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Periodic solutions of a semilinear elliptic equation with a fractional Laplacian

Changfeng Gui, Jie Zhang and Zhuoran Du

Dedicated to Professor Paul Rabinowitz with gratitude and admiration.

Abstract. We consider periodic solutions of the following problem associated with the fractional Laplacian

$$(-\partial_{xx})^s u(x) + F'(u(x)) = 0, \qquad u(x) = u(x+T), \qquad \text{in } \mathbb{R},$$

where $(-\partial_{xx})^s$ denotes the usual fractional Laplace operator with $0 < s < 1$. The primitive function F of the nonlinear term is a smooth double-well potential. We prove the existence of periodic solutions with large period T using variational methods. An estimate of the energy of the periodic solutions is also established.

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1. Introduction

We consider the following problem involving the fractional Laplacian

$$(-\partial_{xx})^s u(x) + F'(u(x)) = 0, \qquad \text{in } \mathbb{R}, \tag{1}$$

where $(-\partial_{xx})^s$, $s \in (0,1)$, denotes the usual fractional Laplace operator, a Fourier multiplier of symbol $|\xi|^{2s}$. The function F is a smooth double-well potential with wells at $+1$ and -1, namely, it satisfies

$$\begin{cases} F(1) = F(-1) = 0 < F(u), & \forall -1 < u < 1, \\ F'(1) = F'(-1) = 0. \end{cases} \tag{2}$$

We also assume that

$$F \text{ is nondecreasing in } (-1,0) \text{ and nonincreasing in } (0,1). \tag{3}$$

Note that conditions (2), (3) mean that $F(0) = \max_{-1\le u\le 1} F(u) > 0$. We may also assume that F is even. A typical example of such double-well even function is $F(u) = \frac{(1-u^2)^2}{4}$. We investigate the existence of odd periodic solutions to Eq. (1), namely the solutions satisfying

$$u(x) = u(x+T), \qquad u(-x) = -u(x), \quad \text{in } \mathbb{R}.$$

We also consider the case that F only satisfies conditions (2), (3), namely, it is not necessary an even function. In this case, we will obtain the existence of periodic solutions (not necessary odd) to Eq. (1).

For the local differential operator case $s = 1$, the corresponding form of Eq. (1) is

$$-u''(x) + F'(u(x)) = 0, \qquad \text{in } \mathbb{R}. \tag{4}$$

A solution u is said to be a layer solution of problem (4) if it satisfies

$$u'(x) > 0, \ \lim_{x\to-\infty} u(x) = -1, \quad \lim_{x\to+\infty} u(x) = 1.$$

The authors in [3] proved that conditions (2) are both necessary and sufficient for the existence of a layer solution to problem (4).

As for periodic solutions to problem (4), one can easily obtain the existence of periodic solutions with large periods. More precisely, there exists $T_0 > 0$ such that problem (4) admits nonconstant periodic solutions with period $T > T_0$. Indeed, for the specific nonlinear function $F(u) = \frac{(1-u^2)^2}{4}$, one can show that $T_0 = 2\pi$(see [1]). For general case F satisfying (2), furthermore assuming $F''(0) < 0$, we can show that $T_0 = 2\pi \times \frac{1}{\sqrt{-F''(0)}}$.

For the fractional Laplacian case, the authors in [4] and [5] proved that (2) are also the necessary and sufficient conditions for the existence of a layer solution to Eq. (1). Moreover, they prove that such layer solution is unique and establish asymptotic behavior of the layer solution. These results also have been proven with different techniques in [8].

What about periodic solutions to Eq. (1)? Plainly, Eq. (1) possesses three constant periodic solutions $u = 1, \ -1, \ 0$, under the conditions (2), (3). In this paper, we will try to find nonconstant periodic solutions.

