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The q-analog of Kostant's partition function and the highest root of the simple Lie algebras

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Abstract

For a given weight of a complex simple Lie algebra, the q-analog of Kostant's partition function is a polynomial valued function in the variable q, where the coefficient of q^k is the number of ways the weight can be written as a nonnegative integral sum of exactly k positive roots. In this paper we determine generating functions for the q-analog of Kostant's partition function when the weight in question is the highest root of the classical Lie algebras of types B, C, and D, and the exceptional Lie algebras of type G_2, F_4, E_6, E_7, and E_8.

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1 Introduction

In this paper we focus on finding an explicit formula for a partition counting problem in the setting of combinatorial representation theory of finite-dimensional simple Lie algebras. The problem of interest involves the q-analog of Kostant's partition function $\wp_q$, a polynomial-valued function in the variable q whose evaluation at $q = 1$ counts the number of ways to write a weight ξ as a sum of positive roots, i.e.,

$$\wp_q(\xi) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots + a_k q^k \tag{1}$$

where a_i is the number of ways to write the weight ξ as a sum of exactly i positive roots [19].

These functions appear in the q-analog of Kostant's weight multiplicity formula, which (after setting $q = 1$) gives the multiplicity of the weight μ in the finite-dimensional complex irreducible representation with highest weight λ; this is given by

$$m_q(\lambda, \mu) = \sum_{w \in W} (-1)^{\ell(w)} \wp_q(w(\lambda + \rho) - (\mu + \rho)), \tag{2}$$

where W denotes the Weyl group, ρ is half the sum of the positive roots, and $\ell(w)$ is the length of $w \in W$ [18]. The following celebrated result involving the q-analog of Kostant's weight multiplicity formula is due to Lusztig [19, Section 10, p. 226]: if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then

$$m_q(\tilde{\alpha}, 0) = q^{\varepsilon_1} + q^{\varepsilon_2} + \cdots + q^{\varepsilon_r}, \tag{3}$$

where $\tilde{\alpha}$ is the highest root and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ are the exponents of $\mathfrak{g}$. This article is motivated by the following question:

Question 1.1. *Can one use purely combinatorial techniques to prove Equation* (3)*?*

The first author resolved this for type A, but the question remains open for other Lie types [15]. One initial step in extending this work to other Lie types is to compute the value of the term indexed by the identity of the Weyl group. This term counts the number of ways to express the highest root of the Lie algebra as a nonnegative integral sum of the positive roots. Our main theorem and the resulting closed formulas provide insight into how to compute all other terms in Equation (3) as $\sigma(\tilde{\alpha} + \rho) - \rho$ has the most partitions into positive roots when $\sigma = 1$.

The main result of this paper resolves this in the classical Lie types (Theorems 3.8, 4.3, and 5.3):

Main Theorem (Generating Functions)**.** The closed formulas for the generating functions $\sum_{r \geq 1} \mathscr{P}_{B_r}(q) x^r$, $\sum_{r \geq 1} \mathscr{P}_{C_r}(q) x^r$, and $\sum_{r \geq 4} \mathscr{P}_{D_r}(q) x^r$, are given by

$$\sum_{r \geq 1} \mathscr{P}_{B_r}(q) x^r = \frac{qx + (-q - q^2)x^2 + q^2 x^3}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2},$$

$$\sum_{r\geq 1} \mathscr{P}_{C_r}(q)x^r = \frac{qx + (-q-q^2)x^2}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},$$
$$\sum_{r\geq 4} \mathscr{P}_{D_r}(q)x^r = \frac{(q+4q^2+6q^3+3q^4+q^5)x^4 - (q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},$$

where $\mathscr{P}_{B_r}(q)$, $\mathscr{P}_{C_r}(q)$, and $\mathscr{P}_{D_r}(q)$ denote $\wp_q(\tilde{\alpha})$ when $\tilde{\alpha}$ is the highest root of the Lie algebras of type B_r, C_r, and D_r, respectively.

We also compute $\wp_q(\tilde{\alpha})$ when $\tilde{\alpha}$ is the highest root of the exceptional Lie algebras of type G_2, F_4, E_6, E_7, and E_8. Letting $\mathscr{P}_{\mathfrak{g}}(q)$ denote $\wp_q(\tilde{\alpha})$ when $\tilde{\alpha}$ is the highest root of the exceptional Lie algebra $\mathfrak{g}$ we find that:

$$\begin{aligned}
\mathscr{P}_{G_2}(q) &= q(1+2q+2q^2+q^3+q^4)\\
\mathscr{P}_{F_4}(q) &= q(q^{10}+3q^9+9q^8+20q^7+40q^6+59q^5+69q^4+53q^3+27q^2+7q+1)\\
\mathscr{P}_{E_6}(q) &= q(q^{10}+5q^9+18q^8+48q^7+97q^6+142q^5+150q^4+105q^3+45q^2+10q+1)\\
\mathscr{P}_{E_7}(q) &= q(q^{16}+6q^{15}+26q^{14}+87q^{13}+247q^{12}+592q^{11}+1216q^{10}+2106q^9\\
&\quad +3054q^8+3617q^7+3420q^6+2488q^5+1340q^4+500q^3+120q^2+16q+1)\\
\mathscr{P}_{E_8}(q) &= q(q^{28}+7q^{27}+35q^{26}+138q^{25}+470q^{24}+1421q^{23}+3913q^{22}+9902q^{21}\\
&\quad +23216q^{20}+50542q^{19}+102283q^{18}+192015q^{17}+333340q^{16}+532288q^{15}\\
&\quad +776864q^{14}+1027773q^{13}+1220804q^{12}+1287007q^{11}+1188475q^{10}\\
&\quad +946515q^9+638680q^8+357494q^7+161931q^6+57540q^5+15435q^4\\
&\quad +2961q^3+378q^2+28q+1).
\end{aligned}$$

From the generating functions in Theorem 1 one can easily extract explicit formulas for $\wp_q(\tilde{\alpha})$ in all Lie types. We do this in Appendix A.

1.1 Connections to existing literature

Our work joins an extensive list of articles exploring combinatorial approaches to computing the value of Kostant's partition function and the associated weight multiplicity formula [6, 7, 10, 11, 13, 14, 15, 16, 22]. The rest of this introduction discusses connections between our main results and the existing literature on vector partition functions as they relate to polytope geometry and multiplex juggling sequences.

Geometric approaches have connected Kostant's partition function to flow polytopes and their volumes [1, 2, 20]. In fact, the formulas for $\wp_q(\tilde{\alpha})$ in our Main Theorem evaluated at $q = 1$ recover the generating functions for the number of lattice points in the Chan-Robbins-Yuen (CRY) polytope of type A, B, C, and D [20]. Two experts on flow polytopes, Baldoni and Vergne, have noted that "[i]n general, it is difficult to give 'concrete' formulae for the partition functions" [2]. This insight is informed by the fact that these geometric approaches deal with the more general theory of *vector partition functions*, i.e., functions computing the number of ways one can express a vector as a linear combination of a fixed set of vectors using nonnegative integral coefficients. Much work has be done to construct efficient programs to compute the values of such functions [1, 9, 21] and part of the challenge is that the computational complexity of doing so is polynomial in the size of the input in some special cases, but NP-hard in general [3, 4, 5].

Applications of polyhedral geometry to such problems usually fix the dimension (here, the Lie algebra's rank) and scale the vector input of the partition function. Our work introduces a novel approach by fixing the weight in question while increasing the rank of the Lie algebra. This gives us a complete picture of the value of Kostant's partition function on the highest root (in all Lie types), and our approach is amenable to generalization by replacing the highest root with other weights. However we suspect this direction will be cumbersome to approach combinatorially and suspect that a polytopal approach might give insight to deal with this in general.

Setting $q = 1$ in the generating functions for the Lie algebras of type B and C recovers the generating functions used by Butler and Graham to count the number of multiplex juggling sequences (OEIS sequence A136775) of length n, base state $\langle 1, 1\rangle$ and hand capacity 2, and the number of periodic multiplex juggling sequences (OEIS sequence A081567) of length n with base state $\langle 2\rangle$ and hand capacity 2, respectively [8]. We list these generating functions in Table 1. While the generating functions agree, we do not know of a combinatorial bijection and pose the following problem.

