

A classification of 3+1D bosonic topological orders (I): the case when pointlike excitations are all bosons

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Topological orders are new phases of matter beyond Landau symmetry breaking. They correspond to patterns of long-range entanglement. In recent years, it was shown that in 1+1D bosonic systems there is no nontrivial topological order, while in 2+1D bosonic systems the topological orders are classified by a pair: a modular tensor category and a chiral central charge. In this paper, following a new line of thinking, we find that in 3+1D the classification is much simpler than it was thought to be; we propose a partial classification of topological orders for 3+1D bosonic systems: If all the pointlike excitations are bosons, then such topological orders are classified by a simpler pair (G, ω_4) : a finite group G and its group 4-cocycle $\omega_4 \in \mathcal{H}^4[G; U(1)]$ (up to group automorphisms). Furthermore, all such 3+1D topological orders can be realized by Dijkgraaf-Witten gauge theories.

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I. INTRODUCTION

In history, we have completely classified some large class of matter states only for a few times. The first time is the classification of all spontaneous symmetry breaking orders^{1,2}. We find that all symmetry breaking orders can be described by a pair:

$$(G_\Psi \subset G_H), \quad (1)$$

where G_H is the symmetry group of the system and G_Ψ , a subgroup of G_H , is the symmetry group of the ground state.

The second time is the classification of all 1-dimensional gapped quantum phases. We find that 1-dimensional gapped quantum phases with on-site symmetry G_H can be classified by a triple (even for strongly interacting bosons/fermions)^{3,4}:

$$[G_\Psi \subset G_H; \text{pRep}(G_\Psi)], \quad (2)$$

where $\text{pRep}(G_\Psi)$ is a *projective representation* of G_Ψ ⁵. We see that all the 1-dimensional gapped quantum phases are described by symmetry breaking plus an addition structure described by $\text{pRep}(G_\Psi)$. The additional structure is the symmetry-protected topological (SPT) order⁶. The $\text{pRep}(G_\Psi)$ structure can equivalently be described by a pair (G_Ψ, ω_2) , where ω_2 is a group 2-cocycle in $\mathcal{H}^2[G_\Psi; U(1)]$.

The third time is the classification of 2-dimensional gapped quantum phases. In the absence of any symmetry, a gapped phase may have a nontrivial topological order⁷⁻⁹. We find that all 2+1D bosonic topological orders are classified by a pair:⁹⁻¹¹

$$(\text{MTC}, c), \quad (3)$$

where MTC is a unitary *modular tensor category* and c is the chiral central charge of the edge states. Physically, the tensor category theory MTC is just a theory that describes the fusion and the braiding of quasiparticle excitations, which correspond to fractional/non-abelian statistics. Modular means that every nontrivial quasiparticle has a nontrivial mutual statistics with some quasiparticles.

The above idea of classification focused on the properties of topological excitations. Later, it was generalized to fermionic systems¹², then included symmetries^{13,14}, and became a fairly complete classification. Traditionally symmetries are described by groups, but even when we focus on only the excitations, no information of the symmetry is missing. This is because excitations carry representations of the symmetry group, from which the group can be uniquely reconstructed (known as the Tanaka duality).

In particular, SPT orders^{15,16} in 2+1D, which were known to be classified by the pair (G, ω_3) where ω_3 are now group 3-cocycles, are also included in this complete classification as a special case. From the excitation point of view, they are equivalently classified by *minimal modular extensions of symmetric fusion categories* (SFC). Here SFC describes the excitations above the symmetric product state, carrying group representations. *Symmetric fusion* emphasizes the nature that these excitations have trivial statistics: they are either bosons or fermions and have trivial mutual braidings. There are only two variants of SFC: one is $\text{Rep}(G)$, the usual representation category, with all excitations being bosons; the other is

$\text{sRep}(G^f)$, for fermionic systems, with some representations assigned Fermi statistics. On the other hand, a minimal modular extension of the SFC just means a smallest MTC that contains the SFC. It corresponds to the *gauged* theory¹⁷ of the SPT order, which is a topological order with no symmetry.

After those fairly complete classification results in 1+1D and 2+1D, in this paper, we are going to study the classification of 3+1D topological orders. We will only deal with the simpler case, the 3+1D topological orders for bosonic systems. 3+1D bosonic topological orders are gapped quantum liquids^{18,19} without any symmetry. However, the physical definition of gapped quantum liquids, which is from the point of view of renormalization group or local unitary transformation, is far from applicable. Instead, similar as the classification in 2+1D which focuses on the properties of topological excitations, we start with some basic properties of 3+1D topological orders:

1. They have pointlike excitations and stringlike excitations, that can move in all directions and have nontrivial braidings among them.
2. The number of types of excitations is finite.
3. The statistics of pointlike excitations is trivial; they are bosons or fermions with trivial mutual statistics. In other words, the pointlike excitations in a 3+1D topological order are described fully by a SFC.
4. The braiding between pointlike and stringlike excitations, and among stringlike excitations must be non-degenerate, enough to detect every type of excitations. This is the principle of remote detectability, which will be elaborated on later.
5. The 2+1D boundary of 3+1D topological order is always gappable. Furthermore, any interface or boundary induced by condensation of bosonic pointlike excitations is gappable. A detailed argument is given in Sec. V A.
6. Just like 2+1D topological orders, the bulk 3+1D topological order is fully determined by its boundary (the so called boundary-bulk duality^{20,21}). Furthermore, one can generalize the Turaev-Viro TQFT²² or Levin-Wen model²³ to construct the explicit model of bulk 3+1D topological order from its gapped boundary. Mathematically this is the generalization of Drinfeld center to higher dimensions.

One may view the above as our axioms or assumptions to mathematically define a 3+1D topological order. This excludes the gapped non-liquid states in 3+1D, such as stacked fractional-quantum Hall layers¹⁸, and fractal/fracton topological states²⁴⁻³⁰ which include Haah's model³¹.

N_{exc}	N_{topAB}	classification: $\mathcal{H}_{\text{Aut}}^4(G)$	examples
1	1	$\mathcal{H}_{\text{Aut}}^4(Z_1) = \mathbb{Z}_1$	trivial product states
2	1	$\mathcal{H}_{\text{Aut}}^4(Z_2) = \mathbb{Z}_1$	Z_2 gauge theory
3	2	$\mathcal{H}_{\text{Aut}}^4(Z_3) = \mathbb{Z}_1, \mathcal{H}_{\text{Aut}}^4(S_3) = \mathbb{Z}_1$	Z_3 and S_3 gauge theories
4	5	$\mathcal{H}_{\text{Aut}}^4(Z_4) = \mathbb{Z}_1, \mathcal{H}_{\text{Aut}}^4(Z_2 \times Z_2) = \mathbb{Z}_2, \mathcal{H}_{\text{Aut}}^4(D_{10}) = \mathbb{Z}_1, \mathcal{H}_{\text{Aut}}^4(A_4) = \mathbb{Z}_1$	
...	

TABLE I. A table of simple 3+1D AB topological orders in bosonic systems, listed according the number N_{exc} of topological types for pointlike excitations. Here, N_{topAB} is the number of distinct AB topological orders and $\mathcal{H}_{\text{Aut}}^4(G)$ is the set of cocycles in $\mathcal{H}^4[G; U(1)]$ up to the automorphisms of G : $\mathcal{H}_{\text{Aut}}^4(G) = \mathcal{H}^4[G; U(1)]/\text{Aut}$. In particular, $\mathcal{H}_{\text{Aut}}^4[Z_2 \times Z_2; U(1)] = \mathbb{Z}_2^2$ and $\mathcal{H}_{\text{Aut}}^4(Z_2 \times Z_2) = \mathbb{Z}_2$.

If the pointlike excitations are all bosons, the corresponding SFC will always have the form $\text{Rep}(G)$ for some finite group G . In other words, the pointlike excitations of a 3+1D topological order can always be viewed as carrying irreducible representations of the group.³² They exactly behave like the quasiparticle excitations above a product state with G symmetry. This is a quite amazing result: a 3+1D topological order whose quasiparticles are all bosonic is always related to a finite group G .³³ We will refer to such (point-like-excitations-are-)all-boson topological order as AB topological order.

One may naturally wonder if a 3+1D AB topological order is always described by a G -gauge theory, since the pointlike excitations in a G -gauge theory are indeed described by $\text{Rep}(G)$. In fact, the above statement is not true. There are 3+1D topological orders arising from the Dijkgraaf-Witten gauge theory³⁴. Dijkgraaf-Witten gauge theories can be viewed as *gauged* SPTs which can be defined in any dimensions. They are determined by, again, a pair (G, ω_{d+1}) : ω_{d+1} is a group cocycle in $\mathcal{H}^{d+1}[G; U(1)]$ where d is the spacial dimension of the system. In 3+1D we need 4-cocycles. In a Dijkgraaf-Witten theory with (G, ω_4) , pointlike excitations are also described by $\text{Rep}(G)$. So we cannot say that all 3+1D topological order with $\text{Rep}(G)$ pointlike excitations are described by the usual G -gauge theory. But, do we have something even more general than Dijkgraaf-Witten gauge theory that also produce $\text{Rep}(G)$ pointlike excitations? In this paper, we like to show that there is nothing more general:

All 3+1D topological orders, whose pointlike excitations are all bosons, are classified by a finite group G and its group 4-cocycle $\omega_4 \in \mathcal{H}^4[G; U(1)]$, up to group automorphisms.

In this paper, “classified” always means a correspondence in a one-to-one fashion. The above result allows us to obtain a table of simple 3+1D AB topological orders (see Table I). Furthermore,

All 3+1D AB topological orders can be realized by Dijkgraaf-Witten gauge theory with a finite gauge group.

For a classification of 3+1D topological orders where some pointlike excitations are fermions, see Ref. 35.

The classification of 3+1D AB topological orders is obtained by condensing all the pointlike excitations in a 3+1D topological order \mathcal{C}^4 to form a new topological order \mathcal{D}^4 (which is possible when all the pointlike excitations are bosons), and argue that

1. The new phase \mathcal{D}^4 is a trivial phase. Therefore, \mathcal{C}^4 has a 2+1D gapped boundary \mathcal{M}^3 induced by such condensation, which carries only stringlike excitations.
2. The above stringlike excitations on the boundary are labeled by the elements of a finite group G , and their fusion rule is given by the group multiplication. It is the same group whose representations are carried by the pointlike excitations in the bulk.
3. The string-only boundary \mathcal{M}^3 form a unitary pointed fusion 2-category whose only nontrivial level are the objects. The different pointed fusion 2-categories are classified by a finite group G and its 4-cocycle ω_4 in $\mathcal{H}^4(G; U(1))$, up to group automorphisms.
4. The bulk topological order \mathcal{C}^4 is the center of the fusion 2-category \mathcal{M}^3 : $\mathcal{C}^4 = Z(\mathcal{M}^3)$ ^{20,21}, which is a Dijkgraaf-Witten gauge theory with (G, ω_4) . Furthermore, each bulk topological order \mathcal{C}^4 corresponds to a unique unitary pointed fusion 2-category \mathcal{M}^3 .

In the following, we will discuss some general properties of 3+1D topological orders. Then we will show the main result of the paper following the above four steps.

II. EXCITATIONS IN TOPOLOGICALLY ORDERED STATE

A. Pointlike excitations

1. Use trap Hamiltonian to define excitations

Consider a bosonic system defined by a local gapped Hamiltonian H_0 in d dimensional space M^d without

boundary. A collection of quasiparticle excitations labeled by p_i and located at \mathbf{x}_i can be produced as *gapped* ground states of $H_0 + \sum_i \delta H_{p_i}$ where δH_{p_i} is non-zero only near \mathbf{x}_i . By choosing different δH_{p_i} 's we can create (or trap) all kinds of pointlike excitations. The gapped ground states of $H_0 + \sum_i \delta H_{p_i}$ may have a degeneracy $D(M^d; p_1, p_2, \dots)$ which depends on the quasiparticle types p_1, p_2, \dots and the topology of the space M^d . The degeneracy is not exact, but becomes exact in the large space and large particle separation limit. We will use $\mathcal{V}(M^d; p_1, p_2, \dots)$ to denote the space of the degenerate ground states, which will also be called fusion space. If the Hamiltonian $H_0 + \sum_i \delta H_{p_i}$ is not gapped, we will say $D(M^d; p_1, p_2, \dots) = 0$ (i.e., $\mathcal{V}(M^d; p_1, p_2, \dots)$ has zero dimension). If $H_0 + \sum_i \delta H_{p_i}$ is gapped, but if δH_{p_i} also creates quasiparticles away from \mathbf{x}_i 's (indicated by the bump in the energy density away from \mathbf{x}_i 's), we will also say $D(M^d; p_1, p_2, \dots) = 0$. (In this case quasiparticles at \mathbf{x}_i 's do not fuse to trivial quasiparticles.) So, if $D(M^d; p_1, p_2, \dots) > 0$, δH_{p_i} only creates/traps quasiparticles at \mathbf{x}_i 's.

For topologically ordered state with no spontaneous symmetry breaking, the fusion space on d -dimensional sphere $M^d = S^d$ with no particles $\mathcal{V}(S^d)$ is always one dimensional. Thus in the presence of pointlike excitations, dimension of the fusion space, $\mathcal{V}(S^d; p_1, p_2, \dots)$ represents the total number of internal degrees of freedom for the quasiparticles p_1, p_2, \dots . To obtain the number of internal degrees of freedom for type- p_i quasiparticle, we consider the dimension $D(S^d; p_i, p_i, \dots, p_i)$ of the fusion space on n type- p_i particles on S^d . In large n limit $D(S^d; p_i, p_i, \dots, p_i)$ has a form

$$\ln D(S^d; p_i, p_i, \dots, p_i) = n(\ln d_{p_i} + o(1/n)). \quad (4)$$

Here d_{p_i} is called the quantum dimension of the type- p_i particle, which describe the internal degrees of freedom the particle. For example, a spin-0 particle has a quantum dimension $d = 1$, while a spin-1 particle has a quantum dimension $d = 3$.

2. Simple type and composite type

Two excitations p (trapped by ΔH_p) and p' (trapped by $\Delta H_{p'}$) are said to have the same type if the corresponding fusion spaces $\mathcal{V}(S^d; p, p_1, \dots)$ and $\mathcal{V}(S^d; p', p_1, \dots)$ can smoothly deform into each other as we change the trap Hamiltonian from ΔH_p to $\Delta H_{p'}$. Two excitations p and p' are of the same type iff they only differ by some local operators. If an excitation can be created by local operators from the ground state, the excitation will be said to have a trivial type, and denoted as **1**.