The fractional Laplace operator $(-\Delta)^s$ can be defined as a Dirichlet-to-Neumann map for a so-called s-harmonic extension problem (see [6]). Given a function ϕ, the solution $\tilde{\phi}$ of the following problem

$$\begin{cases} \operatorname{div}(y^a \nabla \tilde{\phi}) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}, \\ \tilde{\phi}(x,0) = \phi(x) & \text{on } \mathbb{R}^n \end{cases}$$

is called the s-harmonic extension of ϕ. The parameter a is related to the power s of the fractional Laplacian $(-\Delta)^s$ by the formula $a = 1-2s \in (-1,1)$. One has

$$\tilde{\phi}(x,y) = \int_{\mathbb{R}^n} p_s(x-z,y)\phi(z)\mathrm{d}z = \int_{\mathbb{R}^n} p_s(z,y)\phi(x-z)\mathrm{d}z, \tag{5}$$

where $p_s(x,y)$ is the s-Poisson kernel

$$p_s(x,y) = C_{n,s}\frac{y^{2s}}{(|x|^2+|y|^2)^{\frac{n+2s}{2}}},$$

and $C_{n,s}$ is the constant which makes $\int_{\mathbb{R}^n} p_s(x,y)\mathrm{d}x = 1$. The authors in [6] proved that

$$(-\Delta)^s \phi(x) = \frac{\partial \tilde{\phi}}{\partial \nu^a} \qquad \text{in } \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+,$$

where

$$\frac{\partial \tilde{\phi}}{\partial \nu^a} := -\lim_{y \downarrow 0} y^a \frac{\partial \tilde{\phi}}{\partial y}.$$

From (5) and the above formula of s-Poisson kernel, we can easily deduce that the s-harmonic extension $\tilde{u}(x,y)$ of an odd periodic solution $u(x)$ to (1) is also odd and periodic with respect to x with the same period of u.

Equation (1) can be realized in a local manner through the nonlinear boundary value problem

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y > 0\}, \\ \frac{\partial U}{\partial \nu^a} = -F'(U) & \text{on } \mathbb{R}. \end{cases} \tag{6}$$

Problem (6) is related to Eq. (1) in the sense that, if U is a solution of (6), then a positive constant multiple of $u(x) := U(x,0)$ satisfies (1). Then, to obtain the existence of (odd) periodic solutions of Eq. (1), we need to construct (odd) periodic solutions with respect to x of (6). We will obtain such existence by variational methods. Note that (6) is the Euler–Lagrangian equation of a functional J which will be defined in (8) in the next section. In particular, for the case that F is even, we will find odd periodic solutions which minimize the energy functional J. For the case that F is not necessarily even, we will find periodic solutions by using the mountain pass theorem.

Our main results are the followings.

Theorem 1.1. *Let $s \in (0,1)$. Assume F satisfies the assumptions (2)–(3) and is even. Then there exists $T_1 > 0$ such that for any $T > T_1$, Eq. (1) admits an odd periodic solution u_T with period T, and $u_T(x) \in (0,1)$ for $x \in (0, \frac{T}{2})$. Moreover, for any positive number $\sigma < \frac{1}{2}$, there exists $T_\sigma \geq T_1$ such that for any $T > T_\sigma$, we have*

$$J(U_T, \Omega_T) < \sigma F(0) T. \tag{7}$$

Here U_T is the s-harmonic extension of u_T.

Theorem 1.2. *Let $s \in (0,1)$. Assume F satisfies the assumptions (2), (3) and $F''(\pm 1) > 0$. Then there exists $T_2 > 0$ such that for any $T > T_2$, Eq. (1) admits a periodic solution u_T with period T, $|u_T(x)| < 1$ in $\mathbb{R}$ and it changes its sign at least once in a period. Moreover, for any $\sigma < \frac{1}{2}$, there exists $T_\sigma \geq T_2$ such that for any $T > T_\sigma$, the inequality (7) also holds true.*

2. Proof of Theorem 1.1

We assume that F satisfies conditions (2), (3) and is even.