Problem 1. *Find a combinatorial bijection between the set of partitions (into positive roots) of the highest root of the Lie algebra of*

1. *type B_n, and the set of multiplex juggling sequences of length n with base state $\langle 1, 1\rangle$ and hand capacity 2;*
2. *type C_n, and the number of periodic multiplex juggling sequences of length n with base state $\langle 2\rangle$ and hand capacity 2.*

Type	Generating function
B	$\dfrac{x - 2x^2 + x^3}{1 - 5x + 5x^2}$
C	$\dfrac{x - 2x^2}{1 - 5x + 5x^2}$

Table 1: Generating functions for the value of $\wp(\tilde{\alpha})$ in Lie types B and C

This paper is organized as follows: Section 2 contains the necessary background on root systems to make our approach precise. Sections 3 and 4 contain recursive formulas that count the number of partitions of the highest root $\tilde{\alpha}$ in types B_r and C_r (respectively) as sums of positive roots. These recursions are used to derive the closed forms for the generating functions $\sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r$ in Theorem 3.8 and $\sum_{r\geq 1} \mathscr{P}_{C_r}(q)x^r$ in Theorem 4.3. Finally in Section 5 we describe how the total number of partitions of the highest root as sums of positive roots in type D_r can be calculated using the partitions of the highest root in type B_{r-2}, the content of Theorem 5.2. This leads to the derivation of a closed formula for the generating function $\sum_{r\geq 4} \mathscr{P}_{D_r}(q)x^r$ in Theorem 5.3.

2 Background

The structure of classical root systems has been extensively studied and completely classified up to isomorphism. We follow the conventions and choices of vector space bases set forth in Goodman and Wallach's text, [12, Section 2.4.3].

Let Φ be the root system for a Lie algebra $\mathfrak{g}$ of type A_r, B_r, C_r, or D_r.

Definition 2.1. A subset $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset \Phi$ is a set of *simple roots* if every root in $\beta \in \Phi$ can be written uniquely as

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$$

with all the c_i's having the same sign. Choosing a set of simple roots Δ partitions Φ into two disjoint subsets $\Phi = \Phi^+ \cup \Phi^-$ of positive roots Φ^+ and negative roots Φ^-, where Φ^+ is the collection of roots where $c_i \geq 0$ and Φ^- is the set of roots with $c_i \leq 0$ for all $\alpha_i \in \Delta$. If $\Delta \subset \Phi$ is a subset of simple roots and $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r$ is a root, then the *height* of β is

$$ht(\beta) = c_1 + c_2 + \cdots + c_r.$$

Naturally the positive roots are those $\beta \in \Phi$ with $ht(\beta) > 0$. In the classical Lie algebras, there is a unique *highest root* $\tilde{\alpha}$ defined by the property $ht(\tilde{\alpha}) \geq ht(\beta)$ for all $\beta \in \Phi$.

Following the conventions set in [12, Section 2.4.3], we now describe a choice of simple roots, list the positive roots, and specify the corresponding highest root in Lie types B_r, C_r, and D_r.

Type B_r ($\mathfrak{so}_{2r+1}(\mathbb{C})$): Let $r \geq 2$ and let $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$ be a set of simple roots. We describe the set of positive roots Φ^+ by breaking them up into roots we refer to as *hooked* and *nonhooked*. We define the set of *nonhooked positive roots* to be

$$\Phi^+_{B_r^{NH}} = \Delta \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i < j \leq r\},$$

and we define the set of *hooked positive roots* to be

$$\Phi^+_{B_r^H} = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_r \mid 1 \leq i < j \leq r\}.$$

Note the highest root of type B_r is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$ and $\Phi^+ = \Phi^+_{B_r^H} \sqcup \Phi^+_{B_r^{NH}}$.

Type C_r ($\mathfrak{sp}_{2r}(\mathbb{C})$): Let $r \geq 2$ and let $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$ be a set of simple roots. Again we break Φ^+ into the set of hooked and nonhooked roots. We define the set of *nonhooked positive roots* to be

$$\Phi^+_{C_r^{NH}} = \Delta \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \le i < j \le r\},$$

and we define the set of *hooked positive roots* to be

$$\Phi^+_{C_r^H} = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{r-1} + \alpha_r \mid 1 \le i < j \le r-1\}.$$

Note that the highest root is $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r$ and $\Phi^+ = \Phi^+_{C_r^H} \sqcup \Phi^+_{C_r^{NH}}$.

Type D_r ($\mathfrak{so}_{2r}(\mathbb{C})$): Let $r \ge 4$ and let $\Delta = \{\alpha_i \mid 1 \le i \le r\}$ be a choice of a set of simple roots. We define the set of *nonhooked positive roots* to be

$$\Phi^+_{D_r^{NH}} = \Delta \cup \{\alpha_i + \cdots + \alpha_j \mid 1 \le i < j \le r\} \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{r-2} + \alpha_r \mid 1 \le i \le r-2\},$$

and we define the set of *hooked positive roots* to be

$$\Phi^+_{D_r^H} = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{r-2} + \alpha_{r+1} + \alpha_r \mid 1 \le i < j \le r-2\}.$$

Note that the highest root of type D_r is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ and the set of positive roots is the disjoint union $\Phi^+ = \Phi^+_{D_r^H} \sqcup \Phi^+_{D_r^{NH}}$.

Definition 2.2. A *partition* of the highest root $\tilde{\alpha}$ in Φ^+ is a multiset $\{\beta_1, \beta_2, \ldots, \beta_k\} = \Gamma$ such that $\beta_i \in \Phi^+$ for all $1 \le i \le k$ and $\tilde{\alpha} = \beta_1 + \beta_2 + \cdots + \beta_k$. The elements $\beta \in \Gamma$ are called the *parts* of the partition Γ.

2.1 Coordinate Vector Notation

Let $e_i = (0, 0, \ldots, 1, 0, 0, \ldots)$ denote the ith standard basis vector of $\mathbb{C}^\infty$. For each classical Lie algebra $\mathfrak{g}$ with set of roots $\Phi_{\mathfrak{g}}$ we let $\Psi : \Phi_{\mathfrak{g}} \to \mathbb{C}^\infty$ be the map defined on the simple roots of the Lie algebras of type B_r and D_r via $\Psi(\alpha_i) = e_i$ and for the Lie algebra of type C_r via $\Psi(\alpha_i) = e_{r-i+1}$. We extend this map linearly to the roots in $\Phi_{\mathfrak{g}}$. In doing this, we relate the coefficients of $\mathscr{P}_{B_r}(q)$, $\mathscr{P}_{C_r}(q)$, $\mathscr{P}_{D_r}(q)$ respectively to counting the number of ways a specific vector in $\mathbb{C}^\infty$ can be expressed as a linear combination of vectors from a fixed set, where scalars are required to be nonnegative integers. This is presented in the lemmas of this section.

Before presenting these lemmas, consistency of notation with literature must be addressed. The coordinate maps $\Psi(\alpha_i)$ are not standard (see [1, 17, 12, 20]). Our purpose for introducing non-standard coordinate maps is that it makes the combinatorial arguments significantly more manageable to present.

Lemma 2.3 (Type B)**.** *Let $r, k \ge 1$ be integers. The number of partitions of the highest root $\tilde{\alpha}$ of the Lie algebra of type B_r with k parts is equal to the number of ways of writing the vector*

$$\Psi(\tilde{\alpha}) = e_1 + 2e_2 + 2e_3 + \cdots + 2e_r = (\underbrace{1, 2, 2, \ldots, 2}_{r \text{ nonzero entries}}, 0, 0, \ldots) \in \mathbb{C}^\infty$$

as a nonnegative integer combination of k of the following vectors:

- *Nonhooked vectors (parts) of the form* e_i *with* $1 \leq i \leq r$.
- *Nonhooked vectors (parts) of the form* $e_i + e_{i+1} + \cdots + e_j$ *with* $1 \leq i < j \leq r$.
- *Hooked vectors (parts) of the the form* $e_i + e_{i+1} + \cdots + e_{j-1} + 2e_j + 2e_{j+1} + \cdots + 2e_r$ *with* $1 \leq i < j \leq r$.

Lemma 2.4 (Type C)**.** *Let* $r, k \geq 1$ *be integers. The number of partitions of the highest root* $\tilde{\alpha}$ *of the Lie algebra of type* C_r *with* k *parts is equal to the number of ways of writing the vector*

$$\Psi(\tilde{\alpha}) = e_1 + 2e_2 + \cdots + 2e_{r-1} + 2e_r = (\underbrace{1,\ 2,\ 2,\ \ldots,\ 2,}_{r \text{ nonzero entries}} 0, 0, \ldots) \in \mathbb{C}^\infty$$

as a nonnegative integer combination of k *of the following vectors:*

- *Nonhooked vectors (parts) of the form* e_i *with* $1 \leq i \leq r$.
- *Nonhooked vectors (parts) of the form* $e_i + e_{i+1} + \cdots + e_j$ *with* $1 \leq i \leq j \leq r$.
- *Hooked vectors (parts) of the form* $e_1 + 2e_2 + \cdots + 2e_i + e_{i+1} + \cdots + e_j$ *with* $1 < i < j \leq r$.