Even after quotient out the local excitations of trivial type, topological quasiparticle type still have two kinds: *simple type* and *composite type*: If the ground-state degenerate subspace $\mathcal{V}(M^d; p, q, \dots)$ cannot not be splitted by any small local perturbations near ΔH_p , then the par-

ticle p is said to be simple. Otherwise, the particle type p is said to be composite.

When p is composite, the fusion space $\mathcal{V}(M^d; p, q, \dots)$ has a direct sum decomposition (after splitting by a generic perturbation of ΔH_p):

$$\begin{aligned} & \mathcal{V}(M^d; p, q, \dots) \\ &= \mathcal{V}(M^d; p_1, q, \dots) \oplus \mathcal{V}(M^d; p_2, q, \dots) \\ & \quad \oplus \mathcal{V}(M^d; p_3, q, \dots) \oplus \dots \end{aligned} \quad (5)$$

where p_1, p_2, p_3, \dots are simple types. The above decomposition allows us to denote the composite type i_1 as

$$p = p_1 \oplus p_2 \oplus p_3 \oplus \dots \quad (6)$$

3. Fusion of pointlike excitations

When we fuse two simple types of topological particles p_1 and p_2 together, it may become a topological particle of a composite type:

$$p_1 \otimes p_2 = q = p'_3 \oplus p''_3 \oplus \dots \quad (7)$$

Here, we will use an integer tensor $N_{p_3}^{p_1 p_2}$ to describe the quasiparticle fusion, where p_i label simple types:

$$p_1 \otimes p_2 = \bigoplus_{p_3} N_{p_3}^{p_1 p_2} p_3. \quad (8)$$

Such an integer tensor $N_{p_3}^{p_1 p_2}$ is referred as the fusion coefficients of the topological order, which is a universal property of the topologically ordered state.

The internal degrees of freedom (i.e. the quantum dimension d_p) for the type- p simple particle can be calculated directly from $N_{p_3}^{p_1 p_2}$. In fact d_p is the largest eigenvalue of the matrix N_p , whose elements are $(N_p)_{p_2 p_1} = N_{p_2}^{p p_1}$.

B. Stringlike excitations

Similarly, we can also use *gapped* trap Hamiltonians $H_0 + \sum_s \Delta H_s$ to define stringlike excitations, where ΔH_s is no zero only near a loop. The ground state subspace of $H_0 + \sum_s \Delta H_s$ is called the fusion space of strings $\mathcal{V}(M^d; s, t, \dots)$. If the fusion spaces $\mathcal{V}(M^d; s, t, \dots)$ and $\mathcal{V}(M^d; s', t, \dots)$ can smoothly deform into each other, we say the strings s and s' are of the same type.

If the ground-state degenerate subspace $\mathcal{V}(M^d; s, t, \dots)$ cannot be split by any small *non-local* perturbations along the string s , then the string s is said to be simple. Otherwise, the string s is said to be composite. We stress that here we allow non-local perturbations along the string s . In other word, any degrees of freedom near the string can interact no matter how far are they. But the interactions do not

involve degrees of freedom far away from the string. The non-local perturbations is necessary. If we used local perturbations to define string types, we would have too many string types that are not related to the topological orders in the ground state.

The fusion of the string loops is also described by integer tensors

$$s_1 \otimes s_2 = \bigoplus_{s_3} M_{s_3}^{s_1 s_2} s_3. \quad (9)$$

The string loops can also shrink and become pointlike excitations

$$s_i \rightarrow \bigoplus_{p_j} M_{p_j}^{s_i} p_j. \quad (10)$$

We like to conjecture that

If $M_1^s > 1$, then the string s is not simple (*i.e.* s is a direct sum of several strings).

If a simple string satisfy $M_1^s = 1$, we say s is a *pure* simple string.

C. The on-string excitations are always gappable

The strings, as 1D extended objects, may carry excitations that travel along them. Those excitations can some times be gapless. In the following, we like to argue that, if the types of pointlike and stringlike excitations are finite, those on-string excitations can always be gapped by adding proper interactions.

We do this by contradiction. Assuming that on-string excitations cannot be gapped by interactions, based on what we know about 1+1D system, there are only two situations.

1. When the on-string excitations are chiral with a non-zero chiral central charge c . But in this case, when fusing n string together, the new string will have on-string excitations with chiral central charge nc . This means that the fusion will produce infinite types of strings. The finite string-type assumption excludes the possibility of on-string excitations with a non-zero chiral central charge.
2. When the on-string excitations are described by the edge of certain fractional-quantum Hall states (which have chiral central charge $c = 0$)³⁶. In this case, those on-string excitations have a gravitational anomaly described by a non-invertible 2+1D topological order^{20,37}. In this case, the open membrane operator that creates such a string on its boundary must creates a non-invertible 2+1D topological order on the membrane. Multiplying membrane operators corresponds to stacking 2+1D topological orders together, and stacking

non-invertible topological order can never produce a trivial topological order²⁰. Thus fusing n string together will always produce a new non-trivial string. Again, finite string-type assumption exclude this possibility.

We like to remark that ungappable strings can appear in gapped non-liquid states^{18,19}, such as the 3+1D gapped states obtained by stack fractional quantum Hall layers. But in that case, the strings are not mobile in all the directions. It appears that the liquid assumption of topological order^{18,19} makes all strings gappable in 3+1D. In the rest of this paper, we will always assume the on-string excitations to be gapped.

III. SOME GENERAL PROPERTIES OF 3+1D TOPOLOGICAL ORDERS OF BOSON SYSTEMS

A. The group structure in 3+1D topological order

We note that the pointlike excitations in 3+1D topological orders are described by a symmetric fusion category (SFC). Physically, a SFC is just a collection of particles which are all bosons or fermions with trivial mutual statistics.

Mathematically, it has been shown that a SFC must be either $\text{Rep}(G)$ (a braided fusion category (BFC) formed by the representations of G with all the irreducible representations being assigned Bose statistics) or $\text{sRep}(G)$ (a braided fusion category (BFC) formed by the representations of G with some of the irreducible representations being assigned Bose statistics while other irreducible representations being assigned Fermi statistics) for some group G . Such group G is uniquely determined (up to isomorphisms) by the SFC, namely by the pointlike excitations. This is known as the Tannaka duality, which will be further explained in Sec. VC.

The above implies that

Each 3+1D topological order is associated with a group G , where the pointlike excitations (particles) are described by $\text{Rep}(G)$ or $\text{sRep}(G)$.

In this paper, we will use this fact heavily to gain a systematic understanding of 3+1D topological orders. (In fact, each higher dimensional topological order is also related to a group in the same fashion.) In some sense, 3+1D topological orders can all be viewed as gauge theories with some old or new twists.

However, those pointlike excitations have trivial mutual statistics among them. One cannot use the pointlike excitations to detect other pointlike excitations by

remote operations. In general, we believe^{20,36}

The principle of remote detectability: In an anomaly-free topological order, every topological excitation can be detected by other topological excitations via some remote operations. If every topological excitation can be detected by other topological excitations via some remote operations, then the topological order is anomaly-free.

Here “anomaly-free” means realizable by a local bosonic model on lattice in the same dimension³⁷. Thus the remote detectability condition is also the anomaly-free condition.

The above implies that an anomaly-free (*i.e.* realizable) 3+1D topological order must contain stringlike topological excitations, so that every pointlike topological excitation can be detected by some stringlike topological excitations via remote braiding, and every stringlike topological excitation can be detected by some pointlike and/or stringlike topological excitations via remote braiding. We see that the properties of stringlike topological excitations are determined by the pointlike topological excitations (*i.e.* $\text{Rep}(G)$ or $\text{sRep}(G)$) to a certain degree.

B. Dimension reduction of topological orders

To understand better the relation between the pointlike and the stringlike excitations, we will introduce the dimension reduction in the next section, which turns out to be a very useful tool in our approach. We can reduce a $d+1$ D topological order \mathcal{C}^{d+1} on space-time $M^d \times S^1$ to d D topological orders on space-time M^d by making the circle S^1 small (see Fig. 1). In this limit, the $d+1$ D topological order \mathcal{C}^{d+1} can be viewed as several d D topological orders \mathcal{C}_i^d , $i = 1, 2, \dots, N_1^{\text{sec}}$ which happen to have degenerate ground state energy. We denote such a dimensional reduction process by

$$\mathcal{C}^{d+1} = \bigoplus_{i=1}^{N_1^{\text{sec}}} \mathcal{C}_i^d, \quad (11)$$

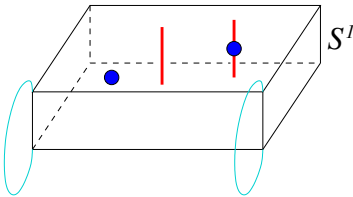


FIG. 1. (Color online) The dimension reduction of 3D space $M^2 \times S^1$ to 2D space M^2 . The top and the bottom surfaces are identified and the vertical direction is the compactified S^1 direction. A 3D pointlike excitation (the blue dot) becomes an anyon particle in 2D. A 3D stringlike excitation wrapping around S^1 (the red line) also becomes an anyon particle in 2D.

TABLE II. The dimension reduction $M^3 \times S^1$ to M^3 of the 3+1D S_3 -gauge theory $\mathcal{C}_{S_3}^4$, where G_χ 's are S_3, Z_2, Z_3 . The 3+1D pointlike excitations p_0, p_1, p_2 becomes the 2+1D pointlike excitations. The 3+1D looplike excitations $s_{\chi q}$, when wrapped around the S^1 , also becomes the 2+1D pointlike excitations. $\mathcal{C}_{S_3}^3$ is the untwisted sector where $\mathbf{1}, A^1, A^2$ correspond to $\text{Rep}(S_3)$. See Ref. 38 and Appendix A.

$\mathcal{C}_{S_3}^4 \rightarrow$	$\mathcal{C}_{S_3}^3$	$\mathcal{C}_{Z_2}^3$	$\mathcal{C}_{Z_3}^3$
Symmetry Breaking	$S_3 \rightarrow S_3$	$S_3 \rightarrow Z_2$	$S_3 \rightarrow Z_3$
$p_0 \rightarrow$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$p_1 \rightarrow$	A^1	e	$\mathbf{1}$
$p_2 \rightarrow$	A^2	$\mathbf{1} \oplus e$	$e_1 \oplus e_2$
$s_{20} \rightarrow$	B	m	-
$s_{21} \rightarrow$	B^1	em	-
$s_{30} \rightarrow$	C	-	$m_1 \oplus m_2$
$s_{31} \rightarrow$	C^1	-	$e_1 m_1 \oplus e_1 m_2$
$s_{32} \rightarrow$	C^2	-	$e_2 m_1 \oplus e_2 m_2$

where N_1^{sec} is the number of sectors produced by the dimensional reduction.

For example, let us use \mathcal{C}_G^{d+1} to denote the $d+1$ D topological order described by the gauge theory with the finite gauge group G . We find that, for $d \geq 3$ (see Table II)^{38,39},

$$\mathcal{C}_G^{d+1} = \bigoplus_{\chi} \mathcal{C}_{G_\chi}^d \quad (12)$$

where \bigoplus_{χ} sums over all different conjugacy classes χ of G , and G_χ is a subgroup of G formed by all the elements that commute with an element in χ . In fact, each dimension reduced d D topological order, $\mathcal{C}_{G_\chi}^d$, is produced by threading a G -gauge flux described by the conjugacy classes χ through the S^1 in the space-time $M^d \times S^1$. The χ -flux breaks the gauge symmetry G down to G_χ . Thus the corresponding d D topological order is a G_χ -gauge theory.

For Dijkgraaf-Witten theories (gauge theories twisted by group-cocycles of the gauge group), the dimension reduction have a form⁴⁰

$$\mathcal{C}_{G, \omega_{d+1}^G}^{d+1} = \bigoplus_{\chi} \mathcal{C}_{G_\chi, \omega_d^{G_\chi}(\chi)}^d, \quad (13)$$

where ω_{d+1}^G is a $(d+1)$ -group-cocycle $\omega_{d+1}^G \in \mathcal{H}^{d+1}(G; U(1))$, $\omega_d^{G_\chi}(\chi)$ is a d -group-cocycle $\omega_d^{G_\chi}(\chi) \in \mathcal{H}^d(G_\chi; U(1))$, and $\mathcal{C}_{G_\chi, \omega_d^{G_\chi}(\chi)}^d$ is the topological order described by Dijkgraaf-Witten theory with gauge group G and cocycle twist ω_{d+1}^G .

To understand the number of sectors N_1^{sec} in the dimension reduction, we note that the different sectors come from the different holonomy of moving pointlike excitations around the S^1 (see Fig. 1). For gauge theory, this so called holonomy comes from the gauge flux going through the compactified S^1 . For more general

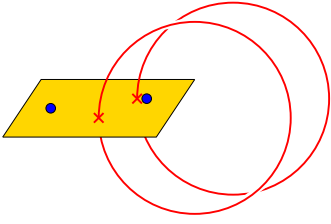


FIG. 2. (Color online) The untwisted sector in the dimension reduction can be realized directly on a 2D sub-manifold in 3D space without compactification.

topological orders, this holonomy comes from threading co-dimension 2 topological excitations through the S^1 .

From this picture, we see that the number of sectors N_1^{sec} is bounded by the number of types of the co-dimension 2 pure topological excitations. Also, if two co-dimension 2 topological excitations cannot be distinguished by their braiding with pointlike excitations, then threading them through the S^1 will not produce different sectors. Thus, the number of sectors N_1^{sec} is the number of the classes of co-dimension 2 topological excitations that can be distinguished by the braiding with the pointlike excitations. In particular, in 3+1D, the number of the sectors N_1^{sec} is the number of the classes of stringlike topological excitations that can be distinguished by the braiding with the pointlike excitations.

From the above discussion, we also see that the dimension reduction always contain a sector where we do not thread any nontrivial string through S^1 and the holonomy of moving any pointlike excitations around the S^1 is trivial. Such a sector will be called the untwisted sector. Also, if a topological order has no nontrivial pointlike excitation, then its dimension reduction contains only one sector – the untwisted sector.

In the untwisted sector, there are three kinds of anyons. The first kind of anyons correspond to the 3+1D pointlike excitations. The second kind of anyons correspond to the 3+1D pure stringlike excitations wrapping around the compactified S^1 . The third kind of anyons are bound states of the first two kinds (see Fig. 1).