Prior to the proof of Theorem 1.1, we introduce a weighted Poincaré inequality in [7], where the authors first give the definition of class S. An open bounded set $D \subset \mathbb{R}^n$ is said to belong to the class S, if there exist α and ρ_0 positive such that for each $\hat{x} \in \partial D$ and for each $\rho < \rho_0$ one has $|B_\rho(\hat{x})\backslash D| \geq \alpha|B_\rho(\hat{x})|$. They establish the following weighted Poincaré inequality.

Proposition 2.1. [7] *Assume D belongs to the class S. Then for all $\hat{x} \in \partial D$, $0 < \rho < \rho_0$, and all $u \in C^1(\overline{B_\rho(\hat{x}) \cap D})$ vanishing on $B_\rho(\hat{x}) \cap \partial D$,*

$$\int_{B_\rho(\hat{x})\cap D} x_n^a u^2(x)\mathrm{d}x \leq C \int_{B_\rho(\hat{x})\cap D} x_n^a |\nabla u(x)|^2 \mathrm{d}x,$$

where $a \in (-1, 1)$ and the positive constant C depends only on ρ, n and a.

Proof of Theorem 1.1. For given positive T, we denote

$$\Omega_T := \left[0, \frac{T}{2}\right] \times [0, +\infty).$$

Problem (6) corresponds to an energy functional

$$J(U, \Omega_T) := \frac{1}{2}\int_{\Omega_T} y^a|\nabla U(x, y)|^2 \mathrm{d}x\mathrm{d}y + \int_0^{\frac{T}{2}} F(U(x, 0))\mathrm{d}x. \tag{8}$$

We denote the admissible set of the energy J as

$$\Lambda_T := \left\{U : 0 \leq U \leq 1, U(0, y) = 0 = U\left(\frac{T}{2}, y\right) \text{ for all } y \geq 0, U \in H^1(\Omega_T, y^a)\right\}.$$

Here

$$H^1(\Omega_T, y^a) := \{U : y^a(U^2 + |\nabla U|^2) \in L^1(\Omega_T)\}.$$

Note that $J(U, \Omega_T) \geq 0$. On the other hand, we have that $0 \in \Lambda_T$ and $J(0, \Omega_T) = F(0)\frac{T}{2} < +\infty$. Hence, there exists a minimizing sequence $\{U_k\} \subseteq \Lambda_T$ of J, namely

$$\lim_{k\to\infty} J(U_k, \Omega_T) = m_T := \inf_{U\in\Lambda_T} J(U, \Omega_T).$$

Due to $F(u) \geq 0$ for $|u| \leq 1$, from the definition of J, we have

$$\int_{\Omega_T} y^a|\nabla U_k(x, y)|^2 \mathrm{d}x\mathrm{d}y \leq 2m_T. \tag{9}$$

From this, Proposition 2.1 and the fact that U_k is bounded, we obtain

$$\int_{\Omega_T} y^a U_k^2(x, y)\mathrm{d}x\mathrm{d}y \leq C < +\infty, \quad \forall\ k. \tag{10}$$

From (9), (10) we deduce that there exists a subsequence of $\{U_k\}$, still denoted as $\{U_k\}$, converging weakly in $H^1(\Omega_T, y^a)$ to a function $U_T \in H^1(\Omega_T, y^a)$. Due to the weak convergence we obtain that

$$\int_{\Omega_T} y^a|\nabla U_T(x, y)|^2 \mathrm{d}x\mathrm{d}y \leq \liminf_{k\to\infty} \int_{\Omega_T} y^a|\nabla U_k(x, y)|^2 \mathrm{d}x\mathrm{d}y.$$

By Fatou's Lemma, we also have

$$\int_0^{\frac{T}{2}} F(U_T(x, 0))\mathrm{d}x \leq \liminf_{k\to\infty} \int_0^{\frac{T}{2}} F(U_k(x, 0))\mathrm{d}x.$$

