega$. If Ω is unbounded, we further assume that $\lim_{y\to\infty} \phi(y) = c$ and $\phi \geq c$ in Ω. Let $\phi^*(r)$ be the equimeasurable rearrangement of ϕ with respect to the measure $e^u dy$ and $e^{U_\lambda} dy$. Then ϕ^* is Lipschitz continuous on $(\epsilon, R-\epsilon)$, for every $\epsilon > 0$, where R is the radius of Ω^*.*

Proof First note that the function ϕ^* is decreasing and the set

$$\mathcal{T} := \{t \geq \min_{\bar{\Omega}} \phi : \quad (\phi^*)^{-1}(t) \ \text{ is not a singleton}\}$$

has Lebesgue measure zero. Indeed $(\phi^*)^{-1}(t)$ is a connected closed interval for all $t \in \mathcal{T}$. Let $0 < r_1 < r_2 < R$ and

$$a(t) = \int_{\{\phi^* > t\}} e^{U_\lambda} dy = \int_{\{\phi > t\}} e^u dy, \quad \forall t > \min_{y\in\overline{\Omega}} \phi.$$

For $\phi^*(r_1), \phi^*(r_2) \notin \mathcal{T}$, we have

$$\begin{aligned} a(\phi^*(r_2)) - a(\phi^*(r_1)) &= \int_{\{\phi^*(|y|)>\phi^*(r_2)\}} e^{U_\lambda} dy - \int_{\{\phi^*(|y|)>\phi^*(r_1)\}} e^{U_\lambda} dy \\ &= \int_{\{\phi(y)>\phi^*(r_2)\}} e^u dy - \int_{\{\phi(y)>\phi^*(r_1)\}} e^u dy \\ &= \int_{\{\phi^*(r_2)<\phi(y)\le\phi^*(r_1)\}} e^u dy \\ &= \int_{\{\phi^*(r_2)<\phi^*(|y|)\le\phi^*(r_1)\}} e^{U_\lambda} dy. \end{aligned}$$

Let $M_1 := \max_{\overline{\Omega^*}} e^{U_\lambda(y)}$, it follows from the above equality that

$$\begin{aligned} a(\phi^*(r_2)) - a(\phi^*(r_1)) &\le M_1 \mu(\{\phi^*(r_2) \le \phi^*(|y|) \le \phi^*(r_1)\}) \\ &= M_1 \mu(r_1 \le |y| \le r_2) = M_1 \pi (r_2^2 - r_1^2) \\ &\le 2\pi R M_1 (r_2 - r_1). \end{aligned}$$

On the other hand, we note that the set $\{y \in \Omega : \phi^*(r_2) < \phi(y)\}$ is bounded. Let

$$\begin{aligned} m(r_2) &:= \inf\{e^{u(y)} : y \in \Omega,\ \phi^*(r_2) < \phi(y)\}, \\ M_2(r_2) &:= \sup\{|\nabla\phi(y)| : y \in \Omega,\ \phi^*(r_2) < \phi(y)\}. \end{aligned}$$

Then

$$\begin{aligned} a(\phi^*(r_2)) - a(\phi^*(r_1)) &\ge m(r_2)\mu(\{\phi^*(r_2) < \phi(y) \le \phi^*(r_1)\}) \\ &\ge \frac{m(r_2)}{M_2(r_2)} \int_{\{\phi^*(r_2)<\phi(y)\le\phi^*(r_1)\}} |\nabla\phi| dy \\ &\ge \frac{m(r_2)}{M_2(r_2)} \int_{\phi^*(r_2)}^{\phi^*(r_1)} \int_{\{\phi^{-1}(t)\}} ds dt \\ &\ge \frac{m(r_2)}{M_2(r_2)} (\phi^*(r_1) - \phi^*(r_2)) K(r_1, r_2), \end{aligned}$$

where

$$K(r_1, r_2) = \min_{\{\phi^*(r_2)\le t\le\phi^*(r_1)\}} \mathcal{H}^1(\phi^{-1}(t)) > 0, \quad 0 < r_1 < r_2 < R.$$

Hence we have

$$0 \leq \frac{\phi^*(r_2) - \phi^*(r_1)}{r_2 - r_1} \leq \frac{2\pi R M_1 M_2(r_2)}{m(r_2) K(r_1, r_2)} < \infty. \tag{2.10}$$

In general, for $0 < \epsilon < r_1^* < r_2^* < R - \epsilon$, we can approximate r_1^*, r_2^* by points r_1, r_2 such that $\phi^*(r_1), \phi^*(r_2) \notin \mathcal{T}$. Then the above estimate also holds for r_1^*, r_2^*. Thus ϕ^* is Lipschitz continuous on $(\epsilon, R - \epsilon)$ for every $\epsilon > 0$. □

Now let

$$j(t) := \int_{\{\phi > t\}} |\nabla \phi|^2 dy, \quad j^*(t) := \int_{\{\phi^* > t\}} |\nabla \phi^*|^2 dy, \ \ \forall t > \min_{y \in \overline{\Omega}} \phi; \tag{2.11}$$

$$J(t) := \int_{\{\phi > t\}} |\nabla \phi| dy, \quad J^*(t) := \int_{\{\phi^* > t\}} |\nabla \phi^*| dy, \ \ \forall t > \min_{y \in \overline{\Omega}} \phi. \tag{2.12}$$

It is easy to see that both $j(t)$ and $J(t)$ are absolutely continuous and non-increasing for $t > \min_{y \in \overline{\Omega}} \phi$. It also follows that $j^*(t)$, $J^*(t)$ are absolutely continuous and non-increasing for $t > \min_{y \in \overline{\Omega}} \phi$. Indeed, all four functions are right-continuous by definition. Furthermore, since ϕ belongs to $C^1(\Omega)$ and ϕ^* is Lipschitz as in Proposition 2.3, observing that $\{y \in \Omega : \phi(y) = t\}$ is bounded for $\{t > \min_{y \in \overline{\Omega}} \phi\}$, we have

$$\begin{aligned} 0 \leq J(t-0) - J(t) &= \int_{\{\phi = t\}} |\nabla \phi| dy = 0, \ \ t > \min_{y \in \overline{\Omega}} \phi, \\ 0 \leq J^*(t-0) - J^*(t) &= \int_{\{\phi^* = t\}} |\nabla \phi^*| dy = 0, \ \ t > \min_{y \in \overline{\Omega}} \phi, \\ 0 \leq j(t-0) - j(t) &= \int_{\{\phi = t\}} |\nabla \phi|^2 dy = 0, \ \ t > \min_{y \in \overline{\Omega}} \phi \end{aligned}$$

and

$$0 \leq j^*(t-0) - j^*(t) = \int_{\{\phi^* = t\}} |\nabla \phi^*|^2 dy = 0, \ \ t > \min_{y \in \overline{\Omega}} \phi.$$

Then all four functions are absolutely continuous and non-increasing.

