# Hybrid Zero Dynamics of Bipedal Robots Under Nonholonomic Virtual Constraints

Jonathan C. Horn, Alireza Mohammadi, Kaveh Akbari Hamed, and Robert D. Gregg, IV

***Abstract*—This letter investigates the hybrid zero dynamics for planar bipedal robots with one degree of underactuation subject to nonholonomic virtual constraints (NHVCs). We first derive the closed-form expression of the bipedal robot zero dynamics under NHVCs. We next present conditions that make the NHVCs invariant with respect to rigid impacts with the ground. Lastly, a reduced dimensionality test, which is independent of the number of degrees of freedom of the bipedal robot, is proposed for checking existence and exponential stability of hybrid periodic orbits under NHVCs. Simulation results using the RABBIT biped robot demonstrate the robustness of the proposed NHVCs against a randomized horizontal push disturbance. A statistical significant difference between the mean number of steps until failure is shown between the NHVC and virtual holonomic constraint control schemes.**

***Index Terms*—Robotics, hybrid systems, nonlinear output feedback.**

## I. Introduction

SINCE their introduction, virtual holonomic constraints (VHCs) have been extensively used for the motion control of biped robots [1]–[6] and powered transfemoral prostheses [7]–[10]. A VHC is a functional relationship of the configuration variables for a mechanical system that can be made invariant via feedback. However, there are some limitations of VHCs; for example, trajectories must be parameterized by a monotonic variable, and kinematic patterns cannot be adjusted in response to large perturbations or environmental changes. A more general class of virtual constraints that depend on configuration velocities, known as nonholonomic virtual constraints (NHVCs), has recently been introduced in [11] and [12]. Prior to [11] and [12], the literature on controlled geometric reduction introduced a form of NHVCs with outputs of relative degree one [13], [14].

Motivated by the recent results of [11] and [12] and the potential kinematic adaptability with NHVCs, this letter investigates the dynamics that result from applying relative degree two NHVCs to biped robots with one degree of underactuation. In particular, we study the zero dynamics of bipedal robots under NHVCs, which depend on the configuration variables as well as the momentum conjugate to the biped unactuated degree of freedom. First, we derive a closed form expression for the bipedal robot swing phase zero dynamics. Next, we present a set of algebraic conditions that ensure that the biped states remain on the zero dynamics manifold after each impact with the ground. Finally, we introduce a reduced order dimensionality test for the resulting system subject to NHVCs. Using this formulation we simulate the RABBIT biped robot (see [1], [6]) under a disturbance that violates the monotonicity assumption under VHCs but recovers under the proposed NHVCs.

*Contributions of This Letter:* Relative degree two NHVCs were first introduced in [11] and experimentally tested in [12]. However, an expression for the swing phase zero dynamics was not derived in [11] and [12]. Theorem 1 in this letter complements the results in [11] and [12] by presenting a closed-form expression for the zero dynamics induced by NHVCs. In order to maintain invariance under impacts, the NHVCs in [11] and [12] depend on a dynamic variable, which gets updated after each impact with the ground. The NHVCs presented in this letter do not depend on such dynamic variables. Rather, hybrid invariance is achieved via the proper choice of NHVC parameters, which satisfy the conditions given in Proposition 2. The reduced order dimensionality test afforded by Theorem 2 in this letter can be done via a Poincaré section analysis of a two dimensional dynamical system, independent of the number of degrees-of-freedom (DOFs). This stability test is a generalization of [6, Th. 5.3]. Finally, the hybrid invariance conditions presented in this letter, from a theoretical perspective, are a generalization of the results in the early VHC literature [2], [6]. From an implementation perspective, unlike [11] and [12], there is no need for updating the NHVC parameters after each impact.

The rest of this letter is organized as follows. Section II presents the necessary preliminaries. In Section III the swing phase zero dynamics as well as conditions for hybrid invariance under NHVCs, and a low dimensional stability test are presented. Section IV presents a simulation study, which illustrates the performance of the nonholonomic walking gait under a backward push disturbance. Finally, Section V

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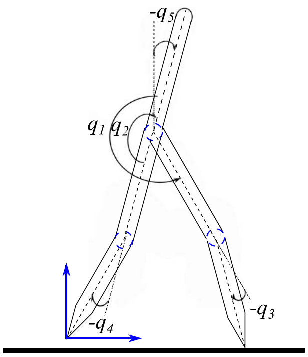


Fig. 1. Schematic of the biped robot RABBIT [1].

provides concluding remarks and potential research for the future.

*Notation:* Given two vectors (matrices) $a$, $b$, we denote by $[a; b]$ the vector (matrix) $[a^T, b^T]^T$ where $(\cdot)^T$ is the transpose operator. Given a matrix $X$, we denote its image by Im($X$) and its kernel by Ker($X$). Given a function $h : \mathcal{X} \to \mathcal{Y}$, we define $h^{-1}(0) := \{x \in \mathcal{X} : h(x) = 0\}$. Given a function $f(\cdot)$ defined on a real interval $\mathcal{I}$, we denote its left and right limits at $x$ by $f^-(x)$ and $f^+(x)$, respectively.

## II. Preliminaries

In this section we present the hybrid dynamics of planar bipedal robots with one degree of underactuation. We also present the notion of periodic solutions and hybrid extension dynamics. Finally, we describe the class of NHVCs, due to Griffin and Grizzle [11], [12], that depend on momenta conjugate to unactuated DOFs. The material in this section is standard and more details can be found in [6], [11], and [12].