Hence $J(U_T, \Omega_T) \leq m_T$. Note that the set Λ_T is weakly closed, so $U_T \in \Lambda_T$. Then $J(U_T, \Omega_T) = m_T$, namely U_T is a minimizer of $J(U, \Omega_T)$ in Λ_T. For any given $\eta \in \Lambda_T$ satisfying $U_T + \sigma\eta \in \Lambda_T$ for all small $|\sigma|$, we construct a real-valued function

$$\varphi(\sigma) := J(U_T + \sigma\eta, \Omega_T).$$

Then

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\varphi(\sigma)|_{\sigma=0} \\ &= \int_{\Omega_T} y^a \nabla U_T(x,y) \cdot \nabla\eta(x,y)\mathrm{d}x\mathrm{d}y + \int_0^{\frac{T}{2}} F'(U_T(x,0))\eta(x,0)\mathrm{d}x \\ &= -\int_{\Omega_T} \eta \mathrm{div}(y^a \nabla U_T)\mathrm{d}x\mathrm{d}y + \int_0^{\frac{T}{2}} \left[\frac{\partial U_T}{\partial \nu^a} + F'(U_T(x,0))\right]\eta(x,0)\mathrm{d}x. \end{aligned}$$

Hence, by the arbitrariness of η, we obtain

$$\begin{cases} \mathrm{div}(y^a \nabla U_T) = 0 & \text{in } \Omega_T, \\ \frac{\partial U_T}{\partial \nu^a} = -F'(U_T) & \text{on } (\partial\Omega_T)_0 := [0, \frac{T}{2}]. \end{cases}$$

Now, we extend U_T oddly (with respect to x) from Ω_T to $[-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty)$. Furthermore, we extend it periodically (with respect to x again) from $[-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty)$ to the whole half space $\overline{\mathbb{R}^2_+}$, and we still denote it as U_T. Then U_T is a weak solution of (6).

The remaining task is to prove that $U_T \not\equiv 0$. For $\sigma \in (0, \frac{1}{2})$, we define the following continuous function

$$h(x) := \begin{cases} \frac{4}{\sigma T}x, & x \in [0, \frac{\sigma T}{4}], \\ 1, & x \in [\frac{\sigma T}{4}, \frac{T}{2} - \frac{\sigma T}{4}], \\ \frac{2}{\sigma} - \frac{4}{\sigma T}x, & x \in [\frac{T}{2} - \frac{\sigma T}{4}, \frac{T}{2}]. \end{cases}$$

Note that $0 \leq h \leq 1$. Then we construct a function $\psi \in \Lambda_T$ as follows

$$\psi(x,y) = \exp\left\{-\frac{y}{2^{b+1}}\right\} h(x),$$

where the parameter b will be determined later. We next compute the energy $J(\psi, \Omega_T)$. From conditions (2), (3) of F, we have

$$\int_0^{\frac{T}{2}} F(\psi(x,0))\mathrm{d}x = \int_0^{\frac{T}{2}} F(h(x))\mathrm{d}x < F(0)\frac{\sigma T}{2}. \tag{11}$$

For the other part of the energy, we have

$$\begin{aligned}
&\int_0^{\frac{T}{2}}\int_0^{\infty} y^a|\nabla\psi(x,y)|^2\mathrm{d}x\mathrm{d}y\\
&\quad=\int_0^{\infty} y^a\exp\left\{-\frac{y}{2^b}\right\}\mathrm{d}y\int_0^{\frac{T}{2}}\left[\frac{h^2(x)}{2^{2b+2}}+(h'(x))^2\right]\mathrm{d}x\\
&\quad\le\left[\frac{1}{2^{2b}}\frac{T}{8}+\frac{8}{\sigma T}\right]\int_0^{\infty} y^a\exp\left\{-\frac{y}{2^b}\right\}\mathrm{d}y\\
&\quad=2^{b(a+1)}\left[\frac{1}{2^{2b}}\frac{T}{8}+\frac{8}{\sigma T}\right]\int_0^{\infty} z^a e^{-z}\mathrm{d}z\\
&\quad=\Gamma(a+1)2^{b(a-1)}\left[\frac{T}{8}+2^{2b}\frac{8}{\sigma T}\right].
\end{aligned}$$