We like to point out that the untwisted sector in the dimension reduction can even be realized directly in 3D space without compactification. Consider a 2D sub-manifold in the 3D space (see Fig. 2), and put the 3D pointlike excitations on the 2D sub-manifold. We can have a loop of string across the 2D sub-manifold which can be viewed as an effective pointlike excitation on the 2D sub-manifold. We can also have a bound state of the above two types of effective pointlike excitations on the 2D sub-manifold. Those effective pointlike excitations on the 2D sub-manifold can fuse and braid just like the anyons in 2+1D. The principle of remote detectability requires those effective pointlike excitations to form a MTC. When we perform dimension reduction, the above MTC becomes the untwisted sector of the dimension reduced 2+1D topological order. We like to men-

tion that the dimension reduction introduce new types of the perturbations that may not be local from 3+1D point of view. But those new perturbations are local in the dimension reduced 2+1D theory. MTC is very rigid which cannot be changed by any 2+1D local perturbations. This is why the untwisted sector is still described by the same MTC that describes the effective pointlike excitations on the 2D sub-manifold.

This way, we show that

The distinct (simple) pointlike excitations in the 3+1D topological order become the distinct anyons (*i.e.* the simple objects) in the dimension reduced 2+1D topological order that corresponds to the untwisted sector.

Table II describes the dimension reduction of a 3+1D topological order $\mathcal{C}_{S_3}^4$ described by S_3 -gauge theory:

$$\mathcal{C}_{S_3}^4 \rightarrow \mathcal{C}_{S_3}^3 \oplus \mathcal{C}_{Z_2}^3 \oplus \mathcal{C}_{Z_3}^3. \quad (14)$$

The 2+1D topological order $\mathcal{C}_{S_3}^3$ is the untwisted sector. The three types of particles in 3+1D $\mathcal{C}_{S_3}^4$, p_0, p_1, p_2 , that form a SFC $\text{Rep}(S_3)$ becomes three types of particles in the untwisted sector $\mathcal{C}_{S_3}^3$, $\mathbf{1}, A^1, A^2$, that also form a $\text{Rep}(S_3)$. We also see that in other sectors, the distinct pointlike excitations in 3+1D may not be reduced to distinct simple objects in the dimension reduced 2+1D topological orders.

Since the dimension reduced 2+1D topological orders must be anomaly-free, they must be described by modular tensor category. Since the untwisted sector always contains $\text{Rep}(G)$, we conclude that

The untwisted sector of a dimension reduced 3+1D topological order is a modular extension of $\text{Rep}(G)$.

In next section, we will show that such a modular extension must be a minimal one.

C. Untwisted sector of dimension reduction is the Drinfeld center of \mathcal{E}

In the following we will show a stronger result, for the untwisted sector. Given a 3+1D bosonic topological order, let the symmetric fusion category formed by the pointlike excitations be \mathcal{E} , $\mathcal{E} = \text{Rep}(G)$ or $\mathcal{E} = \text{sRep}(G^f)$.

The untwisted sector $\mathcal{C}_{\text{untw}}^3$ of dimension reduction of a 3+1D topological orders must be the 2+1D topological order described by Drinfeld center of \mathcal{E} : $\mathcal{C}_{\text{untw}}^3 = Z(\mathcal{E})$.

Note that Drinfeld center $Z(\mathcal{E})$ is the minimal modular extension of \mathcal{E} .

First, let us recall the definition of Drinfeld center. The Drinfeld center $Z(\mathcal{A})$ of a fusion category \mathcal{A} , is a braided fusion category, whose objects are pairs $(A, b_{A,-})$, where A is an object in \mathcal{A} , $b_{A,-}$ is a set of isomorphisms

$b_{A,X} : A \otimes X \cong X \otimes A, \forall X \in \mathcal{A}$, satisfying natural conditions. $b_{A,X}$ is called a half braiding. Morphisms between the pairs $(A, b_{A,-}), (B, b_{B,-})$ is a subset of morphisms between A, B , such that they commute with the half braidings $b_{A,-}, b_{B,-}$. The fusion and braiding of pairs is given by

$$(A, b_{A,-}) \otimes (B, b_{B,-}) = (A \otimes B, (b_{A,-} \otimes \text{id}_B)(\text{id}_A \otimes b_{B,-})),$$

$$c_{(A, b_{A,-}), (B, b_{B,-})} = b_{A,B}. \quad (15)$$

In other words, to half-braid $A \otimes B$, one just half-braids B and A successively, and the braiding between pairs is nothing but the half braiding.

$\mathcal{C}_{\text{untw}}^3 = Z(\mathcal{E})$ is the consequence that the strings in the untwisted sectors are in fact shrinkable. From the effective theory point of view, we can shrink a string s (including bound states of particles with strings, in particular, pointlike excitations viewed as bound states with the trivial string) to a pointlike excitation p_s^{shr} in \mathcal{E}

$$s \rightarrow p_s^{\text{shr}} = p_1 \oplus p_2 \oplus \dots, \quad p_1, p_2, \dots \in \mathcal{E} \quad (16)$$

So if we only consider fusion, the particles s, p in the dimension reduced untwisted sector $\mathcal{C}_{\text{untw}}^3$ can all be viewed as the particles in \mathcal{E} , regardless if they come from the 3D particles or 3D strings. In particular, the particles from the 3+1D strings s can be viewed as composite particles in \mathcal{E} (see eqn. (16)). To obtain the Drinfeld center we need to introduce braiding among those particles p 's and s 's.

In the untwisted sector, the braiding between strings s, s' , denoted by $c_{s,s'}$, requires string s' moving through string s , which prohibits shrinking string s . However, there is no harm to consider the shrinking if we focus on only the initial and end states of the braiding process.

In particular, the braiding between a string s and a particle p , induces an isomorphism between the initial and end states where the string s is shrunk (see Fig. 3)

$$c_{s,p}^{\text{shr}} : p_s^{\text{shr}} \otimes p \cong p \otimes p_s^{\text{shr}} \quad (17)$$

which is automatically a half-braiding on the particle p_s^{shr} . Thus, $(p_s^{\text{shr}}, c_{s,-}^{\text{shr}})$, by definition, is an object in the Drinfeld center $Z(\mathcal{E})$. Shrinking induces a functor

$$\mathcal{C}_{\text{untw}}^3 \rightarrow Z(\mathcal{E})$$

$$s \mapsto (p_s^{\text{shr}}, c_{s,-}^{\text{shr}}) \quad (18)$$

which is obviously monoidal and braided, *i.e.*, preserves fusion and braiding. It is also fully faithful, namely bijective on the morphisms. Physically this means that the local operators on both sides are the same. On the left side, morphisms on a string s are operators acting on near (local to) the string s ; on the right side, morphisms in the Drinfeld center are morphisms on the particle p_s^{shr} which commute with the half braiding $c_{s,-}^{\text{shr}}$. From the shrinking picture, morphisms on p_s^{shr} can be viewed as the operators acting on both near the string s and the interior of the string (namely on a disk D^2). But in order

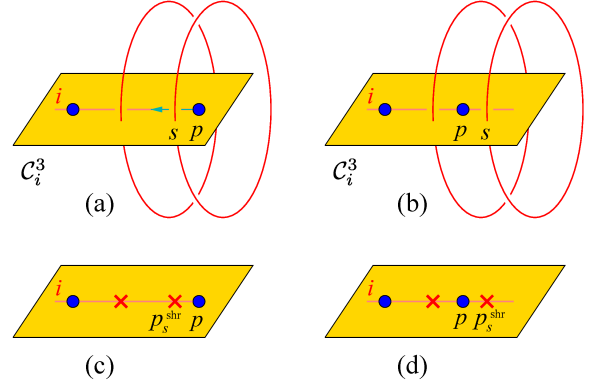


FIG. 3. (Color online) From (a) to (b) is the braiding $c_{s,p}$ in the untwisted sector. (c)(d) are obtained from (a)(b) by shrinking strings. Shrinking thus induces a “half-braiding” isomorphism $c_{s,p}^{\text{shr}}$ from (c) to (d).

to commute with $c_{s,p}$ for all p , which can be represented by string operators for all p going through the interior of the string s (this includes all possible string operators, because string operators for all particles form a basis), we can take only the operators that act trivially on the interior of the string. Therefore, morphisms on the right side are also operators acting on only near the string. This establishes that the functor is fully faithful, thus a braided monoidal embedding functor; in other words, $\mathcal{C}_{\text{untw}}^3$ can be viewed as a full sub-MTC of $Z(\mathcal{E})$. However, $Z(\mathcal{E})$ is already a *minimal* modular extension of \mathcal{E} , which implies that

$$\mathcal{C}_{\text{untw}}^3 = Z(\mathcal{E}). \quad (19)$$

As $Z(\mathcal{E})$ is known well, many properties can be easily extracted. For example, objects in $Z(\mathcal{E})$ have the form (χ, ρ) , where χ is a conjugacy class, ρ is a representation of the subgroup that centralizes χ . One then concludes

1. A looplike excitation in a 3+1D topological order always has an integer quantum dimension, which is $|\chi| \dim \rho$.
2. Pure strings (ρ trivial) always correspond to conjugacy classes of the group.

We also see that a 3+1D bosonic topological order is similar to a gauge theory of a finite group G . The following properties are the same:

- 1) the quantum dimensions of pointlike and stringlike excitations.
- 2) the fusion rule of those excitations.
- 3) particle-loop and two-loop braidings.

IV. CONDENSING ALL THE POINTLIKE EXCITATIONS TO OBTAIN A TRIVIAL TOPOLOGICAL ORDER

Starting from this section, we are going to show the main result of the paper via the four steps outlined at

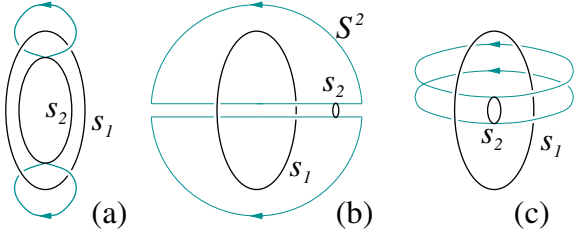


FIG. 4. (Color online) The braiding path of moving the string s_2 around s_1 . (a), (b), (c) described the same kind of the braiding paths that can deform into each other smoothly.

the end of introduction. First, we like to show that condensing all the pointlike excitations in 3+1D always gives us a trivial topological order. To do so, we first like to show the following:

A. There is no 3+1D topological order with only nontrivial stringlike excitations

Such a result can be shown using the principle of remote detectability in Section III A. When there is no nontrivial pointlike excitations, the remote detectability condition requires that a single loop of string can be remotely detected by braiding other stringlike excitations around the loop. Such a braiding is the two-string braiding described by Fig. 4a where a string s_2 is braided around a loop s_1 .

We can also use the dimension reduction picture to show that the anomaly-free condition requires that the two-string braiding must be nontrivial. Since there is no nontrivial pointlike excitations, the dimension reduction contains only the untwisted sector. In the 2+1D dimension reduced topological order, all the nontrivial anyons come from the pure strings in the 3+1D topological order, and correspond to the pure strings wrapping around the compactified S^1 . The 2+1D topological order is anomaly-free and the anyons form a modular tensor category. Physically, it means that any nontrivial anyon must have nontrivial mutual statistics with some anyons. This implies that any nontrivial pure strings in 3+1D must have nontrivial two-string braiding with some strings.

Next we like to show that the two-string braiding is always trivial when there is no nontrivial pointlike excitations. This is because the braiding path of s_2 -string around a s_1 -string in Fig. 4a is a torus wrapped around the loop s_1 . Such a torus can be deformed into a sphere S^2 around s_1 with a thin tube going through its center (see Fig. 4b). If the total space is a 3-sphere S^3 , we can deform the sphere S^2 into a small sphere on the other side of S^3 . This deforms the braiding path of s_2 into a thin torus, that describes a small string s_2 braiding around the loop s_1 (see Fig. 4c). This is like shrinking the string s_2 into a point and let the point braid around the loop s_1 . Since there is no nontrivial pointlike excita-

tions, the point that represents the small s_2 must have the trivial braiding phase around the loop s_1 . This way, we show that the two strings must have trivial braiding around each other when there is no nontrivial pointlike excitations. Therefore,

The 3+1D topological orders with only stringlike excitations cannot exist (*i.e.* they must be anomalous).

B. There is no nontrivial stringlike excitations that have trivial braiding with all pointlike excitations

Let us assume that there is a nontrivial stringlike excitation s , that has trivial braiding with all pointlike excitations. If all the pointlike excitations are bosons, then we can condense all the pointlike excitations to obtain a new 3+1D topological order, which will have no nontrivial pointlike excitations. But since the string s has trivial braiding with all pointlike excitations, it can survive the condensation and become a nontrivial stringlike excitation in the new 3+1D topological order.

However, in the last section, we have shown that 3+1D topological orders with only stringlike excitations cannot exist. This contradiction implies that

There is no nontrivial stringlike excitations with trivial braiding with all pointlike excitations, if all the pointlike excitations are bosons.

This result also implies that

The untwisted sector of a dimension reduced 3+1D topological order is a minimal modular extension of $\text{Rep}(G)$.

This is because, in the untwisted sector, other anyons beside $\text{Rep}(G)$ all come from strings in 3+1D, which all have nontrivial braiding with the particles in $\text{Rep}(G)$. This implies the modular extension to be minimal. However, the above result is weaker than that obtained in Section. III C.

C. Condensing all the pointlike excitations gives rise to a trivial 3+1D topological order

When all the pointlike excitations are bosons, we can obtain a new topological order by condensing all the pointlike excitations. The new topological order has no nontrivial pointlike excitations (since they are all condensed) and has no nontrivial stringlike excitations (they are confined due to the nontrivial braiding with the pointlike excitations). Thus the new topological order must be an invertible topological order. But in 3+1D all the invertible topological orders are the trivial one^{20,41,42}. Hence condensing all the pointlike excitations gives us a trivial 3+1D topological order.

In a gauge theory, condensing all the pointlike excitations corresponds to condensing all the charged excitations, which breaks all the “gauge symmetry”. This will give us an Anderson-Higgs phase, which is a trivial phase with no topological order.

V. STRING-ONLY BOUNDARY OF 3+1D TOPOLOGICAL ORDER

A. The canonical gapped boundary of 3+1D topological order

In this section, we are going to study a particular boundary of 3+1D topological orders. In Ref. 20, it was argued that all 3+1D bosonic topological orders can have a gapped boundary. The argument goes as follows.

