Graphs and Combinatorics (2018) 34:365–372
https://doi.org/10.1007/s00373-018-1881-0

ORIGINAL PAPER

Circuit Decompositions and Shortest Circuit Coverings of Hypergraphs

Liying Kang[1] · Weihua Lu[2] · Yezhou Wu[3] · Dong Ye[4] · Cun-Quan Zhang[5]

Received: 5 November 2017 / Revised: 5 November 2017 / Published online: 23 February 2018

Abstract It is one of fundamental theorems in graph theory that every even graph has a circuit decomposition. This classical result for ordinary graphs is extended in this paper for uniform bridgeless hypergraphs if the degree of every vertex is even. One of major open problems for shortest circuit cover was a conjecture proposed by Itai and Rodeh (Automata, Languages and Programming, Lecture Notes in Computer Science, vol. 62, pp. 289–299. Springer, Berlin, 1978) that every bridgeless graph G

Liying Kang: Partially supported by National Natural Science Foundation of China (Nos. 11471210 and 11571222). Yezhou Wu: Partially supported by National Natural Science Foundation of China (No. 11501504) and Natural Science Foundation of Zhejiang Province (No. LY16A010005). Dong Ye: Partially supported by Simons Foundation (No. 396516). Cun-Quan Zhang: Partially supported by National Security Agency (No. H98230-16-1-0004) and National Science Foundation (No. DMS-1700218).

✉ Cun-Quan Zhang
cqzhang@mail.wvu.edu

Liying Kang
lykang@shu.edu.cn

Weihua Lu
lwh8797@163.com

Yezhou Wu
yezhouwu@zju.edu.cn

1 Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

2 College of Arts and Sciences, Shanghai Maritime University, Shanghai 201306, People's Republic of China

3 Ocean College, Zhejiang University, Zhoushan 316021, Zhejiang, People's Republic of China

4 Department of Mathematical Sciences and Center for Computational Sciences, Middle Tennessee State University, Murfreesboro, TN 37132, USA

5 Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

has a circuit cover of total length at most $|E|+|V|-1$. This conjecture was solved by Fan (J Combin Theory Ser B 74:353–367, 1998) for ordinary graphs, and is extended in this paper for bridgeless hypergraphs.

Keywords Hypergraph · Circuit decomposition · Circuit cover

1 Introduction

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with the vertex set $\mathcal{V}$ and hyperedge set $\mathcal{E}$. Let r be a positive integer. A hypergraph $\mathcal{H}$ is *r-uniform* if every hyperedge of $\mathcal{H}$ contains r vertices. An ordinary graph is a special case of hypergraphs, a 2-uniform hypergraph.

In this paper, some classical results in graph theory about circuit decomposition and circuit covering are extended to r-uniform hypergraphs. We use Berge's definition for a circuit in a hypergraph [3].

Theorem 1.1 (See [6] Theorem 2.1.7 and [24] Proposition 1.2.27) *Let G be a graph. If the degree of every vertex is even, then G has a circuit decomposition.*

Theorem 1.1 is to be generalized for all r-uniform hypergraphs.

Theorem 1.2 *Let $r \geq 2$ be an integer and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an r-uniformed hypergraph. If $\mathcal{H}$ is bridgeless and the degree of every vertex is even, then $\mathcal{H}$ has a circuit decomposition.*

In [2], the circuit decomposition problem was verified for the cases that r is even or $r = 3$. In this paper, Theorem 1.2 (and its stronger version Theorem 3.6) provides a complete solution that extends Theorem 1.1 for hypergraphs.

A family $\mathcal{F}$ of circuits of a graph (or a hypergraph) is a *circuit cover* if every edge (or hyperedge, respectively) is contained in at least one member of $\mathcal{F}$. Find a circuit cover with minimum total length is not only an optimization problem but also has a very closed relation with some mainstream research problems, such as, integer flow theory, circuit double cover problem, etc. (see [1,4,7–11,15,17]).