Note that $a-1<0$, for the purpose that the term $2^{b(a-1)}\Gamma(a+1)$ is small, we can choose sufficiently large b. For chosen b, the other term $2^{2b}\frac{8}{\sigma T}$ is also small provided that T is large enough. Hence there exists $T_1>0$ such that for any $T>T_1$, the following estimate holds true

$$\int_0^{\frac{T}{2}}\int_0^{\infty} y^a|\nabla\psi(x,y)|^2\mathrm{d}x\mathrm{d}y<(1-\sigma)F(0)\frac{T}{2}. \tag{12}$$

From (11), (12), we have

$$J(0,\Omega_T)=F(0)\frac{T}{2}>J(\psi,\Omega_T)\ge J(U_T,\Omega_T),$$

which shows that $U_T\not\equiv 0$.

Now we set

$$u_T(x):=U_T(x,0),$$

then u_T is an odd periodic solution of Eq. (1). In view of $0\le U_T|_{\Omega_T}\le 1$ and $U_T|_{\Omega_T}\not\equiv 0$ or 1, a Hopf principle in [4] shows that $U_T(x,0)=u_T(x)\in(0,1)$ in $(0,\frac{T}{2})$, where we used the fact that $F'(0)=F'(1)=0$.

Indeed, by choosing larger b and T, we can improve the inequality (12) as follows

$$\int_0^{\frac{T}{2}}\int_0^{\infty} y^a|\nabla\psi(x,y)|^2\mathrm{d}x\mathrm{d}y<F(0)\frac{\sigma T}{2},\quad \text{for any } T>T_\sigma, \tag{13}$$

where $T_\sigma\ge T_1$ and $\lim_{\sigma\to 0}T_\sigma\to+\infty$. From (13) and (11), we obtain

$$J(\psi,\Omega_T)<\sigma F(0)T.$$

This and the minimality of U_T with respect to J yield estimate (7). □

3. Proof of Theorem 1.2

We assume that F satisfies conditions (2), (3) and is not necessary even.

Proof of Theorem 1.2. We now define the Hilbert space as

$$\mathcal{H} := \left\{ U : |U| \le 1,\ \|U\|_{\mathcal{H}}^2 := \int_{\overline{\Omega}_T} y^a |\nabla U(x,y)|^2 \mathrm{d}x\mathrm{d}y + \int_{-\frac{T}{2}}^{\frac{T}{2}} U^2(x,0))\mathrm{d}x < \infty \right\},$$

where

$$\overline{\Omega}_T := \left[-\frac{T}{2}, \frac{T}{2}\right] \times [0, +\infty).$$

We consider the corresponding energy functional

$$J(U, \overline{\Omega}_T) = \frac{1}{2} \int_{\overline{\Omega}_T} y^a |\nabla U(x,y)|^2 \mathrm{d}x\mathrm{d}y + \int_{-\frac{T}{2}}^{\frac{T}{2}} F(U(x,0))\mathrm{d}x.$$

Plainly, $J \in C^1(\mathcal{H}, \mathbb{R})$, since F is smooth. Next we verify the Palais–Smale condition. Namely, for any sequence $U_k \in \mathcal{H}$ with

$$J(U_k, \overline{\Omega}_T) \text{ bounded}$$

and

$$J'(U_k, \overline{\Omega}_T) \to 0 \quad \text{in } \mathcal{H},$$

it contains a convergent subsequence. Similar estimates as (9), (10) yield that there exists a subsequence of $\{U_k\}$, still denoted as $\{U_k\}$, converging weakly to a function $\bar{U}$ in $H^1(\overline{\Omega}_T, y^a)$. In view of

$$H^1(\overline{\Omega}_T, y^a) \hookrightarrow H^s\left(-\frac{T}{2}, \frac{T}{2}\right) \hookrightarrow\hookrightarrow L^2\left(-\frac{T}{2}, \frac{T}{2}\right),$$

we have

$$U_k(x,0) \to \bar{U}(x,0) \quad \text{in } L^2\left(-\frac{T}{2}, \frac{T}{2}\right). \tag{14}$$