It can also be shown that

$$\int_{\{\phi = t\}} |\nabla \phi| ds \geq \int_{\{\phi^* = t\}} |\nabla \phi^*| ds, \quad \text{for } a.e. \ t > \min_{y \in \overline{\Omega}} \phi. \tag{2.13}$$

Indeed, since $\phi \equiv c$ on $\partial\Omega$, for every $t \neq c$ with $t > \min_{y \in \overline{\Omega}} \phi$ the level set $\{\phi = t\}$ encloses a bounded subset Ω_t of Ω. Hence it follows from Cauchy–Schwarz and Bol's inequalities that

$$
\begin{aligned}
\int_{\{\phi=t\}} |\nabla\phi| ds &\geq \left(\int_{\{\phi=t\}} e^{\frac{u}{2}}\right)^2 \left(\int_{\{\phi=t\}} \frac{e^u}{|\nabla\phi|}\right)^{-1} \\
&= \left(\int_{\{\phi=t\}} e^{\frac{u}{2}}\right)^2 \left(-\frac{d}{dt}\int_{\Omega_t} e^u\right)^{-1} \\
&\geq \frac{1}{2}\left(\int_{\Omega_t} e^u\right)\left(8\pi - \int_{\Omega_t} e^u\right)\left(-\frac{d}{dt}\int_{\Omega_t} e^u\right)^{-1} \qquad (2.14) \\
&= \frac{1}{2}\left(\int_{\Omega_t^*} e^{U_\lambda}\right)\left(8\pi - \int_{\Omega_t^*} e^{U_\lambda}\right)\left(-\frac{d}{dt}\int_{\Omega_t^*} e^{U_\lambda}\right)^{-1} \\
&= \left(\int_{\{\phi^*=t\}} e^{\frac{U_\lambda}{2}}\right)^2 \left(-\frac{d}{dt}\int_{\Omega_t^*} e^{U_\lambda}\right)^{-1} \\
&= \int_{\{\phi^*=t\}} |\nabla\phi^*| ds, \quad \text{for } a.e.\ t > \min_{y\in\overline{\Omega}} \phi.
\end{aligned}
$$

Therefore we have the following proposition.

Proposition 2.4 *Assume that u, ϕ satisfy the conditions in Proposition 2.3. Let U_λ be given by (2.3). Define the equimeasurable symmetric rearrangement ϕ^* of ϕ, with respect to the measures $e^u dy$ and $e^{U_\lambda} dy$, by (2.9). Then ϕ^* is Lipschitz continuous on $(\epsilon, R-\epsilon)$ for every $\epsilon > 0$, and $j^*(t)$, $J^*(t)$ are absolutely continuous and non-increasing in $t > \min_{y\in\overline{\Omega}}\phi$, where $j^*(t)$ and $J^*(t)$ are defined as in (2.11) and (2.12), respectively. Moreover, (2.13) holds.*

3 The sphere covering inequality

The main objective of this section is to prove the following theorem.

Theorem 3.1 *Let Ω be a simply-connected subset of R^2 which may not necessarily be bounded, and assume $w_i \in C^2(\overline{\Omega})$, $i = 1, 2$ satisfy*

$$
\Delta w_i + e^{w_i} = f_i(y), \quad \int_\Omega e^{w_i} dy \leq 8\pi \qquad (3.1)
$$

where $f_2 \geq f_1 \geq 0$ in Ω. If $w_2 \geq w_1$, $w_2 \not\equiv w_1$ in ω and $w_2 = w_1$ on $\partial\omega$, $\lim_{y\in\omega\to\infty} w_2(y) - w_1(y) = 0$ for some piecewise Liptschitz subdomain $\omega \subset \Omega$, then

$$\int_{\omega} (e^{w_1} + e^{w_2}) dy \geq 8\pi. \tag{3.2}$$

Moreover, the equality only holds when $f_2 \equiv f_1 \equiv 0$ *and there is an analytic function* $\xi = g(y)$ *such that* $g(\omega) = B_1$ *and* $g(\partial\omega) = \partial B_1$, *and* $e^{w_1(y)}dy = e^{U_{\lambda_1}(\xi)}d\xi$, $e^{w_2(y)}dy = e^{U_{\lambda_2}(\xi)}d\xi$ *for some* $\lambda_2 > \lambda_1 > 0$.

Remark 3.2 The domain ω is not required to be simply connected. Neither Ω nor ω is required to be bounded. Also note that if $f_1 \equiv f_2$, the condition $w_2 \geq w_1$, $w_2 \not\equiv w_1$ in the theorem can just be replaced by $w_2 \not\equiv w_1$, since there must be a piecewise Liptschitz subdomain ω_1 of ω such that $w_2 > w_1$ in ω_1 and $w_2 = w_1$ on $\partial\omega_1$ after switching the indices. If $w_2 - w_1$ changes sign in Ω, the inequality has indeed a lower bound of 16π. Note that when $\omega = \Omega$, the integral condition in (3.1) is not needed.

Before proving the above theorem, let us first show that Theorem 3.1 holds when w_1, w_2 are both radial. Choose $\lambda_2 > \lambda_1$ and let $U_{\lambda_1}, U_{\lambda_2}$ be given by (2.3). Suppose $U_{\lambda_1} = U_{\lambda_2}$ on ∂B_R for some $R > 0$. Then

$$\frac{\lambda_1}{1 + \frac{\lambda_1^2 R^2}{8}} = \frac{\lambda_2}{1 + \frac{\lambda_2^2 R^2}{8}} = \kappa.$$

Hence λ_1, λ_2 are positive real roots of the quadratic equation

$$R^2\lambda^2 + 8 = \frac{8}{\kappa}\lambda.$$

This implies $\kappa \leq 2/R^2$,

$$\lambda_1 + \lambda_2 = \frac{8}{\kappa R^2}, \text{ and } \lambda_1\lambda_2 = \frac{8}{R^2}. \tag{3.3}$$

Direct computations yield

$$\begin{aligned}\int_{B_R} \left(e^{U_{\lambda_1}} + e^{U_{\lambda_2}}\right) dy &= 8\pi \left(\frac{\lambda_1^2 R^2}{8 + \lambda_1^2 R^2} + \frac{\lambda_2^2 R^2}{8 + \lambda_2^2 R^2}\right)\\ &= 8\pi \left(\frac{\lambda_1^2 R^2}{\frac{8\lambda_1}{\kappa}} + \frac{\lambda_2^2 R^2}{\frac{8\lambda_2}{\kappa}}\right) = 8\pi \left[\frac{\kappa R^2}{8}(\lambda_1 + \lambda_2)\right]\\ &= 8\pi.\end{aligned}$$

Thus we have the following

Proposition 3.1 *Let $\lambda_2 > \lambda_1$, and U_{λ_1} and U_{λ_2} be radial solutions of the equation*

$$\Delta u + e^u = 0, \tag{3.4}$$

defined in (2.3) with $U_{\lambda_2} > U_{\lambda_1}$ in B_R, and $U_{\lambda_1} = U_{\lambda_2}$ on ∂B_R, for some $R > 0$. Then

$$\int_{B_R} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}}) dy = 8\pi.$$

To understand the above equality geometrically, we may scale the conformal factor by $1/2$ and consider two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with constant Gaussian curvature 1 as follows

$$\mathcal{S}_1 = (B_R, e^{2V_{\lambda_1}} dy) \quad \text{and} \quad \mathcal{S}_2 = (B_R, e^{2V_{\lambda_2}} dy).$$

where $2V_\lambda = U_\lambda - \ln 2$. Notice that the metrics $g_i = e^{2V_{\lambda_i}} dy$ have the same conformal factor on ∂B_R and hence (3.3) holds and

$$\kappa \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) = \frac{\kappa(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} = 1. \tag{3.5}$$