### A. Hybrid Dynamics Underactuated Bipedal Robots

*Continuous Dynamics:* During the swing phase, the biped robot dynamics are given by (see [15]),

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu, \tag{1}$$

the Lagrangian, $\mathcal{L} : TQ \to \mathbb{R}$, is $\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - V(q)$, with $K$ and $V$ representing the biped kinetic and potential energy functions, respectively. In (1), $q := [q_1; \ldots ; q_N] \in Q$ is the vector of generalized coordinates, and the configuration space $Q$ is an open and connected subset of $\mathbb{R}^N$. Therefore, the state $x := [q; \dot{q}]$ of the biped belongs to the state space $\mathcal{X} := Q \times \mathbb{R}^N$. The matrix, $B \in \mathbb{R}^{N\times(N-1)}$, is assumed to be constant and of full rank and the vector of torque inputs, $u$, belongs to an open and connected subset of $\mathbb{R}^{N-1}$. This assumption implies that there exists a row vector $B^\perp \in \mathbb{R}^{1\times N}$ such that $B^\perp B = 0$. The unactuated coordinate is then defined by

$$q_u := B^\perp q. \tag{2}$$

Finally, $M(q), C(q, \dot{q})$, and $G(q)$, denote the inertia and Coriolis/centripetal matrices, and the vector of gravitational forces, respectively. Additionally, we assume the following.

**HM1)** The inertia matrix is not a function of the unactuated DOF. That is, $(\partial M/\partial q_u)(q) = 0$ for all $q \in \mathcal{Q}$. Bipedal robots with point feet such as the compass gait biped and the biped robot RABBIT [1] satisfy this assumption.

*Impact Dynamics:* The vertical height from the ground and the horizontal position of the swing leg end, with respect to an inertial coordinate frame, are denoted by $p_2^v(q)$ and $p_2^h(q)$, respectively. Furthermore, the set

$$\mathcal{S} := \{(q, \dot{q}) \in \mathcal{X} : p_2^v(q) = 0, p_2^h(q) > 0\}, \tag{3}$$

is called the switching surface. In this letter, we assume that the leg impacts with the ground are perfectly inelastic [16]. Under this assumption, the impact is modeled by

$$[q^+; \dot{q}^+] = [\Delta_q(q^-); \Delta_{\dot{q}}(q^-)\dot{q}^-], (q^-, \dot{q}^-) \in \mathcal{S}, \tag{4}$$

where $[q^-; \dot{q}^-]$ and $[q^+; \dot{q}^+]$ are the states of the robot just before and after impact, respectively. The overall biped hybrid dynamics (1) – (4) can be described by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u, & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-), & x^- \in \mathcal{S} \end{cases} \tag{5}$$

where $\Delta(x) := [\Delta_q(q); \Delta_{\dot{q}}(q)\dot{q}]$, $g(x) := [0; M^{-1}(q)B]$, and $f(x) := [I_N;\ M^{-1}(q)(-C(q, \dot{q})\dot{q} - G(q))]$.

### B. Periodic Solutions and Hybrid Extension Dynamics

In this section, we review the notion of solutions, periodic orbits, and hybrid restriction dynamics. A function $\phi(t) : [t_0, t_f) \to \mathcal{X}, t_f \in \mathbb{R} \cup \infty, t_f > t_0$, is a **solution** of (5) if (a) $\phi(t)$ is right continuous on $[t_0, t_f)$, (b) the limit exists (from the left and the right) at each point $t \in (t_0, t_f)$; and (c) there exists a closed discrete subset $\mathcal{T} \subset [t_0, t_f)$ called the **set of impact times** such that, (i) for every $t \notin \mathcal{T}, \phi(t) \notin \mathcal{S}$, and (ii) for $t \in \mathcal{T}, \phi^-(t) \in \mathcal{S}$ and $\phi^+(t) = \Delta(\phi^-(t))$. A solution of (5), $\phi : [t_0, \infty) \to \mathcal{X}$, is **periodic** if there exists a finite $T > 0$ such that $\phi(t + T) = \phi(t)$ for all $t \in [t_0, \infty)$. Finally, $\mathcal{O} \subset \mathcal{X}$ is a **periodic orbit** of (5) if $\mathcal{O} = \{\phi(t)|t \geq t_0\}$, where $\varphi(\cdot)$ is a periodic solution of (5).

Given a set $\mathcal{Z} \subset \mathcal{X}$ that satisfies $\mathcal{Z} \cap \mathcal{S} \neq \varnothing$, we say that $\mathcal{Z}$ is **control invariant** for (5), if there exists a state-feedback control law $u^*(x)$ such that $f_{zero}(z) := f(z) + g(z)u^*(z) \in T_z\mathcal{Z}$, for all $z$ in $\mathcal{Z}$. Furthermore, we say that the control invariant set $\mathcal{Z}$ is **impact invariant** for (5), if $\Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}$. We say that $\mathcal{Z}$ is **hybrid invariant** for the dynamical system given by (5) if it is both control and hybrid invariant. Given a hybrid invariant set $\mathcal{Z}$, the **hybrid restriction dynamics** of the dynamical system in (5) are defined by

$$\Sigma\big|_{\mathcal{Z}} := \begin{cases} \dot{z}(t) = f_{zero}(z(t)), & z^-(t) \notin \mathcal{S} \cap \mathcal{Z} \\ z^+(t) = \Delta|_{\mathcal{S}\cap\mathcal{Z}}(z^-(t)), & z^-(t) \in \mathcal{S} \cap \mathcal{Z}. \end{cases} \tag{6}$$

### C. Noholonomic Virtual Constraints

In this letter, we investigate a class of nonholonomic constraints that involve the momenta conjugate to the unactuated DOF (e.g., $q_5$ for the biped in Fig. 1) and were previously used in [11] and [12]. The momentum conjugate to the unactuated DOF, $q_u$, is defined as

$$\sigma_u(q, \dot{q}) := \frac{\partial \mathcal{L}}{\partial \dot{q}_u}(q, \dot{q}) = B^\perp M(q)\dot{q}, \tag{7}$$

and, from the Euler-Lagrange equations, satisfies

$$\frac{d}{dt}\sigma_u(q, \dot{q}) = \frac{\partial \mathcal{L}}{\partial q_u}(q). \tag{8}$$