The following result was originally proposed by Itai and Rodeh [13] in 1978 as one of two major conjectures for optimal circuit covering problem, and, was later proved by Fan [11] in 1998.

Theorem 1.3 (Fan [11]) *If $G = (V, E)$ is a bridgeless graph, then G has a circuit cover of total length at most $|E| + |V| - 1$.*

Theorem 1.3 is extended to hypergraphs as follows.

Theorem 1.4 *If $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a bridgeless hypergraph, then $\mathcal{H}$ has a circuit cover with total length at most $|\mathcal{E}| + |\mathcal{V}| - 1$.*

2 Preliminary

An ordered pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called a *hypergraph* where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of hyperedges where each $E_i \in \mathcal{E}$ is a subset of $\mathcal{V}$ [3]. Let k be a positive integer. A hypergraph is *k-uniform* if each hyperedge consists of precisely k vertices. Note that a 2-uniform hypergraph is an ordinary graph.

Definition 2.1 Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. A hyperedge E is a *bridge* of $\mathcal{H}$ if $\mathcal{H} - E$ has more components than $\mathcal{H}$.

Definition 2.2 A *circuit* of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is an alternative sequence of vertices and hyperedges $v_0, E_0, v_1, E_1, \ldots, v_{r-1}, E_{r-1}, v_0$ such that

1. $v_i, v_{i+1} \in E_i$ $(mod\ r)$;
2. $v_0, v_1, \ldots, v_{r-1}$ are all distinct vertices;
3. $E_0, E_1, \ldots, E_{r-1}$ are all distinct hyperedges.

And the *length* of a circuit is the number of hyperedges in the alternative sequence.

Definition 2.3 Let $\mathcal{F}$ be a family of circuits of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and k be a positive integer.

1. $\mathcal{F}$ is a *circuit k-cover* of $\mathcal{H}$ if every hyperedge of $\mathcal{H}$ is contained in precisely k members of $\mathcal{F}$. And a 1-cover is also called a circuit decomposition.
2. $\mathcal{F}$ is a *shortest circuit cover* of $\mathcal{H}$ if it is a circuit cover of $\mathcal{H}$ with minimum total length.

Definition 2.4 Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. The *incident graph* of $\mathcal{H}$, denoted by $B(\mathcal{H})$, is a bipartite graph constructed as follows: the bipartition of the vertex set is $\{\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H})\}$ and two vertices $E_i \in \mathcal{E}(\mathcal{H})$ and $v_j \in \mathcal{V}(\mathcal{H})$ are adjacent in $B(\mathcal{H})$ if and only if v_j is contained in the hyperedge E_i in $\mathcal{H}$.

By the construction of $B(\mathcal{H})$, we have the following observation which will be used through our proof.

Observation 2.5 *A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ has a circuit k-cover if and only if its incident graph $B(\mathcal{H})$ has a set of circuits that covers every vertex of $\mathcal{E}$ precisely k times.*

3 Circuit Decomposition of Hypergraphs and Proof of Theorem 1.2

Definition 3.1 Let G be a graph and v be a vertex of G. Suppose that $F \subset E(v)$, then we denote by $G_{[v;F]}$ the graph obtained from G by splitting the edges of F away from v, that is, adding a new vertex v' and changing the end v of the edges in F to be $v^{'}$.

Lemma 3.2 (Fleischner [12]) *Let G be a 2-edge-connected graph. Suppose v is a vertex of G with $d_G(v) \geq 4$ and let $e_0, e_1, e_2 \in E(v)$. Then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is 2-edge-connected or $G_{[v;\{e_0,e_1,e_2\}]}$ has more components than G. (See Fig. 1.)*

Lemma 3.3 (Petersen [19]) *Every $2k$-regular graph is 2-factorable.*

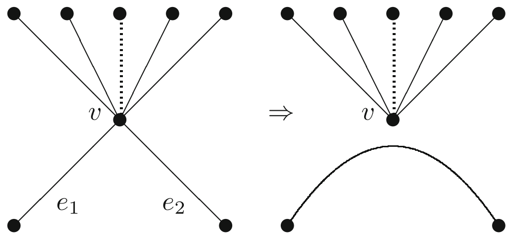

Fig. 1 G and $\overline{G_{[v;\{e_1,e_2\}]}}$: splitting $\{e_1, e_2\}$ away from v and suppressing the degree 2 vertex v'