Next we explain that areas of $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal to the areas of two complementary spherical caps on the unit sphere, and consequently the total area must be

$$A_1 + A_2 = \int_{B_R} e^{2V_{\lambda_1}} dy + \int_{B_R} e^{2V_{\lambda_2}} dy = 4\pi.$$

By scaling in $y \in \mathbb{R}^2$ and using the stereographic projection $\Pi\colon S^2 \to \mathbb{R}^2$ with respect to the north pole $N = (0, 0, 1)$:

$$y = \Pi(x) := \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right),$$

we can see that the surface $\mathcal{S}_1 = (B_R, e^{2V_{\lambda_1}} dy)$ is isometric to

$$\left(B_{\lambda_1 R/\sqrt{8}}, e^{2V_1} dy\right) = \left(B_{\lambda_1 R/\sqrt{8}}, \frac{4}{(1 + |y|^2)^2} dy\right),$$

which in turn is isometric to a disc $\mathcal{C}_1$ around the south pole. Similarly the surface $\mathcal{S}_2 = (B_R, e^{2V_{\lambda_2}} dy)$ is isometric to

$$\big(B_{\lambda_2 R/\sqrt{8}}, e^{2V_2} dy\big) = \big(B_{\lambda_2 R/\sqrt{8}}, \frac{4}{(1+|y|^2)^2} dy\big),$$

which in turn is isometric to a disc $\mathcal{C}_2$ around the south pole.

Using the Kelvin transformation $z = y/|y|^2$ and (3.3), one can see that

$$\big(B_{\lambda_2 R/\sqrt{8}}, \frac{4}{(1+|y|^2)^2} dy\big) \quad \text{is isometric to} \quad \big(\mathbb{R}^2 \setminus B_{\lambda_1 R/\sqrt{8}}, \frac{4}{(1+|z|^2)^2} dz\big)$$

with the latter being isometric to $S^2 \setminus \mathcal{C}_1$. This implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ are indeed isometric to two complementary spherical caps on the unit sphere, and therefore their areas sum to exactly 4π.

Note that the area of the smaller cap $\mathcal{C}_1$ can be arbitrarily close to 0 or 2π. It is therefore important that the two surfaces $\mathcal{S}_1, \mathcal{S}_2$ are not the same, which is geometrically the reason why w_1, w_2 should be distinct.

The following lemma will play a key role in the proof of Theorem 3.1.

Lemma 3.3 *Assume that $\psi \in C^{0,1}(\overline{B_R})$ is a strictly decreasing, radial, Lipschitz function, and satisfies*

$$\int_{\partial B_r} |\nabla \psi| ds \le \int_{B_r} e^{\psi} dy \tag{3.6}$$

a.e. $r \in (0, R)$ and $\psi = U_{\lambda_1} = U_{\lambda_2}$ for some $\lambda_2 > \lambda_1$ on ∂B_R, and $R > 0$. Then

$$\text{either} \quad \int_{B_R} e^{\psi} dy \le \int_{B_R} e^{U_{\lambda_1}} dy \quad \text{or} \quad \int_{B_R} e^{\psi} dy \ge \int_{B_R} e^{U_{\lambda_2}} dy. \tag{3.7}$$

Moreover if the inequality in (3.6) is strict in a set with positive measure in $(0, R)$, then the inequalities in (3.7) are also strict.

Proof Let $m_1 := \int_{B_R} e^{U_{\lambda_1}} dy$, $m_2 := \int_{B_R} e^{U_{\lambda_2}} dy$, and $m := \int_{B_R} e^{\psi} dy$. Also define

$$\beta := \left(\int_{\partial B_R} e^{\frac{\psi}{2}} ds\right)^2 = \left(\int_{\partial B_R} e^{\frac{U_{\lambda_1}}{2}} ds\right)^2 = \left(\int_{\partial B_R} e^{\frac{U_{\lambda_2}}{2}} ds\right)^2.$$

It follows from Proposition 2.2 that

$$\beta \ge \frac{1}{2} m(8\pi - m).$$

On the other hand,

$$\beta = \frac{1}{2}m_1(8\pi - m_1) = \frac{1}{2}m_2(8\pi - m_2).$$

Hence m_1 and m_2 are roots of the quadratic equation

$$x^2 - 8\pi x + 2\beta = 0.$$

Since m satisfies

$$m^2 - 8\pi m + 2\beta \geq 0,$$

we have

$$\text{either} \quad m \leq m_1 \quad \text{or} \quad m \geq m_2.$$

Equality holds only when the equality in (3.6) holds for *a.e.* $r \in (0, R)$. This completes the proof. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1 Suppose w_1 and w_2 satisfy the assumptions of Theorem 3.1. Then

$$\Delta(w_2 - w_1) + e^{w_2} - e^{w_1} = f_2 - f_1 \geq 0.$$

Without loss of generality we may assume $\int_\Omega e^{w_1} dy < 4\pi$. We can choose $\lambda_2 > \lambda_1$ such that U_{λ_1} and U_{λ_2} are as described in Proposition 3.1, and

$$\int_\Omega e^{w_1} dy = \int_{B_1} e^{U_{\lambda_1}} dy. \tag{3.8}$$

Let $\Omega_t = \{y \in \Omega : w_2(y) - w_1(y) > t\}$ and φ be the symmetrization of $w_2 - w_1$ with respect to the measures $e^{w_1}dy$ and $e^{U_{\lambda_1}}dy$. Then using Proposition 2.4, Green's formula, equation (3.1) and Cavalieri's principle we have

$$\begin{aligned}
\int_{\{\varphi=t\}} |\nabla\varphi| ds &\leq \int_{\{w_2-w_1=t\}} |\nabla(w_2 - w_1)| ds \quad \text{[by the inequality (2.13)]} \\
&\leq \int_{\Omega_t} \left(e^{w_2} - e^{w_1}\right) dy \quad \text{[using Green's formula and Eq. (3.1)]} \\
&= \int_{\Omega_t} e^{w_2-w_1} e^{w_1} dy - \int_{\Omega_t} e^{w_1} dy
\end{aligned}$$

$$= \int_{\{\varphi>t\}} e^{\varphi} e^{U_{\lambda_1}} dy - \int_{\{\varphi>t\}} e^{U_{\lambda_1}} dy$$

(using the rearrangement and Cavalieri's principle)

$$= \int_{\{\varphi>t\}} e^{U_{\lambda_1}+\varphi} dy - \int_{\{\varphi=t\}} |\nabla U_{\lambda_1}| ds, \quad \text{for} \quad a.e. t > 0. \tag{3.9}$$

Hence

$$\int_{\{\varphi=t\}} |\nabla(U_{\lambda_1}+\varphi)| ds \leq \int_{\{\varphi>t\}} e^{(U_{\lambda_1}+\varphi)} dy, \quad \text{for} \quad a.e. t > 0. \tag{3.10}$$

Since $\varphi \geq 0$ is decreasing in r, $\psi := U_{\lambda_1} + \varphi$ is a strictly decreasing radial function, and

$$\int_{\partial B_r} |\nabla \psi| ds \leq \int_{B_r} e^{\psi} dy, \quad a.e. \quad r \in (0,1), \tag{3.11}$$

by Proposition 2.4 and the above inequality we know that ψ belongs to $C^{0,1}(B_1)$. It follows from Lemma 3.3 that

$$\int_{B_1} e^{\psi} dy = \int_{B_1} e^{U_{\lambda_1}+\varphi} dy \geq \int_{B_1} e^{U_{\lambda_2}} dy.$$

