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Lattice Point Visibility on Generalized Lines of Sight

Edray H. Goins, Pamela E. Harris, Bethany Kubik, and Aba Mbirika

Abstract. For a fixed $b \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we say that a point (r, s) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ is b-visible from the origin if it lies on the graph of a power function $f(x) = ax^b$ with $a \in \mathbb{Q}$ and no other integer lattice point lies on this curve (i.e., line of sight) between $(0, 0)$ and (r, s). We prove that the proportion of b-visible integer lattice points is given by $1/\zeta(b + 1)$, where $\zeta(s)$ denotes the Riemann zeta function. We also show that even though the proportion of b-visible lattice points approaches 1 as b approaches infinity, there exist arbitrarily large rectangular arrays of b-invisible lattice points for any fixed b. This work specialized to $b = 1$ recovers original results from the classical lattice point visibility setting where the lines of sight are given by linear functions with rational slope through the origin.

1. INTRODUCTION. A point (r, s) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ is said to be visible from the origin if it lies on a straight line through the origin $(0, 0)$ and no other lattice point lies on this line of sight between $(0, 0)$ and (r, s). Given this definition, it is natural to ask what proportion of lattice points are visible from the origin, which is equivalent to computing the probability that two integers are relatively prime. This problem was first addressed in the 1800s by numerous people including: Dirichlet, who proved a weaker form of the problem in 1849 [13]; Cesàro, who is often attributed as having posed this problem in 1881 [8]; and Sylvester, who along with Cesàro gave independent proofs of this result in 1883 [9, 26]. Cesàro proved that the probability that two randomly chosen integers in $\{1, 2, \ldots, n\}$ are coprime is given by $1/\zeta(2)$ as n approaches infinity, where $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ denotes the Riemann zeta function [8]. Thus, the proportion of visible integer lattice points is given by $1/\zeta(2) = 6/\pi^2 \approx .608$.

In 1971, Herzog and Stewart characterized patterns of visible (respectively, invisible) points within the approximately 60% (respectively, 40%) of the lattice containing visible (respectively, invisible) points [16] and their seminal work continues to motivate research in this area [2, 3, 10, 15, 17, 18, 19]. Additionally, it has been shown that the set of lattice points in the plane visible from the origin contains arbitrarily large square arrays of adjacent invisible lattice points [5, Theorem 5.29, p. 119]. This is connected to a celebrated result in number theory regarding the existence of two mutually pairwise coprime sets of consecutive integers. Since then, others have further studied properties of strings of consecutive composite numbers and their connection to integer lattice point visibility [12, 14, 25].

In this work, we fix $b \in \mathbb{N}$ and say that a point (r, s) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ is b-visible from the origin if it lies on the graph of a power function $f(x) = ax^b$ with $a \in \mathbb{Q}$ and no other integer lattice point lies on this curve (i.e., line of sight) between $(0, 0)$ and (r, s). Hence, our work specialized to $b = 1$ recovers the classical setting of lattice point visibility whose lines of sight are given by linear functions $f(x) = ax$ with $a \in \mathbb{Q}$. We remark that throughout this work, following the wording introduced

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by Pólya, we often refer to lattice points as trees and collections of adjacent trees as forests [4, 24].

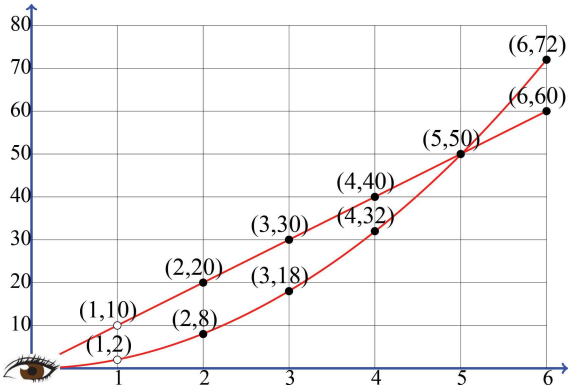


Figure 1. Lines of sight $f(x) = 10x$ and $g(x) = 2x^2$ with visible and invisible points.