Lemma 3.4 (Bollobas, Saito and Wormald [5]) *Let G be an r-regular multigraph of edge-connectivity $\lambda = \lambda(G) \geq 1$. Then G has a k-factor for the following values of k:*

1. *if r is even, all even numbers at least 2 and at most $r - 2$, together with, if $V(G)$ is also even, all odd numbers at least $\frac{r}{\lambda}$ and at most $r - \frac{r}{\lambda}$,*
2. *if r is odd, $\lambda \geq 2$, and $\lambda^* = 2\lfloor\frac{\lambda}{2}\rfloor + 1$, all positive even integers not more than $\frac{(\lambda^*-1)r}{\lambda^*}$ and all odd integers at least $\frac{r}{\lambda^*}$ and at most $r - 2$.*

For no other value of k, $1 \leq k \leq r - 1$, can a graph G be guaranteed to have a k-factor.

Theorem 1.2 and its stronger version Theorem 3.6 are corollaries of the following lemma.

Lemma 3.5 *Let $r \geq 3$ and $G = (V, E)$ be a graph (not necessary bipartite) with a partition $\{U_r, U_r^c\}$ of $V(G)$ such that*

1. *$d_G(v) = r$ if $v \in U_r$,*
2. *$d_G(v)$ is even if $v \in U_r^c$.*

If G is bridgeless, then G has a spanning subgraph F such that

(a) *$d_F(v) = 2$ if $v \in U_r$,*
(b) *$d_F(v)$ is even if $v \in U_r^c$.*

Proof If $U_r^c = \emptyset$, then G is an r-regular bridgeless graph. By Lemma 3.4, the graph G contains a 2-factor F. Hence, we assume that $U_r^c \neq \emptyset$ and prove the lemma by induction on $\sum_{v \in U_r^c} d(v)$.

Let $x \in U_r^c$. If $d(x) = 2$, we may apply the induction to the suppressed graph $\overline{G}$. So, let $d(x) \geq 4$. By Lemma 3.2, one may split a pair of edges e_1, e_2 away from x, and the resulting graph $G_{[x;\{e_1,e_2\}]}$ remains bridgeless. Again, applying induction to the suppressed graph $G' = \overline{G_{[x;\{e_1,e_2\}]}}$, we obtain an even subgraph F' with $d_{F'}(v) = 2$

for every $v \in U_r$. Hence the subgraph F of G induced by edges of F' also has the property $d_F(v) = 2$ for every $v \in U_r$ since the split vertex x is in U_r^c. □

In this section, we prove the following theorem which is stronger than Theorem 1.2.

Theorem 3.6 *Let r be a positive integer $r \geq 2$ and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an r-uniform hypergraph. If the degree of every vertex is even and the incident graph $B(\mathcal{H})$ is bridgeless, then $\mathcal{H}$ has a circuit decomposition.*

Proof We apply Lemma 3.5 to the incident graph $B(\mathcal{H})$, where $B(\mathcal{H})$ is G, $\mathcal{E}$ is U_r and $\mathcal{V}$ is U_r^c. A spanning even subgraph F of $B(\mathcal{H})$ (described in Lemma 3.5) corresponds to a circuit decomposition of $\mathcal{H}$ (by Observation 2.5). □

Note that the condition "bridgeless" of $B(\mathcal{H})$ is not in consideration if r is even since the incident graph $B(\mathcal{H})$ is even and therefore it has no odd-edge-cut.

However, even degree of every vertex of an r-uniform hypergraph is not a necessary condition for circuit decomposition. That is, Theorem 3.6 can be further extended for some hypergraphs with some odd-degree vertices (with a similar proof of Theorem 3.6).

