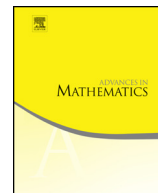




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The measures with an associated square function operator bounded in L^2 [☆]

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ABSTRACT

In this paper we provide an extension of a theorem of David and Semmes [3] to general non-atomic measures. The result provides a geometric characterization of the non-atomic measures for which a certain class of square function operators, or singular integral operators, are bounded in L^2 . The description is given in terms of a modification of Jones' β -coefficients.

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1. Introduction

Fix $d \geq 2$ and $s \in (0, d)$. The aim of this paper is to provide an extension of a theorem of David and Semmes [3] to general non-atomic measures. Their theorem provides a geometric characterization of the s -dimensional Ahlfors–David regular measures¹ for which a certain class of square function operators, or singular integral operators, are bounded in $L^2(\mu)$.

Their description is given in terms of Jones’ β -coefficients, which are defined for $s \in \mathbb{N}$ as

$$\beta_\mu(B(x, r)) = \left(\frac{1}{\mu(B(x, r))} \inf_{L \in \mathcal{P}_s} \int_{B(x, r)} \left(\frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y) \right)^{1/2},$$

where $B(x, r)$ denotes the open ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$, and $\mathcal{P}_s$ denotes the collection of affine s -planes in $\mathbb{R}^d$. Jones introduced these coefficients (with $L^\infty(\mu)$ norm replacing the $L^2(\mu)$ mean) in order to give a new proof of the boundedness of the Cauchy Transform on a Lipschitz curve [10] and to characterize the rectifiable curves in $\mathbb{R}^2$ [11].

Let us now state the David–Semmes theorem in the form most convenient for our purposes.

Theorem A. [3] *Suppose that μ is an s -dimensional Ahlfors–David regular measure. The following three statements are equivalent:*

(i) *For every odd function $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying standard decay estimates,² and $\varepsilon > 0$, the truncated singular integral operator (SIO)*

¹ A measure μ is Ahlfors–David regular if there exists a constant $C > 0$ such that $\frac{1}{C}r^s \leq \mu(B(x, r)) \leq Cr^s$ for every $x \in \text{supp}(\mu)$ and $r > 0$.

² Namely, that for every multi-index α , there is a constant $C_\alpha > 0$ such that $|D^\alpha K(x)| \leq \frac{C_\alpha}{|x|^{s+|\alpha|}}$ for every $x \in \mathbb{R}^d \setminus \{0\}$.

$$T_{\mu,\varepsilon}(f)(\cdot) = \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} K(\cdot - y) f(y) d\mu(y) \quad (1.1)$$

is bounded on $L^2(\mu)$ with an operator norm that can be estimated independently of ε .

(ii) For every odd function $\psi \in C_0^\infty(\mathbb{R}^d)$, the square function operator

$$S_{\mu,\psi}(f)(\cdot) = \left[\int_0^\infty \left| \frac{1}{t^s} \int_{\mathbb{R}^d} \psi\left(\frac{\cdot - y}{t}\right) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right]^{\frac{1}{2}} \quad (1.2)$$

is bounded in $L^2(\mu)$.

(iii) $s \in \mathbb{Z}$ and there exists a constant $C > 0$ such that

$$\int_Q \int_0^{\ell(Q)} \beta_{\mu|_Q}(B(x,r))^2 \frac{dr}{r} d\mu(x) \leq C\mu(Q) \quad (1.3)$$

for every cube $Q \subset \mathbb{R}^d$, where $\mu|_Q$ denotes the restriction of μ to Q .

We shall henceforth refer to (i) as the condition that *all SIOs with smooth odd kernels are bounded in $L^2(\mu)$* .

The path that David and Semmes take to prove Theorem A is to show that condition (ii) implies (iii), and also that (iii) is equivalent to a number of geometric conditions on the support of μ , such as *uniform rectifiability* (see [3] for definitions). One can then apply a theorem of David [2] to conclude that (i) holds. A standard artifice takes us from (i) to (ii) (see Section 1.4 below).