Under Hypothesis HM1, it can be seen that $\partial \mathcal{L}/\partial q_u = (1/2)\dot{q}^T(\partial M/\partial q_u)\dot{q} - \partial V/\partial q_u$ depends only on the configuration variables. Indeed,

$$\frac{\partial \mathcal{L}}{\partial q_u}(q) = -\frac{\partial V}{\partial q_u}(q). \tag{9}$$

A **nonholonomic virtual constraint (NHVC)**, due to Griffin and Grizzle [11], [12], for the biped robot dynamics in (1)–(4) is an output function of the form

$$y = h(q, \sigma_u(q, \dot{q})). \tag{10}$$

A nice feature of such an output for the biped dynamics is that while the output depends on the biped robot joint velocities through $\sigma_u(\cdot)$, the input $u$ will only appear after taking two derivatives of the output in (10). Indeed, taking one derivative of the output $y$ and using (8) yields

$$\dot{y} = \frac{\partial h}{\partial q}(q, \sigma_u(\cdot))\dot{q} + \frac{\partial h}{\partial \sigma_u}(q, \sigma_u(\cdot))\frac{\partial \mathcal{L}}{\partial q_u}(q). \tag{11}$$

Therefore, only after taking two derivatives of the output in (10) the input will appear. By using relative degree two outputs, feedback control can then be used to influence both the joint positions and velocities. We define the following mappings for a given open connected set $I \subset \mathbb{R}$, the biped robot dynamics in (1), and a smooth function $h : Q \times I \to \mathbb{R}^{(N-1)}$, $(q, \sigma) \mapsto h(q, \sigma)$, where $\sigma$ is a variable.

$$\gamma_u : Q \to \mathbb{R}^{1\times N}, q \mapsto B^{\perp}M(q) \tag{12a}$$

$$\mathcal{A} : Q \times I \to \mathbb{R}^{(N-1)\times(N-1)}, (q, \sigma) \mapsto \frac{\partial h}{\partial q}(q, \sigma)M^{-1}(q)B, \tag{12b}$$

$$\mathcal{P} : Q \times I \to \mathbb{R}^{N\times N}, (q, \sigma) \mapsto \left[\frac{\partial h}{\partial q}(q, \sigma);\ \gamma_u(q)\right], \tag{12c}$$

$$w : Q \times I \to \mathbb{R}^{N}, (q, \sigma) \mapsto \left[-\frac{\partial h}{\partial \sigma}(q, \sigma)\frac{\partial \mathcal{L}}{\partial q_u}(q); \sigma\right]. \tag{12d}$$

## III. Hybrid Zero Dynamics Under Nonholonomic Virtual Constraints

This section presents the main results of this letter. First, we derive a closed form expression for the robot swing phase dynamics under NHVCs. Next, we present conditions that guarantee existence of hybrid zero dynamics under NHVCs. Finally, we present a reduced dimensionality test for checking existence and stability of hybrid periodic orbits for underactuated bipedal robots under NHVCs.

### A. Swing Phase Zero Dynamics

*Lemma 1:* Consider the biped robot dynamics in (1). Suppose that a smooth function $h : Q \times I \to \mathbb{R}^{N-1}$, where $I \subset \mathbb{R}$ is an open and connected set, is selected so that

**H1)** $h(q, \sigma) = [h_1(q, \sigma); \dots; h_{N-2}(q, \sigma); h_{N-1}(q, \sigma)]$;

**H2)** there exists an open set $T\tilde{Q} \subset TQ$ such that for each $(q, \dot{q})$ in $T\tilde{Q}$, $h(q, \sigma_u(q, \dot{q}))$ has vector relative degree $\{2, \dots, 2\}$, where $\sigma_u(q, \dot{q})$ is given by (7);

**H3)** there exist two smooth real-valued functions $\theta_1(q)$ and $\theta_2(q, \sigma)$ such that the mapping

$$\Phi : \tilde{Q} \times I \to \mathbb{R}^{N+1}, (q, \sigma) \mapsto [h(q, \sigma); \theta_1(q); \theta_2(q, \sigma)], \tag{13}$$

is a diffeomorphism onto its image;

**H4)** there exists one point $(q, \sigma)$ in $\tilde{Q} \times I$ where $h$ vanishes.

Then, 1) $h^{-1}(0)$ is a smooth two-dimensional embedded submanifold of $\tilde{Q} \times I$ with $(\varrho_1, \varrho_2) = (\theta_1(q), \theta_2(q, \sigma))$ as a valid set of coordinates on $h^{-1}(0)$; and, 2) the decoupling matrix $\mathcal{A}(q, \sigma_u(q, \dot{q}))$ in (12b) and $\mathcal{P}(q, \sigma_u(q, \dot{q}))$ in (12c) are square and invertible on $T\tilde{Q}$; and, 3) given $(q, \sigma)$ in $\tilde{Q} \times I$ and a vector $v \in \mathbb{R}^N$ such that $(\partial h/\partial q)v + (\partial h/\partial \sigma)(\partial \mathcal{L}/\partial q_u) = 0$ and $\gamma_u(q)v = \sigma$, we have

$$v = \mathcal{P}^{-1}(q, \sigma_u(q, \dot{q}))w(q, \sigma_u(q, \dot{q})). \tag{14}$$