Figure 1 shows two examples of lines of sight on which we mark the lattice points that are visible with white nodes and those that are invisible with black nodes. Figure 2 marks the b-invisible lattice points in the square $[0, 50] \times [0, 50]$ for $b = 1, 2, 3, 4$. Note that the number of b-visible points increases substantially relative to a small growth in b even in this small portion of the integer lattice. This observation, presented in Table 1, leads us naturally to our first result.

Table 1. Proportion of b-visible and b-invisible points for $b = 1, 2, 3, 4$ with all values approximated to three decimal places.

b	$\zeta(b+1)$	$\frac{1}{\zeta(b+1)}$	$1 - \frac{1}{\zeta(b+1)}$	Proportion of b-invisible points in 50×50 grid
1	1.644	.608	**.392**	$953/2500 \approx$ **.381**
2	1.202	.832	**.168**	$399/2500 \approx$ **.160**
3	1.082	.924	**.076**	$166/2500 \approx$ **.066**
4	1.036	.964	**.035**	$75/2500 \approx$ **.030**

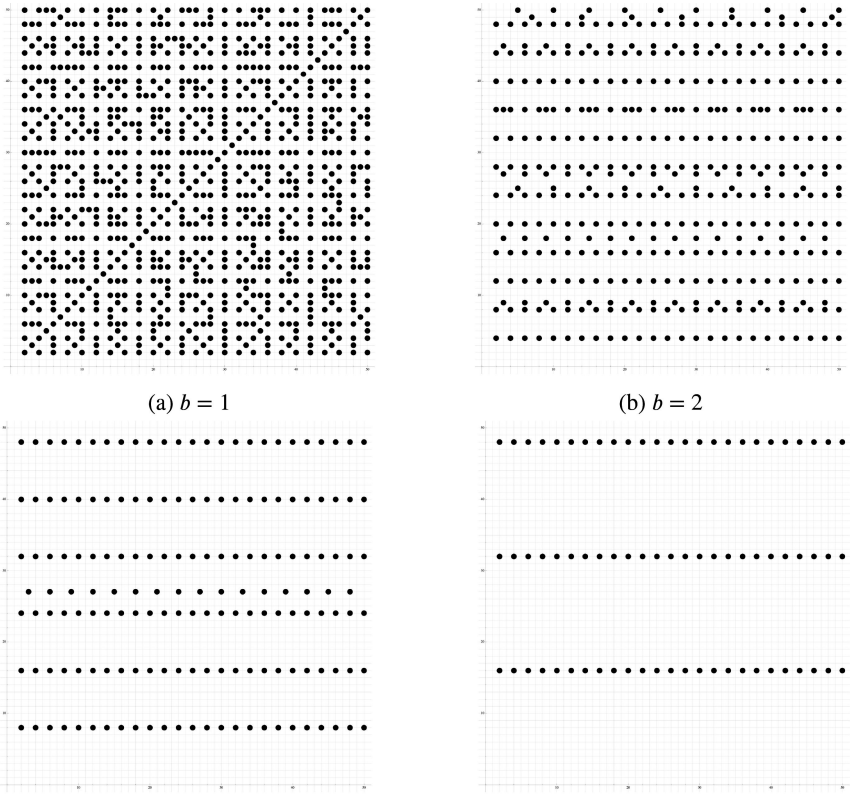
Theorem 1. *Fix an integer $b \in \mathbb{N}$. Then the proportion of points $(r, s) \in \mathbb{N} \times \mathbb{N}$ that are b-visible is* $\dfrac{1}{\zeta(b+1)}$.

Theorem 1 implies that the proportion of b-visible lattice points approaches 1 as b approaches infinity. However, as our next result shows, for any fixed $b \in \mathbb{N}$ there exist arbitrarily large b-invisible rectangular forests, that is, rectangular arrays of adjacent b-invisible integer lattice points.