At this point we should mention that David and Semmes asked whether replacing the condition (i) with just the $L^2(\mu)$ boundedness of the s -Riesz transform – the SIO with kernel $K(x) = \frac{x}{|x|^{s+1}}$ – is already sufficient to conclude that (iii) holds. The fact that $s \in \mathbb{Z}$ under this assumption was proved by Vihtilä [24]. Demonstrating that (1.3) holds if $s \in \mathbb{Z}$ has proven more elusive, and is at present only known when $s = 1$, by the Mattila–Melnikov–Verdera theorem [16], and $s = d - 1$, when it was proved by Nazarov–Tolsa–Volberg [19] (in an equivalent form, building upon a proof scheme introduced in [6]).

In this paper, we do not make any progress on the Riesz transform question, but instead give a complete solution to another problem of David and Semmes referred to (rather generously) in Section 21 of [3] as a “glaring omission” in their theorem. Namely, we provide an analogue of Theorem A for general non-atomic locally finite Borel measures (without any regularity assumptions). Moreover, we do so for the somewhat smaller class of singular integral kernels considered by Mattila and Preiss [15]. When specialized to the case of Ahlfors–David regular measures, our arguments yield a new direct proof of the assertion that (ii) implies (iii) in Theorem A above.

1.1. The non-integer condition: the Wolff energy

The conditions that should replace (iii) in Theorem A when one considers a general measure are by now quite well agreed upon by specialists. This is particularly true when $s \notin \mathbb{Z}$, due to the work of Mateu–Prat–Verdera [12]. It turned out that a well-known object in non-linear potential theory, the *Wolff energy*, provides the key. We define the Wolff energy of a cube $Q \subset \mathbb{R}^d$ by

$$\mathcal{W}(\mu, Q) = \int_Q \int_0^\infty \left(\frac{\mu(Q \cap B(x, r))}{r^s} \right)^2 \frac{dr}{r} d\mu(x).$$

The *Mateu–Prat–Verdera theorem* states that, if $s \in (0, 1)$, then for a non-atomic measure μ , the s -Riesz transform of μ is bounded in $L^2(\mu)$ if and only if the following Wolff energy condition holds:

$$\mathcal{W}(\mu, Q) \leq C\mu(Q) \text{ for every cube } Q \subset \mathbb{R}^d. \quad (1.4)$$

In the proof presented in [12], the necessity of the Wolff energy condition for the boundedness of the s -Riesz transform relied fundamentally on the restriction to $s \in (0, 1)$, as it made use of a variation of the Menger–Melnikov curvature formula. However, the sufficiency of the condition (1.4) relied on neither the particular structure of the s -Riesz kernel $\frac{x}{|x|^{s+1}}$, nor the restriction on s , and by adapting their technique one can prove the following result.

Theorem B (Mateu–Prat–Verdera). *Fix $s \in (0, d)$. If μ is a measure that satisfies (1.4), then all s -dimensional SIOs with all smooth odd kernels are bounded in $L^2(\mu)$ (that is, statement (i) of Theorem A holds).*

To find a proof of this theorem precisely as stated, one can consult Appendix A of [8], the calculation is also essentially carried out in [5]. The *Mateu–Prat–Verdera conjecture* asks whether one may extend the necessity of the condition (1.4) for the $L^2(\mu)$ boundedness of the s -Riesz transform in $L^2(\mu)$ to the range $s > 1$, $s \notin \mathbb{Z}$. This was recently proved in the case when $s \in (d - 1, d)$ by M.-C. Reguera and the three of us [9]. It is an open problem for $s \in (1, d - 1) \setminus \mathbb{Z}$.