Hence

$$\int_{\Omega} (e^{w_1} + e^{w_2}) dy = \int_{B_1} \left(e^{U_{\lambda_1}} + e^{U_{\lambda_1}+\varphi}\right) dy \geq \int_{B_1} \left(e^{U_{\lambda_1}} + e^{U_{\lambda_2}}\right) dy = 8\pi.$$

Moreover, if the equality holds, then

$$\int_{\{w_2-w_1=t\}} |\nabla(w_2 - w_1)| ds = \int_{\Omega_t} \left(e^{w_2} - e^{w_1}\right) dy \quad \text{for} a.e. t > 0,$$

and hence $\int_{\Omega_t} (f_2 - f_1) dy = 0$ for a.e. $t > 0$. Thus $f_2 \equiv f_1$. On the other hand it follows from the computations in (2.14) that the equality in (3.9) for a.e. $t > 0$ leads to the equality in Bol's inequality for w_1 in Ω_t for a.e. $t > 0$. Therefore $f_1 \equiv 0$ and, by the argument right before Proposition 2.2 in Sect. 2, there is an analytic function $\xi = g(y)$ such that $g(\Omega) = B_1$ and $g(\partial\Omega) = \partial B_1$ and $e^{w_1(y)} dy = e^{U_{\lambda_1}(\xi)} d\xi$. Furthermore, we have also $\psi = U_{\lambda_1} + \varphi \equiv U_{\lambda_2}(\xi)$ since it is the only other radial solution of (3.4) with the same boundary condition as U_{λ_1} on ∂B_1. This proof is complete. □

The following consequence of Bol's inequality and the equimeasurable symmetric rearrangement was first proved in [3] and was used frequently in the study of related equations (see also Lemma 3.1 in [26], or Proposition 3.3 in [37] for a proof).

Proposition 3.2 *Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain and assume that $w \in C^2(\overline{\Omega})$ satisfies*

$$\Delta w + e^w \geq 0 \ \ in \ \ \overline{\Omega} \tag{3.12}$$

and $\int_\Omega e^w \leq 8\pi$. Consider an open set $\omega \subset \Omega$ and define the first eigenvalue of the operator $\Delta + e^w$ in $H_0^1(\omega)$ by

$$\lambda_{1,w}(\omega) := \inf_{\{\phi \in H_0^1(\omega) : |\phi|_{L^2(\omega)} = 1\}} \left(\int_\omega |\nabla \phi|^2 - \int_\omega \phi^2 e^w \right),$$

and suppose $\lambda_{1,w}(\omega) \leq 0$. Then $\int_\omega e^w \geq 4\pi$, and the inequality is strict if the inequality in (3.12) is strict at some point in ω.

We would like to point out that the Sphere Covering Inequality is more general than Proposition 3.2 and can deal with the case when w_1 and w_2 are different, while Proposition 3.2 may be regarded as a limiting case when $w_1 \equiv w_2 \equiv w$. To be more precise, when $w_1 \equiv w_2$, the other conditions on w_2, w_1 in the Sphere Covering Inequality become the eigenvalue condition $\lambda_{1,w}(\omega) \leq 0$ in the following sense: suppose that there exist two sequences of solutions w_1^k, w_2^k of (3.1) with f_1^k, f_2^k in Ω, $k = 1, 2, \ldots$ with $w_i^k \to w$ in $C^2(\omega)$, $f_i^k \to f$ in $C^0(\omega)$, $i = 1, 2$ as $k \to \infty$ and the conditions of Theorem 3.1 hold for each k. Then, the standard elliptic theory leads to

$$\varphi := \lim_{k \to \infty} \frac{w_2^k - w_1^k}{||w_2^k - w_1^k||_{L^\infty(\omega)}} > 0$$

and φ is the first eigenfunciton of the linearized operator $\Delta + e^w$ in $H_0^1(\omega)$ with $\lambda_{1,w}(\omega) = 0$. Hence w satisfies the condition in Proposition 3.2. The Sphere Covering Inequality applied to w_1^k, w_2^k, after taking limit $k \to \infty$, gives

$$\int_\omega e^w + e^w = \lim_{k \to \infty} \int_\omega e^{w_1^k} + e^{w_2^k} \geq 8\pi,$$

which leads to the same lower bound $\int_\omega e^w \geq 4\pi$ as in Proposition 3.2. This is why Proposition 3.2 may be regarded as a limiting case of the Sphere Covering Inequality.

Remark 3.4 It can be seen from its proof that Proposition 3.2 has a geometrical interpretation as follows: Given a simply connected region ω on a surface with a conformal metric $dg = e^w dx$ and Gaussian curvature less than $1/2$, if the first eigenvalue of Laplacian in ω with zero Dirichlet boundary condition

$$\lambda_1(\omega):= \inf_{\{\phi\in H_0^1(\omega),\phi\not\equiv 0\}} \frac{\int_\omega |\nabla\phi|^2}{\int_\omega \phi^2 e^w} \le 1,$$

then the area of ω must be bigger than or equal to 4π, which is the area of a standard upper hemisphere $S^+_{\sqrt{2}}$ with radius $\sqrt{2}$ and Gaussian curvature $1/2$ (see Proposition 2 in [43]). Note that such a hemisphere has the first eigenvalue equal to 1 as the height function $\phi_1 = x_3$ is the first eigenfunction due to the fact that $-\Delta_{S_{\sqrt{2}}}\phi_1 = \phi_1$. In other words, Proposition 3.2 is an immediate consequence of the extremal eigenvalue theorem which says that a geodesic disc on the sphere achieves the smallest first eigenvalue of Laplacian among all surfaces with the same area and the same Gaussian curvature upper bound. It would be very interesting to see whether there is an intrinsic geometric explanation of Theorem 3.1.

4 Best constant in a Moser–Trudinger type inequality

Let us consider the functional $J_\alpha(u)$ defined in (1.6) and restricted to the set

$$\mathcal{M}:= \left\{ u \in H^1(S^2) : \int_{S^2} e^u x_j = 0 \ \text{ for } \ j = 1, 2, 3 \right\}.$$

In this section we shall prove that $\inf_{u\in\mathcal{M}} J_\alpha(u) = 0$ for $\alpha \ge \frac{1}{2}$. Critical points of $J_\alpha(u)$, up to an additive constant such that $\int_{S^2} e^u d\omega_1 = 1$, satisfy

$$\frac{\alpha}{2}\Delta u + e^u - 1 = 0 \quad \text{on } \ S^2. \tag{4.1}$$

Throughout this section we shall assume that the volume form is normalized so that $\int_{S^2} e^u d\omega_1 = 1$. In particular, the Lagrange multipliers vanish due to Kazdan–Warner identity [32] (see [16,26] for more details). Following [26], let Π be the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the north pole $N = (0, 0, 1)$:

$$\Pi:= \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

Suppose u is a solution of (4.1), and let

$$\bar{u}(y):=u(\Pi^{-1}(y)) \text{ for } y \in \mathbb{R}^2.$$

Then $\bar{u}$ satisfies

$$\Delta \bar{u} + \frac{8}{\alpha(1+|y|^2)^2}(e^{\bar{u}} - 1) = 0 \text{ in } \mathbb{R}^2 . \tag{4.2}$$

Now if we let

$$v = \bar{u} - \frac{2}{\alpha} \ln(1+|y|^2) + \ln\left(\frac{8}{\alpha}\right), \tag{4.3}$$

then v satisfies

$$\Delta v + (1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^v = 0 \text{ in } \mathbb{R}^2, \tag{4.4}$$

and

$$\int_{R^2} (1+|y|^2)^{2(\frac{1}{\alpha}-1)} e^v dy = \frac{8\pi}{\alpha}. \tag{4.5}$$

4.1 Uniqueness of axially symmetric solutions

For convenience of the reader, we first use a new method to prove Conjecture A for axially symmetric functions, which was originally proven in [30,36].