*Proof:* From Hypotheses H3 and H4, it follows that $\mathrm{rank}(dh)|_{h^{-1}(0)} = N - 1$, where $dh := [\partial h/\partial q,\ \partial h/\partial \sigma]$. Since $h(\cdot)$ is a mapping from $\tilde{Q} \times I$ to $\mathbb{R}^{N-1}$, Statement 1 follows (see [17]). Invertibility of $\mathcal{A}(q, \sigma_u(q, \dot{q}))$ follows immediately from the relative degree condition and the general results in [18]. To show $\mathcal{P}(q, \sigma_u(q, \dot{q}))$ is invertible on $\tilde{Q} \times I$, we need to prove $\mathcal{P}(q, \sigma_u(q, \dot{q}))v = 0$, holds only if $v = 0$. Using Equation (12c), this implies that $[\frac{\partial h}{\partial q}(q, \sigma);\ \gamma_u(q)]v = 0$. Therefore, $\gamma_u(q)v = B^{\perp}M(q)v = 0$. Hence, $M(q)v$ belongs to $\mathrm{Ker}(B^{\perp}) = \mathrm{Im}(B)$. It follows that $v = M^{-1}(q)Bv'$ for some $v'$ in $\mathrm{Im}(B) = \mathbb{R}^{N-1}$. Substituting $v = M^{-1}(q)Bv'$ in $\frac{\partial h}{\partial q}(q, \sigma)v = 0$ yields $\mathcal{A}(q, \sigma_u(q, \dot{q}))v' = 0$. Since $\mathcal{A}(q, \sigma_u(q, \dot{q}))$ is invertible for all $(q, \sigma) \in Q \times I$, we conclude that $v' = 0$. Hence, $\mathcal{P}(\cdot)v = 0$ holds only if $v = 0$. Finally, Statement 3 can be proved by considering the augmented equation $[\frac{\partial h}{\partial q}; \gamma_u(q)]v = [-\frac{\partial h}{\partial \sigma}\frac{\partial \mathcal{L}}{\partial \sigma_u}; \sigma]$. From (12c), (12d), and Statement 2, Equation (14) follows. ■

Since we are interested in making the NHVCs invariant with respect to the biped robot dynamics, we define

$$\mathcal{Z} := \{(q, \dot{q}) \in T\tilde{Q} : h(q, \sigma_u(\cdot)) = 0,\ \frac{\partial h}{\partial q}(q, \sigma_u(\cdot))\dot{q} + \frac{\partial h}{\partial \sigma}(q, \sigma_u(\cdot))\frac{\partial \mathcal{L}}{\partial q_u}(q) = 0\}, \tag{15}$$

where $(\cdot)$ denotes $(q, \dot{q})$. The following proposition provides conditions under which $\mathcal{Z}$ is a control invariant surface.

*Proposition 1:* Consider the hypotheses of Lemma 1 and **H5)** the matrix

$$\Xi(q, \dot{q}) = \begin{bmatrix} \Xi_{11}(q, \dot{q}) & \Xi_{12}(q, \dot{q}) \\ \Xi_{21}(q, \dot{q}) & \Xi_{22}(q, \dot{q}) \end{bmatrix}, \tag{16}$$

where

$$\Xi_{11}(q, \dot{q}) := \frac{\partial h}{\partial q} + \frac{\partial h}{\partial \sigma}\frac{\partial \gamma_u}{\partial q}\dot{q},\ \ \Xi_{12}(q, \dot{q}) := \frac{\partial h}{\partial \sigma}\gamma_u,$$

$$\Xi_{21}(q, \dot{q}) := \frac{\partial}{\partial q}\left(\frac{\partial h}{\partial q}\dot{q}\right) + \frac{\partial^2 h}{\partial q \partial \sigma}\frac{\partial \mathcal{L}}{\partial q_u} + \frac{\partial h}{\partial \sigma}\frac{\partial^2 \mathcal{L}}{\partial q \partial q_u} + \frac{\partial^2 h}{\partial \sigma^2}\frac{\partial \gamma_u}{\partial q}\dot{q}\frac{\partial \mathcal{L}}{\partial q_u},$$

$$\Xi_{22}(q, \dot{q}) := \frac{\partial h}{\partial q} + \frac{\partial^2 h}{\partial \sigma^2}\gamma_u\frac{\partial \mathcal{L}}{\partial q_u} + \frac{\partial^2 h}{\partial \sigma \partial q}\gamma_u\dot{q},$$

is of full row rank $2N - 2$ for all $(q, \dot{q}) \in T\tilde{Q}$. Then,

1) the set $\mathcal{Z}$ given by (15) is a smooth two-dimensional embedded submanifold of $TQ$;

2) there exists a control torque $u^*(q, \dot{q})$ that makes the manifold $\mathcal{Z}$ invariant.

*Proof:* We only provide a sketch of the proof. The matrix $\Xi(q, \dot{q})$ in (16) is indeed the Jacobian $\partial/\partial(q, \dot{q})[h(\cdot); \dot{h}(\cdot)]$. The rank condition $\mathrm{rank}(\Xi(\cdot)) = 2N - 2$ in H5 then proves Statement 1. The proof of Statement 2 directly follows from the general results in [18]. ■

The following theorem provides a valid set of coordinates on $\mathcal{Z}$ and the form of the zero dynamics under NHVCs.

*Theorem 1:* Consider the hypotheses of Lemma 1, Proposition 1, and assume

**H6)** the matrix $[\Xi(q, \dot{q}); \Theta(q, \dot{q})]$ is square and invertible for all $(q, \dot{q})$ in $T\tilde{Q}$, where $\Xi(\cdot)$ is given by (16) and $\Theta(q, \dot{q})$ is the Jacobian matrix $\partial/\partial(q, \dot{q})[\theta_1(q); \theta_2(q, \sigma_u(q, \dot{q}))]$. Then,

1) the mapping $\bar{\Phi} : (q, \dot{q}) \mapsto [h(\cdot); \dot{h}(\cdot); \theta_1(q); \theta_2(q, \sigma_u(\cdot))]$ is a valid coordinate transformation on $T\tilde{Q}$;

2) $(\xi_1; \xi_2) = (\theta_1(q); \theta_2(q, \sigma_u(q, \dot{q})))$ is a valid set of coordinates on $\mathcal{Z}$. Furthermore, in these coordinates,

$$\dot{\xi}_1 = \kappa_1(\xi_1, \xi_2),$$
$$\dot{\xi}_2 = \kappa_2(\xi_1, \xi_2) + \zeta(\xi_1, \xi_2), \tag{17}$$

where $\kappa_1(\xi_1,\xi_2) := \frac{\partial\theta_1}{\partial q}\mathcal{P}^{-1}w|_{\mathcal{Z}}$, $\zeta(\xi_1,\xi_2) := \frac{\partial\theta_2}{\partial q}\mathcal{P}^{-1}w|_{\mathcal{Z}}$, and $\kappa_2(\xi_1,\xi_2) := \frac{\partial\theta_2}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial q_u}|_{\mathcal{Z}}$.

