SPECTRUM OF THE KOHN LAPLACIAN ON THE ROSSI SPHERE

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Abstract. We study the spectrum of the Kohn Laplacian $\Box_b^t$ on the Rossi example $(\mathbb{S}^3, \mathcal{L}_t)$. In particular we show that 0 is in the essential spectrum of $\Box_b^t$, which yields another proof of the global non-embeddability of the Rossi example.

1. Introduction

1.1. **General Setting.** Let $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ : |z_1|^2 + |z_2|^2 = 1\}$ denote the 3-sphere in $\mathbb{C}^2$. $\mathbb{S}^3$ is a three real dimensional manifold and it can be viewed as an abstract CR manifold when one chooses a specific complex vector field that determines the complex tangent vectors. It is a general question whether an abstract CR manifold can be realized as a manifold in $\mathbb{C}^N$, for some N, where the complex tangent spaces coincides with the ones induced from the ambient space. One way of addressing this question is studying a second order differential operator, so-called the Kohn Laplacian, that naturally arises on CR manifolds. Many geometric properties of abstract CR manifolds can be studied by analyzing the properties of this differential operator. In this note we address the embeddability question by studying the spectrum of the Kohn Laplacian on a specific abstract CR manifold. In particular we examine the essential spectrum of the Kohn Laplacian. The essential spectrum of a bounded self-adjoint operator is the subset of the spectrum that contains eigenvalues of infinite multiplicity and the limit points. We refer the readers to [Bog91] and [CS01] for the general theory of CR manifolds to Kohn Laplacian, and to [D95] for spectral theory.

1.2. **Main Problem.** Rossi showed that the CR-manifold $(\mathbb{S}^3, \mathcal{L}_t)$ is not CR-embeddable [Ros65], where

$$\mathcal{L}_t = \overline{z_1}\frac{\partial}{\partial z_2} - \overline{z_2}\frac{\partial}{\partial z_1} + \bar{t}\left(z_1\frac{\partial}{\partial \overline{z_2}} - z_2\frac{\partial}{\partial \overline{z_1}}\right),$$

and $|t| < 1$. In the case of strictly pseudoconvex CR-manifolds Boutet de Monvel proved that if the real dimension of the manifold is at least 5, then it can always be globally CR-embedded into $\mathbb{C}^N$ for some N [BdM75]. Later Burns approached this problem in the $\overline{\partial}$ context and showed that if the tangential operator $\overline{\partial}_{b,t}$ has closed range and the Szegö projection is bounded, then the CR-manifold is CR-embeddable into $\mathbb{C}^N$ [Bur79]. Later in 1986, Kohn showed that CR-embeddability is equivalent to showing that the tangential Cauchy-Riemann operator $\overline{\partial}_{b,t}$ has closed range [Koh85].

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In the setting of the Rossi example, as an application of the closed graph theorem, $\overline{\partial}_{b,t}$ has closed range if and only if the Kohn Laplacian

$$\Box_b^t = -\mathcal{L}_t \frac{1+|t|^2}{(1-|t|^2)^2} \overline{\mathcal{L}}_t$$

has closed range, see [BE90, 0.5]. Furthermore, the closed range property is equivalent to the positivity of the essential spectrum of $\Box_b^t$, see [Fu05] for similar discussion. In this note we tackle the problem of embeddability, from the perspective of spectral analysis. In particular, we show that 0 is in the essential spectrum of $\Box_b^t$, so the Rossi sphere is not globally CR-embeddable into $\mathbb{C}^N$. This provides a different approach to the results in [Bur79, Koh85].

We start our analysis with the spectrum of $\Box_b^t$. We utilize spherical harmonics to construct finite dimensional subspaces of $L^2(\mathbb{S}^3)$ such that $\Box_b^t$ has tridiagonal matrix representations on these subspaces. We then use these matrices to compute eigenvalues of $\Box_b^t$. We also present numerical results obtained by *Mathematica* that motivate most of our theoretical results. We then present an upper bound for small eigenvalues and we exploit this bound to find a sequence of eigenvalues that converge to 0.

In addition to particular results in this note, our approach can be adopted to study possible other perturbations of the standard CR-structure on the 3-sphere, such as in [BE90]. Furthermore, our approach also leads some information on the growth rate of the eigenvalues and possible connections to finite-type (in the sense of commutators) results similar to the ones in [Fu08]. We plan to address these issues in future papers.

2. Analysis of $\Box_b$ on $\mathcal{H}_{p,q}(\mathbb{S}^3)$

2.1. **Spherical Harmonics.** We start with a quick overview of spherical harmonics, we refer to [ABR01] for a detailed discussion. We will state the relevant theorems on $\mathbb{C}^2$ and $\mathbb{S}^3 \subseteq \mathbb{C}^2$. A polynomial in $\mathbb{C}^2$ can be written as

$$p(z, \overline{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{z}^\beta$$

where $z \in \mathbb{C}^2$, each $c_{\alpha,\beta} \in \mathbb{C}$, and α, $\beta \in \mathbb{N}^2$ are multi-indices. That is, $\alpha = (\alpha_1, \alpha_2), z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$, and $|\alpha| = \alpha_1 + \alpha_2$.

We denote the space of all homogeneous polynomials on $\mathbb{C}^2$ of degree m by $\mathcal{P}_m(\mathbb{C}^2)$, and we let $\mathcal{H}_m(\mathbb{C}^2)$ denote the subspace of $\mathcal{P}_m(\mathbb{C}^2)$ that consists of all harmonic homogeneous polynomials on $\mathbb{C}^2$ of degree m. We use $\mathcal{P}_m(\mathbb{S}^3)$ and $\mathcal{H}_m(\mathbb{S}^3)$ to denote the restriction of $\mathcal{P}_m(\mathbb{C}^2)$ and $\mathcal{H}_m(\mathbb{C}^2)$ onto $\mathbb{S}^3$. We denote the space of complex homogenous polynomials on $\mathbb{C}^2$ of bi-degree p, q by $\mathcal{P}_{p,q}(\mathbb{C}^2)$, and those polynomials that are homogeneous and harmonic by $\mathcal{H}_{p,q}(\mathbb{C}^2)$. As before, we denote $\mathcal{P}_{p,q}(\mathbb{S}^3)$ and $\mathcal{H}_{p,q}(\mathbb{S}^3)$ as the polynomials of the previous spaces, but restricted to $\mathbb{S}^3$. We recall that on $\mathbb{C}^2$, the Laplacian is defined as

$$\Delta = 4\left(\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}\right).$$

As an example, the polynomial $z_1\overline{z_2} - 2z_2\overline{z_1} \in \mathcal{P}_{1,1}(\mathbb{C}^2)$, and $z_1\overline{z_2}^2 \in \mathcal{H}_{1,2}(\mathbb{C}^2)$. We take our first step by stating the following decomposition result.