Lemma 2.5 (Type D)**.** *Let* $r, k \geq 1$ *be integers. The number of partitions of the highest root* $\tilde{\alpha}$ *of the Lie algebra of type* D_r *with* k *parts is equal to the number of ways of writing the vector*

$$\Psi(\tilde{\alpha}) = e_1 + 2e_2 + 2e_3 + \cdots + 2e_{r-2} + e_{r-1} + e_r = (\underbrace{1,\ 2,\ 2, \ldots,\ 2,\ 1,\ 1}_{r \text{ nonzero entries}}, 0, 0, \ldots) \in \mathbb{C}^\infty$$

as a nonnegative integer combination of k *of the following vectors:*

- *Nonhooked vectors (parts) of the form* e_i *with* $1 \leq i \leq r$.
- *Nonhooked vectors (parts) of the form* $e_i + e_{i+1} + \cdots + e_j$ *with* $1 \leq i < j \leq r$.
- *Nonhooked vectors (parts) of the form* $e_i + e_{i+1} + \cdots + e_{r-2} + e_r$ *with* $1 \leq i \leq r-2$.
- *Hooked vectors (parts) of the the form*

$$e_i + e_{i+1} + \cdots + e_{j-1} + 2e_j + 2e_{j+1} + \cdots + 2e_{r-2} + e_{r+1} + e_r$$

  with $1 \leq i < j \leq r-2$.

We use the words *vector* and *part* interchangeably throughout the rest of the article, the context will be clear.

We note that the nonhooked vectors of the Lie algebras of type B, C, and D in a fixed rank correspond to the positive roots of the Lie algebra of type A in that rank. Consequently one can use Equation (4) or (9) to find generating functions for Kostant's partition function on the highest root of types B and D, respectively, when the positive roots used are restricted to those of type A (we omitted Type C as that can be done directly).

3 Type B

Let $\mathscr{P}_{B_r}(q) := \wp_q(\tilde{\alpha})$, where $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$ is the highest root of the Lie algebra of type B_r. We can recover $\mathscr{P}_{B_r}(q)$ by determining the number of partitions of $e_1 + 2e_2 + \cdots + 2e_r$ into k parts where the parts are as specified in Lemma 2.3. In this light, throughout this section, a *partition* of the vector $e_1 + 2e_2 + \cdots + 2e_\ell$ in $\mathbb{C}^\infty$ where $2 \le \ell \le r$ is a nonnegative integer combination of k (not necessarily distinct) vectors from the following set:

- Nonhooked vectors (parts) of the form e_i with $1 \le i \le \ell$.
- Nonhooked vectors (parts) of the form $e_i + e_{i+1} + \cdots + e_j$ with $1 \le i < j \le \ell$.
- Hooked vectors (parts) of the the form $e_i + e_{i+1} + \cdots + e_{j-1} + 2e_j + 2e_{j+1} + \cdots + 2e_\ell$ with $1 \le i < j \le \ell$.

The following example illustrates the computation of a partition function.

Example 3.1. In Table 2 we provide the 40 ways in which $\gamma = e_1 + 2e_2 + 2e_3 + 2e_4$ can be expressed as a sum of nonhooked and hooked vectors of type B_4. From this we can compute

$$\mathscr{P}_{B_4}(q) = q^7 + 3q^6 + 8q^5 + 11q^4 + 11q^3 + 5q^2 + q. \quad \square$$

Let P denote a partition of the vector $e_1 + 2e_2 + \cdots + 2e_{r-1}$. If P does not have a part containing $2e_{r-1}$ as a summand, then P contains exactly two parts each containing an e_{r-1} summand. Hence let $\mathbf{u} = e_i + \cdots + e_{r-1}$ and $\mathbf{v} = e_j + \cdots + e_{r-1}$ denote these two parts.

For any partition P containing only nonhooked parts there are exactly four ways to extend these partitions P of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ to a partition of $e_1 + 2e_2 + \cdots + 2e_{r-1} + 2e_r$, where the only parts that change are the ones containing e_{r-1}:

$E_B(1)$: replace $\mathbf{u}$ and $\mathbf{v}$ with $\bar{\mathbf{u}} = \mathbf{u} + e_r$ and $\bar{\mathbf{v}} = \mathbf{v} + e_r$ in P,

$E_B(2)$: replace $\mathbf{u}$ with $\bar{\mathbf{u}} = \mathbf{u} + e_r$ in P, introduce the part e_r into P, and leave $\mathbf{v}$ unchanged,

$E_B(3)$: replace $\mathbf{v}$ with $\bar{\mathbf{v}} = \mathbf{v} + e_r$ in P, introduce the part e_r to P, and leave $\mathbf{u}$ unchanged,

$E_B(4)$: introduce the part e_r twice in P, and leave $\mathbf{u}$ and $\mathbf{v}$ unchanged.

From the definition of a partition and the extensions above the following is immediate.

Proposition 3.2. *Let P be a partition of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ that does not contain a hooked vector as a part, and for $1 \le \ell \le 4$ let $P(\ell)$ be the result of applying extension $E_B(\ell)$ to P. Then for $\ell \ne k$, $P(\ell) = P(k)$ if and only if $\{\ell, k\} = \{2, 3\}$. Furthermore, if $P(\ell) = P(k)$, then $\mathbf{u} = \mathbf{v}$.*

n	Partitions of γ using n vectors
7	$\{e_1,\ e_2,\ e_2,\ e_3,\ e_3,\ e_4,\ e_4\}$
6	$\{e_1+e_2,\ e_2,\ e_3,\ e_3,\ e_4,\ e_4\}$
	$\{e_1,\ e_2,\ e_2+e_3,\ e_3,\ e_4,\ e_4\}$
	$\{e_1,\ e_2,\ e_2,\ e_3,\ e_3+e_4,\ e_4\}$
5	$\{e_1,\ e_2,\ e_2,\ e_3,\ e_3+2e_4\}$
	$\{e_1,\ e_2,\ e_3,\ e_4,\ e_2+e_3+e_4\}$
	$\{e_2,\ e_3,\ e_4,\ e_4,\ e_1+e_2+e_3\}$
	$\{e_1,\ e_2,\ e_2+e_3,\ e_4,\ e_3+e_4\}$
	$\{e_1,\ e_2,\ e_2,\ e_3+e_4,\ e_3+e_4\}$
	$\{e_1+e_2,\ e_2,\ e_3,\ e_4,\ e_3+e_4\}$
	$\{e_1+e_2,\ e_3,\ e_4,\ e_4,\ e_2+e_3\}$
	$\{e_1,\ e_2+e_3,\ e_4,\ e_4,\ e_2+e_3\}$
4	$\{e_1,\ e_2,\ e_3,\ e_2+e_3+2e_4\}$
	$\{e_1,\ e_2,\ e_2+e_3,\ e_3+2e_4\}$
	$\{e_1+e_2,\ e_2,\ e_3,\ e_3+2e_4\}$
	$\{e_2,\ e_3,\ e_4,\ e_1+e_2+e_3+e_4\}$
	$\{e_1,\ e_2+e_3,\ e_4,\ e_2+e_3+e_4\}$
	$\{e_1,\ e_2,\ e_3+e_4,\ e_2+e_3+e_4\}$
	$\{e_1+e_2,\ e_3,\ e_4,\ e_2+e_3+e_4\}$
	$\{e_2+e_3,\ e_4,\ e_4,\ e_1+e_2+e_3\}$
	$\{e_2,\ e_3+e_4,\ e_4,\ e_1+e_2+e_3\}$
	$\{e_2,\ e_1+e_2,\ e_3+e_4,\ e_3+e_4\}$
	$\{e_1+e_2,\ e_3+e_4,\ e_4,\ e_3+e_4\}$
3	$\{e_1,\ e_2,\ e_2+2e_3+2e_4\}$
	$\{e_2,\ e_3,\ e_1+e_2+e_3+2e_4\}$
	$\{e_1,\ e_2+e_3,\ e_2+e_3+4e_4\}$
	$\{e_1+e_2,\ e_3,\ e_2+e_3+2e_4\}$
	$\{e_1+e_2,\ e_2+e_3,\ e_3+2e_4\}$
	$\{e_1+e_2+e_3,\ e_2,\ e_3+2e_4\}$
	$\{e_2,\ e_3+e_4,\ e_1+e_2+e_3+e_4\}$
	$\{e_2+e_3,\ e_4,\ e_1+e_2+e_3+e_4\}$
	$\{e_1,\ e_2+e_3+e_4,\ e_2+e_3+e_4\}$
	$\{e_1+e_2+e_3,\ e_4,\ e_2+e_3+e_4\}$
	$\{e_1+e_2,\ e_3+e_4,\ e_2+e_3+e_4\}$
2	$\{e_1+e_2+e_3+e_4,\ e_2+e_3+e_4\}$
	$\{e_1+e_2+e_3,\ e_2+e_3+2e_4\}$
	$\{e_1+e_2,\ e_2+2e_3+2e_4\}$
	$\{e_2+e_3,\ e_1+e_2+e_3+2e_4\}$
	$\{e_2,\ e_1+e_2+2e_3+2e_4\}$
1	$\{e_1+2e_2+2e_3+2e_4\}$

Table 2: Partitions of $\tilde{\alpha}$ in type B_4

It is important to note that if we began with two distinct partitions P and P' of the vector $e_1+2e_2+\cdots+2e_{r-1}$ which do not have a hooked vector as a part, then by the definition of the extensions $E_B(1)$, $E_B(2)$, $E_B(3)$, and $E_B(4)$, the extensions of P and P' do not yield the same final partition as these extensions only affect the parts that involve e_r.