1.2. The integer condition: the Jones energy

For the case of integer s , we introduce the Jones energy of a cube $Q \subset \mathbb{R}^d$:

$$\mathcal{J}(\mu, Q) = \int_Q \int_0^\infty \left[\beta_{\mu|_Q}(B(x, r))^2 \left(\frac{\mu(Q \cap B(x, r))}{r^s} \right)^2 \right] \frac{dr}{r} d\mu(x). \quad (1.5)$$

Here $\mu|_Q$ denotes the restriction of μ to Q . This square function appears in Azzam–Tolsa [1], where amongst other things, the following theorem is proved.

Theorem C. [1] *Let μ be a non-atomic measure on $\mathbb{C}$. Then the Cauchy transform, the one dimensional SIO with kernel $K(z) = \frac{1}{z}$ in $\mathbb{C}$, is bounded in $L^2(\mu)$, if and only if $\sup_{z \in \mathbb{C}, r > 0} \frac{\mu(B(z, r))}{r^s} \leq C$ and*

$$\mathcal{J}(\mu, Q) \leq C\mu(Q) \text{ for every cube } Q \subset \mathbb{C}.$$

This theorem makes essential use of the relationship between the L^2 -norm of the Cauchy transform of a measure, and the curvature of a measure. Nevertheless, by combining the techniques of [1] with those in [21], Girela–Sarrión [7] succeeded in proving the sufficiency of the Jones energy condition for the boundedness of SIOs in greater generality:

Theorem D. [7] *Fix $s \in \mathbb{Z}$, $s \in (0, d)$. Suppose that there is a constant $C > 0$ such that $\sup_{x \in \mathbb{R}^d} \frac{\mu(B(x, r))}{r^s} \leq C$ and*

$$\mathcal{J}(\mu, Q) \leq C\mu(Q) \text{ for every cube } Q \subset \mathbb{R}^d. \quad (1.6)$$

Then all s -dimensional SIOs with smooth odd kernels are bounded in $L^2(\mu)$.

1.3. Statement of results

Choose a non-negative non-increasing function $\varphi \in C^\infty([0, \infty))$, such that $\text{supp}(\varphi) \subset [0, 2)$ and $\varphi \equiv 1$ on $[0, 1)$. We form the square function operator

$$\mathcal{S}_\mu(f)(x) = \left(\int_0^\infty \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

We shall prove the following two results:

Theorem 1.1. *Fix $s \notin \mathbb{Z}$. Let μ be a non-atomic locally finite Borel measure. If the square function operator $\mathcal{S}_\mu$ is bounded in $L^2(\mu)$, then there is a constant $C > 0$ such that*

$$\mathcal{W}(\mu, Q) \leq C\mu(Q) \quad (1.7)$$

for every cube $Q \subset \mathbb{R}^d$.

Theorem 1.2. *Fix $s \in \mathbb{Z}$. Let μ be a non-atomic locally finite Borel measure. If the square function operator $\mathcal{S}_\mu$ is bounded in $L^2(\mu)$, then there is a constant $C > 0$ such that $\frac{\mu(B(x, r))}{r^s} \leq C$ for every $x \in \mathbb{R}^d$, $r > 0$, and*

$$\mathcal{J}(\mu, Q) \leq C\mu(Q) \quad (1.8)$$

for every cube $Q \subset \mathbb{R}^d$.

1.4. Singular integrals and square functions

When combined with the theorems of Mateu–Prat–Verdera [16] and Girela–Sarrión [7] (Theorems B and D above), our theorems yield the following result.