Proposition 2.1. [ABR01, Theorem 5.12] $L^2(\mathbb{S}^3) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{S}^3)$.

The spherical harmonics form an orthogonal basis on $\mathbb{S}^3$ similar to the Fourier series on the unit circle $\mathbb{S}^1$. They are also the eigenfunctions of the Laplacian on $\mathbb{S}^3$. The summation above is understood as the orthogonal direct sum of Hilbert spaces. This statement is essential to the spectral analysis of $\Box_b^t$ on $L^2(\mathbb{S}^3)$ since it decomposes the infinite dimensional space $L^2(\mathbb{S}^3)$ into finite dimensional pieces, which is necessary for obtaining the matrix representation of $\Box_b^t$ (a special case of the general spectral theory of compact operators). In order to get such a matrix representation, we need a method for obtaining a basis for $\mathcal{H}_k(\mathbb{S}^3)$. Proposition 2.3 presents a method to do so for $\mathcal{H}_m(\mathbb{C}^2)$ and Proposition 2.5 presents a method for $\mathcal{H}_{p,q}(\mathbb{C}^2)$. The dimension of the matrix representation on a particular $\mathcal{H}_m(\mathbb{S}^3)$ is the dimension of the subspace $\mathcal{H}_m(\mathbb{S}^3)$, which is given below and analogously given for $\mathcal{H}_{p,q}(\mathbb{C}^2)$.

Proposition 2.2. [ABR01, Proposition 5.8] *For $k, p, q \geq 2$,*

$$\dim \mathcal{P}_{p,q}(\mathbb{C}^2) = (p+1)(q+1),$$

$$\dim \mathcal{H}_{p,q}(\mathbb{C}^2) = p + q + 1$$
$$\dim \mathcal{H}_k(\mathbb{C}^2) = (k+1)^2.$$

Now we present a method to obtain explicit bases of spaces of spherical harmonics. These bases play an essential role in explicit calculations in the next section. Here, K denotes the Kelvin transform,

$$K[g](z) = |z|^{-2} g\left(\frac{z}{|z|^2}\right).$$

For multi-indices $\alpha, \beta \in \mathbb{N}^2$, D^α and $\overline{D}^\beta$ denote the differential operators

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial^{\alpha_1} z_1)(\partial^{\alpha_2} z_2)} \text{ and } \overline{D}^\beta = \frac{\partial^{|\beta|}}{(\partial^{\beta_1} \overline{z_1})(\partial^{\beta_2} \overline{z_2})}.$$

Proposition 2.3. [ABR01, Theorem 5.25] *The set*

$$\{K[D^\alpha |z|^{-2}] : |\alpha| = m \text{ and } \alpha_1 \leq 1\}$$

is a vector space basis of $\mathcal{H}_m(\mathbb{C}^2)$, and the set

$$\{D^\alpha |z|^{-2} : |\alpha| = m \text{ and } \alpha_1 \leq 1\}$$

is a vector space basis of $\mathcal{H}_m(\mathbb{S}^3)$.

Homogenous polynomials of degree k can be written as the sum of polynomials of bi-degree p, q such that $p + q = k$.

Proposition 2.4. $\mathcal{P}_k(\mathbb{C}^2) = \bigoplus_{p+q=k} \mathcal{P}_{p,q}(\mathbb{C}^2)$.

Analogous to the version in Proposition 2.3, we use the following method to construct an orthogonal basis for $\mathcal{H}_{p,q}(\mathbb{C}^2)$ and $\mathcal{H}_{p,q}(\mathbb{S}^3)$. The proof pretty much follows the proof of [ABR01, Theorem 5.25], with changes from single index to double index.

Proposition 2.5. *The set*

$$\left\{ K[\overline{D}^\alpha D^\beta |z|^{-2}] \;\middle|\; |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\}$$

is a basis for $\mathcal{H}_{p,q}(\mathbb{C}^2)$*, and the set*

$$\left\{ \overline{D}^{\alpha} D^{\beta} |z|^{-2} \;\middle|\; |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_1 = 0 \right\}$$

is an orthogonal basis for $\mathcal{H}_{p,q}(\mathbb{S}^3)$*.*

2.2. $\Box_b$ **on** $\mathcal{H}_{p,q}(\mathbb{S}^3)$**.** Before we study the operator $\Box_b^t$, we first need some background on a simpler operator we call $\Box_b$. It arises from the CR-manifold $(\mathbb{S}^3, \mathcal{L})$, and is defined as

$$\Box_b = -\mathcal{L}\overline{\mathcal{L}}.$$

Here, $\mathcal{L} = \mathcal{L}_0 = \overline{z_1}\frac{\partial}{\partial z_2} - \overline{z_2}\frac{\partial}{\partial z_1}$ the standard $(1,0)$ vector field from the ambient space. We note that this CR-structure is induced from $\mathbb{C}^2$ and this manifold is naturally embedded. By the machinery above we can compute the eigenvalues of $\Box_b$, see also [Fol72] for a more general discussion.