Definition 3.3. Let $r \geq 2$, the let

$$\mathscr{P}^{H}_{B_r}(q) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i is the number of partitions of $e_1+2e_2+\cdots+2e_r$ with i parts where one part is a hooked vector. Similarly, let

$$\mathscr{P}^{NH}_{B_r}(q) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i is the number of ways to write $e_1+2e_2+\cdots+2e_r$ as a sum of exactly i parts where no part is a hooked vector.

We remark that the function $\mathscr{P}^{NH}_{B_r}(q)$ is the restriction of Kostant partition function to the positive roots of type A_r. A similar phenomenon holds for $\mathscr{P}^{NH}_{C_r}(q)$ and $\mathscr{P}^{NH}_{D_r}(q)$.

Lemma 3.4. *For $r \geq 3$,*

$$\mathscr{P}_{B_r}(q) = \mathscr{P}^{H}_{B_r}(q) + \mathscr{P}^{NH}_{B_r}(q).$$

Proof. This follows directly from Lemma 2.3 and the fact that any partition of $e_1 + 2e_2 + \cdots + 2e_r$ has at most one part that is a hooked vector. □

Proposition 3.5. *For $r \geq 3$, the polynomials $\{\mathscr{P}^{NH}_{B_i}(q)\}$ satisfy the following recursion*

$$\mathscr{P}^{NH}_{B_r}(q) = (1+q)^2 \mathscr{P}^{NH}_{B_{r-1}}(q) - q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}^{NH}_{B_i}(q) \right). \tag{4}$$

Proof. Recall that every partition P of $e_1 + 2e_2 + \cdots + 2e_r$ with no hooked parts comes from an extension of a partition P' of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ with no hooked parts via at least one of the four extensions $E_B(1)$, $E_B(2)$, $E_B(3)$, and $E_B(4)$. These extensions respectively add zero, one, one, and two summands.

Hence, the polynomial whose coefficients encode the total of number of said extensions is

$$(1 + 2q + q^2)\mathscr{P}^{NH}_{B_{r-1}}(q) = (1+q)^2 \mathscr{P}^{NH}_{B_{r-1}}(q).$$

However, by Proposition 3.2, $(1+q)^2\mathscr{P}^{NH}_{B_{r-1}}(q)$ double counts the contribution of partitions P of $e_1 + 2e_2 + \cdots + 2e_r$ obtained from extensions $E_B(2)$ and $E_B(3)$ of partitions P' of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ that contain two equal parts $\mathbf{u}$ and $\mathbf{v}$ with

$$\mathbf{u} = \mathbf{v} = e_{i+1} + \cdots + e_{r-1}$$

for some i with $1 \leq i \leq r-2$. Consider these partitions for a fixed i. By removing $\mathbf{u}$ and $\mathbf{v}$ from such a partition P', we see that these partitions are in bijection with the set of partitions of $e_1 + 2e_2 + \cdots + 2e_i$. Thus, the polynomial whose coefficients encode the total number of such partitions is given by $q^3\mathscr{P}^{NH}_{B_i}(q)$, where we multiply by q^3 because $\bar{P}(2)$ and $\bar{P}(3)$ each have three more parts than P'. By ranging over all possible i we have that the polynomial encoding the double counted partitions is

$$q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}^{NH}_{B_i}(q) \right).$$

Thus

$$\mathscr{P}^{NH}_{B_r}(q) = (1+q)^2 \mathscr{P}^{NH}_{B_{r-1}}(q) - q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}^{NH}_{B_i}(q) \right).$$ □

Proposition 3.6. *The closed form for the generating function $\sum_{r \geq 1} \mathscr{P}^{NH}_{B_r}(q) x^r$ is*

$$\frac{qx + (-2q - q^2)x^2 + (q+q^2)x^3}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2}.$$

Proof. For simplicity, let $S(q,x) = \sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r$. Then

$$\begin{aligned} S(q,x) &= \mathscr{P}^{NH}_{B_1}(q)x + \mathscr{P}^{NH}_{B_2}(q)x^2 + \sum_{r\geq 3} \mathscr{P}^{NH}_{B_r}(q)x^r \\ &= qx + (q^3+q^2)x^2 + \sum_{r\geq 3}\left[(1+q)^2\mathscr{P}^{NH}_{B_{r-1}}(q) - q^3\left(\sum_{i=1}^{r-2}\mathscr{P}^{NH}_{B_i}(q)\right)\right]x^r \\ &= qx + (q^3+q^2)x^2 + x(1+q)^2\left(\sum_{r\geq 3}\mathscr{P}^{NH}_{B_{r-1}}(q)x^{r-1}\right) \\ &\qquad - q^3x^2\sum_{r\geq 3}\left(\sum_{i=1}^{r-2}\mathscr{P}^{NH}_{B_i}(q)\right)x^{r-2} \\ &= qx + (q^3+q^2)x^2 + x(1+q)^2\left(S(q,x) - qx\right) - \frac{q^3x^2}{1-x}S(q,x). \end{aligned}$$

The second equality given above comes from Proposition 3.5. Hence,

$$S(q,x)\cdot\left(1 - x(1+q)^2 + \frac{q^3x^2}{1-x}\right) = qx + (q^3+q^2)x^2 - qx^2(1+q)^2,$$

so

$$S(q,x) = \frac{qx + (-2q-q^2)x^2 + (q+q^2)x^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2}. \qquad \square$$

Proposition 3.7. *For $r \geq 3$, the polynomials $\{\mathscr{P}^H_{B_i}(q)\}, \{\mathscr{P}^{NH}_{B_i}(q)\}$ satisfy the following recurrence*

$$\mathscr{P}^H_{B_r}(q) = q + 2\left(\sum_{i=2}^{r-1}\mathscr{P}^{NH}_{B_i}(q)\right) - q^2\left(\sum_{i=1}^{r-2}\sum_{j=1}^{i}\mathscr{P}^{NH}_{B_j}(q)\right).$$

Proof. Let $i \in \{2,3\ldots,r-1\}$. Let P be a partition of $e_1+2e_2+\cdots+2e_i$ that does not have any hooked part. Then P has two parts that contain e_i as a summand. Let these two parts be denoted $\mathbf{u} = e_j+\cdots+e_i$ and $\mathbf{v} = e_k+\cdots+e_i$. We can extend P to a partition of B_r that has a hooked part: either replace $\mathbf{u}$ by $e_j+\cdots+e_i+2e_{i+1}+\cdots+2e_r$ or replace $\mathbf{v}$ with $e_k+\cdots+e_i+2e_{i+1}+\cdots+2e_r$. Ignoring the full partition $e_1+2e_2+\cdots+2e_r$, every partition of $e_1+2e_2+\cdots+2e_r$ that has a hooked part can be constructed by extending a partition P of $e_1+2e_2+\cdots+2e_i$ for some i using the aforementioned process. Indeed, suppose Q is a partition of $e_1+2e_2+\cdots+2e_r$ that has a hooked part, say $e_j+e_{j+1}+\cdots+e_i+2e_{i+1}+\cdots+2e_r$. Then the partition P that has all the same parts as Q but replaces $e_j+e_{j+1}+\cdots+e_i+2e_{i+1}+\cdots+2e_r$ with $e_j+e_{j+1}+\cdots+e_i$ is a partition of $e_1+2e_2+\cdots+2e_i$ that extends to Q. Consequently, we have

$$\mathscr{P}^H_{B_r}(q) = q + 2\left(\sum_{i=2}^{r-1}\mathscr{P}^{NH}_{B_i}(q)\right) - G(q)$$

where $G(q)$ subtracts the contribution of partitions of $e_1 + 2e_2 + \cdots + 2e_r$ that can arise from partitions of $e_1 + 2e_2,\ e_1 + 2e_2 + 2e_3, \ldots, e_1 + 2e_2 + \cdots + 2e_{r-1}$ (that do not have any hooked part) via the above extension in multiple ways.