Note that

$$\begin{aligned} &\int_{\overline{\Omega}_T} y^a |\nabla (U_k(x,y) - \bar{U}(x,y))|^2 \mathrm{d}x\mathrm{d}y \\ &\quad = \langle J'(U_k, \overline{\Omega}_T) - J'(\bar{U}, \overline{\Omega}_T), U_k - \bar{U}\rangle \\ &\qquad - \int_{-\frac{T}{2}}^{\frac{T}{2}} [F'(U_k(x,0)) - F'(\bar{U}(x,0))](U_k(x,0)) - \bar{U}(x,0))\mathrm{d}x. \end{aligned}$$

Clearly,

$$\langle J'(U_k, \overline{\Omega}_T) - J'(\bar{U}, \overline{\Omega}_T), U_k - \bar{U}\rangle \to 0.$$

We also have

$$\begin{aligned} &\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} [F'(U_k(x,0)) - F'(\bar{U}(x,0))](U_k(x,0)) - \bar{U}(x,0))\mathrm{d}x \right| \\ &\quad \le C \int_{-\frac{T}{2}}^{\frac{T}{2}} |U_k(x,0) - \bar{U}(x,0)|^2 \mathrm{d}x \to 0, \end{aligned}$$

where the convergence result follows from (14). Hence

$$\int_{\overline{\Omega}_T} y^a |\nabla(U_k(x,y) - \bar{U}(x,y))|^2 \mathrm{d}x\mathrm{d}y \to 0.$$

This and (14) give that

$$U_k \to \bar{U} \text{ in } \mathcal{H}.$$

We have obtained the Palais–Smale condition.

We set

$$\mathbf{\Gamma} := \{g \in C([0,1];\mathcal{H}) : g(0) = -1,\ g(1) = 1\}.$$

Note that

$$J(1, \overline{\Omega}_T) = J(-1, \overline{\Omega}_T) = 0 \le J(v, \overline{\Omega}_T), \qquad \forall\, v \in \mathcal{H},$$

and J is stable at 1 and -1, namely

$$\int_{\overline{\Omega}_T} y^a |\nabla\varphi|^2 \mathrm{d}x\mathrm{d}y + \int_{-\frac{T}{2}}^{\frac{T}{2}} F''(\pm 1)\varphi^2(x,0)\mathrm{d}x > 0, \quad \text{for all } \varphi \ne 0 \in \mathcal{H}.$$

Hence we have

$$\delta_T := \inf_{g\in\mathbf{\Gamma}} \sup_{t\in[0,1]} J(g(t), \overline{\Omega}_T) > 0.$$

We set

$$J(U_T, \overline{\Omega}_T) = \delta_T, \qquad \text{where } U_T = g(t_0) \text{ for some } g \in \mathbf{\Gamma} \text{ and some } t_0 \in (0,1).$$

Now we extend U_T periodically (with respect to x) from $\overline{\Omega}_T$ to the whole half space $\overline{\mathbb{R}^2_+}$, and we still denote it as U_T. From the mountain pass theory in [2], we know that U_T is a weak solution of (6).

Next we show that $U_T \not\equiv 0$. We choose a similar function $\psi \in \mathcal{H}$ as in the above section

$$\psi(x,y) = \exp\left\{-\frac{y}{2^{b+1}}\right\} \tilde{h}(x),$$

where $\tilde{h}$ is the odd extension of the function h (defined in the above section) from $(0, \frac{T}{2})$ onto $(-\frac{T}{2}, \frac{T}{2})$. We construct a path as

$$\bar{g}(t) = \begin{cases} 2t\psi + (1-2t)(-1), & \text{for } 0 \le t \le \frac{1}{2}, \\ (2-2t)\psi + (2t-1), & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Clearly, $\bar{g} \in \mathbf{\Gamma}$. We have

$$\int_{\overline{\Omega}_T} y^a |\nabla \bar{g}|^2 \mathrm{d}x\mathrm{d}y \le \int_{\overline{\Omega}_T} y^a |\nabla\psi(x,y)|^2 \mathrm{d}x\mathrm{d}y.$$