Theorem 2.1. *Suppose* $f \in \mathcal{H}_{p,q}(\mathbb{S}^3)$*. Then*

$$\Box_b f = (pq + q)f.$$

Proof. Expanding the definition, we get

$$\begin{aligned}
\Box_b &= -\left(\overline{z_2}\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial z_2}\right)\left(z_2\frac{\partial}{\partial \overline{z_1}} - z_1\frac{\partial}{\partial \overline{z_2}}\right) \\
&= -\overline{z_2}\frac{\partial}{\partial z_1}\left(z_2\frac{\partial}{\partial \overline{z_1}} - z_1\frac{\partial}{\partial \overline{z_2}}\right) + \overline{z_1}\frac{\partial}{\partial z_2}\left(z_2\frac{\partial}{\partial \overline{z_1}} - z_1\frac{\partial}{\partial \overline{z_2}}\right) \\
&= -z_2\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \overline{z_2}\frac{\partial}{\partial \overline{z_2}} + z_1\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} \\
&\quad - z_1\overline{z_1}\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + \overline{z_1}\frac{\partial}{\partial \overline{z_1}} + z_2\overline{z_1}\frac{\partial^2}{\partial z_2 \partial \overline{z_1}}
\end{aligned}$$

Now, let $f \in \mathcal{H}_{p,q}(\mathbb{S}^3)$. Since f is harmonic, we know that $\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} = -\frac{\partial^2}{\partial z_2 \partial \overline{z_2}}$. Substituting, we get

$$\begin{aligned}
&= z_2\overline{z_2}\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + \overline{z_2}\frac{\partial}{\partial \overline{z_2}} + z_1\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} \\
&\quad + z_1\overline{z_1}\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \overline{z_1}\frac{\partial}{\partial \overline{z_1}} + z_2\overline{z_1}\frac{\partial^2}{\partial z_2 \partial \overline{z_1}}
\end{aligned}$$

Since f is a polynomial and $\Box_b$ is linear, it suffices to show that if $f = z^{\alpha}\overline{z}^{\beta} = z_1^{\alpha_1} z_2^{\alpha_2} \overline{z_1}^{\beta_1} \overline{z_2}^{\beta_2}$, where $\alpha_1 + \alpha_2 = p$ and $\beta_1 + \beta_2 = q$, then the claim holds. Using the expansion above, each derivative simply becomes a multiple of f, and we have

$$\begin{aligned}
\Box_b f &= (\alpha_2\beta_2 + \beta_2 + \alpha_1\beta_2 + \alpha_1\beta_1 + \beta_1 + \alpha_2\beta_1)f \\
&= ((\alpha_1 + \alpha_2)(\beta_1 + \beta_2) + (\beta_1 + \beta_2))f \\
&= (pq + q)f.
\end{aligned}$$

□

In a similar manner, we can show that $-\overline{\mathcal{L}}\mathcal{L}f = (pq+p)f$. For $\Box_b$, we actually have that $\operatorname{spec}(\Box_b) = \{pq + q \mid p, q \in \mathbb{N}\}$, therefore $0 \notin \operatorname{essspec}(\Box_b)$ since it is not an accumulation point of the set above.

3. Experimental Results in Mathematica

Using the symbolic computation environment provided by *Mathematica*, we are able to write a program to streamline our calculations[1]. We implement the algorithm provided in Proposition 2.5 to construct the vector space basis of $\mathcal{H}_k(\mathbb{S}^3)$ for a specified k. As an example, our code produces the following basis of $\mathcal{H}_3(\mathbb{S}^3)$:

$$\{-6\overline{z_2}^3, -6\overline{z_1}\,\overline{z_2}^2, -6\overline{z_1}^2\overline{z_2}, -6\overline{z_1}^3, 4z_1\overline{z_1}\,\overline{z_2} - 2z_2\overline{z_2}^2, 2z_1\overline{z_1}^2 - 4z_2\overline{z_1}\,\overline{z_2}, -6z_2\overline{z_1}^2, -6z_1\overline{z_2}^2,$$
$$4z_1z_2\overline{z_1} - 2z_2^2\overline{z_2}, -6z_2^2\overline{z_1}, 2z_1^2\overline{z_1} - 4z_1z_2\overline{z_2}, -6z_1^2\overline{z_2}, -6z_2^3, -6z_1z_2^2, -6z_1^2z_2, -6z_1^3\}.$$

Now, with the basis for $\mathcal{H}_k(\mathbb{S}^3)$, the matrix representation of $\Box_b^t$ on $\mathcal{H}_k(\mathbb{S}^3)$ can be computed for each k. In particular, we use this program to construct the matrix representations for $1 \le k \le 12$. For a specific k, the code applies $\Box_b^t$ to each basis element of $\mathcal{H}_k(\mathbb{S}^3)$ obtained by the results in the previous sections. Then, using the inner product defined by,

$$\langle f, g\rangle = \int_{\mathbb{S}^3} f\overline{g}\,d\sigma,$$

where σ is the standard surface area measure, the software computes $\langle \Box_b^t f_i, f_j\rangle$, where f_i, f_j are basis vectors for $\mathcal{H}_k(\mathbb{S}^3)$. With these results, *Mathematica* yields the matrix representation for the imputed value of k. For example, the program produces the matrix representation for $k = 3$ seen in Figure 3.1. Since each entry has a common normalization factor,

$$h = \frac{1+|t|^2}{(1-|t|^2)^2},$$

this constant has been factored out. With *Mathematica's* Eigenvalue function, the eigenval-

$$h\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6\bar{t} & 0 & 0 & 0 & 0\\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\bar{t} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -6\bar{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -6\bar{t} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 4+3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\bar{t} & 0\\
0 & 0 & 0 & 0 & 0 & 4+3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\bar{t} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 4+3|t|^2 & 0 & 0 & 0 & 0 & 0 & -2\bar{t} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4+3|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\bar{t}\\
0 & 0 & -2t & 0 & 0 & 0 & 0 & 0 & 3+4|t|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2t & 0 & 0 & 0 & 0 & 0 & 3+4|t|^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3+4|t|^2 & 0 & 0 & 0 & 0 & 0\\
-2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3+4|t|^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 3|t|^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2 & 0 & 0\\
0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3|t|^2
\end{pmatrix}$$

Figure 3.1. Matrix Representation of $\Box_b^t$ on $\mathcal{H}_3(\mathbb{S}^3)$

ues are then calculated for these matrix representations. Our numerical results suggest that the smallest non-zero eigenvalue of $\Box_b^t$ on $\mathcal{H}_{2k-1}(\mathbb{S}^3)$ decreases as k increases. Conversely,

[1]Our code for this and the other symbolic computations described below is available on our website at `https://sites.google.com/a/umich.edu/zeytuncu/home/publ`

the smallest non-zero eigenvalue of $\Box_b^t$ on $\mathcal{H}_{2k}(\mathbb{S}^3)$ increases with k. The smallest eigenvalue of $\mathcal{H}_{2k-1}(\mathbb{S}^3)$ is plotted for $1 \leq k \leq 5$ and $0 < |t| < 1$ in Figure 3.2. It is apparent that $\lambda_{\min,1} \leq \lambda_{\min,3} \leq \lambda_{\min,5} \leq \lambda_{\min,7} \leq \lambda_{\min,9}$ where $\lambda_{\min,k}$ denotes the smallest non-zero eigenvalue of $\Box_b^t$ on $\mathcal{H}_k(\mathbb{S}^3)$. These initial numerical results suggest that $\lim_{k\to\infty} \lambda_{\min,2k-1} = 0$ for $0 < |t| < 1$, which agrees with our final result.