Lemma 4.1 *Let $\alpha \geq \frac{1}{2}$ and $u \in \mathcal{M}$ be a solution of (4.1). If u is axially symmetric, then $u \equiv 0$.*

Proof We may assume that u is symmetric about the x_3-axis, i.e. $u = g(x_3)$, $x_3 \in [-1, 1]$. Since $\int_{S^2} e^u x_3 d\omega = 0$, g could not be monotone in x_3 unless it is identically equal to a constant C. Therefore, if $u \not\equiv C$, then it must have either a local minimum or local maximum at some point $x_3^0 \in (-1, 1)$. Without loss of generality we can assume $x_3^0 \geq 0$. Now choose some point $p = (x_1^p, 0, x_3^p) \in S^2$ with $x_3^0 < x_3^p < 1$ and let $u_p(x) = u(R^{-1}(x))$ for some $R \in SO(3)$ with $R(p) = (0, 0, 1)$. Define $\bar{u}_p = u_p(\Pi^{-1})$ and let

$$v_p = \bar{u}_p - \frac{2}{\alpha} \ln(1+|y|^2) + \ln\left(\frac{8}{\alpha}\right).$$

the v_p satisfies (4.4) and (4.5). Now let

$$\varphi_p(y):=y_2\frac{\partial v_p}{\partial y_1}-y_1\frac{\partial v_p}{\partial y_2}.$$

Note that the set of critical points of $\bar{u}_p$ contains a closed simple curve $\mathcal{C}\subset\mathbb{R}^2$ which contains the origin in its interior. On the other hand v_p is evenly symmetric about the y_1-axis, therefore

$$\mathcal{C}\cup\{y=(y_1,0): y\in\mathbb{R}^2\}\subset\varphi_p^{-1}(0).$$

Hence $\varphi_p^{-1}(0)$ divides $\mathbb{R}^2$ into at least four simply-connected regions Ω_i, $i=1,2,3,4$. Now let $w_p:=\ln((1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^{v_p})$. Then w_p satisfies

$$\Delta w_p+e^{w_p}=\frac{8(\frac{1}{\alpha}-1)}{(1+|y|^2)^2}>0 \ \text{ in } \ \mathbb{R}^2\,.$$

On the other hand φ_p satisfies

$$\Delta\varphi_p+e^{w_p}\varphi_p=0 \ \text{ in } \ \mathbb{R}^2\,.$$

Note that if $\varphi_p\equiv 0$ in an open subset of $\mathbb{R}^2$, then $\varphi_p\equiv 0$ in $\mathbb{R}^2$. Thus it follows from Proposition 3.2 that

$$\frac{8\pi}{\alpha}=\int_{\mathbb{R}^2}(1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^{v_p}dy=\sum_{i=1}^{4}\int_{\Omega_i}e^{w_p}dy>4\times 4\pi=16\pi.$$

This implies $\alpha<\frac{1}{2}$, which is a contradiction. Therefore $\varphi_p\equiv 0$ and consequently u is also axially symmetric about the line passing through p and the origin. Since $p\neq(0,0,1)$, u must be identically equal to a constant, and therefore must be zero. □

Next we prove that if u is evenly symmetric about a plane passing through the origin, then u is axially symmetric. Note that this result was remarked by Ghoussoub and Lin [26], we provide the details here since it is needed in the proof of the main result.

Lemma 4.2 *Let $\alpha\geq\frac{1}{2}$ and u be a solution of (4.1). If u is evenly symmetric about a plane passing through the origin, then u is axially symmetric.*

Proof The proof is similar to the proof of Lemma 4.1. We may assume that u is evenly symmetric about x_1x_3-plane. Let u_0 be the restriction of u to $\{x\in S^2:\ \ x_2=0\}$ and assume that $p\in S^2$ is a maximum point of u_0. Since u

is symmetric about the x_1x_3-plane, p is also a critical point of u on S^2. Without loss of generality we may assume $p = (0, 0, -1)$. We claim that

$$\varphi(y_1, y_2) = y_2\frac{\partial v}{\partial y_1} - y_1\frac{\partial v}{\partial y_2} \equiv 0,$$

where v is defined by (4.3). Suppose $\varphi \not\equiv 0$. Since v has a critical point at the origin, the nodal line of φ is piecewise smooth and divides a neighborhood of the origin into at least four regions (see Theorem 2.5 of [22], or [8,31]). On the other hand φ is anti-symmetric (odd) with respect to the y_1-axis and the nodal line of φ contains the y_1-axis. Therefore the nodal line of φ divides $\mathbb{R}^2$ into at least 4 simply-connected regions Ω_i, $i = 1, 2, 3, 4$. As before, we can show that $\varphi \equiv 0$ and consequently u is axially symmetric about the line passing through p and the origin. □

4.2 The general case

We shall prove the even symmetry of a solution to (4.1).

Theorem 4.3 *Let $\alpha \geq \frac{1}{2}$ and assume u to be a solution of* (4.1)*. Then u is evenly symmetric about any plane passing through the origin and a critical point of u. Therefore u must be axially symmetric and consequently $u \equiv 0$.*

Proof Without loss of generality we may assume that $(1, 0, 0)$ is a critical point of u, and that u is not symmetric about the x_1x_2-plane. To finish the proof, it is enough to prove that u is symmetric about the x_1x_2-plane. Define $u^*(x_1, x_2, x_3){:=}u(x_1, x_2, -x_3)$ and $\tilde{u}(x) = u(x) - u^*(x)$. Notice that $\tilde{u}(x_1, x_2, 0) = 0$, for all $(x_1, x_2, 0) \in S^2$. Then $\tilde{u}$ satisfies

$$\frac{\alpha}{2}\Delta\tilde{u} + c(x)\tilde{u} = 0, \quad \text{on} \ \ S^2, \tag{4.6}$$

where

$$c(x){:=}\frac{e^u - e^{u^*}}{u - u^*}.$$

Since $(1, 0, 0)$ is a critical point of u, it follows from Hopf's lemma that $\tilde{u}$ must change sign in $S^+{:=}\{x \in S^2 : x_3 > 0\}$. Indeed, classic results imply that the nodal set of $\tilde{u}$ consists of finite many immmersed smooth curves which have only finite many self-intersecting points which are critical points of $\tilde{u}$. Moreovr, $\tilde{u}$ behaves like a harmonic polynomial near critical points, and its nodal set locally looks like straight lines with equal angles near critical points. (See, e.g. Theorem 2.5 of [22], or [8,31].) Recall that $\tilde{u} = 0$ on ∂S^+. Therefore the nodal lines of $\tilde{u}$ divides S^+ into at least two simply-connected regions and there exists $S_+^+, S_-^+ \subset S^+$ such that $u = u^*$ on $\partial(S_+^+ \cup S_-^+)$,

$$u > u^* \ \text{ on } \ S_+^+ \ \text{ and } \ u < u^* \ \text{ on } \ S_-^+.$$