Theorem 2. *Let $b \in \mathbb{N}$. For any integers $n, m > 0$, there exists a lattice point (r, s) such that every point $(r + i, s + j)$, where $0 \le i < n$ and $0 \le j < m$, is b-invisible from the origin.*

Although we present a proof that arbitrarily large b-invisible rectangular forests exist for all values $b \in \mathbb{N}$, our work does not construct forests close to the origin. In the

classical $b = 1$ case, the work of Herzog and Stewart used prime matrices and the Chinese remainder theorem to compute invisible square forests and they presented 2×2 and 3×3 invisible forests shown in Figure 3 [**16**].



(a) $b = 1$ (b) $b = 2$

(c) $b = 3$ (d) $b = 4$

Figure 2. The b-invisible lattice points in $[0, 50] \times [0, 50]$ when $b = 1, 2, 3, 4$.

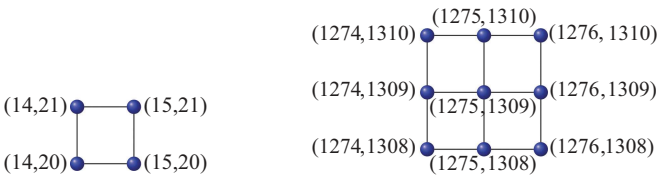


Figure 3. The 2×2 and 3×3 invisible forests lying closest to the origin.

It is easily verified that every point (r, s) in the forests of Figure 3 satisfies the condition $\gcd(r, s) > 1$. It turns out that, up to symmetry, these are the closest invisible square forests of size $n = 2$ and $n = 3$. In a brief remark, Wolfram claims to have found the closest 4×4 invisible forest, being located approximately 12 million units from the origin [**27**, p. 1093]. However, this has yet to be confirmed in the literature. Although to date no one knows the closest $n \times n$ invisible square forests for $n \geq 5$, recently bounds have been given on where invisible square forests might exist in the integer lattice [**17**, **22**]. Finding such bounds in our generalized setting remains an open problem.

Our paper is organized as follows. Section 2 contains the necessary definitions to make our approach precise. Section 3 provides a proof of Theorem 1. Section 4 gives a construction of arbitrarily large rectangular b-invisible forests, thereby proving Theorem 2.

2. BACKGROUND. The results presented in this paper are limited to the first quadrant of the plane, and, due to the symmetry of the plane, our results can be easily extended to apply to all of $\mathbb{Z} \times \mathbb{Z}$.

Definition 1. Fix $b \in \mathbb{N}$. A point $(r, s) \in \mathbb{N} \times \mathbb{N}$ is said to be *b-invisible* if the following two conditions hold:

1. The point (r, s) lies on the graph of $f(x) = ax^b$ for some $a \in \mathbb{Q}$. That is, $s = ar^b$.
2. There exists an integer $k > 1$ such that k divides r and k^b divides s.

The point is said to be *b-visible* if it satisfies Condition 1, but fails to satisfy Condition 2.

When we say that a point is b-invisible or b-visible, it is always with respect to the origin. If $(r, s) \in \mathbb{N} \times \mathbb{N}$ is b-invisible and Condition 1 is satisfied by the function $f(x) = ax^b$, then $(-r, s)$, $(-r, -s)$, and $(r, -s)$ are b-invisible under the functions $a(-x)^b$, $-a(-x)^b$, and $-ax^b$, respectively, and likewise for b-visible points. Thus, in our study, it suffices to determine the b-visibility (meaning, whether the point is b-visible or b-invisible) of the lattice points in $\mathbb{N} \times \mathbb{N}$.

To speak about the b-visibility of a lattice point in this new setting, we develop a generalization of the greatest common divisor.

Definition 2. Fix $b \in \mathbb{N}$. The *generalized greatest common divisor* of r and s with respect to b is denoted $\gcd_b$ and is defined as

$$\gcd_b(r, s) := \max\{k \in \mathbb{N} \mid k \text{ divides } r \text{ and } k^b \text{ divides } s\}.$$

Observe that $\gcd_b$ coincides with the classical greatest common divisor when b equals 1. Moreover, from the lattice point visibility language, the new generalized greatest common divisor implies that for a fixed $b \in \mathbb{N}$ the point (r, s) is *b-visible* if there exists a function $f(x) = ax^b$ with $a \in \mathbb{Q}$ such that (r, s) is on the graph of f and is the first integral point on the graph of f from the origin. The following result gives a necessary and sufficient condition to determine b-visibility.