Theorem 3.7 *For a given odd integer $r \geq 3$, let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an r-uniform hypergraph. If $B(\mathcal{H})$ is bridgeless and, for every $v \in \mathcal{V}$, either $d_{\mathcal{H}}(v)$ is even or at least r, then $\mathcal{H}$ has a circuit decomposition.*

Similar to the proof of Theorems 3.6 and 3.7 is an immediate corollary of the following lemma.

Lemma 3.8 *Let $r \geq 3$ and odd. And let $G = (V, E)$ be a graph with a partition $\{U_r, U_r^c\}$ of $V(G)$ such that*

1. *$d_G(v) = r$ if $v \in U_r$,*
2. *$d_G(v)$ is either even or at least r if $v \in U_r^c$.*

If G is bridgeless, then G has a spanning subgraph F such that

(a) *$d_F(v) = 2$ if $v \in U_r$,*
(b) *$d_F(v)$ is even if $v \in U_r^c$.*

Proof The proof of the lemma is similar to Lemma 3.5, in which the induction is on the total degree of the following vertex subset $\{v \in U_r^c : d_G(v) \neq r\}$ instead of the entire subset U_r^c. □

Together with Theorems 3.6 and 3.7 generalizes some earlier result in [2] for r-regular r-uniformed hypergraphs.

4 Shortest Circuit Cover of Hypergraph and Proof of Theorem 1.4

It is obvious that a hypergraph admitting a circuit decomposition has a circuit cover with total length $|\mathcal{E}|$, which is certainly the best bound for shortest circuit cover problem. Note that not every hypergraph admits a circuit decomposition. Theorem 1.3 is an attempt to estimate the upper bound.

It is obvious that a circuit cover of a hypergraph is equivalent to a family of circuits that contains all vertices of $\mathcal{E}(\mathcal{H})$ in the incident graph $B(\mathcal{H})$.

Lemma 4.1 (Fan [11]) *Let G be a 2-edge-connected graph. The vertices of G can be covered by circuits of total length at most* $2(|V(G)|-1)$.

The following result is stronger than Theorem 1.4.

Theorem 4.2 *Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a hypergraph such that the corresponding incident graph $B(\mathcal{H})$ is 2-edge-connected. Then $\mathcal{H}$ has a circuit cover with total length at most $|\mathcal{E}|+|\mathcal{V}|-1$.*

Proof Since $B(\mathcal{H})$ is 2-edge-connected, by Lemma 4.1, the incident graph $B(\mathcal{H})$ has a family $\mathcal{F}_B$ of circuits covering all vertices with total length at most $2(|V(B(\mathcal{H}))|-1)$. Note that each $2t$-circuit C_B of the incident graph $B(\mathcal{H})$ corresponds to a t-circuit of the hypergraph $\mathcal{H}$. Hence, $\mathcal{F}_B$ corresponds a circuit cover $\mathcal{F}_{\mathcal{H}}$ of $\mathcal{H}$, which is of total length at most $|V(B(\mathcal{H}))|-1=|\mathcal{E}|+|\mathcal{V}|-1$. □

For shortest circuit cover problems of ordinary graphs, it was proved independently by Bermond et al. [4] and Alon and Tarsi [1] that every bridgeless graph G has a circuit cover of total length at most $\frac{5}{3}|E(G)|$.

Beyond Theorem 1.3, we have an analog result for hypergraphs.

Theorem 4.3 *Let $\mathcal{H}=(\mathcal{V},\mathcal{E})$ be a hypergraph. If $B(\mathcal{H})$ is bridgeless, then $\mathcal{H}$ has a circuit cover of total length at most* $2(|\mathcal{E}(\mathcal{H})|-1)$.

Proof Let $B(\mathcal{H})=(X,Y)$ be the incident graph with the partition $\{X=\mathcal{E}(\mathcal{H}),\ Y=\mathcal{V}(\mathcal{H})\}$.