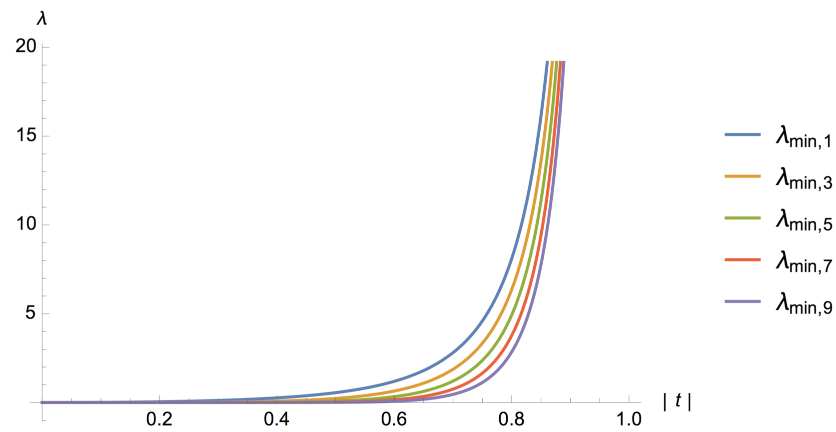

FIGURE 3.2. Smallest Non-Zero Eigenvalues for $k = 1, 3, 5, 7, 9$.

4. INVARIANT SUBSPACES OF $\mathcal{H}_{2k-1}(\mathbb{S}^3)$ UNDER $\Box_b^t$

In this section we fix $k \geq 1$ and work on $\mathcal{H}_{2k-1}(\mathbb{S}^3)$. As we have seen, $\Box_b^t$ can be expanded in the following way:

$$\begin{aligned} \Box_b^t &= -(\mathcal{L} + \bar{t}\overline{\mathcal{L}})\frac{1+|t|^2}{(1-|t|^2)^2}(\overline{\mathcal{L}} + t\mathcal{L}) \\ &= -h(\mathcal{L}\overline{\mathcal{L}} + |t|^2\overline{\mathcal{L}}\mathcal{L} + t\mathcal{L}^2 + \bar{t}\overline{\mathcal{L}}^2). \end{aligned} \tag{1}$$

This is because of the linearity of $\mathcal{L}$ and $\overline{\mathcal{L}}$. Now, we need the following property.

Lemma 4.1. *If $\langle f_i, f_j \rangle = 0$ and $f_i, f_j \in \mathcal{H}_{0,2k-1}(\mathbb{S}^3)$, then $\langle \overline{\mathcal{L}}^\sigma f_i, \overline{\mathcal{L}}^\sigma f_j \rangle = 0$ for $0 \leq \sigma \leq 2k-1$.*

Proof. Choose f_i and f_j in $\mathcal{H}_{0,2k-1}(\mathbb{S}^3)$ and $\langle f_i, f_j \rangle = 0$. We show that for $0 \leq \sigma \leq 2k-1$, $\overline{\mathcal{L}}^\sigma f_i$ and $\overline{\mathcal{L}}^\sigma f_j$ are orthogonal. To do this we use induction on σ. Suppose $\langle \overline{\mathcal{L}}^{\sigma-1} f_i, \overline{\mathcal{L}}^{\sigma-1} f_j \rangle = 0$, and we show that $\langle \overline{\mathcal{L}}^\sigma f_i, \overline{\mathcal{L}}^\sigma f_j \rangle = 0$. Note that, the adjoint of $\overline{\mathcal{L}}$ is $-\mathcal{L}$ and

$$\begin{aligned} \langle \overline{\mathcal{L}}^\sigma f_i, \overline{\mathcal{L}}^\sigma f_j \rangle &= \langle \overline{\mathcal{L}}^{\sigma-1} f_i, -\mathcal{L}\overline{\mathcal{L}}^\sigma f_j \rangle \\ &= \langle \overline{\mathcal{L}}^{\sigma-1} f_i, -(\mathcal{L}\overline{\mathcal{L}})\overline{\mathcal{L}}^{\sigma-1} f_j \rangle \\ &= \langle \overline{\mathcal{L}}^{\sigma-1} f_i, -\Box_b\overline{\mathcal{L}}^{\sigma-1} f_j \rangle. \end{aligned}$$

However[2] since $\overline{\mathcal{L}}^{\sigma-1} f_j \in \mathcal{H}_{\sigma-1,2k-1-\sigma+1}(\mathbb{S}^3)$, we know that $\Box_b \overline{\mathcal{L}}^{\sigma-1} f_j = (\sigma)(2k-\sigma-2)\overline{\mathcal{L}}^{\sigma-1} f_j$. Therefore,

$$\begin{aligned}\langle \overline{\mathcal{L}}^{\sigma-1} f_i, -\Box_b \overline{\mathcal{L}}^{\sigma-1} f_j \rangle &= \langle \overline{\mathcal{L}}^{\sigma-1} f_i, -(\sigma)(2k-\sigma-2)\overline{\mathcal{L}}^{\sigma-1} f_j \rangle \\ &= -(\sigma)(2k-\sigma-2)\langle \overline{\mathcal{L}}^{\sigma-1} f_i, \overline{\mathcal{L}}^{\sigma-1} f_j \rangle \\ &= 0,\end{aligned}$$

by our induction hypothesis as desired. □

With this, we first note that if $\{f_0, \dots, f_{2k-1}\}$ is an orthogonal basis for $\mathcal{H}_{0,2k-1}(\mathbb{S}^3)$, then $\{\overline{\mathcal{L}}^{\sigma} f_0, \dots, \overline{\mathcal{L}}^{\sigma} f_{2k-1}\}$ is an orthogonal basis for $\mathcal{H}_{\sigma,2k-1-\sigma}(\mathbb{S}^3)$. Now, we define the following subspaces of $\mathcal{H}_{2k-1}(\mathbb{S}^3)$.