*Proof:* Since $[\Xi(\cdot);\Theta(\cdot)]$ is the Jacobian of $\bar{\Phi}$, the rank condition in H6 proves Statement 1. Consider a given pair $(q,\dot{q})\in\mathcal{Z}$ and let $\sigma_0 := \sigma_u(q,\dot{q})$. Since on $\mathcal{Z}$, we have $h(q,\sigma_0)=0$, from Lemma 1 it follows that

$$[q;\sigma_0] = \Upsilon([\xi_1;\xi_2]), \tag{18}$$

where $\Upsilon([\xi_1;\xi_2]) := \Phi^{-1}([0;\xi_1;\xi_2])$, $\xi_1 := \theta_1(q)$ and $\xi_2 := \theta_2(q,\omega_0)$. Furthermore, $\gamma_u(q)\dot{q}=\sigma_0$ and from definition of $\mathcal{Z}$, we have $(\partial h/\partial q)\dot{q} + (\partial h/\partial\sigma)(\partial\mathcal{L}/\partial q_u) = 0$. From Lemma 1 and (18), it follows that

$$\dot{q} = \Omega([\xi_1;\xi_2]), \tag{19}$$

where $\Omega([\xi_1;\xi_2]) := \mathcal{P}^{-1}(\Upsilon([\xi_1;\xi_2]))w(\Upsilon([\xi_1;\xi_2]))$. Taking the derivative of $[\theta_1(q);\theta_2(q,\sigma_u(\cdot))]$ and evaluating on $\mathcal{Z}$, we obtain $\dot{\theta}_i = L_f\theta_i + L_g\theta_i u^\star$. Since

$$L_g\theta_i(q,\dot{q}) = \left[\frac{\partial\theta_i}{\partial q}\ \ \frac{\partial\theta_i}{\partial\dot{q}}\right]\begin{bmatrix}0\\ -M^{-1}(q)B\end{bmatrix},$$

$L_g\theta_1(q,\dot{q})=0$ holds. Also, $L_g\theta_2(q,\sigma_u(q,\dot{q}))$ is equal to

$$\frac{\partial\theta_2}{\partial\dot{q}}M^{-1}(q)B = \frac{\partial\theta_2}{\partial\sigma_u}|_{\mathcal{Z}}B^{\perp}M(q)M^{-1}(q)B = 0. \tag{20}$$

Thus, $\dot{\theta}_i = L_f|_{\mathcal{Z}}\theta_i(q,\dot{q})$, $i=1,2$. Hence, from (18), (19), the zero dynamics form in (17) follows. ■

*Remark 1:* The zero dynamics expression in (17) is a generalization of this letter presented in [6]. In particular, if $h(q,\sigma) = h(q)$, then $\zeta(\xi_1,\xi_2) = 0$ in (17) and the zero dynamics presented in [6] will be retrieved.

The following lemma provides a sufficient condition for the rank condition in hypothesis H6 to hold.

*Lemma 2:* Consider the hypotheses of Lemma 1 and Proposition 1. A sufficient condition for H6 to hold is that $\text{rank}([\Xi_{11}(q,\dot{q});\partial\theta_1/\partial q]) = N$ and $\text{rank}([\Xi_{22}(q,\dot{q});(\partial\theta_2/\partial\sigma)\gamma_u(q)]) = N$ for all $(q,\dot{q})\in T\tilde{Q}$.

*Proof:* From the special form of $[\Xi(q,\dot{q});\Theta(q,\dot{q})]$ where $\Theta(q,\dot{q}) = [\partial\theta_1/\partial q\ \ ,0;\ (\star)\ \ ,(\partial\theta_2/\partial\sigma)\gamma_u(q)]$, it follows that $[\Xi(\cdot);\Theta(\cdot)]$ has full rank $2N$ on $T\tilde{Q}$ if $\text{rank}([\Xi_{11}(\cdot);\partial\theta_1/\partial q]) = N$ and $\text{rank}([\Xi_{22}(\cdot);(\partial\theta_2/\partial\sigma)\gamma_u(q)]) = N$. ■

## B. Hybrid Zero Dynamics

Existence of hybrid zero dynamics means that the zero dynamics manifold must be invariant under impact, that is $\Delta(\mathcal{S}\cap\mathcal{Z})\subset\mathcal{Z}$. In this section, we first investigate the topology of $\mathcal{S}\cap\mathcal{Z}$. Next, we provide conditions for existence of hybrid zero dynamics under NHVCs.