Let G_1 be a subgraph of $B(\mathcal{H})$ (by deleting edges) such that

(a) G_1 remains bridgeless,
(b) $V(G_1)=V(B(\mathcal{H}))$ (with the same bipartition $\{X,Y\}$), and $d_{G_1}(v)\geq 2$ for every $v\in X$,
(c) $|E(G_1)|$ is minimized.

Then, repeatedly apply Lemma 3.2 to every vertex of $\{v\in Y: d_{G_1}(v)\geq 4\}$ in G_1. Let G_2 be the resulting graph. The suppressed graph $\overline{G_2}$ has the following properties.

1. $\overline{G_2}$ remains bridgeless,
2. The vertex set $V(\overline{G_2})=X\cup Y_3$ where $Y_3\subseteq Y$ and $d_{\overline{G_2}}(v)=3$ for every $v\in Y_3$.

By Lemma 4.1, let $\mathcal{F}_{\overline{G_2}}$ be a family of circuits of G covering all vertices of $\overline{G_2}$ with total length is at most $2(|V(\overline{G_2})|-1)=2(|X|+|Y_3|-1)$ because $V(\overline{G_2})=X\cup Y_3$.

We claim that *each vertex of Y_3 is passed through by at least two members of $\mathcal{F}_{\overline{G_2}}$*. If not, the uncovered edge can be deleted from G_1. This contradicts (a) (the choice of G_1).

Let $\mathcal{F}$ be the circuit cover of $\mathcal{H}$ corresponds to $\mathcal{F}_{\overline{G_2}}$. Since, in the calculation of total length of the circuit cover of $\mathcal{F}$, these vertices of Y_3 are not included. By our claim, the total length of circuit cover of $\mathcal{F}$ is at most

$$2(|V(\overline{G_2})|-1)-2|Y_3|=2(|X|+|Y_3|-1)-2|Y_3|=2(|X|-1)=2(|\mathcal{E}(\mathcal{H})|-1)$$

because $X=\mathcal{E}(\mathcal{H})$ □

5 Remarks

Note that Theorem 1.1 is an "if and only if" statement for ordinary graphs, while Theorem 3.6 is not an "if and only if" statement for hypergraphs. Theorem 3.7 indicates that *even degree of every vertex of a hypergraph is not a necessary condition* for circuit decomposition. It is a natural question to ask the following problem.

Problem 5.1 *Find a necessary and sufficient condition for circuit decomposition of hypergraphs.*

For ordinary graphs, "bridgeless" is a necessary condition for circuit covering. For hypergraphs, we adapted the same definition of "bridge" in Definition 2.1. However, for circuit covering problem, a different definition of bridge might be more appropriate.

Definition 5.2 An hyperedge of $\mathcal{H}$ is an *acyclic bridge* if it is not contained any circuit of $\mathcal{H}$.

It is evident that, for ordinary graphs, the concepts of "bridge" and "acyclic bridge" are equivalent. However, for hypergraphs, a bridge may not be acyclic.

The following is a straightforward observation for acyclic bridges of a hypergraph.

(*) For a hypergraph $\mathcal{H}$, a hyperedge E_i is an acyclic bridge if and only if the vertex E_i is a star-cut of the incident graph $B(\mathcal{H})$ (where a vertex v is called a star-cut *of a graph G if (1) the vertex v is a cut-vertex of G, and (2) every edge incident with v is a cut-edge of G.)*

For shortest circuit cover problems of ordinary graphs, Alon and Tarsi [1] conjectured that the total length can be further reduced to $\frac{7}{5}|E(G)|$. Beyond Theorem 4.3, we would like to propose an analog problem for hypergraphs.

Problem 5.3 *Find a constant $c < 2$ such that every hypergraph $\mathcal{H}$ without acyclic bridge has a circuit cover of total length at most $c|\mathcal{E}(\mathcal{H})|$. Particularly, is $c \leq \frac{7}{5}$?*

The famous circuit double cover conjecture was proposed in [13,21–23] for bridgeless ordinary graphs. It is natural to propose an analog problem for hypergraphs.