To compute $G(q)$, we suppose P and P' are partitions of $e_1 + 2e_2 + \cdots + 2e_i$ and $e_1 + 2e_2 + \cdots + 2e_{i'}$ respectively for which the above extension leads to the same partition Q of $e_1 + 2e_2 + \cdots + 2e_r$. Only one part of the partition achieved after extending P (similarly P') to Q contains a hooked part, and it is of the form $e_k + \cdots + e_i + 2e_{i+1} + \cdots + 2e_r$ (similarly $e_{k'} + \cdots + e_{i'} + 2e_{i'+1} + \cdots + 2e_r$). Since P and P' both extend to Q, this implies

$$e_k + \cdots + e_i + 2e_{i+1} + \cdots + 2e_r = e_{k'} + \cdots + e_{i'} + 2e_{i'+1} + \cdots + 2e_r$$

and hence $i = i'$ and $e_k + \cdots + e_i = e_{k'} + \cdots + e_{i'}$. Since all other parts of P and P' are the same as the parts of Q besides the part containing a $2e_r$ summand, we deduce $P = P'$.

Thus, in order to determine $G(q)$, we need to determine when applying the two different aforementioned extensions to a partition P (not containing a hooked part) of $e_1+2e_2+\cdots+2e_i$ for some $i \in \{2, \ldots, r-1\}$ can result in the same partition. Let P be a partition of $e_1+2e_2+\cdots+2e_i$ not containing a hooked part. Then P has exactly two parts containing e_i as a summand. Call these $\mathbf{u} = e_k+\cdots+e_i$ and $\mathbf{v} = e_{k'}+\cdots+e_i$. When applying our extension, these are replaced by $e_k + \cdots + e_i + 2e_{i+1} + \cdots + 2e_r$ and $e_{k'} + \cdots + e_i + 2e_{i+1} + \cdots + 2e_r$ respectively, and all other summands remain the same, so the two extensions are equal if and only if $\mathbf{u} = \mathbf{v}$ (i.e. $k = k'$).

We can now compute $G(q)$. Fix $i \in \{2, \ldots, r-1\}$. We determine the contribution of the set of partitions P of $e_1 + 2e_2 + \cdots + 2e_i$ (not containing a hooked part) to $G(q)$. From the previous paragraph, we obtain a contribution to $G(q)$ for every such partition P in which its two parts $\mathbf{u}, \mathbf{v}$ containing e_i are the same, say $\mathbf{u} = \mathbf{v} = e_{j+1} + \cdots + e_i$. The remainder of the partition P can then range over any partition of $e_1 + 2e_2 + \cdots + 2e_j$ with $1 \leq j < i$ not containing a hooked part. For each such partition of $e_1 + 2e_2 + \cdots + 2e_j$, the partition P (and hence its extension) has two more parts than it, accounting for $\mathbf{u}$ and $\mathbf{v}$. Thus, the combined contribution to $G(q)$ arising from partitions of $e_1 + 2e_2 + \cdots + 2e_i$ is given by

$$\sum_{j=1}^{i-1} q^2 \mathscr{P}_{B_j}^{NH}(q).$$

Ranging over all i gives us

$$G(q) = \sum_{i=2}^{r-1}\sum_{j=1}^{i-1} q^2 \mathscr{P}_{B_j}^{NH}(q) = \sum_{i=1}^{r-2}\sum_{j=1}^{i} q^2 \mathscr{P}_{B_j}^{NH}(q). \qquad \square$$

Theorem 3.8. *The closed formula for the generating function* $\sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r$ *is given by*

$$\sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r = \frac{qx + (-q - q^2)x^2 + q^2x^3}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2}.$$

Proof. For simplicity, let $T(q,x) = \sum_{r\geq 1} \mathscr{P}^{H}_{B_r}(q)x^r$, and $S(q,x) = \sum_{r\geq 1} \mathscr{P}^{NH}_{B_r}(q)x^r$. Then

$$\begin{aligned} T(q,x) &= \mathscr{P}^{H}_{B_1}(q)x + \mathscr{P}^{H}_{B_2}(q)x^2 + \sum_{r\geq 3} \mathscr{P}^{H}_{B_r}(q)x^r \\ &= qx^2 + \sum_{r\geq 3}\left(q + 2\left(\sum_{i=2}^{r-1} \mathscr{P}^{NH}_{B_i}(q)\right) - q^2\left(\sum_{i=1}^{r-2}\sum_{j=1}^{i} \mathscr{P}^{NH}_{B_j}(q)\right)\right)x^r \\ &= qx^2 + \sum_{r\geq 3} qx^r + \sum_{r\geq 3} 2\left(\sum_{i=2}^{r-1} \mathscr{P}^{NH}_{B_i}(q)\right)x^r - q^2\sum_{r\geq 3}\left(\sum_{i=1}^{r-2}\sum_{j=1}^{i} \mathscr{P}^{NH}_{B_j}(q)\right)x^r \\ &= qx^2 + \frac{qx^3}{1-x} + \frac{2x}{1-x}(S(q,x) - qx) - \frac{q^2x^2}{(1-x)^2}S(q,x). \end{aligned}$$

The second equality follows from Proposition 3.7. Hence,

$$\begin{aligned} \sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r &= S(q,x) + T(q,x) \\ &= S(q,x) + qx^2 + \frac{qx^3}{1-x} - \frac{2qx^2}{1-x} + \frac{2x}{1-x}S(q,x) - \frac{q^2x^2}{(1-x)^2}S(q,x), \end{aligned}$$

the first equality following from Lemma 3.4. It follows then that

$$\sum_{r\geq 1} \mathscr{P}_{B_r}(q)x^r = \frac{qx + (-q-q^2)x^2 + q^2x^3}{1 - (2+2q+q^2)x + (1+2q+q^2+q^3)x^2}. \qquad \square$$

4 Type C

Let $\mathscr{P}_{C_r}(q) := \wp_q(\tilde{\alpha})$, where $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r$ is the highest root of the Lie algebra of type C_r. By Lemma 2.4, we can recover $\mathscr{P}_{C_r}(q)$ by determining the number of partitions of $e_1 + 2e_2 + \cdots + 2e_r$ into k parts where the parts are as specified in Lemma 2.4. In this light, throughout this section, a *partition* of the vector $e_1+2e_2+\cdots+2e_\ell$ in $\mathbb{C}^\infty$ where $2 \leq \ell \leq r$ is a nonnegative integer combination of k (not necessarily distinct) vectors from the following set:

- Nonhooked vectors (parts) of the form $e_i + e_{i+1} + \cdots + e_j$ with $1 \leq i \leq j \leq \ell$.
- Hooked vectors (parts) of the form $e_1 + 2e_2 + \cdots + 2e_i + e_{i+1} + \cdots + e_j$ with $1 < i < j \leq \ell$.

Let P be a partition of $\Psi(\tilde{\alpha}) = e_1 + 2e_2 + 2e_3 + \cdots + 2e_{r-1} \in \mathbb{C}^\infty$. If P has more than one part, then exactly two of the parts of P must contain e_{r-1} as a summand. Let $\mathbf{u} = e_i + \cdots + e_{r-1}$ and $\mathbf{v} = e_j + \cdots + e_{r-1}$ be these two parts. For any partition P besides the partition $\{e_1+2e_2+\cdots+2e_{r-1}\}$ there are exactly four ways to extend a partition P of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ to a partition of $e_1 + 2e_2 + \cdots + 2e_{r-1} + 2e_r$, where the parts that do not contain e_{r-1} remain the same:

$E_C(1)$: introduce two e_r parts to P to get $P \cup \{e_r, e_r\}$,

$E_C(2)$: replace $\mathbf{u}$ by $\bar{\mathbf{u}} = e_i + \cdots + e_{r-1} + e_r$, introduce the part e_r to P, and leave $\mathbf{v}$ unchanged,

$E_C(3)$: replace $\mathbf{v}$ by $\bar{\mathbf{v}} = e_j + \cdots + e_{r-1} + e_r$, introduce the part e_r to P, and leave $\mathbf{u}$ unchanged,

$E_C(4)$: replace both $\mathbf{u}$ and $\mathbf{v}$ by $\bar{\mathbf{u}} = e_i + \cdots + e_{r-1} + e_r$ and $\bar{\mathbf{v}} = e_j + \cdots + e_{r-1} + e_r$ respectively.