*Lemma 3:* Consider the hypotheses of Lemma 1 and Proposition 1. Let $q_0^-$ satisfy $[h(q_0^-,\omega);p_2^v(q_0^-)] = 0$, $p_2^h(q_0^-) > 0$, for all $\omega\in\mathcal{I}$, where $\mathcal{I}$ is an open interval in $\mathbb{R}$. Then, $\mathcal{S}\cap\mathcal{Z}$ is a smooth embedded one-dimensional submanifold of $T\tilde{Q}$ if $\mathcal{S}\cap\mathcal{Z}\neq\varnothing$ and $\text{rank}([h;p_2^v]) = N$ on $\mathcal{S}\cap\mathcal{Z}$. Furthermore, the connected component of $\mathcal{S}\cap\mathcal{Z}$ containing $q_0^-$ is diffeomorphic to $\mathbb{R}$ per $\bar{\lambda}:\mathcal{I}\to\mathcal{S}\cap\mathcal{Z}$, where

$$\begin{aligned}\bar{\lambda}(\omega) &:= [\bar{\lambda}_q;\bar{\lambda}_{\dot{q}}(\omega)],\ \ \bar{\lambda}_q := q_0^-,\\ \bar{\lambda}_{\dot{q}}(\omega) &:= \mathcal{P}^{-1}(q_0^-,\omega)w(q_0^-,\omega).\end{aligned} \tag{21}$$

*Proof:* The rank condition $\text{rank}([h;p_2^v]) = N$ on $\mathcal{S}\cap\mathcal{Z}$ implies that the map $[h;\dot{h};p_2^v]$ has constant rank $2N-1$ on $\mathcal{S}\cap\mathcal{Z}$. Therefore, $\mathcal{S}\cap\mathcal{Z}$ is a smooth embedded one-dimensional submanifold of $T\tilde{Q}$. Now, consider an arbitrary point $(q_0^-,\dot{q})$ in $\mathcal{S}\cap\mathcal{Z}$. By (19), it can be seen that $\dot{q} = \mathcal{P}^{-1}(q_0^-,\omega)w(q_0^-,\omega)$. ■

*Proposition 2:* Consider the hybrid dynamics given by (5) and a NHVC $h(\cdot)$ satisfying the hypotheses of Lemma 1. Then the following statements are equivalent,

1) $\Delta(\mathcal{S}\cap\mathcal{Z})\subset\mathcal{Z}$; and,
2) for every $(q_0^-,\dot{q}_0^-)\in\mathcal{S}\cap\mathcal{Z}$, we have

$$\begin{aligned}&h(\Delta_q(q_0^-),\gamma_{0u}^-\Delta_{\dot{q}}^-\dot{q}_0^-) = 0,\\ &\mathcal{P}^{-1}(\Delta_q(q_0^-),\gamma_{0u}^-\Delta_{\dot{q}}^-\dot{q}_0^-)w(\Delta_q q_0^-,\gamma_{0u}^-\Delta_{\dot{q}}^-\dot{q}_0^-) = \Delta_{\dot{q}}^-\dot{q}_0^-,\end{aligned}$$

where $\gamma_{0u}^- := \gamma_u(\Delta_q q_0^-)$ and $\Delta_{\dot{q}}^- := \Delta_{\dot{q}}(q_0^-)$.

*Proof:* From the hypotheses of Lemma 1, definition of the zero dynamics manifold $\mathcal{Z}$, and (19), it follows that $\dot{q} = \mathcal{P}^{-1}(q,\gamma_u(q)\dot{q})w(q,\gamma_u(q)\dot{q})$ for all $(q,\dot{q})\in\mathcal{Z}$. Hence, the equivalence of Statements 1 and 2 follows from the definition of $\mathcal{Z}$ and the rigid impact model in (4). ■

*Remark 2:* The rank conditions in H5, H6, and Lemma 2 can be verified using the parameterizations of the robot states restricted to the zero dynamics manifold $\mathcal{Z}$. In particular, using the coordinates in Theorem 1, (18) and (19), we have $[q;\sigma_0] = \Upsilon([\xi_1;\xi_2])$ and $\dot{q} = \Omega([\xi_1;\xi_2])$. Then, the rank condition verification can be carried out on a planar region where $\xi_1$ and $\xi_2$ evolve. This is due to the fact that the full rank condition holds if and only if certain square submatrices of the matrices in H5, H6, and Lemma 2 are invertible matrix functions on $T\tilde{Q}$. The continuity property of determinant of square matrices guarantees that if these rank conditions hold on the zero dynamics manifold, then they hold in a neighborhood of the zero dynamics manifold.

## C. Reduced Dimensionality Test Under NHVCs

The existence and stability of periodic orbits of the hybrid dynamical system in (5) under hybrid invariant NHVCs can be completely determined on the basis of the zero dynamics in (17) and the restriction of the impact model $\Delta(\cdot)$ to $\mathcal{Z}$. Indeed, under the hypotheses of Theorem 1 and Lemma 3, if we choose the Poincaré section to be $\mathcal{S}\cap\mathcal{Z}$ and in coordinates $(\xi_1,\xi_2) = (\theta_1,\theta_2)$, $\mathcal{S}\cap\mathcal{Z}$ and the impact map $\Delta:(\xi_1^-,\xi_2^-)\to(\xi_1^+,\xi_2^+)$ simplify to

$$\mathcal{S}\cap\mathcal{Z} = \{(\xi_1^-,\xi_2^-):\xi_1^- = \theta_1(q_0^-),\ \xi_2^-\in\mathbb{R}\}, \tag{22}$$

$$\xi_1^+ = \theta_1\circ\Delta_q(q_0^-), \tag{23}$$

$$\xi_2^+ = \theta_2(\Delta_q(q_0^-),\gamma_{0u}^-\Delta_{\dot{q}}^-\Upsilon([\xi_1^-;\xi_2^-])), \tag{24}$$

where $\gamma_{0u}^- := \gamma_u(\Delta_q q_0^-)$, $\Delta_{\dot{q}}^- := \Delta_{\dot{q}}(q_0^-)$, and $\Upsilon(\cdot)$ is given in (18). For the hybrid restriction dynamics given by the zero dynamics in (17) and the impact in (23), (24), we define the Poincaré map to be the partial map $\rho:\mathcal{S}\cap\mathcal{Z}\to\mathcal{S}\cap\mathcal{Z}$, $z\mapsto\varphi(T_I\circ\Delta(z),\Delta(z))$ with $T_I(\cdot)$ and $\varphi(\cdot,z_0)$ being the time to impact function restricted to $\mathcal{Z}$ and the solution to the zero dynamics in (17) with initial condition $z_0$, respectively.