Proposition 3. *A point $(r, s) \in \mathbb{N} \times \mathbb{N}$ is b-visible if and only if $gcd_b(r, s) = 1$.*

Proof. By Definition 1, a point $(r, s) \in \mathbb{N} \times \mathbb{N}$ is b-visible if $s = ar^b$ for some $a \in \mathbb{Q}$ and there does not exist an integer $k > 1$ such that k divides r and k^b divides s. Hence, the largest positive integer that satisfies the visibility criterion is 1. Thus, $\gcd_b(r, s) = 1$.

For the other direction, suppose that $\gcd_b(r, s) = 1$. Then $k = 1$ is the largest integer such that k divides r and k^b divides s and the point (r, s) does not satisfy Condition 2 of Definition 1. Also, note that for every pair (r, s), there exists a unique $a = s/r^b \in \mathbb{Q}$ such that $s = ar^b$. Hence, (r, s) is b-visible. ∎

Note that in the classical $b = 1$ setting of lattice point visibility, a point (r, s) is visible if and only if $\gcd(r, s) = 1$. Hence, Proposition 3 generalizes the condition for a lattice point to be b-visible via the generalized greatest common divisor $\gcd_b$ as stated in Definition 2. We also remark that the same integer lattice point can be b-visible and b'-invisible for distinct b and b'. We illustrate this in the following example.

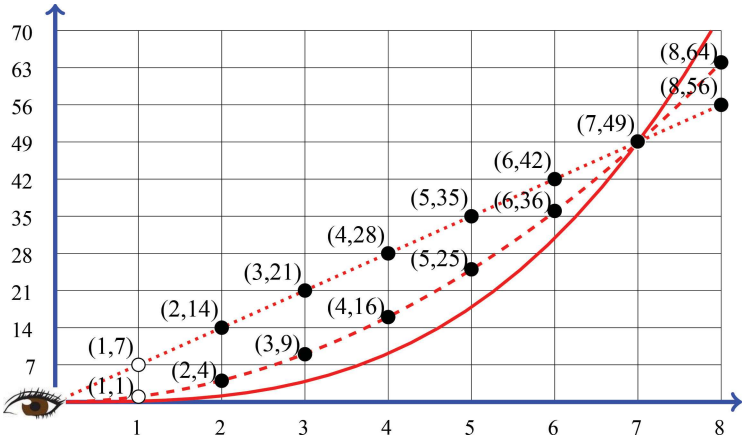


Figure 4. Invisible and visible points under different lines of sights.

Example 4. In Figure 4 the dotted curve is $f(x) = 7x$, the dashed curve is $g(x) = x^2$, and the solid curve is $h(x) = \frac{1}{7}x^3$. A white node denotes a visible point, while a black node denotes an invisible point. In particular, the white–black point at $(7, 49)$ is not 1-visible since $\gcd(7, 49) = 7$ and is not 2-visible since $\gcd_2(7, 49) = 7$. However it is 3-visible since $\gcd_3(7, 49) = 1$.

3. PROPORTION OF *b*–VISIBLE LATTICE POINTS. The literature on lattice point visibility presents rigorous proofs of the $b = 1$ case of Theorem 1, in particular in MONTHLY articles by Casey and Sadler [**7**, Theorem 1] and Christopher [**11**, Theorem 1]. Other recent proofs (see [**1**, **6**]) give illuminating plausibility arguments but are merely heuristic sketches as there is no uniform probability distribution on the natural numbers and these arguments gloss over this important fact. However, these proofs can be made rigorous by the methods presented by Pinsky [**23**]. Following an analogous method, we now present a proof of our result regarding the proportion of b-visible points in the lattice, for $b \geq 1$.