Definition 4.1. Suppose $\{f_0, \dots, f_{2k-1}\}$ is an orthogonal basis for $\mathcal{H}_{0,2k-1}(\mathbb{S}^3)$. Then we define

$$\begin{aligned}V_i &= \operatorname{span}\{f_i, \overline{\mathcal{L}}^2 f_i, \dots, \overline{\mathcal{L}}^{2j-2} f_i, \dots, \overline{\mathcal{L}}^{2k-2} f_i\}, \\ W_i &= \operatorname{span}\{\overline{\mathcal{L}} f_i, \overline{\mathcal{L}}^3 f_i, \dots, \overline{\mathcal{L}}^{2j-1} f_i, \dots, \overline{\mathcal{L}}^{2k-1} f_i\}.\end{aligned}$$

Denote the basis elements of V_i by $v_{i,1}, \dots, v_{i,k}$ and for W_i by $w_{i,1}, \dots, w_{i,k}$. We first note that since each bidegree space $\mathcal{H}_{p,q}(\mathbb{S}^3) \subseteq \mathcal{H}_{2k-1}(\mathbb{S}^3)$ has $2k$ elements, we have $2k$ V_i spaces and $2k$ W_i spaces. We now note the following fact.

Theorem 4.1. $\displaystyle\bigoplus_{i=0}^{2k-1} V_i \oplus W_i = \mathcal{H}_{2k-1}(\mathbb{S}^3)$.

Proof. We first note that by Proposition 2.4,

$$\mathcal{H}_{2k-1}(\mathbb{S}^3) = \bigoplus_{i=0}^{2k-1} \mathcal{H}_{i,2k-1-i}(\mathbb{S}^3)$$

but by Lemma 4.1, we see that this is really just

$$= \bigoplus_{i=0}^{2k-1} \overline{\mathcal{L}}^i f_0 \oplus \dots \oplus \overline{\mathcal{L}}^i f_{2k-1}.$$

Manipulating this, we have

$$\begin{aligned}&= \bigoplus_{i=0}^{2k-1} f_i \oplus \overline{\mathcal{L}} f_i \cdots \oplus \overline{\mathcal{L}}^{2k-1} f_i \\ &= \bigoplus_{i=0}^{2k-1} f_i \oplus \overline{\mathcal{L}}^2 f_i \oplus \dots \oplus \overline{\mathcal{L}}^{2k-2} f_i \oplus \overline{\mathcal{L}} f_i \oplus \overline{\mathcal{L}}^3 f_i \oplus \dots \oplus \overline{\mathcal{L}}^{2k-1} f_i \\ &= \bigoplus_{i=0}^{2k-1} V_i \oplus W_i,\end{aligned}$$

which is our goal. □

The advantage of constructing these spaces in the first place is due to the following fact.

[2]For $f \in \mathcal{H}_{i,j}(\mathbb{S}^3)$ by counting degrees we notice $\overline{\mathcal{L}} f \in \mathcal{H}_{i-1,j+1}(\mathbb{S}^3)$.

Theorem 4.2. *For $0 \leq i \leq 2k-1$, the subspaces V_i and W_i are invariant under $\Box_b^t$.*

Proof. By equation (1), we have that

$$\Box_b^t = -h(\mathcal{L}\overline{\mathcal{L}} + |t|^2\overline{\mathcal{L}}\mathcal{L} + t\mathcal{L}^2 + \bar{t}\overline{\mathcal{L}}^2)$$

Since the fraction in front is a constant, we can ignore it and only consider the expression in the parentheses. Let $f \in \mathcal{H}_{0,2k-1}(\mathbb{S}^3)$, and define $v_\sigma = \overline{\mathcal{L}}^\sigma f$ to be a basis element of either V_i or W_i, since they have the same form. We first note that $v_\sigma \in \mathcal{H}_{\sigma,2k-1-\sigma}(\mathbb{S}^3)$. Then by our expansion we have that

$$\Box_b^t v_\sigma = -h(\mathcal{L}\overline{\mathcal{L}}v_\sigma + |t|^2\overline{\mathcal{L}}\mathcal{L}v_\sigma + t\mathcal{L}^2 v_\sigma + \bar{t}\overline{\mathcal{L}}^2 v_\sigma)$$

We already know $\mathcal{L}\overline{\mathcal{L}}v_\sigma$ and $\overline{\mathcal{L}}\mathcal{L}v_\sigma$ will simply be a multiple of v_σ, so we consider $\mathcal{L}^2 v_\sigma$ and $\overline{\mathcal{L}}^2 v_\sigma$.

$$\begin{aligned}
\mathcal{L}^2 v_\sigma &= \mathcal{L}^2\overline{\mathcal{L}}^\sigma f \\
&= \mathcal{L}\left[\mathcal{L}\overline{\mathcal{L}}\left[\overline{\mathcal{L}}^{\sigma-1} f\right]\right] \\
&= -(\sigma)(2k-\sigma)\mathcal{L}\overline{\mathcal{L}}\left[\overline{\mathcal{L}}^{\sigma-2} f\right] \\
&= (\sigma)(\sigma-1)(2k+1-\sigma)(2k-\sigma)\overline{\mathcal{L}}^{\sigma-2} f \\
&= (\sigma)(\sigma-1)(2k+1-\sigma)(2k-\sigma)v_{\sigma-2} && \text{(2a)} \\
\overline{\mathcal{L}}^2 v_\sigma &= \overline{\mathcal{L}}^2\left[\overline{\mathcal{L}}^\sigma f\right] \\
&= \overline{\mathcal{L}}^{\sigma+2} f \\
&= v_{\sigma+2} && \text{(2b)}
\end{aligned}$$

so we get multiples of $v_{\sigma-2}$ and $v_{\sigma+2}$. Relating this back to V_i and W_i, we see that if $\sigma = 2j-2$, then $\mathcal{L}^2 v_{i,j}$ is a multiple of $v_{i,j-1}$, and $\overline{\mathcal{L}}^2 v_{i,j}$ is a multiple of $v_{i,j+1}$. If $\sigma = 2j-1$, we get a similar result for $w_{i,j}$. So we indeed have that both subspaces V_i and W_i are invariant under $\Box_b^t$, and we are done. □

In light of this fact, we can consider $\Box_b^t$ not on the whole space $L^2(\mathbb{S}^3)$ or $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, but rather on these V_i and W_i spaces. In fact, we actually have a representation of $\Box_b^t$ on these spaces with respect to the orthogonal bases for V_i and W_i as in Definition 4.1.