First, topological orders have an important topological invariant: the ground state degeneracy on closed spaces⁴³. To obtain a more detailed characterization of topological orders, we consider degenerate ground states on closed spaces with different metrics. The set of closed spaces with different metrics forms the moduli space. So the degenerate ground states on closed spaces with different metrics form a vector bundle over the moduli space⁷⁻⁹. Such a vector bundle should fully characterize topological orders.

We can view the space as a boundary of the space-time. If the above vector bundle admits a flat connection, it means that the corresponding partition function on space-time with a boundary can be independent of perturbation of space-time metrics. This implies that for such a special realization of the topological order, the correlations of local operators on the boundary are short ranged, *i.e.* the boundary is gapped.

If the vector bundle does not admit a flat connection (*i.e.* when the Chern class is non-trivial), then the non-flat connection must be a $U(1)$ connection, since the ground state degeneracy cannot be lifted by any perturbations. Such a non-flat $U(1)$ connection is described by a gravitational Chern-Simons term in the partition function on closed space-time (for more details, see Ref. 20). But in 3+1D, there is no gravitational Chern-Simons term. So the vector bundle always admits a flat connection. Thus all 3+1D bosonic topological orders can have a gapped boundary. We call such a gapped boundary an anomalous 2+1D topological order.

It is known in the theory of 2+1D topological orders, that the interface induced by boson condensation, between the original phase and the condensed phase, is always gappable. We believe that this is also true in higher dimensional topological orders. The idea is as follows. In the most clean case, where we do nothing more than forcing the desired bosonic pointlike excitations to condense, new excitations in the condensed phase and on the interface in-between should all come from the original phase. More precisely, those excitations that have non-trivial braiding with the condensed boson can no longer

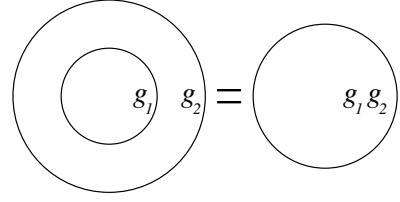


FIG. 5. The fusion of boundary stringlike excitations $s_{g_1}^{\text{bdry}} \otimes s_{g_2}^{\text{bdry}} = s_{g_1 g_2}^{\text{bdry}}$ which can be abbreviated as $g_1 \otimes g_2 = g_1 g_2$.

move freely in the condensed phase; they are confined and stuck on the interface. On the other hand, those excitations that have trivial braiding with the condensed boson survive and become excitations in the condensed phase. Since all the excitations in the original phase are gapped, we believe that both the condensed phase and the interface are also gapped. When the condensed phase is a trivial, the interface is a gapped boundary.

From the last section, we see that every 3+1D AB topological order (where all pointlike excitations are bosons) has a boundary obtained by condensing all the pointlike excitations. Such a boundary has no topologically non-trivial pointlike excitations, and has only stringlike excitations. In light of the above argument, we believe that such a string-only boundary is gapped:

Every 3+1D AB topological order has a string-only gapped boundary.

B. Unitary pointed fusion 2-category

We will show that the string-only gapped boundary is described by a so called *unitary pointed fusion 2-category*. But what is a fusion 2-categories? In general, a fusion category describes the fusion of codimension-1 excitations, *i.e.* domain-wall excitations. In 1-dimensional space, the domain-wall excitations are pointlike. The fusion of those pointlike excitations in 1D space is described by a fusion 1-category (which is also called fusion category). In 2-dimensional space, the domain-wall excitations are stringlike. The fusion of those stringlike excitations in 2D space is described by a fusion 2-category.

We like to point out that the fusion 2-categories that describes the string-only boundary of 3+1D AB topological order are very special: (1) the stringlike excitations on the boundary are labeled by the group elements of G : s_g^{bdry} , $g \in G$. (2) The fusion of the boundary stringlike excitations (see Fig. 5) is very simple and is given by the group multiplication

$$s_{g_1}^{\text{bdry}} \otimes s_{g_2}^{\text{bdry}} = s_{g_1 g_2}^{\text{bdry}}. \quad (20)$$

The fusion 2-categories with the above type of fusion rule are called pointed fusion 2-categories. Such an amazing result is a consequence of condensing all the pointlike excitations described by $\text{Rep}(G)$ on the boundary.

One way to show the above result is to consider the untwisted sector of the dimension reduction, which is a 2+1D topological order. We have shown that the untwisted sector is a minimal modular extension of $\text{Rep}(G)$ in Sections III C and IV B. The 2+1D boundary with only strings corresponds to a 1+1D boundary with only particles of the untwisted sector in the dimension reduced 2+1D topological order. Such a 1+1D boundary is obtained by condensing all the anyons in $\text{Rep}(G)$. The corresponding mathematical problem has already been solved, see for example Ref. 44; we reorganized the related mathematical results and provided physical interpretations in Ref. 14 (Section VI D). We find that the particles on such 1+1D boundary of the untwisted sector are labeled by group elements in G with a fusion given by group multiplication. Those 1+1D boundary particles correspond to the strings on the 2+1D boundary (see Fig. 1), this allows us to show eqn. (20). In the next a few sections, we will give a different argument without using dimension reduction.

C. Tannaka Duality in more explicit language

Our argument relies heavily on the Tannaka duality, or Tannaka reconstruction theorem for group representations. It is exactly how we extract the group G from an abstract symmetric fusion category (SFC). A naive example is that for an abelian group, the tensor product of its irreducible representations, has exactly the same group structure, which can be viewed as a Fourier transformation.

In more general cases, one can reconstruct a group G from its representation category $\text{Rep}(G)$, by the automorphisms of a fiber functor, namely a functor F from $\text{Rep}(G)$ to the category of vector spaces Vect , that preserves the fusion and braiding. We know that the category of vector spaces Vect describes particles in a trivial phase (*i.e.* in a product state with no symmetry). So one way to physically realize a fiber functor is by condensing (or other ways such as symmetry breaking) a nontrivial phase to a trivial phase. With a fiber functor F , we have

$$G \cong \text{Aut}(F : \text{Rep}(G) \rightarrow \text{Vect}). \quad (21)$$

To understand the physical significance of the above amazing result, let us consider a physical problem: given a system with a symmetry whose ground state is a product state with the symmetry, if we only measure the system via probes that do not break the symmetry, can we determine the symmetry group of the system? Here symmetric probes correspond to operators O that commute with all group actions, $gOg^{-1} = O, \forall g$. Generic group actions are not symmetric probes, unless they are in the center of the group. On the other hand, the fusion and braiding of the pointlike excitations above the ground state correspond to symmetric operation. The representation category $\text{Rep}(G)$ contains only those symmetric

probes. Tannaka duality tells us that we can indeed determine the symmetry group via only symmetric probes. Although the fiber functor seems to break the symmetry if we realize it physically, mathematically it is proven that such fiber functor always exists and is unique up to isomorphisms. Therefore, from the data of symmetric probes (fusion and braiding) in $\text{Rep}(G)$, we can obtain (formally calculate) the group G up to isomorphisms, without really breaking the symmetry of the system.

Now let us try to break the abstract theorem into more explicit terms. Firstly, the fiber functor means nothing but realizing the abstract fusion and braiding in $\text{Rep}(G)$ category with the tensor product and (trivial) braiding of concrete Hilbert spaces in a quantum system. It is helpful to consider how we build $\text{Rep}(G)$ in Vect : a group representation is a vector space V equipped with a group action $\rho_V : G \rightarrow \text{GL}(V)$. Moreover, there is a monoidal structure for the representations, which is taking the tensor product of the vector spaces $V \otimes_{\mathbb{C}} W$ and the new group action is $\rho_{V \otimes_{\mathbb{C}} W}(g) = \rho_V(g) \otimes_{\mathbb{C}} \rho_W(g)$ (which is called the fusion of group representations).

The Tannaka duality goes exactly the other direction. Assuming that we know a representation category $\text{Rep}(G)$, which contains only information on symmetric operations such as how the representations fuse with each other, can we obtain the group actions and also the group? The answer of the theorem then goes:

1. If we have a collection of invertible linear maps α_X for each irreducible representation X , acting on the vector space $F(X)$ assigned to X by a fiber functor F , such that
2. They are compatible with the fusion, in the sense that $\alpha_{X \otimes Y} = \alpha_X \otimes_{\mathbb{C}} \alpha_Y$,

$$\begin{array}{ccc} F(X \otimes Y) & \xlongequal{\sim} & F(X) \otimes_{\mathbb{C}} F(Y) \\ \downarrow \alpha_{X \otimes Y} & & \downarrow \alpha_X \otimes_{\mathbb{C}} \alpha_Y \\ F(X \otimes Y) & \xlongequal{\sim} & F(X) \otimes_{\mathbb{C}} F(Y) \end{array} \quad (22)$$

It is possible that $X \otimes Y$ is a reducible representation. We extend the linear maps α_W to W being reducible representations by direct sums, *i.e.* if W is the direct sum of irreducible representations W_i , $W = \bigoplus_i W_i$, α_W is given by the corresponding direct sum $\alpha_W = \bigoplus_i \alpha_{W_i}$.

This collection of invertible linear maps α_X , must correspond to the action of some group element $g \in G$, $\alpha_X = \rho_{F(X)}(g)$.

Moreover, take all collections of such invertible linear maps, they form a group under composition, namely the automorphism group of the fiber functor, $\text{Aut}(F : \text{Rep}(G) \rightarrow \text{Vect})$. It is isomorphic to G . In other words, if at the beginning we are given an abstract bosonic SFC \mathcal{E} , with a fiber functor $F : \mathcal{E} \rightarrow \text{Vect}$, we can use the above reconstruction to extract the group underlying \mathcal{E} , via $\mathcal{E} \cong \text{Rep}(\text{Aut}(F : \mathcal{E} \rightarrow \text{Vect}))$.

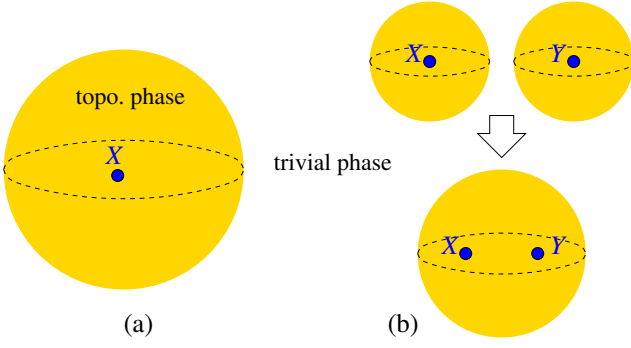


FIG. 6. (Color online) (a) The fusion space $F(X)$ for a 3-disk D^3 containing only one particle X . (b) Merging two 3-disks to one 3-disk induces an isomorphism $F(X) \otimes_{\mathbb{C}} F(Y) \cong F(X \otimes Y)$.

D. Fusion of boundary strings recover the group

Let us focus on the loop excitations on the string-only boundary. A loop excitation shrunk to a point may become a direct sum of pointlike excitations (see eqn. (A20))

$$s = n\mathbf{1} \oplus \dots \quad (23)$$

where $\mathbf{1}$ is the trivial pointlike excitation and \dots represent other possible nontrivial pointlike excitations. When $n = 0$, the string is not pure. Another possibility is that $n > 1$. In this case the string is unstable; it has accidental degeneracy which can be lifted by perturbations. So the pure simple strings have $n = 1$.

Since there is only trivial particle on the boundary, when we shrink a loop on the boundary, it must become a multiple of the trivial particle, $n\mathbf{1}$. Thus, it suffices to consider only the *simple* loops ($n = 1$) on the boundary, which shrink to the trivial particle $\mathbf{1}$. In other words, simple loops on the boundary shrink to *nothing*; this is an essential property in the following discussions. We note that such simple loops have a quantum dimension $d = 1$, and their fusion is grouplike. For the moment, we denote the group formed by the simple loops on the boundary under fusion (see Fig. 5), by H .

To apply the Tannaka duality, we need a physical realization of the fiber functor. Consider a simple topology for a string-only boundary: put the 3+1D topological order \mathcal{C}^4 in a 3-disk D^3 , the boundary on $\partial D^3 = S^2$, and outside is the trivial phase \mathcal{D}^4 . When there is only a particle X in the 3-disk, with no string and no other particles, we associate the corresponding fusion space (the physical states with such a configuration) to the particle X , and denote this fusion space by $F(X)$ (see Fig. 6). Viewed from very far away, a 3-disk containing a particle X is like a “local excitation” in the trivial phase, thus $F(X)$ mimics a local Hilbert space. When there are two 3-disks, each containing only one particle, X and Y respectively, the fusion space is $F(X) \otimes_{\mathbb{C}} F(Y)$. Moreover, as adiabatically deforming the system will not change

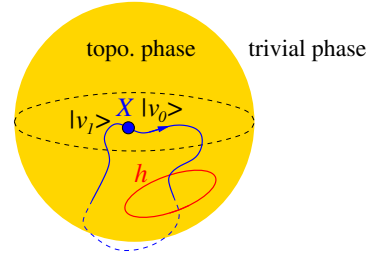


FIG. 7. (Color online) Moving a particle (blue) around a loop excitation (red) on the boundary. The solid line is a half-braiding path. The dashed line is the complementing path in the trivial phase.

the fusion space, we can “merge” the two 3-disks to obtain one 3-disk containing one particle $X \otimes Y$. Therefore $F(X) \otimes_{\mathbb{C}} F(Y) \cong F(X \otimes Y)$. Similarly, F also preserves the braiding of particles. In other words, the assignment $X \rightarrow F(X)$ gives rise to a fiber functor. By Tannaka duality, we can, at least formally, reconstruct a group $G = \text{Aut}(F)$, such that the particles in the bulk \mathcal{C}^4 are identified with $\text{Rep}(G)$. Our goal is to show that the fusion group H of the simple loops on the boundary, is the same as G .

To do this we consider the process of adiabatically moving a particle X around a simple loop $h \in H$ on the boundary, as shown in Fig. 7. As the simple loop shrinks to nothing, inserting simple loops will not change the fusion space. But an initial state $|v_0\rangle \in F(X)$, after such an adiabatically moving process, can evolve into a different end state $|v_1\rangle \in F(X)$. Thus, braiding X around h induces an invertible (since we can always move X backwards) linear map on the fusion space $F(X)$, $\alpha_{X,h} : |v_0\rangle \mapsto |v_1\rangle$.

Next, consider that we have two particles X, Y in the bulk. If we braid them together (fusing them to one particle $X \otimes Y$) around the simple loop h , we obtain the linear map $\alpha_{X \otimes Y, h}$. If the fusion of the bulk particles is given by $X \otimes Y = \bigoplus_i W_i$, we can split $X \otimes Y$ to the irreducible representations W_i , and braid W_i with h ; in other words, $\alpha_{X \otimes Y, h} = \bigoplus_i \alpha_{W_i, h}$.