Note that

$$\bar{g}(t) = \begin{cases} 2t\psi + (1-2t)(-1) \in [-1, \psi], & \text{for } 0 \le t \le \frac{1}{2}, \\ (2-2t)\psi + (2t-1) \in [\psi, 1], & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

We denote $\bar{g}(t)$ as $\bar{g}_t(x,y)$ to emphasize the dependence of $\bar{g}$ on (x,y). Then for $0 \le t \le \frac{1}{2}$, we have

$$\begin{aligned}\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\bar{g}_t(x,0))\mathrm{d}x &= \int_{-\frac{T}{2}}^{0} F(\bar{g}_t(x,0))\mathrm{d}x + \int_{0}^{\frac{T}{2}} F(\bar{g}_t(x,0))\mathrm{d}x \\ &\le \int_{-\frac{T}{2}}^{0} F(\psi(x,0))\mathrm{d}x + F(0)\frac{T}{2}.\end{aligned}$$

Similarly, for $\frac{1}{2} \le t \le 1$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\bar{g}_t(x,0))\mathrm{d}x \le \int_{0}^{\frac{T}{2}} F(\psi(x,0))\mathrm{d}x + F(0)\frac{T}{2}.$$

Hence for any $t \in [0,1]$, one has

$$\begin{aligned}J(\bar{g}_t, \overline{\Omega}_T) \ &\le \frac{1}{2}\int_{\overline{\Omega}_T} y^a |\nabla\psi(x,y)|^2 \mathrm{d}x\mathrm{d}y \\ &\quad + \max\left\{\int_{-\frac{T}{2}}^{0} F(\psi(x,0))\mathrm{d}x, \int_{0}^{\frac{T}{2}} F(\psi(x,0))\mathrm{d}x\right\} + F(0)\frac{T}{2}.\end{aligned} \tag{15}$$

Then similar computation as (11) and (12) in the preceding section shows that there exists $T_2 > 0$ such that for $T > T_2$ we have

$$J(\bar{g}_t, \overline{\Omega}_T) < F(0)T = J(0, \overline{\Omega}_T), \text{ for } \forall\ t \in [0,1].$$

Hence

$$J(U_T, \overline{\Omega}_T) = \delta_T \le \max_{t\in[0,1]} J(\bar{g}_t, \overline{\Omega}_T) < J(0, \overline{\Omega}_T),$$

which gives that $U_T \not\equiv 0$. Then $u_T(x) := U_T(x,0)$ is the desired solution of Eq. (1). Plainly $u_T(x)$ must change its sign. Hence $u_T(x)$ change its sign at least once in a period. A Hopf principle in [4] gives again that $|u_T(x)| = |U_T(x,0)| < 1$.

Finally, we show estimate (7). To this end, for any given integer $m > 1$, we define $2m-1$ continuous functions $h_i (1 \le i \le 2m-1)$ as follows

$$h_i(x) = \begin{cases} -\frac{8m}{T}x + 4m, & \text{for } x \in [\frac{T}{2} - \frac{T}{8m}, \frac{T}{2}], \\ 1, & \text{for } x \in [\frac{T}{2} - \frac{iT}{2m} + \frac{T}{8m}, \frac{T}{2} - \frac{T}{8m}], \\ \frac{8m}{T}x - 4(m-i), & \text{for } x \in [\frac{T}{2} - \frac{iT}{2m} - \frac{T}{8m}, \frac{T}{2} - \frac{iT}{2m} + \frac{T}{8m}], \\ -1 & \text{for } x \in [-\frac{T}{2} + \frac{T}{8m}, \frac{T}{2} - \frac{iT}{2m} - \frac{T}{8m}], \\ -\frac{8m}{T}x - 4m, & \text{for } x \in [-\frac{T}{2}, -\frac{T}{2} + \frac{T}{8m}]. \end{cases}$$