We remark that the only remaining partitions P of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ in $\mathbb{C}^\infty$ which are not formed from the above extensions are the following three partitions:

$$P_1 = \{e_1 + 2e_2 + \cdots + 2e_r\} \tag{5}$$
$$P_2 = \{e_1 + 2e_2 + \cdots + 2e_{r-1} + e_r, e_r\} \tag{6}$$
$$P_3 = \{e_1 + 2e_2 + \cdots + 2e_{r-1}, e_r, e_r\} \tag{7}$$

which contain one, two, and three parts respectively. From the definition of a partition and the extensions above the following is immediate.

Proposition 4.1. *Let P be a partition of $\Psi(\tilde{\alpha}) = e_1 + 2e_2 + \cdots + 2e_{r-1} \in \mathbb{C}^\infty$ distinct from the partition with only one part $\{e_1 + 2e_2 + \cdots + 2e_{r-1}\}$ and for $1 \le \ell \le 4$ let $P(\ell)$ be the result of applying extension $E_C(\ell)$ to P. Then for $\ell \neq k$, $P(\ell) = P(k)$ if and only if $\{\ell, k\} = \{2, 3\}$. Furthermore, if $P(\ell) = P(k)$, then $\mathbf{u} = \mathbf{v}$.*

Note that the extensions $E_C(1)$, $E_C(2)$, $E_C(3)$, and $E_C(4)$ only affect parts that contain e_{r-1} as a summand, hence if you start with two distinct partitions P and P' of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_{r-1}$ in $\mathbb{C}^\infty$ then the extensions $P(\ell) \neq P'(j)$ for any $1 \le \ell, j \le 4$.

Theorem 4.2. *If $r \ge 3$, then the polynomials $\{\mathscr{P}_{C_i}(q)\}$ satisfy the following recurrence*

$$\mathscr{P}_{C_r}(q) = (1+q)^2(\mathscr{P}_{C_{r-1}}(q) - q) - q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}_{C_i}(q) \right) + (q + q^2 + q^3).$$

Proof. Recall that every partition P (except for the three partitions P_1, P_2 and P_3 listed in Equations (5), (6), and (7), respectively) of the vector $e_1 + 2e_2 + \cdots + 2e_{r-1} + 2e_r$ comes from an extension of a partition P' of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ except the full partition $\{e_1 + 2e_2 + \cdots + 2e_{r-1}\}$ via the four extensions $E_C(1)$, $E_C(2)$, $E_C(3)$, and $E_C(4)$. These extensions respectively add two, one, one, and zero parts. The polynomial whose coefficients encode the total of number of said extensions is

$$(1 + 2q + q^2)(\mathscr{P}_{C_{r-1}}(q) - q) = (1+q)^2(\mathscr{P}_{C_{r-1}}(q) - q).$$

However, by Proposition 4.1, $(1+q)^2(\mathscr{P}_{C_{r-1}}(q) - q)$ double counts the contribution of the partitions P of $e_1 + 2e_2 + \cdots + 2e_{r-1} + 2e_r$ obtained from extensions $E_C(2)$

and $E_C(3)$ of partitions P' of $e_1 + 2e_2 + \cdots + 2e_{r-1}$ which contain two equal parts $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{u} = \mathbf{v} = e_{i+1} + \cdots + e_{r-1}$$

for some i with $1 \leq i \leq r-2$. Consider such partitions P' for a fixed i. The set of such partitions correspond bijectively with the partitions of $e_1 + 2e_2 + \cdots + 2e_i$. Thus the polynomial encoding such double counted partitions is given by $q^3 \mathscr{P}_{C_i}(q)$, where we multiply by q^3 because $P'(2)$ and $P'(3)$ each have three more parts than P'. By ranging over all possible i we have that the total number of double counted partitions is

$$q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}_{C_i}(q) \right).$$

Thus

$$\mathscr{P}_{C_r}(q) = (1+q)^2(\mathscr{P}_{C_{r-1}}(q) - q) - q^3 \left(\sum_{i=1}^{r-2} \mathscr{P}_{C_i}(q) \right) + (q + q^2 + q^3),$$

where the term $(q + q^2 + q^3)$ is the contribution from the three partitions P_1, P_2, and P_3 given in Equations (5), (6), and (7). □

Theorem 4.3. *The closed formula for the generating function* $\sum_{k \geq 1} \mathscr{P}_{C_k}(q)x^k$ *is given by*

$$\sum_{k \geq 1} \mathscr{P}_{C_k}(q)x^k = \frac{qx + (-q^2 - q)x^2}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2}.$$

Proof. Let

$$\sum_{k \geq 1} f_k(q)x^k = \frac{qx + (-q^2 - q)x^2}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2}. \tag{8}$$

We will show that $f_k(q) = \mathscr{P}_{C_k}(q)$ for all $k \geq 1$. We proceed by induction, and we observe that

$$\begin{aligned} f_1(q) &= \mathscr{P}_{C_1}(q) = q, \\ f_2(q) &= \mathscr{P}_{C_2}(q) = q(q^2 + q + 1), \text{and} \\ f_3(q) &= \mathscr{P}_{C_3}(q) = q(q^4 + 2q^3 + 4q^2 + 2q + 1). \end{aligned}$$

By induction we can assume that $f_{k-1}(q) = \mathscr{P}_{C_{k-1}}(q)$ and $f_{k-2}(q) = \mathscr{P}_{C_{k-2}}(q)$. From the rational expression of the generating formula given in Equation (8) we know that

$$\begin{aligned} f_k(q) &= (2 + 2q + q^2)f_{k-1}(q) - (1 + 2q + q^2 + q^3)f_{k-2}(q) \\ &= (1+q)^2 f_{k-1}(q) - q^3 f_{k-2}(q) + f_{k-1}(q) - (1+q)^2 f_{k-2}(q), \\ &\qquad \text{and by induction hypothesis} \\ &= (1+q)^2 \mathscr{P}_{C_{k-1}}(q) - q^3 \mathscr{P}_{C_{k-2}}(q) + \mathscr{P}_{C_{k-1}}(q) - (1+q)^2 \mathscr{P}_{C_{k-2}}(q). \end{aligned}$$

Using Theorem 4.2 we note that

$$(1+q)^2\mathscr{P}_{C_{k-1}}(q) - q^3\mathscr{P}_{C_{k-2}}(q) = q^2 + \mathscr{P}_{C_k}(q) + q^3\left(\sum_{i=1}^{k-3}\mathscr{P}_{C_i}(q)\right)$$

and

$$\mathscr{P}_{C_{k-1}}(q) - (1+q)^2\mathscr{P}_{C_{k-2}}(q) = -q^2 - q^3\left(\sum_{i=1}^{k-3}\mathscr{P}_{C_i}(q)\right).$$

Adding the last two equalities yields the desired result. □

5 Type D

Let $\mathscr{P}_{D_r}(q) := \wp_q(\tilde{\alpha})$, where $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ is the highest root of the Lie algebra of type D_r. By Lemma 2.5, we can recover $\mathscr{P}_{D_r}(q)$ by determining the number of partitions of $e_1 + 2e_2 + \cdots + 2e_{r-2} + e_{r-1} + e_r$ into k parts where the parts are as specified in Lemma 2.5. In this light, throughout this section, for any $\ell \geq 5$ we refer to a *partition* of $e_1 + 2e_2 + \cdots + 2e_{\ell-2} + e_{\ell-1} + e_\ell \in \mathbb{C}^\infty$, or equivalently a partition of the highest root in type D_ℓ, as a nonnegative integer combination of k (not necessarily distinct) vectors from the following set:

- Nonhooked vectors (parts) of the form $e_i + e_{i+1} + \cdots + e_j$ with $1 \leq i \leq j \leq \ell$.
- Nonhooked vectors (parts) of the form $e_i + e_{i+1} + \cdots + e_{\ell-2} + e_\ell$ with $1 \leq i \leq \ell-2$.
- Hooked vectors (parts) of the the form

  $$e_i + e_{i+1} + \cdots + e_{j-1} + 2e_j + 2e_{j+1} + \cdots + 2e_{\ell-2} + e_{\ell-1} + e_\ell$$

  with $1 \leq i < j \leq \ell - 2$.