The following theorem, whose proof directly follows the results in [6], provides conditions for existence and stability of hybrid periodic orbits under NHVCs.

*Theorem 2:* Assume the hypotheses of Theorem 1 with a hybrid invariant NHVC. Consider the hybrid restriction dynamics given by the zero dynamics in (17) and the impact in (23), (24). Consider the Poincaré map $\rho:\mathcal{S}\cap\mathcal{Z}\to\mathcal{S}\cap\mathcal{Z}$,

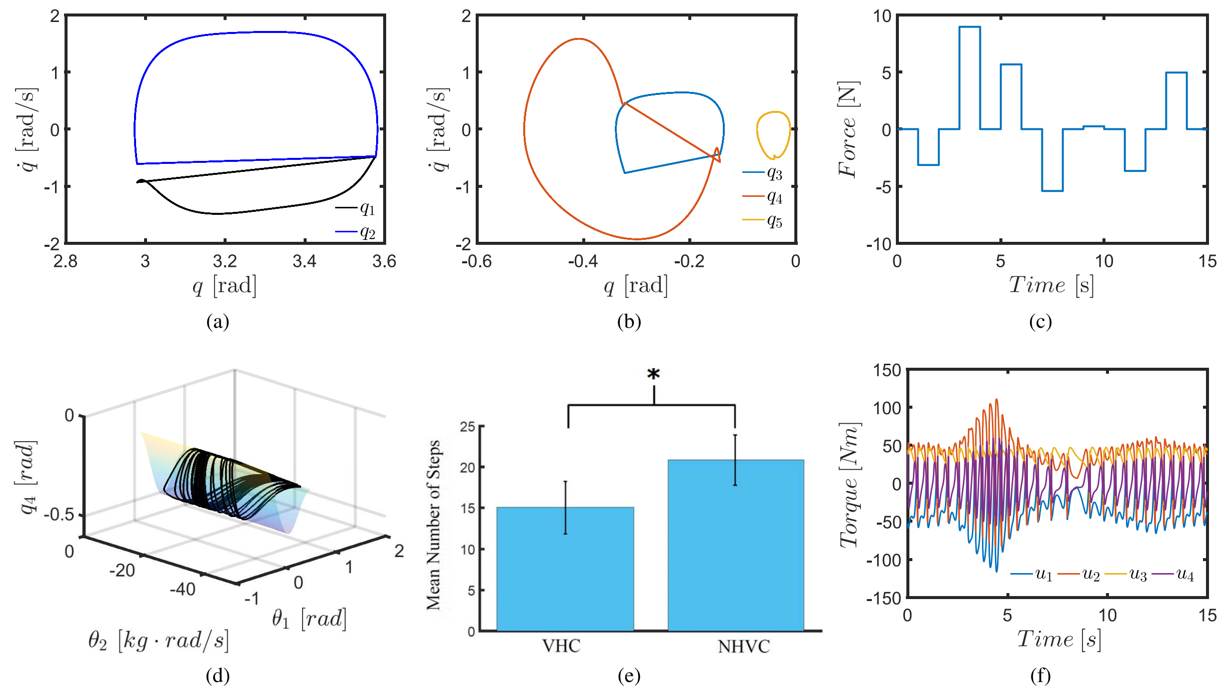


Fig. 2. The simulation results for the biped: (a,b) the nominal periodic orbits of the biped subject to NHVCs, (c) an example randomly generated force profile applied to the biped for both controllers, (d) the joint angle $q_4$ plotted against $\theta_1 = \tau$ and $\theta_2 = \sigma_u$ subject to a push disturbance for the NHVC controller, (e) the mean number of steps before failure using VHC and NHVCs ($*$ indicates $p < 0.05$), and (f) the torque profiles for each of the joints for the NHVC controller.

$z \mapsto \varphi(T_I \circ \Delta(z), \Delta(z))$. If there exists $z^* \in \mathcal{S} \cap \mathcal{Z}$ such that $z^* = \rho(z^*)$ and the discrete time system $\delta z[k+1] = A\delta z[k]$, where $\delta z[k] := z[k] - z^*$, and $A := \frac{\partial \rho}{\partial z}(z^*)$ is exponentially stable, then the orbit $\varphi(\cdot, \Delta(z^*))$ is an exponentially stable hybrid periodic orbit for (5).

### D. Parameterizing NHVCs

Although the results in the previous sections are general, we only consider the specific class

$$y = h_{\mathrm{h}}(q, \tau(q)) - h_{\mathrm{nh}}(\sigma_u(q, \dot{q})), \tag{25}$$

of NHVCs, which is given by the affine sum of a holonomic function, $h_{\mathrm{h}}(q, \tau(q)) := H_0 q - h_d \circ \tau(q)$, and a nonholonomic Bézier polynomial

$$h_{\mathrm{nh}}(\sigma_u) := \sum_{n=0}^{M} \kappa_n \frac{M!}{n!(M-n)!} \sigma_u^n (1-\sigma_u)^{M-n}. \tag{26}$$

The holonomic part $h_{\mathrm{h}}(\cdot)$ is a VHC that encodes stable walking gaits of planar bipedal robots using the normalized monotonic gait phasing variable $\tau(q)$, which takes values in the interval $[0, 1)$. Additionally, we choose the zero dynamics coordinates such that $\theta_1 = \tau(q)$ and $\theta_2 = \sigma_u$ which satisfy Theorem 1. Unlike $\tau(q)$, $\sigma_u$ is not strictly monotonic, and the minimum and maximum values can not be known a priori. Therefore, we can not use a normalized representation of $\sigma_u$. By Lemma 3, we require that $h_{\mathrm{nh}}$ is designed such that it vanishes when $\mathcal{S} \cap \mathcal{Z}$. In this letter, we assume that $h_{\mathrm{h}}(\cdot)$ is given and the gait design is only done for $h_{\mathrm{nh}}(\cdot)$. In particular, we find the coefficients of the polynomial $h_{\mathrm{nh}}(\cdot)$ by solving the following optimization problem,

$$\begin{aligned} \kappa^* = \underset{\kappa \in \mathbb{R}^{M+1}}{\arg\min} \ & \frac{1}{\text{step length}} \int_0^T ||u_\kappa^*(t)||_2^2 dt \\ \text{s.t.} \ & \Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}, \text{Physical Constraints}. \end{aligned} \tag{27}$$