Theorem 1.3. *Suppose that μ is a non-atomic locally finite Borel measure. The following statements are equivalent.*

- (i) *All SIOs with smooth odd kernels are bounded in $L^2(\mu)$.*
- (ii) *All SIOs of Mattila–Preiss type are bounded in $L^2(\mu)$. These are the SIOs with kernels that have the form $K(x) = \frac{x}{|x|^{s+1}}\psi(|x|)$ for $\psi \in C^\infty([0, \infty))$ satisfying*

$$|\psi^{(k)}(t)| \leq C_k |t|^{-k} \text{ for every } t \in [0, \infty) \text{ and every } k \geq 0.$$

- (iii) *The square function operator S_μ is bounded in $L^2(\mu)$.*

(iv) *Either*

- *$s \notin \mathbb{Z}$ and the Wolff energy condition (1.7) holds,*

or

- *$s \in \mathbb{Z}$ and the Jones energy condition (1.8) holds.*

That (iii) implies (iv), is merely a restatement of Theorems 1.1 and 1.2, while Theorems B and D imply that (iv) implies (i). That (i) implies (ii) is trivial as every SIO of Mattila–Preiss type is a SIO with smooth odd kernel. Thus we only need to show that (ii) implies (iii). This is a standard argument, already present in [3,15]. To sketch the idea, let us fix a sequence ε_k of independent mean zero ± 1 -valued random variables (on some probability space Ω). For $\omega \in \Omega$, $t \in [1, 2)$, and $k_0 \in \mathbb{N}$, consider the following SIO of Mattila–Preiss type

$$T_{t,k_0,\omega}(f)(x) = \int_{\mathbb{R}^d} \left[\sum_{k \in \mathbb{Z}, |k| \leq k_0} \varepsilon_k(\omega) \frac{x-y}{(2^k t)^{(s+1)}} \varphi\left(\frac{|x-y|}{2^k t}\right) \right] f(y) d\mu(y).$$

Following Section 3 of [3], one obtains that

$$\|S_\mu(f)\|_{L^2(\mu)}^2 \leq C \sup_{k_0 \in \mathbb{N}} \int_1^2 \mathbb{E}_\omega \|T_{t,k_0,\omega}(f)\|_{L^2(\mu)}^2 \frac{dt}{t} \leq C \|f\|_{L^2(\mu)}^2,$$

since all SIOs of Mattila–Preiss type are bounded in $L^2(\mu)$.

Proving that (iii) implies (ii) or (ii) implies (i) without going through (iv) appears to be non-trivial. (At least we do not know how to do that.)

1.5. The particular choice of the bump function φ doesn't matter too much

It is natural to wonder the extent to which the mapping properties of $\mathcal{S}_\mu$ depend on the particular choice of the bump function φ . Here we make three remarks in this regard, with the particular aim of convincing the reader that Theorems 1.1 and 1.2 remain valid if one instead defines the square function operator in a more customary way with a (perhaps only bounded measurable) bump function that is supported away from 0.

(1) Suppose that $\psi \in C^\infty([0, \infty))$ is a non-negative function that has bounded support and is identically equal to 1 near 0. Then the proofs of Theorems 1.1 and 1.2 can be adapted so that the same conclusions are reached with the $L^2(\mu)$ boundedness of $\mathcal{S}_\mu$ replaced by that of the operator

$$\mathcal{S}_{\mu,\psi}(f)(x) = \left(\int_0^\infty \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \psi\left(\frac{|x-y|}{t}\right) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

(2) For non-negative functions ψ and g , define the multiplicative convolution

$$\psi_g(t) = \int_0^\infty \psi\left(\frac{t}{u}\right) g(u) \frac{du}{u}.$$

From a change of variable and Minkowski's inequality we infer that

$$\|\mathcal{S}_{\mu,\psi_g}(f)\|_{L^2(\mu)} \leq \left[\int_0^\infty u^{s+1} g(u) \frac{du}{u} \right] \|\mathcal{S}_{\mu,\psi}(f)\|_{L^2(\mu)},$$

and as such, if $\mathcal{S}_{\mu,\psi}$ is bounded in $L^2(\mu)$, and $\int_0^\infty u^s g(u) du < \infty$, then $\mathcal{S}_{\mu,\psi_g}$ is bounded in $L^2(\mu)$.