Proof of Theorem 1. Fix $N, b \in \mathbb{N}$. Let $[N] := \{1, 2, \ldots, N\}$. Let r, s be two numbers picked independently with uniform probability in $[N]$ and fix a prime p in $[N]$. By Proposition 3, a point $(r, s) \in \mathbb{N} \times \mathbb{N}$ is b-visible if and only if $\gcd_b(r, s) = 1$. Let P_N denote the probability that p divides r and p^b divides s. There are $\left\lfloor \frac{N}{p} \right\rfloor$ integers in $[N]$ that are divisible by p, namely $p, 2p, \ldots, \left\lfloor \frac{N}{p} \right\rfloor p$. Thus, the probability that p divides r is $\frac{1}{N}\left\lfloor \frac{N}{p} \right\rfloor$. Similarly, the probability that p^b divides s is $\frac{1}{N}\left\lfloor \frac{N}{p^b} \right\rfloor$. By mutual independence, the probability that p divides r and that p^b divides s is $P_N = \frac{1}{N^2}\left\lfloor \frac{N}{p} \right\rfloor \left\lfloor \frac{N}{p^b} \right\rfloor$. Therefore, the probability that p does not divide r or that p^b does not divide s is $1 - P_N$. Since $P_N \to \frac{1}{p^{b+1}}$ as $N \to \infty$, by multiplying over all of the primes we have that the probability that p does not divide r or that p^b does not divide s given that p is prime is

$$\lim_{N\to\infty} \prod_{\substack{p \text{ prime} \\ p \leq N}} (1 - P_N) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{b+1}}\right) = \frac{1}{\zeta(b+1)},$$

where $\zeta(s) = \prod_{p \text{ prime}} (1 - 1/p^s)^{-1}$. ∎

4. ARBITRARILY LARGE *b*–INVISIBLE FORESTS. We exploit the Chinese remainder theorem to prove that arbitrarily large $m \times n$ arrays of adjacent b-invisible integer lattice points in the plane exist for every $b \in \mathbb{N}$. We call such arrays of points b-invisible rectangular forests of size $m \times n$.

Proof of Theorem 2. It suffices to show that there exists a pair $(r, s) \in \mathbb{N} \times \mathbb{N}$ such that $\gcd_b(r + i, s + j) \neq 1$ for all $0 \leq i < n$ and $0 \leq j < m$. To obtain a pair (r, s), we first choose mn distinct primes and label them $p_{i,j}$ where $0 \leq i < n$ and $0 \leq j < m$. Place the primes in a matrix as follows:

$$P_{m\times n} = \begin{pmatrix} p_{0,m-1} & p_{1,m-1} & \cdots & p_{n-1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{0,1} & p_{1,1} & \cdots & p_{n-1,1} \\ p_{0,0} & p_{1,0} & \cdots & p_{n-1,0} \end{pmatrix}.$$

The choice of the nonstandard indexing of the entries in the matrix $P_{m\times n}$ will become clear at the proof's conclusion. Set $C_i = \prod_{j=0}^{m-1} p_{i,j}$ and $R_j = \prod_{i=0}^{n-1} p_{i,j}$ and consider the following systems of linear congruences:

$$\begin{cases} r + 0 \equiv 0 \pmod{C_0} \\ r + 1 \equiv 0 \pmod{C_1} \\ \vdots \\ r + (n-1) \equiv 0 \pmod{C_{n-1}} \end{cases} \quad \text{and} \quad \begin{cases} s + 0 \equiv 0 \pmod{R_0^b} \\ s + 1 \equiv 0 \pmod{R_1^b} \\ \vdots \\ s + (m-1) \equiv 0 \pmod{R_{m-1}^b}. \end{cases}$$

The integers in the set $\{C_i\}_{i=0}^{n-1}$ are pairwise relatively prime. Thus, by the Chinese remainder theorem, there exists a unique solution $r \pmod{\prod_{i=0}^{n-1} C_i}$. Similarly the integers in the set $\{R_j\}_{j=0}^{m-1}$ are pairwise relatively prime and hence there is a unique solution $s \pmod{\prod_{j=0}^{m-1} R_j^b}$.