Define S_+^-, S_-^- to be the reflections of S_+^+, S_-^+ with respect to the x_1x_2-plane. Then we also have $u = u^*$ on $\partial(S_+^- \cup S_-^-)$,

$$u < u^* \ \text{ on } \ S_+^- \ \text{ and } \ u > u^* \ \text{ on } \ S_-^-.$$

Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4 \subset \mathbb{R}^2$ be the images of $S_-^-, S_+^-, S_-^+, S_+^+ \subset S^2$ under the stereographic projection, respectively. Define v_1, v_2 as follows

$$v_1(y) = u(\Pi^{-1}(y)) - \frac{2}{\alpha}\ln(1+|y|^2) + \ln\left(\frac{8}{\alpha}\right)$$

and

$$v_2(y) = u^*(\Pi^{-1}(y)) - \frac{2}{\alpha}\ln(1+|y|^2) + \ln\left(\frac{8}{\alpha}\right).$$

Then v_1 and v_2 both satisfy (4.4) and w_i defined by

$$w_i := \ln((1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^{v_i})$$

satisfies

$$\Delta w_i + e^{w_i} = \frac{8(\frac{1}{\alpha}-1)}{(1+|y|^2)^2} \geq 0 \ \text{ in } \ \mathbb{R}^2, \ \ i = 1, 2.$$

Moreover $w_1 = w_2$ on $\partial\Omega_i$, $i = 1, 2, 3, 4$. Applying the Sphere Covering Inequality (Theorem 3.1) in Ω_i, $i = 1, 2, 3, 4$, we obtain that

$$\begin{aligned} 2 \times \frac{8\pi}{\alpha} &= \int_{\mathbb{R}^2}(1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^{v_1}dy + \int_{\mathbb{R}^2}(1+|y|^2)^{2(\frac{1}{\alpha}-1)}e^{v_2}dy \\ &\geq \sum_{i=1}^{4}\int_{\Omega_i}(e^{w_1}+e^{w_2})dy > 4 \times 8\pi. \end{aligned}$$

Hence $\alpha < \frac{1}{2}$, which is a contradiction. Thus u is evenly symmetric about the x_1x_2-plane and the proof is complete. □

5 Radial symmetry of solutions in $\mathbb{R}^2$

In this section, we shall consider solutions to a general class of equations in $\mathbb{R}^2$ and prove radial symmetry of the solutions. Assume $u \in C^2(\mathbb{R}^2)$ satisfies

$$\Delta u + k(|y|)e^u = 0 \ \text{ in } \ \mathbb{R}^2, \tag{5.1}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} k(|y|)e^u dy = \beta < \infty, \tag{5.2}$$

where $K(y) = k(|y|) \in C^2(\mathbb{R}^2)$ is a non constant positive function satisfying

$$\begin{aligned} &(K1) \qquad \Delta \ln(k(|y|)) \geq 0, \quad y \in \mathbb{R}^2 \\ &(K2) \qquad k(|y|) \leq C(1+|y|)^m, \quad y \in \mathbb{R}^2 \end{aligned}$$

for some constant $C, m > 0$. It is easy to see that $(K1)$ implies that both $k(r)$ and $\frac{rk'(r)}{k(r)}$ are nondecreasing. Let

$$2l = \lim_{r\to\infty} \frac{rk'(r)}{k(r)}.$$

From $(K2)$ we know that $0 \leq 2l \leq m$ and hence for any $\epsilon > 0$ there exists a positive constant $C_\epsilon > 0$ such that

$$C_\epsilon(1+|y|^2)^{l-\epsilon} \leq k(|y|) \leq C(1+|y|^2)^l, \quad y \in \mathbb{R}^2.$$

Without loss of generality we may assume that $m = 2l$. Then it follows from Theorem 1.1 in [21] that

$$\beta \geq 2l + 2.$$

Following [26] and using Pohazaev's identity, we can obtain the following result.

Proposition 5.1 *Suppose u is a solution to (5.1)–(5.2), where K is not a constant and $(K1)$–$(K2)$ hold with $m = 2l$. Then, if $\beta > 2l + 2$, there holds*

$$4 < \beta < 4l + 4. \tag{5.3}$$

Proof By Theorem 1.1 in [21], we have

$$u(y) = -\beta \ln(|y|) + C + O(|y|^{-\gamma}) \tag{5.4}$$

for some constants C and $\gamma > 0$ as $y \to \infty$, if $\beta > 2l+2$. Also if $\beta = 2l+2$, then for any $\epsilon > 0$ there exists $R(\epsilon) > 0$ such that

$$-\beta \ln(|y|) - C \leq u(y) \leq (\epsilon - \beta) \ln(|y|), \quad |y| \geq R(\epsilon)$$

for some constant C. On the other hand, it is easy to see that when $\beta > 2l+2$, we have

$$\nabla u = (-\beta + o(1)) \frac{y}{|y|^2}, \quad \text{as } y \to \infty.$$

Multiplying (5.1) by $y \cdot \nabla u$ and integrating by parts on $B_R = \{y : |y| \leq R\}$, we obtain

$$\begin{aligned} &\textstyle\int_{\partial B_R} (y \cdot \nabla u) \frac{\partial u}{\partial \nu} ds - \frac{1}{2} \int_{\partial B_R} (y \cdot \nu) |\nabla u|^2 ds = - \int_{B_R} k(|y|) y \cdot \nabla e^u dy \\ &\quad = \textstyle\int_{B_R} (2k(|y|) + k'(|y|)|y|) e^u dy - \int_{\partial B_R} (y \cdot \nu) k(|y|) e^u ds. \end{aligned}$$

Letting $R \to \infty$ and using (5.4), we obtain that

$$\int_{\mathbb{R}^2} \big(2k(|y|) + k'(|y|)|y|\big) e^u dy = \pi \beta^2.$$

Hence we derive (5.3) from

$$2k(|y|) \leq 2k(|y|) + k'(|y|)|y| \leq (2l+2) k(|y|), \quad y \in \mathbb{R}^2,$$

and the fact that equality holds everywhere in any of the above inequalities only when $l = 0$ and k equals to a constant. Note that by our assumptions, $k(|y|) = |y|^{2l}$ is not allowed for $l > 0$ since $k(0) = 0$. The proof is complete. □

Remark 5.1 In all applications considered in this paper, it holds that $\beta > 2l+2$. We wonder whether $\beta > 2l+2$ is always true for all solutions to (5.1)–(5.2) under the general conditions $(K1)$–$(K2)$.

It is shown in [36] that

Proposition 5.2 *If* $0 < l \leq 1$, *there exists a radially symmetric solution* u_β *to* (5.1) *if and only if* $\beta \in (4, 4l+4)$. *The radial solution is also unique in this case. If* $l > 1$, *there exists a unique radially symmetric solution* u_β *to* (5.1) *for* $\beta \in (4l, 4l+4)$. *In the latter case, there exists* $\beta(l) \in (4, 4l)$ *such that there is no radial solution for* $\beta < \beta(l)$ *but there are at least two radial solutions for* $\beta \in (\beta(l), 4l)$.

Now we are ready to prove the following general theorem.