Theorem 4.3. *The matrix representation of $\Box_b^t$, $m(\Box_b^t)$, on V_i and W_i is tridiagonal, where $m(\Box_b^t)$ on V_i is*

$$m(\Box_b^t) = h\begin{pmatrix} d_1 & u_1 & & & \\ -\bar{t} & d_2 & u_2 & & \\ & -\bar{t} & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & -\bar{t} & d_k \end{pmatrix}$$

where $u_j = -t \cdot (2j)(2j-1)(2k-2j)(2k-1-2j)$ *and* $d_j = (2j-1)(2k+1-2j) + |t|^2 \cdot (2j-2)(2k+2-2j)$. *For* W_i, *we get something similar:*

$$m(\Box_b^t) = h \begin{pmatrix} d_1 & u_1 & & & \\ -\bar{t} & d_2 & u_2 & & \\ & -\bar{t} & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & -\bar{t} & d_k \end{pmatrix}$$

where $u_j = -t \cdot (2j+1)(2j)(2k-2j)(2k-1-2j)$ *and* $d_j = (2j)(2k-2j) + |t|^2 \cdot (2j-1)(2k+1-2j)$.

We note that the above definitions don't depend on i; in other words, each of these matrices are the same on V_i and W_i, regardless of the choice of i.

Proof. Using equations (2a) and (2b), along with Theorem 2.1, we can entirely describe the action of each piece of $\Box_b^t$ on a basis element $v_{i,j}$ or $w_{i,j}$:

$$-\mathcal{L}\overline{\mathcal{L}}v_{i,j} = (2j-1)(2k+1-2j)v_{i,j}$$
$$-\overline{\mathcal{L}}\mathcal{L}v_{i,j} = (2j-2)(2k+2-2j)v_{i,j}$$
$$-\mathcal{L}^2 v_{i,j} = -(2j-2)(2j-3)(2k+3-2j)(2k+2-2j)v_{i,j-1}$$
$$-\overline{\mathcal{L}}^2 v_{i,j} = -v_{i,j+1}$$

$$-\mathcal{L}\overline{\mathcal{L}}w_{i,j} = (2j)(2k-2j)w_{i,j}$$
$$-\overline{\mathcal{L}}\mathcal{L}w_{i,j} = (2j-1)(2k+1-2j)w_{i,j}$$
$$-\mathcal{L}^2 w_{i,j} = -(2j-1)(2j-2)(2k+2-2j)(2k+1-2j)w_{i,j-1}$$
$$-\overline{\mathcal{L}}^2 w_{i,j} = -w_{i,j+1}.$$

By looking at it this way, we notice the tridiagonal structure. So with these observations, we can state that

$$\begin{aligned} \Box_b^t v_{i,j} = h\big(& -t \cdot (2j-2)(2j-3)(2k+3-2j)(2k+2-2j)v_{i,j-1} \\ & + \big((2j-1)(2k+1-2j) + |t|^2 \cdot (2j-2)(2k+2-2j)\big)v_{i,j} \\ & - \bar{t} \cdot v_{i,j+1} \big) \\ \Box_b^t w_{i,j} = h\big(& -t \cdot (2j-1)(2j-2)(2k+2-2j)(2k+1-2j)w_{i,j-1} \\ & + \big((2j)(2k-2j) + |t|^2 \cdot (2j-1)(2k-1-2j)\big)w_{i,j} \\ & - \bar{t} \cdot w_{i,j+1} \big). \end{aligned}$$

Now that we have this formula, we can find $m(\Box_b^t)$ on V_i and W_i by computing their effect on the basis vectors $v_{i,j}$ and $w_{i,j}$: when we do this for V_i, we get

$$\begin{aligned} d_j &= (2j-1)(2k+1-2j) + |t|^2 \cdot (2j-2)(2k+2-2j), \\ u_{j-1} &= -t \cdot (2j-2)(2j-3)(2k+3-2j)(2k+2-2j), \\ \text{hence } u_j &= -t \cdot (2j)(2j-1)(2k-2j)(2k-1-2j) \end{aligned}$$

and for W_i, we get

$$\begin{aligned} d_j &= (2j)(2k-2j) + |t|^2 \cdot (2j-1)(2k-1-2j), \\ u_{j-1} &= -t \cdot (2j-1)(2j-2)(2k+2-2j)(2k+1-2j), \\ \text{hence } u_j &= -t \cdot (2j+1)(2j)(2k-2j)(2k-1-2j). \end{aligned}$$

Finally, by factoring out h and simply substituting each portion in we obtain the matrix representations above. $\square$

An immediate consequence of this is that each V_i subspace contributes the same set of eigenvalues to the spectrum of $\Box_b^t$, and similarly for each W_i. Furthermore, we note that the matrices are rank k (by the tridiagonal structure it is at least rank $k-1$ and by Proposition 5.2 the determinant is non-zero, hence rank k). Since the choice of i does not change $m(\Box_b^t)$ on these spaces, we will fix an arbitrary i and call the spaces V and W instead.

5. Bottom of the Spectrum of $\Box_b^t$

Now that we have a matrix representation for $\Box_b^t$ on these V and W spaces inside $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, we can begin to analyze their eigenvalues as k varies. First, we go over some facts about tridiagonal matrices.

Proposition 5.1. *Suppose A is a tridiagonal matrix,*

$$A = \begin{pmatrix} d_1 & u_1 & & & \\ l_1 & d_2 & u_2 & & \\ & l_2 & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & l_{k-1} & d_k \end{pmatrix}$$

and the products $u_i l_i > 0$ for $1 \le i < k$, then A is similar to a symmetric tridiagonal matrix.