But it is also equivalent if we move X, Y one after the other. More precisely, we can first separate Y into another 3-disk, braid X with h , and then merge Y back to the original 3-disk. Thus moving X alone corresponds to the linear map $\alpha_{X,h} \otimes_{\mathbb{C}} \text{id}_{F(Y)}$. Similarly, moving Y alone corresponds to $\text{id}_{F(X)} \otimes_{\mathbb{C}} \alpha_{Y,h}$ and in total we have the linear map $\alpha_{X,h} \otimes_{\mathbb{C}} \alpha_{Y,h}$. Therefore, $\alpha_{X \otimes Y, h} = \alpha_{X,h} \otimes_{\mathbb{C}} \alpha_{Y,h}$, or using only irreducible representations,

$$\alpha_{X,h} \otimes_{\mathbb{C}} \alpha_{Y,h} = \bigoplus_i \alpha_{W_i, h}. \quad (24)$$

These linear maps are compatible with the fusion of bulk particles.

Moreover, the simple loop h provides such an invertible linear map $\alpha_{X,h}$ for *each* particle $X \in \text{Rep}(G)$ in \mathcal{C}^4 , thus by Tannaka duality, these linear maps must corre-

spond to the action of certain group element $\varphi(h) \in G$, $\alpha_{X,h} = \rho_{F(X)}(\varphi(h))$. In other words, we obtain a map φ from the simple loops H to G , $\varphi : H \rightarrow G$. It is compatible with the fusion of simple loops, because the path of braiding around two concentric simple loops, g_1, g_2 (as in Fig. 5), separately, can be continuously deform to the braiding path around the two loops together, or around their fusion $g_1 \otimes g_2 = g_1 g_2$. This implies that $\varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)$, namely, φ is a group homomorphism.

What we really want is that φ is an isomorphism and $H = G$. This is a consequence of the remote detectability condition. Before proving it, we explain in detail the principle of remote detectability near the string-only boundary. The general idea is the same, that everything must be detectable remotely. Near a string-only boundary, the only way to perform remote detection is the *half-braiding* between bulk particles and boundary strings^{20,36}. Therefore,

- (1) there is no nontrivial boundary string that has trivial half-braiding with all the bulk particles (boundary strings are detectable by bulk particles).
- (2) there is no nontrivial bulk particle that has trivial half-braiding with all the boundary string.

One may have doubts in (2): even if bulk particles can not be detected by boundary strings, we may still have bulk strings to detect them. The reason for (2) is that we believe generalized boundary-bulk duality, that the bulk strings can always be viewed as certain “lift” of boundary strings to the bulk.^{20,21} If a bulk particle has trivial half-braiding with all boundary strings, it also has trivial braiding with all the “lift” of boundary strings, *i.e.* all the bulk strings, which conflicts with the remote detectability condition in the bulk.

A typical half-braiding path is shown in Fig. 7. It is important to note that the (non-Abelian) geometric phase depends on the half-braiding path; however, we can extract a universal path-independent half-braiding invariant, by complementing the half-braiding path into a full loop with another half loop of path in the trivial phase outside the boundary. Different half loop of path in the trivial phase with the same starting and end points on the boundary always contribute the same geometric phase (because closed paths in the trivial phase has no geometric phase). This way we obtain the half-braiding invariant as the expectation value of such whole loop adiabatically moving process (half in the bulk, half in the trivial phase).⁴⁵ Trivial half-braiding means that such half-braiding invariant is trivial. Immediately we see that the linear maps $\alpha_{X,h}$ are directly related to the half-braidings, in the sense that $\langle \alpha_{X,h} \rangle$ gives the above half-braiding invariant. If $\alpha_{X,h}$ is the identity map, it implies trivial half-braiding between X and h .

Now, we are ready to show that $\varphi : H \rightarrow G$ is an isomorphism:

1. φ is injective. Consider $\ker \varphi$, namely the simple

loops that induce just identity linear maps on all bulk particles. In other words, $\ker \varphi$ consists of simple loops that have trivial half-braiding with all bulk particles. By the remote detectability condition (1), $\ker \varphi$ must be trivial, which means φ is injective.

2. φ is surjective. We already showed that $\varphi : H \rightarrow G$ is injective, so we can view H as a subgroup of G .

Now consider a special particle in the bulk, which carries the representation $\text{Fun}(G/H)$, linear functions on the right cosets G/H . More precisely, $\text{Fun}(G/H)$ consists of all linear functions on G , $f : G \rightarrow \mathbb{C}$, such that $f(hx) = f(x)$, $\forall h \in H, x \in G$ (takes the same value on a coset). The group action is the usual one on functions, $\rho_{\text{Fun}(G/H)}(g) : f(x) \mapsto f(g^{-1}x)$.

The linear maps $\alpha_{X,h}$ induced by the simple loops are all actions of group elements in H , and they are all identity maps on the special particle $\text{Fun}(G/H)$. In other words, the bulk particle $\text{Fun}(G/H)$ has trivial half-braiding with all the boundary strings. By the remote detectability condition (2), it must be the trivial particle carrying the trivial representation. In other words, we have $G = H$.

To conclude, the simple loop excitations on the string-only boundary, forms a group under fusion. It is exactly the same group whose representations are carried by the pointlike excitations in the bulk.

If we insert a bulk loop excitation in the 3-disk and perform a similar braiding process, it also induces a linear map on the underlining fusion space. One may wonder if this also associates group elements to the bulk strings. This is in general not true. Unlike the boundary simple loops, inserting a bulk string, even if it is pure and simple, will enlarge the fusion space of only particles, as long as the quantum dimension of such string is greater than 1. As a result, only those bulk strings with quantum dimension $d = 1$ can be associated with group elements. In section III C we have shown that all the bulk strings can be associated with conjugacy classes of the group G (even if topological order is not a G -gauge theory). Some further discussions can be found in Appendix C.

VI. THE CLASSIFICATION OF UNITARY POINTED FUSION 2-CATEGORIES

A. A mathematical formulation

First, let us consider the so-called unitary pointed fusion (1-)categories. A pointed fusion category consists of a finite number of simple objects. A simple object x is an object such that $\text{Hom}(x, x) = \mathbb{C}$. For each simple object x , there is a simple object y such that $x \otimes y = 1$, where 1 is the tensor unit, *i.e.* $1 \otimes x = x = x \otimes 1$. In other words,

the set of simple objects form a finite group G . We will also denote the simple object by g_1, g_2, g_3 , etc.

In this case, the only not-yet-fixed structure is the associator isomorphism:

$$(g_1 \otimes g_2) \otimes g_3 \rightarrow g_1 \otimes (g_2 \otimes g_3). \quad (25)$$

Note that both domain and target are the same simple object. But the associator isomorphisms can be non-trivial. By the simpleness, the isomorphism is just a non-zero complex number $\omega_3(g_1, g_2, g_3)$. If the theory is unitary, one needs to further require that this number is a phase (in $U(1)$). Then the pentagon condition implies that $\omega_3(g_1, g_2, g_3)$ is a 3-cocycle. Different cocycles give equivalent fusion categories if they differ by a 3-coboundary. Moreover, we may permute the simple objects by group automorphisms, thus two different cocycles also give equivalent fusion categories if they got mapped to each other by group automorphisms. In conclusion, unitary pointed fusion categories, are classified by (G, ω_3) where $\omega_3 \in \mathcal{H}^3[G; U(1)]$ up to group automorphisms.

Now let us consider a pointed unitary fusion 2-category⁴⁶. We will not define it in full detail here, but only describe some physically relevant ingredients of it. It has only finite number of simple objects. A generic object is a direct sum of simple objects. For two simple objects x, y , we have $\text{Hom}(x, x) = \text{Vect}$ and $\text{Hom}(x, y) = 0$ for $x \neq y$, where 0 is the category consisting of only the 0-vector space. The tensor unit 1 is simple. For each simple object x , there is a simple object y such that $x \otimes y = 1$, where 1 is the tensor unit. So, again the set of simple objects is a finite group G . We will denote simple objects by group elements g_1, g_2, g_3 , etc.

For a simple object g , the identity 1-morphism id_g is \mathbb{C} (the only invertible object in Vect). The 2-morphisms form $\text{Hom}(\text{id}_g, \text{id}_g) = \text{Hom}_{\text{Vect}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$, and there are unit 1-isomorphisms and associator 1-isomorphisms: (1) The unit 1-isomorphisms: $1 \otimes g = g \rightarrow g$ is just the identity morphism $\text{id}_g = \mathbb{C}$. (2) The associator 1-isomorphism:

$$(g_1 \otimes g_2) \otimes g_3 \rightarrow g_1 \otimes (g_2 \otimes g_3) \quad (26)$$

is also still the identity 1-morphism $\text{id}_{g_1 g_2 g_3} = \mathbb{C}$. There are two ways to go from $((g_1 g_2) g_3) g_4$ to $g_1 (g_2 (g_3 g_4))$ via identity 1-morphisms. Therefore, two paths both give the identity map $\text{id}_{g_1 g_2 g_3 g_4} = \mathbb{C}$. So the commutative of the pentagon diagram is clear. But we can introduce for each pentagon a 2-isomorphism: $\mathbb{C} \rightarrow \mathbb{C}$, which is a phase, denoted by $\omega_4(g_1, g_2, g_3, g_4)$. These 2-isomorphisms need satisfy a higher coherence relation. Then this coherence relation implies that $\omega_4(g_1, g_2, g_3, g_4)$ is a 4-cocycle. Again, 4-cocycles differing only by a coboundary give equivalent pointed fusion 2-categories. One can do the same for the triangle relation. Namely, one can introduce a 2-isomorphism for each triangle. We believe that these 2-isomorphisms should give the same unitary fusion 2-category.

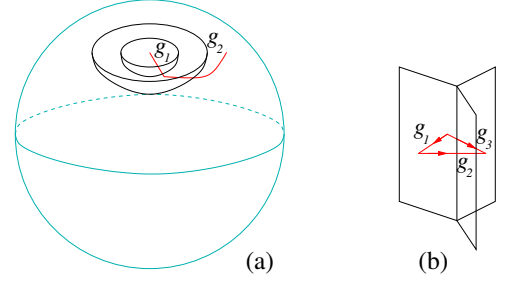


FIG. 8. (Color online) (a) Two boundary strings $s_{g_1}^{\text{bdry}}$ and $s_{g_2}^{\text{bdry}}$ on the surface S^2 of D^3 . The process of fusing the two strings to a no string state is described by a membrane-net in D^3 (which is formed by two hemispheres in this case). The dual of the membrane-net is a string-net (which is formed by two strings in this case). (b) The intersection of three membranes is dual to a triangle of strings. The fusion rule of the boundary strings corresponds to the flat connection condition eqn. (28) on the string-net.

The same structure is discussed in Ref. 47, under a different name, G -graded 2-vector spaces $2\text{Vec}_G^{\omega_4}$, where they also believe that (G, ω_4) is enough to determine a unitary pointed fusion 2-category.

Note that the equivalence between unitary pointed fusion 2-categories must preserve the tensor product of simple objects, thus must correspond to some group automorphism $\phi: G \cong G$. Such automorphism also acts on the cocycles (necessarily change the cocycle if it is an outer automorphism, namely, not of the form $x \mapsto gxg^{-1}$ for some $g \in G$). Under such automorphism ϕ , (G, ω_4) and $(G, \phi(\omega_4))$, where ω_4 is a 4-cocycle, correspond to the same pointed unitary fusion 2-category. Therefore, we believe that pointed unitary fusion 2-categories one-to-one correspond to the pairs (G, ω_4) where $\omega_4 \in \mathcal{H}^4[G; U(1)]$, up to group automorphisms.

B. A physical argument

In the following, we will try to understand the above mathematical result from a physical point of view. Let the 3-dimensional space to be a 3-disk D^3 . Consider the boundary strings $s_{g_1}^{\text{bdry}}$ and $s_{g_2}^{\text{bdry}}$ on the surface of the 3-disk $S^2 = \partial D^3$ (see Fig. 8a). The process for the boundary strings to fuse to a non-string state can be represented by a membrane-net in D^3 (see also Ref. 47). The same boundary strings can fuse to a non-string state through a different process which is represented by another membrane-net in D^3 . To compare the two processes, we can glue the boundary of the above two membrane together along the S^2 , to form a membrane-net in S^3 . Such a membrane-net in S^3 describe the process of creating boundary strings from a no-string state, and then fuse those boundary strings to no-string state. Such a process induce a $U(1)$ geometric phase $e^{i\theta}$, since the fusion space of the boundary strings is always 1-dimensional. So we assign such a $U(1)$ geometric phase

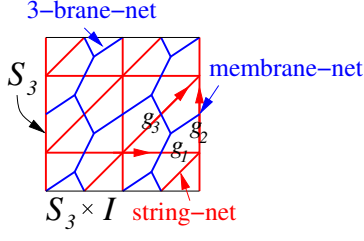


FIG. 9. (Color online) Two processes are described by membrane-net on the two boundaries of $S^3 \times I$. The change between the two processes is described by a 3-brane-net (the blue lines) on $S^3 \times I$ (with 2 of the 3 dimensions suppressed). The red lines form the string-net which is dual to the 3-brane-net.

$e^{i\theta}$ to the membrane-net on S^3 .

But such a $U(1)$ geometric phase may not have a local expression. Let us assume that the membrane-net on S^3 is formed by the 2-simplices of a triangulation of S^3 . The vertices of the triangulation are labeled by I, J, K, \dots . “Having no local expression” means that we cannot assign a phase factor $\omega_3(IJKL)$ to each 3-simplex $\langle IJKL \rangle$ of the triangulation to express the total $U(1)$ geometric phase $e^{i\theta}$ as a product of those local phases:

$$e^{i\theta} \neq \prod_{\langle IJKL \rangle} \omega_3(IJKL). \quad (27)$$

We see that the process of creating some boundary strings from nothing and then fusing them to nothing can be represented by a membrane-net on space S^3 . Such a process correspond to a phase factor $e^{i\theta}$. Two different processes of creating some boundary strings from nothing and then fusing them to nothing give rise to two phase factors $e^{i\theta}$ and $e^{i\theta'}$. The two processes can be compared by a “time”-evolution from the membrane-net on S^3 that correspond to the first process, to the membrane-net on S^3 that correspond to the second process. In other words, the comparison of the two processes is represented by a 3-brane-net on $S^3 \times I$, where S^3 is the space and the segment I represents the “time” direction (see Fig. 9). The first process corresponds to the membrane-net on one boundary of $S^3 \times I$ which is one boundary of the 3-brane-net on $S^3 \times I$. The second process corresponds to the membrane-net on the other boundary of $S^3 \times I$ which is the other boundary of the 3-brane-net on $S^3 \times I$.