Theorem 5.2 *Assume that* $K(y) = k(|y|) > 0$ *is not constant and satisfies* $(K1)$–$(K2)$*, and* u *is a solution to* (5.1)–(5.2) *with* $2l + 2 < \beta \leq 8$*. Then* u *must be radially symmetric.*

Proof It follows from (5.4) that $\lim_{|y|\to\infty} u(y) = -\infty$, and hence u has a maximum point $p \in \mathbb{R}^2$. We first prove that u is evenly symmetric about the line passing through the origin and p. In particular if $p = (0, 0)$, then the following argument guarantees that u is evenly symmetric about any line passing through the origin and hence u must be radially symmetric. Without loss of generality we may assume that p lies on y_1-axis. Define

$$v(y_1, y_2) = u(y_1, y_2) - u(y_1, -y_2). \tag{5.5}$$

Suppose $v \not\equiv 0$. Then the nodal line of v, $v^{-1}(0)$, contains the y_1-axis. On the other hand since the critical point p lies on y_1-axis, the nodal line of v divides every small neighborhood of p into at least four regions. Therefore the nodal line of v divides $\mathbb{R}^2$ into at least four simply-connected regions Ω_i, $i = 1, 2, 3, 4$. Now notice that on each Ω_i the equation

$$\Delta u + k(|y|)e^u = 0 \quad y \in \Omega_i$$

has two solutions $u_i^1(y_1, y_2) = u(y_1, y_2)$ and $u_i^2(y_1, y_2) = u(y_1, -y_2)$ with $u_i^1|_{\partial\Omega} = u_i^2|_{\partial\Omega_i}$. Define $w{:=}u + \ln(k(|y|))$. Then w satisfies

$$\Delta w + e^w = \Delta(\ln(k(|y|))) \geq 0. \tag{5.6}$$

Thus on each Ω_i, the above equation has two solutions w_i^1, w_i^2 with $w_i^1|_{\partial\Omega_i} = w_i^2|_{\partial\Omega}$, $i = 1, 2, 3, 4$. Hence it follows from Theorem 3.1 that

$$\begin{aligned} 4\pi\beta &= 2\int_{\mathbb{R}^2} k(|y|)e^u dy = \int_{\mathbb{R}^2} k(|y|)e^{u(y_1,y_2)}dy + \int_{\mathbb{R}^2} k(|y|)e^{u(y_1,-y_2)}dy \\ &\geq \sum_{i=1}^{4}\int_{\Omega_i}(e^{w_1} + e^{w_2})dy > 4 \times 8\pi = 32\pi. \end{aligned}$$

Consequently $\beta > 8$, which is a contradiction, and therefore u is evenly symmetric about the y_1-axis.

Next we shall prove that u is indeed axially symmetric. Let $\phi = y_2 \cdot u_{y_1} - y_1 \cdot u_{y_2}$. Then ϕ satisfies

$$\Delta\phi + K(y)e^u\phi = 0, \quad y \in \mathbb{R}^2. \tag{5.7}$$

On the other hand, u_{y_2} satisfies

$$\Delta u_{y_2} + K(y)e^u u_{y_2} = -y_2 \frac{k'(|y|)}{|y|} e^u, \quad y \in \mathbb{R}^2. \tag{5.8}$$

Note that both u_{y_2} and ϕ are odd function in y_2. Let us multiply Eq. (5.7) by u_{y_2} and Eq. (5.8) by ϕ and subtract. Then, integrating the resulting equation in $B_R^+ = \{y : y_2 > 0, |y| \le R\}$, we obtain

$$\int_{\partial B_R^+} \left(\phi \frac{\partial u_{y_2}}{\partial \nu} - u_{y_2} \frac{\partial \phi}{\partial \nu} \right) ds = - \int_{B_R^+} y_2 \frac{k'(|y|)}{|y|} e^u \phi dy.$$

Applying standard Schauder estimates for the elliptic equation satisfied by $u + \beta \ln(|y|)$ and using the fact that $\beta > 2l + 2$, we obtain

$$|\nabla u(y)| \le \frac{C}{|y|}, \qquad |\nabla^2 u(y)| \le \frac{C}{|y|^2}, \quad |y| > 1$$

for some constant C. Letting $R \to \infty$, we derive

$$\int_{\mathbb{R}^2{}_+} y_2 \frac{k'(|y|)}{|y|} e^u \phi dy = 0.$$

We claim that $\phi \equiv 0$ in $\mathbb{R}^2$. Assume the contrary.

Since $k(r)$ is not constant, by $(K1)$ we have two cases: either $k'(r) > 0$ for $r > 0$ or there exists $r_0 > 0$ such that $k(r) = k(0)$ for $r \in [0, r_0]$ and $k'(r) > 0$ for $r > r_0$. In the first case, since $y_2 \frac{k'(|y|)}{|y|} e^u > 0$ in $\mathbb{R}^2{}_+$, there exist at least two regions $\Omega_1, \Omega_2 \subset \mathbb{R}^2{}_+$ such that $\phi > 0$ in Ω_1 and $\phi < 0$ in Ω_2 and $\phi = 0$ on $\partial\Omega_i, i = 1, 2$. Applying Proposition 3.2 to $\Omega_i, i = 1, 2$, we conclude that

$$\int_{\mathbb{R}^2{}_+} k(|y|)e^u dy \ge \int_{\Omega_1} e^w dy + \int_{\Omega_2} e^w dy > 8\pi,$$

and therefore $\beta > 8$. This contradiction shows $\phi \equiv 0$ in $\mathbb{R}^2$, and hence $u(y)$ is radially symmetric.

In the second case, we have

$$\int_{\mathbb{R}^2{}_+ \cap B_{r_0}^c} y_2 \frac{k'(|y|)}{|y|} e^u \phi dy = 0.$$

If ϕ changes signs in $B_{r_0}^c$, we can argue as above that ϕ must be identically 0 in $\mathbb{R}^2$. If ϕ does not change sign in $B_{r_0}^c$, then $\phi \equiv 0$ in $B_{r_0}^c$. Choose $k_1(r) = e^{sr^3}$ and define

$$\Gamma(r) = \Delta\big(\ln k(r) - \ln k_1(r)\big),$$

Let $w = u + \ln k(r) - \ln k_1(r)$. Then

$$\Delta w(y) + k_1(|y|)e^w = \Gamma(|y|) \text{ in } \mathbb{R}^2. \tag{5.9}$$

and

$$\Delta w_{y_2}(y) + k_1(|y|)e^w w_{y_2} = \frac{y_2}{|y|}\big(\Gamma'(|y|) - k_1'(|y|)e^w\big) \text{ in } \mathbb{R}^2. \tag{5.10}$$

where, for sufficiently large s,

$$\Gamma'(|y|) = -9s + \frac{d}{dr}[\Delta\big(\ln k(r)\big)] < 0, \quad \forall r \in [0, r_0].$$