Throughout our proof, we will relate the polynomials $\{\mathscr{P}_{D_i}(q)\}$ to the polynomials $\{\mathscr{P}^H_{B_i}(q)\}$ and $\{\mathscr{P}^{NH}_{B_i}(q)\}$ from Section 3. As such, for $r \geq 3$, we refer to a *partition* of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_\ell \in \mathbb{C}^\infty$, or equivalently a partition of the highest root in type B_ℓ, as a nonnegative integer combination of k (not necessarily distinct) vectors from the following set:

- Nonhooked vectors (parts) of the form $e_i + e_{i+1} + \cdots + e_j$ with $1 \leq i \leq j \leq \ell$.
- Hooked vectors (parts) of the the form $e_i + e_{i+1} + \cdots + e_{j-1} + 2e_j + 2e_{j+1} + \cdots + 2e_\ell$ with $1 \leq i < j \leq \ell$.

In order to determine the generating function for the sequence $\{P_{D_r}(q)\}_{r\geq 4}$, we explictly relate this sequence to the polynomials for the Lie algebras of type B. Before stating the first result we provide the following.

Definition 5.1. Let $r \geq 2$, the let

$$\mathscr{P}^{H}_{D_r}(q) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i is the number of partitions of $e_1 + 2e_2 + \cdots + 2e_{r-2} + e_{r-1} + e_r$ with i parts where one part is a hooked vector (of type D_r). Similarly, let

$$\mathscr{P}^{NH}_{D_r}(q) = c_0 + c_1 q + c_2 q^2 + \cdots + c_k q^k,$$

where c_i is the number of ways to write $e_1 + 2e_2 + \cdots + 2e_{r-2} + e_{r-1} + e_r$ as a sum of exactly i parts where no parts are hooked vectors (of type D_r).

Theorem 5.2. *For $r \geq 2$,*

$$\mathscr{P}_{D_{r+2}}(q) = \mathscr{P}^{H}_{B_r}(q) + q^2 \mathscr{P}^{NH}_{B_r}(q) + (2q+2)\left(2\mathscr{P}^{NH}_{B_r}(q) - q^2 \sum_{i=1}^{r-1} \mathscr{P}^{NH}_{B_i}(q)\right).$$

Proof. We first observe there is a bijection between partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ that have a hooked part and partitions of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ that have a hooked part that preserves the number of parts in each partition. Indeed, this bijection takes any partition of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ and removes the $e_{r+1} + e_{r+2}$ from its hooked part. From this we deduce $\mathscr{P}^{H}_{D_{r+2}}(q) = \mathscr{P}^{H}_{B_r}(q)$. It therefore remains to show that

$$\mathscr{P}^{NH}_{D_{r+2}}(q) = q^2 \mathscr{P}^{NH}_{B_r}(q) + (2q+2)\left(2\mathscr{P}^{NH}_{B_r}(q) - q^2 \sum_{i=1}^{r-1} \mathscr{P}^{NH}_{B_i}(q)\right). \tag{9}$$

We split this into five cases depending on the partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ and the parts which contain the summands e_{r+1} and e_{r+2}.

Case 1: This case considers partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ containing e_{r+1} and e_{r+2} as parts. The polynomial encoding the count for all such partitions is $q^2 \mathscr{P}^{NH}_{B_r}(q)$ because each of these is uniquely obtained by introducing the parts $\{e_{r+1}, e_{r+2}\}$ to a partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ (in type B_r) that itself has no hooked part.

Case 2: This case considers partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ containing e_{r+2} as a part, but in which e_{r+1} is not a part. Every partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ without a hooked part can be extended in two ways to get such a partition by adding e_{r+1} to a summand involving e_r. On the level of polynomials, this gives $2q\mathscr{P}^{NH}_{B_r}(q)$ (the q comes from introducing the lone e_{r+2} part to the partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$). However, this double counts the contributions of partitions of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ whose two parts containing e_r are the same. The over count is given by $q\left(q^2 \sum_{i=1}^{r-1} \mathscr{P}^{NH}_{B_i}(q)\right)$, the q for the lone e_{r+2} part, and the q^2 for the two summands containing e_r extended from a partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_i$ for some $i \in \{1, 2 \ldots, r-1\}$. On the level of polynomials this gives us

$$q\left(2\mathscr{P}^{NH}_{B_r}(q) - q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q)\right).$$

Case 3: This case considers partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$ containing e_{r+1} as a part, but in which e_{r+2} is not a part. This argument follows directly from the argument in the previous case and gives us

$$q\left(2\mathscr{P}^{NH}_{B_r}(q) - q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q)\right).$$

Case 4: This case considers the partitions of $e_1+2e_2+\cdots+2e_r+e_{r+1}+e_{r+2}$ with e_{r+1} and e_{r+2} as summands in different parts. Any partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ that does not have a hooked part can be extended to such a partition in two ways: by adding e_{r+1} to one part containing e_r and adding e_{r+2} to the other part containing e_r. This does not change the number of parts so on the level of polynomials we get $2\mathscr{P}^{NH}_{B_r}(q)$, but we must subtract double counts which come from those partitions of

$$e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$$

whose two parts containing e_r are the same. By a similar argument to the previous case, this double count is accounted for by $q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q)$. Hence the polynomial encoding the count for such partitions is

$$2\mathscr{P}^{NH}_{B_r}(q) - q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q).$$

Case 5: This case considers the remaining partitions of $e_1 + 2e_2 + \cdots + 2e_r + e_{r+1} + e_{r+2}$: those with e_{r+1} and e_{r+2} as summands of the same part. Any partition of $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ not containing a hooked part can be extended to such a partition by adding $e_{r+1} + e_{r+2}$ to a summand containing e_r. This does not change the number of parts so the total we get on the level of polynomials is $2\mathscr{P}^{NH}_{B_r}(q)$. But we must subtract double counts which come from those partitions in $e_1 + 2e_2 + 2e_3 + \cdots + 2e_r$ whose two summands containing e_r are the same. This is accounted for by $q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q)$. So the polynomial encoding the count for such partitions is given by

$$2\mathscr{P}^{NH}_{B_r}(q) - q^2\sum_{i=1}^{r-1}\mathscr{P}^{NH}_{B_i}(q).$$

Adding these five cases yields the desired result. □

Theorem 5.3. *The closed form for the generating series* $\sum_{r\geq 4}\mathscr{P}_{D_r}(q)x^r$ *is*

$$\sum_{r\geq 4}\mathscr{P}_{D_r}(q)x^r = \frac{(q+4q^2+6q^3+3q^4+q^5)x^4 - (q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2}.$$

Proof. Observe that $\sum_{r\geq 4} \mathscr{P}_{D_r}(q)x^r$

$$
\begin{aligned}
&= \sum_{r\geq 2} \mathscr{P}_{D_{r+2}}(q)x^{r+2} \\
&= \sum_{r\geq 2} \left(\mathscr{P}^{H}_{B_r}(q) + q^2\mathscr{P}^{NH}_{B_r}(q) + (2q+2)\left(2\mathscr{P}^{NH}_{B_r}(q) - q^2 \sum_{i=1}^{r-1} \mathscr{P}^{NH}_{B_i}(q) \right) \right) x^{r+2} \\
&= \sum_{r\geq 2} \mathscr{P}^{H}_{B_r}(q)x^{r+2} + q^2 \sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r+2} + (4q+4)\sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r+2} \\
&\qquad - q^2(2q+2)\sum_{r\geq 2}\left(\sum_{i=1}^{r-1} \mathscr{P}^{NH}_{B_i}(q) \right) x^{r+2} \\
&= x^2 \sum_{r\geq 2} \mathscr{P}^{H}_{B_r}(q)x^{r} + q^2x^2 \sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r} + (4q+4)x^2\sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r} \\
&\qquad - \frac{q^2(2q+2)x^3}{1-x} \sum_{r\geq 1} \mathscr{P}^{NH}_{B_r}(q)x^{r}.
\end{aligned}
$$

Now for simplicity, let

$$S(q,x) = \sum_{r\geq 1} \mathscr{P}^{H}_{B_r}(q)x^r, \quad T(q,x) = \sum_{r\geq 1} \mathscr{P}^{NH}_{B_r}(q)x^r.$$

Directly from Proposition 3.6, we have

$$T(q,x) = \frac{qx + (-2q-q^2)x^2 + (q+q^2)x^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2}$$

and Lemma 3.4 together with Theorem 3.8 implies

$$
\begin{aligned}
S(q,x) &= \frac{qx + (-q-q^2)x^2 + q^2x^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2} - T(q,x) \\
&= \frac{qx + (-q-q^2)x^2 + q^2x^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2} \\
&\qquad - \frac{qx + (-2q-q^2)x^2 + (q+q^2)x^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2} \\
&= \frac{qx^2 - qx^3}{1-(2+2q+q^2)x + (1+2q+q^2+q^3)x^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{r\geq 4} \mathscr{P}_{D_r}(q)x^r &= x^2 \sum_{r\geq 2} \mathscr{P}^{H}_{B_r}(q)x^{r} + q^2x^2 \sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r} + (4q+4)x^2\sum_{r\geq 2} \mathscr{P}^{NH}_{B_r}(q)x^{r} \\
&\qquad - \frac{q^2(2q+2)x^3}{1-x} \sum_{r\geq 1} \mathscr{P}^{NH}_{B_r}(q)x^{r}
\end{aligned}
$$

$$= x^2S(q,x) + q^2x^2(T(q,x) - qx) + (4q+4)x^2(T(q,x) - qx)$$
$$- \frac{q^2(2q+2)x^3}{1-x}T(q,x)$$
$$= \frac{(q+4q^2+6q^3+3q^4+q^5)x^4 - (q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2}. \qquad \square$$

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A Appendix: Explicit Formulas

Using classical techniques from generating functions we extract the following explicit formulas.