For each $0 \leq i < n$ and $0 \leq j < m$, we have by construction that C_i divides $r + i$ and R_j^b divides $s + j$, and thus $p_{i,j}$ divides $r + i$ and $p_{i,j}^b$ divides $s + j$. Hence $p_{i,j}$ divides $\gcd_b(r + i, s + j)$ and so $\gcd_b(r + i, s + j) \neq 1$. Hence every point $(r + i,\ s + j) \in \mathbb{N} \times \mathbb{N}$ with $0 \leq i < n$ and $0 \leq j < m$ is b-invisible, as desired. ∎

The proof of Theorem 2 constructs b-invisible forests of any dimension. We illustrate this process below by constructing a 2-invisible forest of size 2×3.

Example 5. Consider the prime matrix

$$P_{2\times 3} = \begin{pmatrix} 7 & 11 & 13 \\ 2 & 3 & 5 \end{pmatrix}.$$

Using the technique described in Theorem 2, we compute the unique solution $r_0 \pmod N$ and $s_0 \pmod{N^2}$, where $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, to the required system of linear congruences

$$\begin{aligned} r_0 &= r + 0 = 27818 = 2 \cdot 7 \cdot 1987 \\ r_1 &= r + 1 = 27819 = 3^2 \cdot 11 \cdot 281 \\ r_2 &= r + 2 = 27820 = 2^2 \cdot 5 \cdot 13 \cdot 107 \\ s_0 &= s + 0 = 602202600 = 2^3 \cdot 3^5 \cdot 5^2 \cdot 12391 \\ s_1 &= s + 1 = 602202601 = 7^2 \cdot 11^2 \cdot 13^2 \cdot 601. \end{aligned}$$

The forest we have constructed is shown in Figure 5 with each corresponding value $\gcd_2(r_i, s_j)$ noted in red. One can easily verify that each of the six lattice points is 2-invisible; indeed as the proof of Theorem 2 states, each prime $p_{i,j}$ in the prime matrix $P_{2\times 3}$ divides the corresponding point (r_i, s_j).

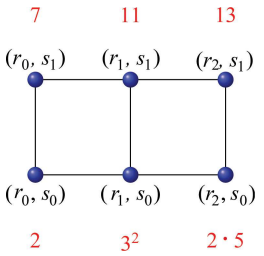


Figure 5. A 2-invisible forest of size 2×3.

Although Theorem 2 provides a way to find b-invisible forests of an arbitrary size, it does not necessarily indicate which ones will be close to the origin. Finding the closest known invisible square forests (when $b = 1$) was explored by Goodrich, Mbirika, and Nielsen [**15**]. In fact, using techniques from [**15**], we find a closer hidden forest with $(r, s) = (440, 38024)$. An exhaustive computer implementation confirms that this is the closest 2-invisible forest of size 2×3 in the first quadrant. We end by posing the following b-visibility problem: For fixed values $b, n, m \in \mathbb{N}$, find the nearest b-invisible forest of dimension $n \times m$.

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is always Iowa because in his graduate school years there, Iowa City became his favorite place on Earth.

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The Paul R. Halmos–Lester R. Ford Awards for 2017

The Paul R. Halmos–Lester R. Ford Awards, established in 1964, are made annually to authors of outstanding expository papers in the MONTHLY. The award is named for Paul R. Halmos and Lester R. Ford, Sr., both distinguished mathematicians and former editors of the MONTHLY. Winners of the Halmos–Ford Awards for expository papers appearing in Volume 124 (2017) of the MONTHLY are as follows.

- Paul E. Becker, Martin Derka, Sheridan Houghten, and Jennifer Ulrich (2017). Build a Sporadic Group in Your Basement. *Amer. Math. Monthly* 124(4): 291–305.
- Maria Deijfen, Alexander E. Holroyd, and James B. Martin (2017). Friendly Frogs, Stable Marriage, and the Magic of Invariance. *Amer. Math. Monthly* 124(5): 387–402.
- Francis E. Su (2017). Mathematics for Human Flourishing. *Amer. Math. Monthly* 124(6): 483–493.
- Michael Barnsley and Andrew Vince (2017). Self-Similar Polygonal Tiling. *Amer. Math. Monthly* 124(10): 905–921.