In 4-dimensions, a 3-brane-net is dual to a string-net where each 3-brane in the 3-brane-net intersects with a string in the string-net (see Fig. 9 and 8). So the strings in the string-net is also labeled by g_i . In the 3-brane-net, only the 3-branes that satisfy the fusion rule eqn. (20) can intersect along a line (see Fig. 9 and 8b). This means that the labels of the strings in the string-net satisfies

$$g_1 g_2 = g_3. \quad (28)$$

[Note the same string with opposite orientations is labeled by g and g^{-1} respectively. The orientation of

strings in the string-net is chosen to from a branching structure (see Appendix E) of the string-net.] The above happen to be the flat connection condition if we view g_i on a string as the gauge connect between the two vertices connected by the string. So the evolution from one process to the other can be represented by a string-net on $S^3 \times I$.

The two different processes may differ by a phase factor $e^{i(\theta' - \theta)}$. So we can assign the string-net on $S^3 \times I$ by such a phase factor, to represent the difference of the two processes. Let us assume the string-net on $S^3 \times I$ is formed by the edges of a triangulation of $S^3 \times I$. Then starting from one boundary of $S^3 \times I$, we can build the whole triangulation of $S^3 \times I$ by adding one pentachoron (*i.e.* one 4-simplex) at a time. We note that adding a pentachoron corresponds to change one process to its neighboring process. The difference of the two neighboring processes is described by the added pentachoron with edges labeled by g_{IJ} , $I, J = 0, 1, 2, 3, 4$ (where $I = 0, 1, 2, 3, 4$ label the five vertices of the pentachoron). We may assign the phase difference of the two neighboring processes to the added pentachoron. So each pentachoron is assigned to a phase factor $\omega_4(\{g_{IJ}\})$. Due to the flat connection condition eqn. (28), g_{IJ} 's for the pentachoron are not independent. So $\omega_4(\{g_{IJ}\})$ can be rewritten as $\omega_4(g_{01}, g_{12}, g_{23}, g_{34})$. Such a 4-variable function on G can be viewed as a group 4-cochain.

So the total phase difference of the two processes can be written as

$$e^{i(\theta' - \theta)} = \prod_{\langle IJKLM \rangle \in S^3 \times I} \omega_4^{s_{IJKLM}}(g_{IJ}, g_{JK}, g_{KL}, g_{LM}) \quad (29)$$

where $\prod_{\langle IJKLM \rangle \in S^3 \times I}$ multiply over all the pentachorons $\langle IJKLM \rangle$ in the triangulation of $S^3 \times I$, and $s_{IJKLM} = \pm 1$ describes the two different orientations of the pentachorons $\langle IJKLM \rangle$ which arises from the branching structure (see Appendix E).

If we choose the two processes described by the boundary of $S^3 \times I$ to be the do nothing process that the leave the no-string state unchanged, then the string-net on $S^3 \times I$ can be viewed as a string-net on S^4 . The total phase difference of the two do-nothing processes (which is actually the same process) should be zero:

$$1 = \prod_{\langle IJKLM \rangle \in S^4} \omega_4^{s_{IJKLM}}(g_{IJ}, g_{JK}, g_{KL}, g_{LM}), \quad (30)$$

and the above should hold for any triangulation of S^4 and any assignment of the label g_{IJ} on the edges (as long as the flat connection condition eqn. (28) is satisfied). It implies that $\omega_4(g_{01}, g_{12}, g_{23}, g_{34})$ is a group 4-cocycle in $\mathcal{H}^4(G; U(1))$. This is a physical way to explain why

Unitary pointed fusion 2-categories are classified by a pair (G, ω_4) up to group automorphisms, where G is a finite group and ω_4 its group 4-cohomology class: $\omega_4 \in \mathcal{H}^4(G; U(1))$.

VII. FROM BOUNDARY TO BULK

We have shown that all 3+1D AB bosonic topological orders can have a boundary described by pointed unitary fusion 2-category \mathcal{M}^3 whose fusion is given by the group G . It is believed the boundary anomalous topological order completely determine the bulk topological order^{20,21}. More precisely, the bulk topological order should be given by the center $Z(\mathcal{M}^3)$, which can be explicitly defined by the 2-category $\mathcal{F}un_{\mathcal{M}^3|\mathcal{M}^3}(\mathcal{M}^3, \mathcal{M}^3)$ of \mathcal{M}^3 - \mathcal{M}^3 -bimodule 2-functors.

But their relation can be many-to-one: several different anomalous boundary topological orders may correspond to the same bulk topological order; in other words, the same bulk topological order can have several different gapped boundaries. Since 3+1D topological orders always have gapped boundaries, all 3+1D topological orders are determined by some anomalous 2+1D boundary topological orders. Mathematically, we say that there is surjective map from the set of anomalous 2+1D boundary topological orders to the set of 3+1D topological orders

$$\begin{aligned} & \text{2+1D boundary anomalous topological orders} \\ \rightarrow & \text{3+1D topological orders.} \end{aligned} \quad (31)$$

Furthermore, since all 3+1D AB topological orders have a string-only boundary described by unitary pointed fusion 2-categories, we further have

$$\begin{aligned} & \text{Unitary pointed fusion 2-categories} \\ \rightarrow & \text{3+1D AB topological orders.} \end{aligned} \quad (32)$$

In this paper the string-only boundary is obtained by condensing all the pointlike particles that form $\text{Rep}(G)$. A natural question is whether such condensation process, and also the resulting boundary, are unique or not.

Firstly, we believe that the condensation of particles in 3+1D follows the same rule as that for condensation of anyons in 2+1D (at least if we restrict the 3+1D condensation to a 2+1D sub-manifold, see Fig. 2). Anyon condensation in 2+1D has been thoroughly studied. It is fully controlled by the so-called *condensable algebra*⁴⁸ in the category of anyons. In other words, the condensable algebra completely determines the condensed phase and the domain wall/boundary between the old phase and the condensed phase.

Thus, we should focus on the condensable algebras in $\text{Rep}(G)$. They are already classified in Theorem 2.2 in Ref. 49 (see also Theorem 3.7 in Ref. 50). There is a unique condensable algebra that condenses all the particles in $\text{Rep}(G)$. It is given by $\text{Fun}(G)$, the algebra of all functions on G . Therefore, there is only one way to condense all particles $\text{Rep}(G)$. We obtain a unique condensed phase, which is trivial. As result, there is a unique, also canonical, string-only boundary.

This way, we got an even stronger result. Each 3+1D AB topological order only have a unique boundary that corresponds to the condensation of all pointlike excita-

tions. In other words, each 3+1D topological order corresponds to a unique unitary pointed fusion 2-category.

Unitary pointed fusion 2-categories classify all 3+1D AB topological orders in a one-to-one fashion.

Such a result is similar to a result in one lower dimension:

Unitary fusion categories describes all 2+1D topological orders with gappable boundary (but in a many-to-one way)^{51,52}.

Let us briefly explain why the approach used in this paper for 3+1D topological orders does not apply in 2+1D, which gives a flavour why we can obtain a stronger result in 3+1D. In 3+1D, all pointlike excitations have trivial statistics; if they are all bosons, it is a natural and canonical choice to condense all of them and we obtain a unique string-only 2+1D gapped boundary. In 2+1D, there are only pointlike excitations with non-trivial statistics between them. One can choose a subset of quasiparticles to condense; if the subset is big enough one can also obtain a gapped 1+1D boundary. However, in general there are several such subsets to condense, among which none is special. As a result, there is no canonical gapped 1+1D boundary. This essential difference makes the classification of topological orders in 3+1D simpler than those in 2+1D.

VIII. REALIZATION BY DIJKGRAAF-WITTEN MODELS

Combining the results from the last a few sections, we obtain that

3+1D AB topological orders are classified by a finite group G and its group 4-cocycle $\omega_4 \in \mathcal{H}^4(G; U(1))$, up to group automorphisms.

A finite group G and its group 4-cocycle happen to be the data needed to construct the Dijkgraaf-Witten model. In fact all the 3+1D AB topological orders can be realized by Dijkgraaf-Witten models.

We note that 3+1D Dijkgraaf-Witten models³⁴ are defined on a 4-dimensional simplicial complex with branching structure (see Appendix E). Let us use I, J, \dots to label the vertices of the complex. The degrees of freedoms live on the links of the complex, which are labeled by $g_{IJ} \in G$ where G is a finite group. g_{IJ} 's satisfies a flat-connection condition

$$g_{IJ}g_{JK} = g_{IK}, \quad (33)$$

for any triangles $\langle IJK \rangle$. The Dijkgraaf-Witten models are defined via a path integral

$$Z = \sum_{\{g_{IJ}\}} \prod_{\langle IJKLM \rangle} \omega_4^{s_{IJKLM}}(g_{IJ}, g_{JK}, g_{KL}, g_{LM}), \quad (34)$$

where $\prod_{\langle IJKLM \rangle}$ multiply over all the 4-cells $\langle IJKLM \rangle$ whose vertices are ordered as $I < J < K < L < M$. Also, $s_{IJKLM} = \pm 1$ describes the orientation of the 4-cell $\langle IJKLM \rangle$ (see Appendix E), and ω_4 is a group 4-cocycle $\omega_4 \in \mathcal{H}^4[G; U(1)]$.

When the space-time has a boundary, we can obtain an exactly soluble boundary by setting $g_{IJ} = 1$ on all the links $\langle IJ \rangle$ on the boundary. Such an exactly soluble boundary is actually the string-only boundary discussed in this paper. The world-lines of topological pointlike excitations are described by Wilson lines in the bulk

$$\prod_{\langle IJ \rangle} R(g_{IJ}) \quad (35)$$

where R is a representation of G . But on the boundary, $g_{IJ} = 1$ and $R(g_{IJ} = 1)$ is an identity matrix. All the different topological pointlike excitations becomes the same trivial excitation on the boundary. However, there are non-trivial stringlike excitations on the boundary. The world-sheet of those boundary stringlike excitations is given by the following: Draw a membrane on the 3-dimension boundary of space-time. Change g_{IJ} on the links that intersect the membrane from $g_{IJ} = 1$ to $g_{IJ} = h$. Such a change still satisfy the flat-connection condition. We see that different boundary strings are labeled by the group elements and their fusion is given by the group multiplication. Therefore, Dijkgraaf-Witten models can realize all unitary pointed fusion 2-category on the boundary. Using the boundary-bulk relation^{20,21}, we can show that Dijkgraaf-Witten models can realize all 3+1D AB topological orders.

IX. RELATION TO 3+1D BOSONIC SPT ORDERS

There are two kinds of SPT orders when the symmetry group is unitary and finite: the ones whose boundary have a pure gauge-anomaly will be called pure SPT orders^{15,16}, and the ones whose boundary have a mixed gauge-gravity-anomaly will be called mixed SPT orders⁵³. In 3+1D space-time, the pure SPT orders are classified by group cohomology $\mathcal{H}^4[G; U(1)]$, while all the mixed SPT orders are described by *some* elements in⁵³

$$\begin{aligned} & \mathcal{H}^1(G; \mathcal{H}^3[SO(\infty); U(1)]) \oplus \mathcal{H}^2(G; \mathcal{H}^2[SO(\infty); U(1)]) \\ &= \mathcal{H}^1(G; \mathbb{Z}) \oplus \mathcal{H}^2(G; \mathbb{Z}_2) = \mathcal{H}^2(G; \mathbb{Z}_2). \end{aligned} \quad (36)$$

For many groups, $\mathcal{H}^2(G; \mathbb{Z}_2) \neq 0$. But a non zero $\mathcal{H}^2(G; \mathbb{Z}_2)$ does not implies the existence of mixed SPT, since not all the elements in $\mathcal{H}^2(G; \mathbb{Z}_2)$ correspond to exist SPT orders.

Since 3+1D AB topological orders can be obtained by gauging¹⁷ the symmetry of 3+1D bosonic SPT states, and since Dijkgraaf-Witten models only correspond to gauging the pure SPT states, we see that the classifica-

tion results in this paper implies that

In 3+1D, there is no mixed bosonic SPT order for unitary finite symmetry group G .

In fact, using SPT invariant, we can directly show that for unitary finite symmetry group G , there is no mixed SPT orders in 3+1D. (However, if G contains time reversal, there are mixed SPT orders in 3+1D^{41,54}.) To obtain SPT invariant, we gauge the symmetry and put a flat-connection A on a closed orientable space-time M^4 . The partition function of the system on M^4 with a fixed flat-connection A is the so called SPT invariant⁵⁵⁻⁵⁷. If there is a mixed 3+1D SPT order described by $\mathcal{H}^2(G; \mathbb{Z}_2) = \oplus \mathbb{Z}_2$, its SPT invariant will have a form

$$Z(M^4, A) = e^{i\pi \int_{M^4} \omega_1(A) \smile w_3 + \omega_2(A) \smile w_2}, \quad (37)$$

where $\omega_n(A)$, w_n are topological n -cocycles in $H^n(M^4; \mathbb{Z}_2)$, and w_n is also the n^{th} Stiefel-Whitney class. There are many relations between Stiefel-Whitney classes and cocycles $\omega_n(A)$. For example, by calculating $Sq^1(\omega_1(A) \smile w_2)$ in two different ways, we find that on orientable M^4 , $\omega_1(A) \smile w_3 = \omega_1(A) \smile \omega_1(A) \smile w_2$ ^{53,58}. (Here Sq^n is the Steenrod operation.) Thus

$$Z(M^4, A) = e^{i\pi \int_{M^4} [\omega_2(A) + \omega_1(A) \smile \omega_1(A)] \smile w_2}, \quad (38)$$

Similarly, $[\omega_2(A) + \omega_1(A) \smile \omega_1(A)] \smile w_2 = Sq^2[\omega_2(A) + \omega_1(A) \smile \omega_1(A)] = [\omega_2(A) + \omega_1(A) \smile \omega_1(A)] \smile [\omega_2(A) + \omega_1(A) \smile \omega_1(A)]$. Thus

$$Z(M^4, A) = e^{i\pi \int [\omega_2(A) + \omega_1(A) \smile \omega_1(A)] \smile [\omega_2(A) + \omega_1(A) \smile \omega_1(A)]}. \quad (39)$$

We see that the SPT order described by the above SPT invariant is actually a pure SPT order described by $\mathcal{H}^4[G; U(1)]$ ^{41,53} and hence, there is no mixed SPT order in 3+1D for unitary finite symmetry group. This result supports our classification of 3+1D AB topological orders in terms of Dijkgraaf-Witten models. In 4+1D, there is a mixed bosonic \mathbb{Z}_2 SPT state⁵³. Gauging such a mixed \mathbb{Z}_2 SPT state will produce a 4+1D AB topological order that is beyond Dijkgraaf-Witten theory.