Corollary (Explicit Formulas)**.** The following are explicit formulas for the value of the q-analog of Kostant's partition function on the highest root of the classical Lie algebras:

$$\begin{aligned}
\text{Type } B_r\ (r \geq 2): \quad \mathscr{P}_{B_r}(q) &= b_1(q)\cdot(f_1(q))^{r-2} + b_2(q)\cdot(f_2(q))^{r-2},\\
\text{Type } C_r\ (r \geq 1): \quad \mathscr{P}_{C_r}(q) &= c_1(q)\cdot(f_1(q))^{r-1} + c_2(q)\cdot(f_2(q))^{r-1},\\
\text{Type } D_r\ (r \geq 4): \quad \mathscr{P}_{D_r}(q) &= d_1(q)\cdot(f_1(q))^{r-4} + d_2(q)\cdot(f_2(q))^{r-4},
\end{aligned}$$

where

$$f_1(q) = \frac{(q^2+2q+2)+q\sqrt{q^2+4}}{2}, \quad f_2(q) = \frac{(q^2+2q+2)-q\sqrt{q^2+4}}{2}$$

and

$$b_1(q) = \frac{(q^5+q^4+5q^3+4q^2+4q)+(q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$
$$b_2(q) = \frac{(q^5+q^4+5q^3+4q^2+4q)-(q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$
$$c_1(q) = \frac{(q^3+4q)+q^2\sqrt{q^2+4}}{2(q^2+4)}, \quad c_2(q) = \frac{(q^3+4q)-q^2\sqrt{q^2+4}}{2(q^2+4)},$$

$$d_1(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q)+(q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$

$$d_2(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q)-(q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}.$$

Proof. Prior to proceeding to each Lie type individually we recall the generating functions for $\sum_{r\geq 1}\mathscr{P}_{B_r}(q)x^r$, $\sum_{r\geq 1}\mathscr{P}_{C_r}(q)x^r$ and $\sum_{r\geq 4}\mathscr{P}_{D_r}(q)x^r$ satisfy the rational expressions

$$\sum_{r\geq 1}\mathscr{P}_{B_r}(q)x^r = \frac{qx+(-q-q^2)x^2+q^2x^3}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2} \tag{10}$$

$$\sum_{r\geq 1}\mathscr{P}_{C_r}(q)x^r = \frac{qx+(-q-q^2)x^2}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2} \tag{11}$$

$$\sum_{r\geq 4}\mathscr{P}_{D_r}(q)x^r = \frac{(q+4q^2+6q^3+3q^4+q^5)x^4-(q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2}. \tag{12}$$

Thus, the sequences $\{\mathscr{P}_{B_r}(q)\}$, $\{\mathscr{P}_{C_r}(q)\}$ and $\{\mathscr{P}_{D_r}(q)\}$ satisfy the recurrence relations

$$\begin{aligned}
\mathscr{P}_{B_r}(q) &= (2+2q+q^2)\mathscr{P}_{B_{r-1}}(q)-(1+2q+q^2+q^3)\mathscr{P}_{B_{r-2}}(q), \text{ for } r\geq 4,\\
\mathscr{P}_{C_r}(q) &= (2+2q+q^2)\mathscr{P}_{C_{r-1}}(q)-(1+2q+q^2+q^3)\mathscr{P}_{C_{r-2}}(q), \text{ for } r\geq 3,\\
\mathscr{P}_{D_r}(q) &= (2+2q+q^2)\mathscr{P}_{D_{r-1}}(q)-(1+2q+q^2+q^3)\mathscr{P}_{D_{r-2}}(q), \text{ for } r\geq 6.
\end{aligned}$$

Consequently, there are functions $b_1(q)$, $b_2(q)$, $c_1(q)$, $c_2(q)$, $d_1(q)$, $d_2(q)$, $f_1(q)$, $f_2(q)$ such that

$$\begin{aligned}
\mathscr{P}_{B_r}(q) &= b_1(q)\cdot(f_1(q))^{r-2}+b_2(q)\cdot(f_2(q))^{r-2} \text{ for every } r\geq 2,\\
\mathscr{P}_{C_r}(q) &= c_1(q)\cdot(f_1(q))^{r-1}+c_2(q)\cdot(f_2(q))^{r-1} \text{ for every } r\geq 1,\\
\mathscr{P}_{D_r}(q) &= d_1(q)\cdot(f_1(q))^{r-4}+d_2(q)\cdot(f_2(q))^{r-4} \text{ for every } r\geq 4.
\end{aligned}$$

To find the explicit formulas for $\mathscr{P}_{B_r}(q)$, $\mathscr{P}_{C_r}(q)$ and $\mathscr{P}_{D_r}(q)$ in terms of r, we determine the functions $b_1(q)$, $b_2(q)$, $c_1(q)$, $c_2(q)$, $d_1(q)$, $d_2(q)$, $f_1(q)$, $f_2(q)$.

From the generating functions in Equations (10), (11) and (12), $f_1(q)$ and $f_2(q)$ are roots, in the variable λ in terms of q, of the polynomial $f_\lambda(q) = \lambda^2-(2+2q+q^2)\lambda+(1+2q+q^2+q^3)$. Hence, without loss of generality,

$$f_1(q) = \frac{(q^2+2q+2)+q\sqrt{q^2+4}}{2}, \quad f_2(q) = \frac{(q^2+2q+2)-q\sqrt{q^2+4}}{2}.$$

We now continue on a case by case basis.
Type B: Since $\mathscr{P}_{B_2}(q) = q^3+q^2+q$ and $\mathscr{P}_{B_3}(q) = q^5+2q^4+4q^3+3q^2+q$, we can determine $b_1(q), b_2(q)$ by solving the system

$$q^3+q^2+q = b_1(q)+b_2(q)$$

$$q^5 + 2q^4 + 4q^3 + 3q^2 + q = b_1(q) \cdot f_1(q) + b_2(q) \cdot f_2(q).$$

This yields

$$b_1(q) = \frac{(q^5 + q^4 + 5q^3 + 4q^2 + 4q) + (q^4 + q^3 + 3q^2 + 2q)\sqrt{q^2+4}}{2(q^2+4)}$$
$$b_2(q) = \frac{(q^5 + q^4 + 5q^3 + 4q^2 + 4q) - (q^4 + q^3 + 3q^2 + 2q)\sqrt{q^2+4}}{2(q^2+4)}.$$

Type C: Since $\mathscr{P}_{C_1}(q) = q$ and $\mathscr{P}_{C_2}(q) = q^3 + q^2 + q$, we can determine $c_1(q), c_2(q)$ by solving the system

$$q = c_1(q) + c_2(q)$$
$$q^3 + q^2 + q = c_1(q) \cdot f_1(q) + c_2(q) \cdot f_2(q).$$

This yields

$$c_1(q) = \frac{(q^4 + 4q) + q^2\sqrt{q^2+4}}{2(q^2+4)}, \quad c_2(q) = \frac{(q^4 + 4q) - q^2\sqrt{q^2+4}}{2(q^2+4)}.$$

Type D: Since $\mathscr{P}_{D_4}(q) = q^5 + 3q^4 + 6q^3 + 4q^2 + q$ and $\mathscr{P}_{D_5}(q) = q^7 + 4q^6 + 11q^5 + 17q^4 + 15q^3 + 6q^2 + q$, we can determine $d_1(q), d_2(q)$ by solving the system

$$q^5 + 3q^4 + 6q^3 + 4q^2 + q = d_1(q) + d_2(q)$$
$$q^7 + 4q^6 + 11q^5 + 17q^4 + 15q^3 + 6q^2 + q = d_1(q) \cdot f_1(q) + d_2(q) \cdot f_2(q).$$

This yields

$$d_1(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q)+(q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$
$$d_2(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q)-(q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}.$$

□

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