Note that $h_i \in [0,1]$. Similarly we define $\psi_i \in \mathcal{H}(1 \le i \le 2m-1)$

$$\psi_i(x,y) = \exp\left\{-\frac{y}{2^{b+1}}\right\} h_i(x).$$

Now we construct a path as

$$\hat{g}(t) = \begin{cases} 2mt\psi_1 + (1-2mt)(-1), & \text{for } 0 \le t \le \frac{1}{2m}, \\ ((i+1)-2mt)\psi_i + (2mt-i)\psi_{i+1}, & \text{for } \frac{i}{2m} \le t \le \frac{i+1}{2m}, \\ & \quad 1 \le i \le 2m-2, \\ (2m-2mt)\psi_{2m-1} + (2mt-(2m-1)), & \text{for } \frac{2m-1}{2m} \le t \le 1. \end{cases}$$

Clearly, $\hat{g} \in \mathbf{\Gamma}$.

For $0 \le t \le \frac{1}{2m}$, from the definitions of $\hat{g}$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x = \int_{\frac{T}{2}-\frac{T}{2m}-\frac{T}{8m}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x \le F(0)\frac{5T}{8m}.$$

Similarly, for $\frac{2m-1}{2m} \le t \le 1$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x = \int_{0}^{\frac{T}{2m}+\frac{T}{8m}} F(\hat{g}_t(x,0))\mathrm{d}x \le F(0)\frac{5T}{8m}.$$

For the cases $\frac{i}{2m} \le t \le \frac{i+1}{2m}(1 \le i \le 2m-2)$, we have

$$\begin{aligned} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x &= \int_{0}^{\frac{T}{8m}} F(\hat{g}_t(x,0))\mathrm{d}x + \int_{\frac{T}{2}-\frac{T}{8m}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x \\ &\quad + \int_{\frac{T}{2}-\frac{(i+1)T}{2m}-\frac{T}{8m}}^{\frac{T}{2}-\frac{iT}{2m}+\frac{T}{8m}} F(\hat{g}_t(x,0))\mathrm{d}x \\ &\le F(0)\frac{T}{m}. \end{aligned}$$

Therefore,

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\hat{g}_t(x,0))\mathrm{d}x \le F(0)\frac{T}{m}, \ \forall\, t \in [0,1]. \tag{16}$$

For the other part of the energy, we have

$$\int_{\Omega_T} y^a|\nabla \hat{g}|^2 \mathrm{d}x\mathrm{d}y \le \max_{1\le i\le 2m-1} 2\int_{\Omega_T} y^a|\nabla\psi_i(x,y)|^2\mathrm{d}x\mathrm{d}y.$$

Similarly, by choosing enough large b and T, we obtain

$$\max_{1\le i\le 2m-1} 2\int_{\Omega_T} y^a|\nabla\psi_i(x,y)|^2\mathrm{d}x\mathrm{d}y \le F(0)\frac{T}{m}, \quad \text{for any } T > T_m, \tag{17}$$

where $T_m \ge T_2$ and $\lim_{m\to+\infty} T_m \to +\infty$. Inequalities (16), (17) give that

$$\max_{t\in[0,1]} J(\hat{g}_t, \overline{\Omega}_T) \le F(0)\frac{2T}{m}, \quad \text{for any } T > T_m.$$

Hence for any $0 < \sigma < \frac{1}{2}$, we can take large $m = m(\sigma)$ such that

$$J(U_T, \overline{\Omega}_T) \le \max_{t\in[0,1]} J(\hat{g}_t, \overline{\Omega}_T) \le F(0)\frac{2T}{m} < \sigma F(0)T, \quad \text{for any } T > T_\sigma,$$

which is the desired estimate (7). Here $T_\sigma := T_{m(\sigma)} \to +\infty$ as $\sigma \to 0$. $\square$

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Changfeng Gui, Jie Zhang and Zhuoran Du
College of Mathematics and Econometrics
Hunan University
Changsha 410082
People's Republic of China

Changfeng Gui
Department of Mathematics
University of Texas at San Antonio
San Antonio TX 78249
USA
e-mail: changfeng.gui@utsa.edu