X. WALKER-WANG MODELS AND PARTICLE-ONLY BOUNDARIES

We like to remark that Walker-Wang models⁵⁹⁻⁶² is another quite systematic way to construct 3+1D bosonic topological orders. In fact, Walker-Wang models realize all 3+1D bosonic topological orders who have a particle-only boundary, which is described by a premodular tensor category. Such particle-only boundary can exist for a 3+1D topological order, if condensing the maximum set of strings that have trivial mutual braiding will change the 3+1D topological order to a trivial phase.

It is known that Walker-Wang models (and the related 3+1D string-net models⁵¹) can realize 3+1D bosonic topological orders with emergent fermionic pointlike excitations.⁶⁰ It appears that Walker-Wang models cannot realize all 3+1D Dijkgraaf-Witten models (*i.e.* not all 3+1D bosonic topological orders whose pointlike excitations are all bosons).

XI. SUMMARY

3+1D topological orders contain both pointlike and stringlike excitations. At first, it appears that 3+1D topological orders, with all the fusion and braiding of those pointlike and stringlike excitations, have a very complicated structure, which may be hard to classify. However, in this paper, we obtain a very simple classification of 3+1D topological orders for bosonic systems, when all the pointlike excitations are bosons: they are classified by unitary pointed fusion 2-categories, which in turn are classified by pairs (G, ω_4) up to group automorphisms. This gives us hope that the 3+1D topological orders may not be that complicated. We may get a simple classification even for the general case when some pointlike excitations are emergent fermions. We hope that the arguments developed in this paper are helpful for such a task, which we plan to carry out in a forthcoming work.

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Appendix A: An example: 3+1D G -gauge theory

To gain an intuitive understanding of 3+1D topological orders and to introduce the related concepts, let us study an exactly soluble local bosonic model whose ground state has a topological order described by a 3+1D gauge theory of a finite group G . Our lattice bosonic model is defined on a 3D spatial lattice whose sites are labeled by I . The degrees of freedom live on the links labeled by IJ . On an oriented link IJ , such degrees of freedom are labeled by $g_{IJ} \in G$. g_{IJ} 's on links with opposite orientations satisfy

$$g_{IJ} = g_{JI}^{-1} \quad (\text{A1})$$

The Hamiltonian of the exactly soluble model is expressed in terms of string operators and membrane operators.

1. The string operators

The string operators are labeled by i 's, the irreducible representations $R_i(g_{IJ})$ of the gauge group G (where $R_i(g_{IJ})$ is the matrix of the irreducible representation):

$$\begin{aligned} B_i|\{g_{IJ}\}\rangle &= \left[\text{Tr} \prod_{IJ \in \text{string}} R_i(g_{IJ}) \right] |\{g_{IJ}\}\rangle \\ &= \left[\text{Tr} R_i \left(\prod_{IJ \in \text{string}} g_{IJ} \right) \right] |\{g_{IJ}\}\rangle \end{aligned} \quad (\text{A2})$$

We note that

$$B_i B_j = \text{Tr} \prod_{I \in \text{string}} R_i(g_{IJ}) \otimes_{\mathbb{C}} R_j(g_{IJ}). \quad (\text{A3})$$

We use $\otimes_{\mathbb{C}}$ to denote the usual tensor product of matrices or vector spaces over the complex numbers \mathbb{C} , while \otimes to denote the fusion of excitations. Using

$$R_i(g) \otimes_{\mathbb{C}} R_j(g) = \bigoplus_k N_k^{ij} R_k(g) \quad (\text{A4})$$

we see that

$$B_i B_j = \sum_k N_k^{ij} B_k. \quad (\text{A5})$$

The ends of the strings are pointlike topological excitations and the above N_k^{ij} are the fusion coefficients of those topological excitations. Let d_i be the quantum dimension of those topological excitations which satisfy

$$\sum_j N_k^{ij} d_j = d_i d_k, \quad (\text{A6})$$

and let

$$B = \sum_i \frac{d_i}{D^2} B_i, \quad D^2 = \sum_i d_i^2. \quad (\text{A7})$$

We have

$$\begin{aligned} B^2 &= \sum_{i,j} \frac{d_i d_j}{D^4} B_i B_j = \sum_{i,j,k} \frac{d_i d_j}{D^4} N_k^{ij} B_k \\ &= \sum_{i,k} \frac{d_i d_i}{D^4} d_k B_k = B. \end{aligned} \quad (\text{A8})$$

Thus, B is a projection operator. In fact, it is a projection operator into the subspace with $\prod_{IJ \in \text{string}} g_{IJ} = 1$.

2. The membrane operators

A membrane is formed by the faces of the dual lattice, which is also a cubic lattice. The faces of the dual lattice correspond to the links in the original lattice and are also labeled by IJ .

TABLE III. Fusion rules of pointlike and stringlike excitations in 3+1D S_3 gauge theory. Here p_0 corresponds to the trivial pointlike excitations which is also the trivial stringlike excitations. p_1 and p_2 are nontrivial pointlike excitations corresponding to the 1D and 2D representation of S_3 (*i.e.* the charged particles). s_{20} and s_{30} correspond to pure stringlike excitations labeled by conjugacy classes χ_2 and χ_3 , and s_{21} , s_{31} and s_{32} are charge and string bound state, as one can see from the fusion rules. See Ref. 38.

\otimes	p_0	p_1	p_2	s_{20} (pure)	s_{21}	s_{30} (pure)	s_{31}	s_{32}
p_0	p_0	p_1	p_2	s_{20}	s_{21}	s_{30}	s_{31}	s_{32}
p_1	p_1	p_0	p_2	s_{21}	s_{20}	s_{30}	s_{31}	s_{32}
p_2	p_2	p_2	$p_0 \oplus p_1 \oplus p_2$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$s_{31} \oplus s_{32}$	$s_{30} \oplus s_{32}$	$s_{30} \oplus s_{31}$
s_{20}	s_{20}	s_{21}	$s_{20} \oplus s_{21}$	$p_0 \oplus p_2 \oplus s_{30} \oplus s_{31} \oplus s_{32}$	$p_1 \oplus p_2 \oplus s_{30} \oplus s_{31} \oplus s_{32}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$
s_{21}	s_{21}	s_{20}	$s_{20} \oplus s_{21}$	$p_1 \oplus p_2 \oplus s_{30} \oplus s_{31} \oplus s_{32}$	$p_0 \oplus p_2 \oplus s_{30} \oplus s_{31} \oplus s_{32}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$
s_{30}	s_{30}	s_{30}	$s_{31} \oplus s_{32}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$p_0 \oplus p_1 \oplus s_{30}$	$s_{32} \oplus p_2$	$s_{31} \oplus p_2$
s_{31}	s_{31}	s_{31}	$s_{30} \oplus s_{32}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$s_{32} \oplus p_2$	$p_0 \oplus p_1 \oplus s_{31}$	$s_{30} \oplus p_2$
s_{32}	s_{32}	s_{32}	$s_{30} \oplus s_{31}$	$s_{20} \oplus s_{21}$	$s_{20} \oplus s_{21}$	$s_{31} \oplus p_2$	$s_{30} \oplus p_2$	$p_0 \oplus p_1 \oplus s_{32}$

A membrane operator is given by

$$Q_a = \sum_{h \in \chi_a} \prod_{IJ \in \text{membrane}} \hat{A}_{IJ}(h). \quad (\text{A9})$$

where the operator $\hat{A}_{IJ}(h)$ is defined as

$$\hat{A}_{IJ}(h)|g_{IJ}\rangle = |hg_{IJ}\rangle, \quad (\text{A10})$$

and χ_a is the a^{th} conjugacy class of G . Also I 's are on one side of the membrane and J 's are on the other side of the membrane,

We note that

$$Q_a Q_b = \sum_{h \in \chi_a} \sum_{\tilde{h} \in \chi_b} \prod_{IJ \in \text{membrane}} \hat{A}_{IJ}(h\tilde{h}) = \sum_c M_c^{ab} Q_c, \quad (\text{A11})$$

The above expression allows us to see that M_c^{ab} are non-negative integers. Clearly $Q_a Q_b = Q_b Q_a$ and $(Q_a Q_b) Q_c = Q_a (Q_b Q_c)$, which imply that

$$M_c^{ab} = M_c^{ba}, \quad \sum_d M_d^{ab} M_e^{dc} = \sum_d M_e^{ad} M_d^{bc} \quad (\text{A12})$$

Let $(M_a)_{cb} = M_c^{ab}$, and we can rewrite the second equation in the above as

$$M_c M_a = M_a M_c. \quad (\text{A13})$$

For example, the permutation group of three elements $S_3 = \{(123), (132), (321), (213), (231), (312)\}$ has three conjugacy classes: $\chi_1 = \{(123)\}$, $\chi_2 = \{(132), (321), (213)\}$, and $\chi_3 = \{(231), (312)\}$. We find that

$$\begin{aligned} Q_1 Q_a &= Q_a, & Q_2 Q_2 &= 3Q_1 + 3Q_3, \\ Q_3 Q_3 &= 2Q_1 + Q_3, & Q_2 Q_3 &= 2Q_2. \end{aligned} \quad (\text{A14})$$

Let \mathbf{c} be a common eigenvector of M_a whose components are all non-negative. (Such common eigenvector exists since the matrix elements of M_a are all non-negative.)

The eigenvalue of such a eigenvector is λ_a for M_a . We choose the scaling factor of \mathbf{c} to satisfy

$$\sum_a \lambda_a c_a = 1. \quad (\text{A15})$$

In this case

$$Q^2 = Q, \quad Q = \sum_a c_a Q_a. \quad (\text{A16})$$

3. A commuting-projector Hamiltonian

Let $Q_{I,a}$ be the smallest membrane operator that creates a small membrane corresponding to the surface of a cube in the dual lattice. Such a membrane wraps a site I in the original cubic lattice. We note that $Q_{I,a}$ is a sum of gauge transformation operators $g_{IJ} \rightarrow h g_{IJ}$. Since the string operators are gauge invariant, we have

$$[B_i, Q_{I,a}] = 0. \quad (\text{A17})$$

Therefore, we can construct the following commuting projector Hamiltonian^{63,64}

$$H = \sum_I (1 - Q_I) + \sum_{\langle IJKL \rangle} (1 - B_{\langle IJKL \rangle}), \quad (\text{A18})$$

where

$$Q_I = \sum_a c_a Q_{I,a}, \quad B_{\langle IJKL \rangle} = \sum_i \frac{d_i}{D} B_{\langle IJKL \rangle, i} \quad (\text{A19})$$

and $\langle IJKL \rangle$ labels the loops around the squares of the original cubic lattice.

The ground state of the above exactly soluble Hamiltonian has a nontrivial topological order. The low energy effective theory is the G -gauge theory.

4. The pointlike and stringlike excitations

What are the excitations for the above Hamiltonian? There are local pointlike excitations created by local operators. There are also topological pointlike excitations that cannot be created by local operators. Two topological pointlike excitations are said to be equivalent if they differ by local pointlike excitations. The equivalent topological pointlike excitations are said to have the same type.

The different types of topological pointlike excitations are created at the ends of the open string operators that we discussed before. Thus we see that types of topological pointlike excitations one-to-one correspond to the irreducible representations of G . In other words, topological pointlike excitations are described by $\text{Rep}(G)$ in a G -gauge theory.

Similarly, there are also topological stringlike excitations. They are created at the boundary of the open membrane operators. However, the types of membrane operators are not one-to-one correspond to the types of stringlike excitations. There are *pure* stringlike excitations which one-to-one correspond to the conjugacy classes of G . There are also *mixed* stringlike excitations which are bound state of pure stringlike excitations and pointlike excitations^{20,47,65}. In general, the types (pure and mixed) of stringlike excitations in a G -gauge theory are labeled by a pair $\chi, R(G_\chi)$, where χ is a conjugacy class of G , $R(G_\chi)$ is a representation of G_χ , and G_χ is a subgroup of G whose elements all commute with a fixed element in χ (a centralizer subgroup).

For example, in S_3 -gauge theory, the χ_2 -flux-loop breaks the S_3 gauge “symmetry” down to Z_2 gauge symmetry. So there are two types of χ_2 -flux-loop excitations, one carries no Z_2 charge (which is the pure one denoted by s_{20}) and the other carries Z_2 charge 1 (denoted by s_{21}). Similarly, the χ_3 -flux-loop breaks the S_3 gauge “symmetry” down to Z_3 gauge symmetry. So there are three types of χ_3 -flux-loop excitations, carrying Z_3 charge $q = 0, 1, 2$ (denoted by s_{3q}). Thus, the string excitations in S_3 -gauge theory are given by $s_{20} = (\chi_2, R_0(Z_2))$; $s_{21} = (\chi_2, R_1(Z_2))$; $s_{30} = (\chi_3, R_0(Z_3))$; $s_{31} = (\chi_3, R_1(Z_3))$; $s_{32} = (\chi_3, R_2(Z_3))$. (Note that $G_{\chi_2} = Z_2$ and $G_{\chi_3} = Z_3$.) Those string like excitations also have a shrinking rule: if we shrink a string to a point, it will behave like a pointlike excitation:

$$\begin{aligned} s_{20} &\rightarrow p_0 \oplus p_2, & s_{21} &\rightarrow p_1 \oplus p_2, \\ s_{30} &\rightarrow p_0 \oplus p_1, & s_{31} &\rightarrow p_2, & s_{32} &\rightarrow p_2. \end{aligned} \quad (\text{A20})$$

In general, the string-excitations whose shrinking rule contain the trivial excitation p_0 are the pure string excitations, which are s_{20} and s_{30} in the S_3 -gauge example. We see that the types of pure string excitations are labeled by χ , the conjugacy classes of G .