Proof. One can verify that if

$$S = \begin{pmatrix} 1 & & & & \\ & \sqrt{\frac{u_1}{l_1}} & & & \\ & & \sqrt{\frac{u_1 u_2}{l_1 l_2}} & & \\ & & & \ddots & \\ & & & & \sqrt{\frac{u_1 \cdots u_{k-1}}{l_1 \cdots l_{k-1}}} \end{pmatrix}$$

then $A = S^{-1} B S$, where

$$B = \begin{pmatrix} d_1 & \sqrt{u_1 l_1} & & & \\ \sqrt{u_1 l_1} & d_2 & \sqrt{u_2 l_2} & & \\ & \sqrt{u_2 l_2} & d_3 & \ddots & \\ & & \ddots & \ddots & \sqrt{u_{k-1} l_{k-1}} \\ & & & \sqrt{u_{k-1} l_{k-1}} & d_k \end{pmatrix}.$$

Therefore, A is similar to a symmetric tridiagonal matrix. $\square$

Another special property of tridiagonal matrices is the continuant.

Definition 5.1. Let A be a tridiagonal matrix, like the above. Then we define the *continuant* of A to be a recursive sequence: $f_1 = d_1$, and $f_i = d_i f_{i-1} - u_{i-1} l_{i-1} f_{i-2}$, where $f_0 = 1$.

The reason we define this is because $\det(A) = f_k$. In addition, if we denote A_i to mean the square sub-matrix of A formed by the first i rows and i columns, then $\det(A_i) = f_i$. With this background, we will now start analyzing $\square_b^t$ on W.

To get bounds on the eigenvalues, we will invoke the Cauchy interlacing theorem, see [Hwa04].

Theorem 5.1 (Cauchy Interlacing Theorem)**.** *Suppose E is an $n \times n$ Hermitian matrix of rank n, and F is an $n-1 \times n-1$ matrix minor of E. If the eigenvalues of E are $\lambda_1 \leq \cdots \leq \lambda_n$ and the eigenvalues of F are $\nu_1 \leq \cdots \leq \nu_{n-1}$, then the eigenvalues of E and F interlace:*

$$0 < \lambda_1 \leq \nu_1 \leq \lambda_2 \leq \nu_2 \leq \cdots \leq \lambda_{n-1} \leq \nu_{n-1} \leq \lambda_n$$

Now, we can get an intermediate bound on the smallest eigenvalue.

Theorem 5.2. *Suppose A is the Hermitian matrix of rank k, like the above, and $\lambda_1 \leq \cdots \leq \lambda_k$ are its eigenvalues. Then*

$$\lambda_1 \leq \frac{\det(A)}{\det(A_{k-1})}$$

where A_{k-1} is A without the last row and column.

Proof. Since A_{k-1} is a $k-1 \times k-1$ matrix minor of A, we can apply the Cauchy interlacing theorem. If the eigenvalues of A_{k-1} are $\nu_1 \leq \cdots \leq \nu_{k-1}$, then

$$\lambda_1 \leq \nu_1 \leq \lambda_2 \leq \nu_2 \leq \cdots \leq \lambda_{n-1} \leq \nu_{n-1} \leq \lambda_n$$

Now, we claim that

$$\lambda_1 \det(A_{k-1}) \leq \det(A)$$

To see why this is true, first observe that the determinant of a matrix is simply the product of all its eigenvalues. In particular,

$$\lambda_1 \det(A_{k-1}) = \lambda_1 \nu_1 \cdots \nu_{k-1}$$

But we can simply apply the Cauchy interlacing theorem: since $\nu_1 \leq \lambda_2$, $\nu_2 \leq \lambda_3$, and so on, we get

$$\begin{aligned} \lambda_1 \nu_1 \cdots \nu_{k-1} &\leq \lambda_1 \lambda_2 \cdots \lambda_k \\ &= \det(A). \end{aligned}$$

Now, dividing both sides by $\det A_{k-1}$,

$$\lambda_1 \leq \frac{\det(A)}{\det(A_{k-1})}$$

as desired. $\square$

Since $m(\Box_b^t)$ on W satisfies the conditions of Proposition 5.1, we find it is similar to this Hermitian tridiagonal matrix:

$$A = \begin{pmatrix} a_1 + b_1|t|^2 & c_1|t| & & & \\ c_1|t| & a_2 + b_2|t|^2 & c_2|t| & & \\ & c_2|t| & a_3 + b_3|t|^2 & \ddots & \\ & & \ddots & \ddots & c_{k-1}|t| \\ & & & c_{k-1}|t| & a_k + b_k|t|^2 \end{pmatrix}$$

where $a_i = (2i)(2k-2i)$, $b_i = (2i-1)(2k+1-2i)$, and $c_i = \sqrt{(2i+1)(2i)(2k-2i)(2k-1-2i)}$. Note that we are ignoring the constant h for now, which we will add back later. If we can find $\det(A_i)$, then by Theorem 5.2 we can get a closed form for the bound on the smallest eigenvalue. With the following lemma, this is possible:

Lemma 5.1. $a_i b_{i+1} = c_i^2$

Proof. This is easily verified using the formulas for a_i, b_{i+1} and c_i. Recall $a_i = (2i)(2k-2i)$, $b_{i+1} = (2i+1)(2k-1-2i)$, and $c_i^2 = (2i+1)(2i)(2k-2i)(2k-1-2i)$, so the products match up. □

Proposition 5.2. *The determinant of A_i is*

$$\begin{aligned} \det(A_i) = {} & a_1 a_2 \cdots a_{i-1} a_i \\ & + b_1 a_2 \cdots a_{i-1} a_i |t|^2 \\ & + \cdots \\ & + b_1 b_2 \cdots b_{i-1} a_i |t|^{2i-2} \\ & + b_1 b_2 \cdots b_{i-1} b_i |t|^{2i} \end{aligned}$$

In each row, we replace a particular a_j with b_j, and multiply by $|t|^2$. Note that if $i = k$, then $a_k = 0$ and all terms but the last term are 0.

Proof. We will prove this using strong induction on i. We start with the base case is $i = 1$, where $\det(A_1) = a_1 + b_1|t|^2$, which does indeed match up with our formula. Next we go over the case $i = 2$, and $\det(A_2) = (a_1 + b_1|t|^2)(a_2 + b_2|t|^2) - c_1^2|t|^2$. By Lemma 5.1 we obtain the desired formula.