We note that ϕ satisfies

$$\Delta\phi + k_1(y)e^w\phi = 0, \quad y \in \mathbb{R}^2. \tag{5.11}$$

Multiply Eq. (5.11) by w_{y_2} and Eq. (5.9) by ϕ and subtract. Then, integrating the resulting equation in $B_{r_0}^+ = \{y : y_2 > 0, |y| \le r_0\}$, we obtain

$$0 = \int_{\partial B_{r_0}^+} (\phi\frac{\partial w_{y_2}}{\partial \nu} - w_{y_2}\frac{\partial\phi}{\partial\nu})ds = \int_{B_{r_0}^+} \frac{y_2}{|y|}\big(\Gamma'(|y|) - k_1'(|y|)e^w\big)\phi dy$$

where

$$\Gamma'(|y|) - k_1'(|y|)e^w < 0.$$

This implies that ϕ must change sign at least once in $\mathbb{R}_+^2 \cap B_{r_0}$ if it is not identically 0. The same arguments as in the first case lead to a contradiction. Therefore, we have $\phi \equiv 0$ in $\mathbb{R}^2$, and hence $u(y)$ is radially symmetric. □

Now we consider several special cases of (5.1)–(5.2). First, if $K(y) = (1 + |y|^2)^l$ for some $l \ge 0$, then (5.1)–(5.2) read as

$$\Delta v + (1 + |y|^2)^l e^v = 0 \text{ in } \mathbb{R}^2, \tag{5.12}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}(1+|y|^2)^l e^v dy=\beta. \tag{5.13}$$

The following result is conjectured in [26].

Conjecture D For $0 < l \leq 2$ and $\beta = (2l+4)$, solutions to (5.12)–(5.13) must be radially symmetric.

For $-2 < l \leq 0$, the radial symmetry of solutions to (5.12)–(5.13) was shown in [17,19] by the moving plane method; while for $l > 0$ the moving plane method does not seem to work, the conjecture was shown in [26] for $0 < l \leq 1$ by using the Alexandrov-Bol inequality. For $2 < l \neq (k-1)(k+2)$, where $k \geq 2$ is an integer, it is pointed out by Lin in [36] that there is a non-radial solution to (5.12)–(5.13) . A direct application of Theorem 5.2 to (5.12)–(5.13) leads to an affirmative answer to Conjecture D. Indeed, all solutions to (5.12)–(5.13) must be radially symmetric as long as $\beta \leq 8$.

Another example is the following equation from the study of self- gravitating strings for a massive W-boson model coupled to Einstein theory in account of gravitational effects ([42,46]).

$$\Delta v+(1+|y|^{2l})e^v=0 \ \text{ in } \ \mathbb{R}^2, \tag{5.14}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}(1+|y|^{2l})e^v dy=\beta, \tag{5.15}$$

where $l > 0$. It is shown in [42] that (5.14)–(5.15) admit a radial solution if and only if

$$\beta\in\big(4\max\{1,l\},4(l+1)\big)$$

and the corresponding radial solution is unique. Furthermore, for $0 < l \leq 1$, the interval above is also optimal for the solvability of (5.14)–(5.15) among non-radial functions. The main known difference between (5.14) and (5.12) is that the latter possesses radial solutions for a larger range of β which is at least $(2l+2, 4l+4)$, and has multiple radial solutions when $l > 2$ and $\beta \in (\beta_l, 4l)$ for some $\beta_l \in (2l+2, 2l+4)$, which also implies the existence of non-radial solutions for (5.12) for $l > 2$ (see [23,36]). While the former has a radial solution only for $\beta \in \big(4\max\{1,l\}, 4(l+1)\big)$, which is also unique. In particular, no non-radial solution is known in this case.

Theorem 5.2 implies that solutions to (5.14)–(5.15) must be radially symmetric when $\beta \leq 8$. As a consequence, the solvability range of β among non-radial functions must be $\beta > 4\max\{1, l\}$ when $l \leq 2$.

Now we are ready to consider (1.8).

Proof of Theorem 1.3. Let u be a solution of (1.8) and $\bar{\alpha} \neq 0$. Without loss of generality we may assume $P = (0, 0, 1)$. Now let $\Pi : S^2 \to R^2$ be the stereographic projection with north pole at $P = (0, 0, 1)$. Similar to (2.10) in [6], we define

$$v(y) = u(\Pi^{-1}y) - \ln\left(\int_{S^2} e^u d\omega_2\right) + \ln(4\lambda) - \left(\frac{\lambda}{4\pi} - \bar{\alpha}\right)\ln(1 + |y|^2), \tag{5.16}$$

where $y = \Pi(x)$. Then u is a solution of (1.8) if and only if v satisfies

$$\begin{cases} \Delta v + (1 + |y|^2)^{\frac{\lambda}{4\pi} - \bar{\alpha} - 2} e^v = 0 \text{ in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} (1 + |y|^2)^{\frac{\lambda}{4\pi} - \bar{\alpha} - 2} e^v dy = \lambda. \end{cases}$$

Due to the uniqueness result for $\lambda \in (0, 8\pi(1 + \alpha_-))$ and nonexistence result for $\lambda \in [8\pi(1 + \alpha_-), 8\pi(1 + \alpha_+)]$, we only need to consider the case when $\lambda > 8\pi(1 + \alpha_+)$. Since $\frac{\lambda}{4\pi} - \bar{\alpha} - 2 > 0$ in this case, it follows from Theorem 5.2 that v is radially symmetric about the origin. Hence u is axially symmetric with respect to $\overrightarrow{OP}$ and the proof is complete. □

We note that Theorem 1.4 follows immediately from Theorem 1.3.

Proof of Theorem 1.5. Without loss of generality we may assume that $\mathbf{n} = (0, 0, 1)$. Let $\Pi : S^2 \to R^2$ be the stereographic projection with north pole at $\mathbf{n} = (0, 0, 1)$. Define

$$v(y) = u(\Pi^{-1}(y)) \text{ for } y \in \mathbb{R}^2.$$

Then v satisfies

$$\Delta v + \frac{J^2(y)\exp(\tilde{\alpha}v - \gamma\psi(y))}{\int_{\mathbb{R}^2} J^2(y)\exp(\tilde{\alpha}v - \gamma\psi(y))dy} - \frac{J^2(y)}{4\pi} = 0 \text{ for } y \in \mathbb{R}^2, \tag{5.17}$$

where

$$J(y) = \frac{2}{1 + |y|^2} \text{ and } \psi(y) = \frac{|y|^2 - 1}{|y^2| + 1}.$$

Now define

$$w(y):=\tilde{\alpha}\left(v(y)-\frac{1}{4\pi}\ln(1+|y|^2)\right)-c,$$

with

$$c=\gamma+\ln\left(\frac{2}{\tilde{\alpha}}\int_{\mathbb{R}^2}J^2(y)e^{\tilde{\alpha}v-\gamma\psi}\right).$$

Then we have

$$\Delta w(y)+K(y)e^{w}=0 \text{ in } \mathbb{R}^2, \tag{5.18}$$

and

$$\int_{\mathbb{R}^2}K(y)e^{w(y)}dy=\tilde{\alpha},$$

where

$$K(y)=8(1+|y|^2)^{\left(-2+\frac{\tilde{\alpha}}{4\pi}\right)}e^{\gamma J(y)}. \tag{5.19}$$

Now we compute

$$\Delta(\ln K(y))=\frac{4(-2+\frac{\tilde{\alpha}}{4\pi})}{(1+|y|^2)^2}+\frac{8\gamma(|y|^2-1)}{(1+|y|^2)^3}.$$

Since the right hand side of the above equation is nonnegative for $0\leq\gamma\leq\frac{\tilde{\alpha}}{8\pi}-1$, it follows from Theorem 5.2 that w is radially symmetric about the origin. □

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