Here, $u_\kappa^*(t)$ is the feedback control that renders the zero dynamics manifold invariant. It can be obtained via computing the feedforward term of the input-output feedback linearizing control law on the zero dynamics manifold. The invertibility of the decoupling matrix is essentially guaranteed whenever $\int_0^T ||u_\kappa^*(t)||_2^2 dt$ is finite, since singularities in the decoupling matrix will normally result in $u_\kappa^*(t)$ taking on unbounded values. Also, the "Physical Constraints" refer to torque saturation, foot clearance, actuator range of motion, and the nonlinear inequality constraints (presented in [6]) required for a solution to exist. The three nonlinear inequality constraints are as follows: 1) minimum normal ground reaction force is $F_1^N > 0$; 2) maximum ratio of tangential to normal ground reaction forces is less than the static friction limit, $|F_1^T / F_1^N| < \mu_s$; 3) $\mathcal{S} \cap \mathcal{Z}$ occurs only at the end of the step. The result of the optimization is the vector of coefficients $\kappa^*$ that yields a hybrid invariant nonholonomic walking gait.

*Remark 3:* Due to the specific format of the NHVC in (25) and for sufficiently small nonholonomic coefficient vector $\kappa^*$, the rank condition in Lemma 2 automatically holds. In particular, $[\Xi_{11}(q, \dot{q}); \partial\theta_1/\partial q] = [\partial h_{\mathrm{h}}/\partial q + (\partial h_{\mathrm{nh}}/\partial \sigma)(\partial \gamma_u/\partial q)\dot{q}; (\partial\theta_1/\partial q)]$, and

$$\begin{bmatrix} \Xi_{22}(q, \dot{q}) \\ (\partial\theta_2/\partial\sigma)\gamma_u(q) \end{bmatrix} = \begin{bmatrix} \partial h_{\mathrm{h}}/\partial q + (\partial^2 h_{\mathrm{nh}}/\partial\sigma^2)\gamma_u(\partial\mathcal{L}/\partial q_u) \\ \gamma_u(q) \end{bmatrix}.$$

Since $[\partial h_{\mathrm{h}}/\partial q; (\partial\theta_1/\partial q)]$ and $[\partial h_{\mathrm{h}}/\partial q; \gamma_u(q)]$ are square and invertible (due to the specific properties of the base holonomic constraint $h_h(\cdot)$), and from the continuity property of determinant of square matrices with respect to their entries, it follows that the rank condition in Lemma 2 holds.

## IV. Simulation Results

Having derived the zero dynamics and hybrid invariance conditions, we now explore the performance of NHVCs through simulation.

### A. Nonholonomic Gait

The computations were performed using MATLAB's `fmincon` function for the biped robot RABBIT [1], [6]. The physical parameters for the robot are taken from [6] and have been omitted here for brevity. In Fig. 2 (a,b) we see the resulting periodic orbit of the optimization. Using Theorem 2 on the resulting walking gait, the eigenvalue of the linearized restricted Poincaré map was computed as $\lambda_1 = 0.6877$. Since the eigenvalue is of magnitude less than one, the controller is locally exponentially stable.

### B. Push Disturbance

We now present simulation results for push recovery of the biped robot, RABBIT (see Fig. 1). We apply a push disturbance of the form $F_d = [f_x; 0]$ at the midpoint of the torso of the robot. In this letter, we apply a randomly generated force ($-10\ \text{N} \leq f_x \leq 10\ \text{N}$) every two seconds for a duration of $\Delta t = 1$ second and simulate the biped for 30 steps or until failure. We performed this experiment 20 times for each controller, and an example of the randomly generated force profile can be seen in Fig. 2 (c). For each trial, the same randomly generated force profile was applied to both controllers.

Fig. 2 (d) shows the dynamic change in $q_4(t)$ plotted against $\theta_1 = \tau(q)$ and $\theta_2 = \sigma_u$ (the zero dynamics coordinates). This figure illustrates that the zero dynamics of the NHVC controller now evolve on a surface parameterized by $\theta_1$ and $\theta_2$, rather than a two-dimensional curve for the VHC controller. Fig. 2 (e) shows the mean number of steps the biped was able to complete before failure [19] for the VHC and NHVC control schemes. In fact, with mean number of steps $\mu_{VHC} = 15 \pm 3.15$ and $\mu_{NHVC} = 20.9 \pm 3.05$ (Mean $\pm$ Standard Deviation), This finding reinforces the results of [11] and [12], but also implies that statistically the two controllers do not belong to the same population with 95% confidence. Fig. 2 (f) shows the torque profiles for each of the joints. The NHVC dynamically changes the torque applied to the system based on the evolution of the momenta conjugate.

## V. Conclusion and Future Research

The central focus of this letter was to derive the hybrid zero dynamics for a one degree of underactuation biped walker with NHVCs. A closed form expression of the zero dynamics was derived for the biped robot during the swing phase of walking. Conditions required to maintain a hybrid invariant walking gait were also presented. Lastly, nonholonomic virtual constraints were used to simulate the RABBIT biped robot with a horizontal disturbance. In future research, we plan to leverage NHVCs for application to powered prosthesis control and investigate methods of designing NHVCs to leverage kinematic adaptability properties.

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