Now, assume the formula works for A_{i-1} and A_i. We need to show that the formula works for A_{i+1}. Using the formula for the continuant, we get

$$\det(A_{i+1}) = (a_{i+1} + b_{i+1}|t|^2)\det(A_i) - c_i^2|t|^2 \det(A_{i-1})$$

Now, use Lemma 5.1:

$$= (a_{i+1} + b_{i+1}|t|^2)\det(A_i) - a_i b_{i+1}|t|^2 \det(A_{i-1})$$

Now, we use our induction hypothesis:

$$
\begin{aligned}
&= (a_{i+1} + b_{i+1}|t|^2)(a_1a_2\cdots a_i + b_1a_2\cdots a_i|t|^2 + \cdots + b_1b_2\cdots b_i|t|^{2i}) \\
&\quad - a_ib_{i+1}|t|^2(a_1a_2\cdots a_{i-1} + b_1a_2\cdots a_{i-1}|t|^2 + \cdots + b_1b_2\cdots b_{i-1}|t|^{2i-2}) \\
&= a_1a_2\cdots a_{i+1} + b_1a_2\cdots a_{i+1}|t|^2 + \cdots + b_1b_2\cdots b_ia_{i+1}|t|^{2i} \\
&\quad + a_1a_2\cdots a_ib_{i+1}|t|^2 + b_1a_2\cdots a_ib_{i+1}|t|^4 + \cdots + b_1b_2\cdots b_{i-1}a_ib_{i+1}|t|^{2i+2} + b_1b_2\cdots b_{i+1}|t|^{2i+2} \\
&\quad - a_1a_2\cdots a_ib_{i+1}|t|^2 - b_1a_2\cdots a_ib_{i+1}|t|^4 - \cdots - b_1b_2\cdots b_{i-1}a_ib_{i+1}|t|^{2i+2} \\
&= a_1a_2\cdots a_{i+1} + b_1a_2\cdots a_{i+1}|t|^2 + \cdots + b_1b_2\cdots b_ia_{i+1}|t|^{2i} + b_1b_2\cdots b_{i+1}|t|^{2i+2}
\end{aligned}
$$

which is the formula for A_{i+1}, and we are done. □

With this knowledge, we are finally able to prove our main result.

Theorem 5.3. $0 \in \mathrm{essspec}(\Box_b^t)$.

Proof. By Proposition 5.1, we have that on W in $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, $m(\Box_b^t)$ is similar to

$$
A = h\begin{pmatrix}
a_1 + b_1|t|^2 & c_1|t| & & & \\
c_1|t| & a_2 + b_2|t|^2 & c_2|t| & & \\
 & c_2|t| & a_3 + b_3|t|^2 & \ddots & \\
 & & \ddots & \ddots & c_{k-1}|t| \\
 & & & c_{k-1}|t| & a_k + b_k|t|^2
\end{pmatrix},
$$

where $a_j = (2j)(2k-2j)$, $b_j = (2j-1)(2k+1-2j)$, and $c_j = \sqrt{(2j+1)(2j)(2k-2j)(2k-1-2j)}$. Now, by Theorem 5.2 we know that

$$
\lambda_{\min} \leq \frac{\det(A)}{\det(A_{k-1})}.
$$

Recall that A_{k-1} denotes the submatrix formed by deleting the last row and column of the $k \times k$ matrix A. To show that $0 \in \mathrm{essspec}(\Box_b^t)$, we want to show that $\det(A)/\det(A_{k-1}) \to 0$ as $k \to \infty$. For this purpose we find an upper bound for $\det(A)/\det(A_{k-1})$ and show that this converges to 0. Notice that Proposition 5.2 implies that,

$$
\begin{aligned}
\frac{\det(A)}{\det(A_{k-1})} &= h\frac{b_1b_2\dots b_{k-1}b_k|t|^{2k}}{a_1a_2\dots a_{k-1} + b_1a_2\dots a_{k-1}|t|^2 + b_1b_2\dots a_{k-1}|t|^4 + \dots b_1b_2\dots b_{k-1}|t|^{2k-2}} \\
&\leq h\frac{b_1b_2\dots b_{k-1}b_k|t|^{2k}}{a_1a_2\dots a_{k-1}}. \qquad (3)
\end{aligned}
$$

since, a_j, b_j, and $|t| > 0$. Now using the formulas for a_j and b_j, notice that (3) can be written as

$$
h(2k-1)|t|^{2k}\prod_{j=1}^{k-1}\frac{(2j+1)(2k-2j-1)}{(2j)(2k-2j)}.
$$

However, we know that for all k and $1 \leq j \leq k-1$,

$$
\frac{(2k-2j-1)}{(2k-2j)} < 1,
$$

and so,

$$h(2k-1)|t|^{2k}\prod_{j=1}^{k-1}\frac{(2j+1)(2k-2j-1)}{(2j)(2k-2j)} \leq h(2k-1)|t|^{2k}\prod_{j=1}^{k-1}\frac{(2j+1)}{(2j)} = h(2k-1)|t|^{2k}\prod_{j=1}^{k-1}1+\frac{1}{2j}.$$

Furthermore, we have

$$h(2k-1)|t|^{2k}\prod_{j=1}^{k-1}1+\frac{1}{2j} \leq h(2k-1)|t|^{2k}\exp\left(\sum_{j=1}^{k-1}\frac{1}{2j}\right).$$

Note that

$$\sum_{j=1}^{k-1}\frac{1}{2j} \leq \frac{1}{2}\ln k + 1$$

so our expression becomes

$$\frac{\det(A)}{\det(A_{k-1})} \leq h(2k-1)|t|^{2k}\exp\left(1+\frac{1}{2}\ln k\right) = eh(2k-1)\sqrt{k}\,|t|^{2k}$$

and our problem reduces to showing that $\lim_{k\to\infty} eh(2k-1)\sqrt{k}|t|^{2k} = 0$. We note that h is a constant and $|t| < 1$; therefore, by L'Hospital's rule the last expression indeed goes to 0.

Finally, we have,

$$0 \leq \lim_{k\to\infty}\lambda_{\min} \leq \lim_{k\to\infty}\frac{\det(A)}{\det(A_{k-1})} \leq \lim_{k\to\infty} eh(2k-1)\sqrt{k}\,|t|^{2k} = 0,$$

and so $\lambda_{\min} \to 0$. Hence $0 \in \operatorname{essspec}(\Box_b^t)$. $\square$

We note that by the discussion in the introduction, this means that the CR-manifold $(\mathcal{L}_t, \mathbb{S}^3)$ is not embeddable into any $\mathbb{C}^N$.

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