Problem 5.4 *Is it true that every hypergraph $\mathcal{H}$ without acyclic bridge has a circuit double cover (i.e. circuit 2-cover)?*

References

1. Alon, N., Tarsi, M.: Covering multigraphs by simple circuits. SIAM J. Algebraic Discrete Methods **6**, 345–350 (1985)
2. Bahmanian, M.A., Šajna, M.: Quasi-Eulerian Hypergraphs. Electron. J. Comb. **24**(3), 3–30 (2017)
3. Berge, C.: Graphs and Hypergraphs. North-Holland, Amsterdam, London (1973)
4. Bermond, J.C., Jackson, B., Jaeger, F.: Shortest covering of graphs with cycles. J. Combin. Theory Ser. B **35**, 297–308 (1983)
5. Bollobas, B., Saito, A., Wormald, N.C.: Regular factors of regular graphs. J. Graph Theory **9**, 97–103 (1985)
6. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, New York (2008)
7. Fan, G.-H.: Integer flows and cycle covers. J. Combin. Theory Ser. B **54**, 113–122 (1992)
8. Fan, G.-H.: Covering graphs by cycles. SIAM J. Discrete Math. **5**, 491–496 (1992)

9. Fan, G.-H.: Short cycle covers of cubic graphs. J. Graph Theory **18**, 131–141 (1994)
10. Fan, G.-H., Raspaud, A.: Fulkerson's conjecture and circuits covers. J. Combin. Theory Ser. B **61**, 133–138 (1994)
11. Fan, G.-H.: Proofs of two minimum circuit cover conjectures. J. Combin. Theory Ser. B **74**, 353–367 (1998)
12. Fleischner, H.: Eine gemeinsame Basis für die Theorie der eulerschen Graphen und den Satz von Petersen. Monatsh. Math. **81**, 267–278 (1976)
13. Itai, A., Rodeh, M.: Covering a graph by circuits. In: "Automata, Languages and Programming," Lecture Notes in Computer Science, Vol. 62, pp. 289-299. Springer, Berlin (1978)
14. Jaeger, F.: Flows and generalized coloring theorems in graphs. J. Combin. Theory Ser. B **26**, 205–216 (1979)
15. Jamshy, U., Raspaud, A., Tarsi, M.: Short circuit covers for regular matroids with nowhere-zero 5-flow. J. Combin. Theory Ser. B **43**, 354–357 (1987)
16. Lonc, Z., Naroski, P.: On tours that contain all edges of a hypergraph. Electron. J. Comb. **17**(R144), 1 (2010)
17. Máčajová, E., Raspaud, A., Tarsi, M., Zhu, X.-D.: Short cycle covers of graphs and nowhere-zero flows. J. Graph Theory **68**, 340–348 (2011)
18. Máčajová, E., Raspaud, A., Rollová, E., Škoviera, M.: Circuit covers of signed graphs. J. Graph Theory **81**, 120–133 (2016)
19. Petersen, J.: Die Theorie der regulären Graphs. Acta Math. **15**, 193–220 (1891)
20. Sajna, M.: Eulerian-type properties of hypregraphs (slides) presented at Contributed Minisymposia "Hypergraphs(CM22)". CanaDAM 2013—Canadian Mathematical Society (2013)
21. Seymour, P.D.: On multi-colorings of cubic graphs and the conjecture of Fulkerson and Tutte. Proc. Lond. Math. Soc. **s3–38**, 423–460 (1979)
22. Szekeres, G.: Polyhedral decompositions of cubic graphs. Bull. Austr. Math. Soc. **8**, 367–387 (1973)
23. Tutte, W.T.: Personal correspondence with H. Fleischner (July 22nd, 1987)
24. West, D.B.: Introduction to Graph Theory, 2nd edn. Prentice Hall, Upper Saddle River (2001)
25. Zhang, C.-Q.: Circular flows of nearly eulerian graphs and vertex splitting. J. Graph Theory **40**, 147–161 (2002)