The χ_1 -flux-loop (*i.e.* trivial flux-loop) does not break the S_3 gauge. So there are three types of χ_1 -flux-loop excitations, carrying a trivial representation (denoted by

s_{10} or p_0), a nontrivial 1D irreducible representation (denoted by s_{11} or p_1), or a 2D irreducible representation (denoted by s_{12} or p_2) of S_3 . In fact, those χ_1 -flux-loop excitations (or trivial-string excitations) correspond to the pointlike excitations. The fusions of all those stringlike excitations are given by Table III.

We may regard looplike excitations (χ, q) with the same conjugacy class χ but different representations q as equivalent and introduce a notion of pure-type: the looplike excitation (χ, q) is said to have an pure-type χ . So the fusion of the membrane operators correspond to the fusion of pure-types, which is closely related to the fusion of stringlike excitations, after we quotient out the G_χ -representations q , by identifying

$$\begin{aligned} p_0 &= Q_1, & p_1 &= Q_1, & p_2 &= 2Q_1, \\ s_{2q} &= Q_2, & s_{3q} &= Q_3, \end{aligned} \quad (\text{A21})$$

in the fusion rule table III. The general identification formula is

$$s_{\chi q} = \dim(q)Q_\chi, \quad (\text{A22})$$

where $\dim(q)$ is the dimension of the G_χ -representations q .

Appendix B: A general discussion of string and membrane operators in 3+1D topological orders

1. String operators in 3+1D topological orders

For a generic 3+1D topological order, the type- i particlelike excitations are still described by string operators

$$B_i = \sum_{a_1 a_2 a_3 \dots} \hat{O}_i^{a a_1}(I_1) \hat{O}_i^{a_1 a_2}(I_2) \hat{O}_i^{a_2 a_3}(I_3) \dots \quad (\text{B1})$$

Here the string operators are matrix product operators. Those $\hat{O}_i^{a a_1}(I)$ are operators acting on the degrees of freedom on site- I . We may view the superscripts a, a_1, a_2, \dots as the indices of the matrix elements for an operator-valued matrix. We say that *two closed string operators are equivalent if they differ by a local unitary transformation*. More precisely, two closed string operators are equivalent iff they can deform into each other while keep all the local operators having short ranged correlations. We also normalize the string operators such that $\langle B_i \rangle$ is independent of string length when the string is closed. Such normalized string operators satisfy the following fusion algebra

$$B_i B_j = N_k^{ij} B_k. \quad (\text{B2})$$

We can show that N_k^{ij} are non-negative integers, by viewing the string operators as the world-lines in time direction.

2. Membrane operators in 3+1D topological orders

Similarly, the type- a stringlike excitations are described by membrane operators

$$Q_a = \sum_{\{a_I\}} \hat{O}_a^{a_1 a_2 a_3 a_4}(I_1) \hat{O}_a^{a_4 a_5 a_6 a_7}(I_2) \hat{O}_a^{a_7 a_8 a_9 a_{10}}(I_3) \dots, \quad (\text{B3})$$

which are tensor network operators. $\hat{O}_a(I)$ are operator-valued tensors and the superscripts are tensor indices. Also *two closed membrane operators are equivalent if they differ by a local unitary transformation*. More precisely, two closed membrane operators are equivalent iff they can deform into each other while keep all the local operators having short ranged correlations.

The equivalent classes of the membrane operators can be different, for membrane operators with different topology. The equivalent classes of the spherical closed membrane operators, $Q_a^{S^2}$, correspond to the pure membrane types. The pure membrane type corresponds to the type for pure stringlike excitations. For toric closed membrane operators, $Q_a^{T^2}$, the number of the equivalent classes will in general be different from the number of the equivalent classes of spherical closed membrane operators. This is because toric closed membrane operators may contain closed string operators wrapping around the non-contractible loops, which generate different equivalent classes. Clearly, if we do not have nontrivial pointlike excitations, then there will be a one-to-one correspondence between the spherical membrane operators and toric membrane operators.

Since the loop of string operator on S^2 is always contractible, the spherical membrane operators does not contain loops of string operator. Thus the spherical membrane operators are labeled by the conjugacy classes only. The spherical membrane operators $Q_\chi^{S^2}$ also satisfy the following fusion algebra

$$Q_{\chi_1}^{S^2} Q_{\chi_2}^{S^2} = \sum_k M_{S^2; \chi_3}^{\chi_1 \chi_2} Q_{\chi_3}^{S^2}. \quad (\text{B4})$$

In particular

$$M_{S^2; \chi_3}^{\chi_1 \chi_2} = M_{\chi_3}^{\chi_1 \chi_2} \quad (\text{B5})$$

which can be calculated from the fusion of the conjugacy classes χ_1 and χ_2 (see eqn. (A11)).

Appendix C: More general properties of stringlike excitations in 3+1D topological orders

1. Pure stringlike excitations and sectors in dimension reduction

In Section III B, we have shown that in 3+1D, the number of the sectors N_1^{sec} in the dimension reduction is the

number of the classes of stringlike topological excitations that can be distinguished by the braiding with the pointlike excitations. But in Section IV B, we have shown that all stringlike topological excitations can be distinguished from each other via their braiding properties with the pointlike excitations. Therefore,

The number of the sectors N_1^{sec} is the number of type of pure stringlike topological excitations, if all the pointlike excitations are bosons.

Let $\text{GSD}_{\mathcal{C}^{d+1}}(M_{\text{space}}^d)$ be the ground state degeneracy of a $d+1$ D topological order \mathcal{C}^{d+1} on a closed d -dimensional space manifold M_{space}^d . We note that

$$\text{GSD}_{\mathcal{C}^{d+1}}(S_{\text{space}}^d) = 1. \quad (\text{C1})$$

Now, let us consider a more general dimension reduction where we reduce d -dimensional space $M_{\text{space}}^d = L_{\text{space}}^{d-n} \times S^n$ to $(d-n)$ -dimensional space L_{space}^{d-n} by shrinking the S^n . N_n^{sec} be the number of sectors of the dimension reduced topological orders. We find that

$$N_n^{\text{sec}} = \text{GSD}_{\mathcal{C}^{d+1}}(S^{d-n} \times S^n). \quad (\text{C2})$$

We see that

$$N_n^{\text{sec}} = N_{d-n}^{\text{sec}}. \quad (\text{C3})$$

2. Pure stringlike excitations are labeled by the conjugacy classes of G

From last section, we see that for 3+1D AB topological order

$$N_1^{\text{sec}} = N_2^{\text{sec}} = \text{number of types of pure strings}. \quad (\text{C4})$$

In the following, we like to show that

$$N_2^{\text{sec}} = \text{number of types of pointlike excitations}. \quad (\text{C5})$$

We consider the GSD on $S^1 \times S^2$. We note that the path integral on space-time $S^1 \times D^3$ gives us a particular ground state on $S^1 \times S^2$. To obtain other ground states on $S^1 \times S^2$ we insert a type- i string operator B_i along the S^1 in $S^1 \times D^3$. The string operator is inserted at a particular point on D^3 . The insertion of different string operators generate linear independent states. This is because the pointlike excitations represented by the string operators have non-degenerate braiding with the pure stringlike excitations.

The braiding between the inserted pointlike excitations and the stringlike excitations is described by creating a small loop of strings on $S^2 = \partial D^3$. Then we enlarge the loop and let the loop wrap around the S^2 . Such a braiding process is equivalent to applying the sphere membrane operator Q_a on $S^2 = \partial D^3$. The eigenvalues of the sphere membrane operators Q_a should distinguish

all the state created by inserting the string operators B_i . Therefore,

The number of the sectors N_2^{sec} in the S^2 -dimension reduction of a 3+1D topological order is the number of type of pointlike topological excitations.

This allows us to show that

For 3+1D topological orders, the number of types of pointlike excitations and is the same as the number of types of pure stringlike excitations.

Moreover, from the fact that the untwisted sector of dimension reduction is the Drinfeld center $Z[\text{Rep}(G)]$, we know that

The *pure* stringlike excitations in a generic 3+1D AB topological orders are labeled by the conjugacy classes of G .

Now, the dimension reduction of a generic bosonic 3+1D topological order \mathcal{C}^4 can be written as

$$\mathcal{C}^4 = \bigoplus_{\chi} \mathcal{C}_{\chi}^3 \quad (\text{C6})$$

where χ is the conjugacy class of the group G whose representations form the SFC of \mathcal{C}^4 , and \sum_{χ} sums over all the conjugacy classes of G .

From the dimension reduction eqn. (C6), we can also compute the ground state degeneracy on 3-torus

$$\text{GSD}_{\mathcal{C}^4}(T^3) = \sum_{\chi} \text{GSD}_{\mathcal{C}_{\chi}^3}(T^2) \quad (\text{C7})$$

Those degenerate ground states form a representation of the mapping class group of T^3 , which is $SL(3, \mathbb{Z})$.

The dimensional reduction leads to reduction of the representation of $SL(3, \mathbb{Z})$ to the representations of $SL(2, \mathbb{Z})$ that characterize the 2+1D dimension reduced topological orders \mathcal{C}_{χ}^3 . We consider $SL(2, \mathbb{Z}) \subset SL(3, \mathbb{Z})$ subgroup and the reduction of the $SL(3, \mathbb{Z})$ representation $R_{\mathcal{C}^4}$ to the $SL(2, \mathbb{Z})$ representations $R_{\mathcal{C}_{\chi}^3}$:

$$R_{\mathcal{C}^4} = \bigoplus_{\chi} R_{\mathcal{C}_{\chi}^3}. \quad (\text{C8})$$

The $SL(3, \mathbb{Z})$ representation $R_{\mathcal{C}^4}$ describes the 3+1D topological order \mathcal{C}^4 and the $SL(2, \mathbb{Z})$ representations $R_{\mathcal{C}_{\chi}^3}$ describe the 2+1D topological orders \mathcal{C}_{χ}^3 . The decomposition eqn. (C8) gives us the dimensional reduction eqn. (C6).

3. Stringlike excitations are G -flux, even in generic 3+1D AB topological orders

From the fact that the untwisted sector of dimension reduction is the Drinfeld center $Z[\text{Rep}(G)]$, we even know

that,

The *pure* stringlike excitations in a generic 3+1D AB topological order have the same fusion ring as the gauge theory with the corresponding gauge group G .

Alternatively, we can argue the above claim using the results for the string-only boundary. This is because the bulk stringlike excitations can be obtained by lifting the boundary stringlike excitations. Since a bulk stringlike excitation s_i can braid around a boundary stringlike excitation s_g^{bdry} , their fusion satisfy

$$s_i \otimes s_g^{\text{bdry}} = s_g^{\text{bdry}} \otimes s_i. \quad (\text{C9})$$

This allows us to show that

$$s_i = \bigoplus_{g \in \chi} s_g^{\text{bdry}} \equiv s_{\chi}, \quad (\text{C10})$$

where χ is a conjugacy class of G . Therefore, even in a generic 3+1D topological order, we may still view stringlike excitations as the G -gauge flux which is described the conjugacy classes of the group G . In particular, the bulk pure stringlike excitations fuse like the conjugacy classes (see eqn. (A11)). As a result, the quantum dimension of a pure stringlike excitation is given by the size of the conjugacy class: $d_{s_{\chi}} = |\chi|$. This is one of the key result of this paper.

There is a simple physical way to understand the relation between the bulk and boundary stringlike excitations. Since the boundary is induced by the condensation of all the pointlike excitations, it corresponds to breaks all the “gauge symmetry”. So there is no gauge equivalence of $g \sim hgh^{-1}$, and there is no degeneracy between the flux-loop that induce g monodromy and hgh^{-1} monodromy. So, if we bring a bulk string s_{χ} near the boundary, it will split $s_{\chi} \rightarrow \bigoplus_{g \in \chi} s_g^{\text{bdry}}$ (see eqn. (C10)).

Appendix D: More about dimension reduction

We argued that the stringlike excitations are G -flux, even in generic 3+1D AB bosonic topological orders. Now we can say more about the 2+1D topological orders \mathcal{C}_{χ}^3 that appear in the dimension reduction eqn. (C6). We first note that the pointlike excitations in 3+1D topological order \mathcal{C}^4 are described by $\text{Rep}(G)$ for a group G . In the dimension reduction, those 3+1D pointlike excitations becomes the 2+1D pointlike excitations with trivial mutual statistics between them; they form symmetric fusion subcategories \mathcal{E}_{χ} of the 2+1D dimension reduced topological orders \mathcal{C}_{χ}^3 .

For the conjugacy class $\chi = \{1\}$, *i.e.* the untwisted sector, $\mathcal{E}_{\chi=\{1\}} = \text{Rep}(G)$ and $\mathcal{C}_{\chi=\{1\}}^3 = Z[\text{Rep}(G)]$ is a minimal modular extension of $\text{Rep}(G)$. But what about the other conjugacy classes?

Because \mathcal{C}_{χ}^3 is induced by threading a G -flux described by χ through S^1 , such a G -flux will break the “gauge

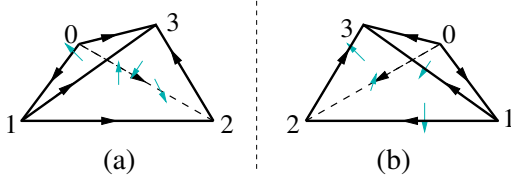


FIG. 10. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

symmetry” from G to G_χ , where G_χ is a subgroup of G that commutes with a fixed element in conjugacy class χ . Therefore, the SFC \mathcal{E}_χ in \mathcal{C}_χ^3 is given by $\text{Rep}(G_\chi)$. The 3+1D pointlike excitations described by $\text{Rep}(G)$ will split into 2+1D pointlike excitations described by $\text{Rep}(G_\chi)$ in each sector (see Table II).

Similarly for the other sectors we have

The dimension reduced 2+1D topological orders \mathcal{C}_χ^3 are minimal modular extensions of $\text{Rep}(G_\chi)$.

The *minimal modular extension* means that the anyons in \mathcal{C}_χ^3 that are not in $\text{Rep}(G_\chi)$ all have nontrivial mutual statistics with the bosons in $\text{Rep}(G_\chi)$. This condition comes from the result in Section IV B. Note that unlike the untwisted sector, \mathcal{C}_χ^3 is in general not the Drinfeld

center of $\text{Rep}(G_\chi)$.

Appendix E: The branching structure of space-time lattice

In order to define a generic lattice theory on the space-time complex M_{latt}^d using local tensors, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.^{15,16,66} A branching structure is a choice of orientation of each link in the d -dimensional complex so that there is no oriented loop on any triangle (see Fig. 10).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, *etc*. So the simplex in Fig. 10a has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its sub-simplices) a canonical orientation. Fig. 10 illustrates two 3-simplices with opposite canonical orientations compared with the 3-dimension space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

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