(3) Finally, suppose that ψ is non-negative, bounded, measurable, and compactly supported in $(0, \infty)$ (so $0 \notin \text{supp}(\psi)$), with $\mathcal{S}_{\mu,\psi}$ bounded on $L^2(\mu)$.

Writing $\text{supp}(\psi) \subset [a, A]$ for some $a, A > 0$, we choose a function $g \in C^\infty([0, \infty))$ supported on $[0, \frac{2}{a}]$ that takes the value $(\int_0^\infty \psi(\frac{1}{u}) \frac{du}{u})^{-1}$ on the interval $[0, \frac{1}{a}]$. Then the function $\psi_g \in C^\infty([0, \infty))$ has support contained in $[0, \frac{2A}{a}]$ and $\psi_g \equiv 1$ on $[0, 1]$. From remark (2) we have that $\mathcal{S}_{\mu,\psi_g}$ is bounded on $L^2(\mu)$.

1.6. The Mayboroda–Volberg theorem

Building on the tools developed in [22,21,20], Mayboroda and Volberg [17,18] proved that if μ is a non-trivial finite measure with $\mathcal{H}^s(\text{supp}(\mu)) < \infty$, and $\mathcal{S}_\mu(1) < \infty$ μ -almost everywhere, then $s \in \mathbb{Z}$ and $\text{supp}(\mu)$ is s -rectifiable (see Section 2.6 below for the definition). When combined with Theorem 1.1 of Azzam–Tolsa [1], Theorems 1.1 and 1.2 above provide another demonstration of this result. We sketch the argument here.

One begins with a standard $T(1)$ -theorem argument which involves finding a compact subset $E \subset \text{supp}(\mu)$ whose μ measure is as close to $\mu(\mathbb{R}^d)$ as we wish, for which $\mathcal{S}_{\mu'}$ is bounded in $L^2(\mu')$ with $\mu' = \mu|_E$. This utilizes the method of suppressed kernels, see for instance Proposition 3.2 of [17]. But since μ' is supported on a set of finite $\mathcal{H}^s$ measure, the conclusion of Theorem 1.1 cannot hold unless $\mu' \equiv 0$, and so $s \in \mathbb{Z}$ and the conclusion of Theorem 1.2 holds. Theorem 1.1 in [1] then yields that $\text{supp}(\mu')$ is rectifiable. From this we conclude that $\text{supp}(\mu)$ is rectifiable.

2. Preliminaries

2.1. Notation

- By $C > 0$ we denote a constant that may change from line to line. Any constant may depend on d and s without mention. If a constant depends on parameters other than d and s , then these parameters are indicated in parentheses after the constant.
- We denote the closure of a set E by $\overline{E}$.
- For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the open ball centred at x with radius r .
- By a measure, we shall always mean a non-negative locally finite Borel measure.
- We denote by $\text{Lip}(\mathbb{R}^d)$ the collection of Lipschitz continuous functions on $\mathbb{R}^d$. For an open set U , we denote by $\text{Lip}_0(U)$ the subset of $\text{Lip}(\mathbb{R}^d)$ consisting of those Lipschitz continuous functions with compact support in U . We define the homogeneous Lipschitz semi-norm

$$\|f\|_{\text{Lip}} = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

- We denote by $\text{supp}(\mu)$ the closed support of μ , that is,

$$\text{supp}(\mu) = \mathbb{R}^d \setminus \left\{ \bigcup B : B \text{ is an open ball with } \mu(B) = 0 \right\}.$$

- For a closed set E , we shall denote by $\mu|_E$ the restriction of the measure μ to E , that is, $\mu|_E(A) = \mu(A \cap E)$ for a Borel set A .
- For $n \geq 0$, we denote by $\mathcal{H}^n$ the n -dimensional Hausdorff measure. When restricted to an n -plane, $\mathcal{H}^n$ is equal to a constant multiple of the n -dimensional Lebesgue measure m_n .
- For a cube $Q \subset \mathbb{R}^d$, $\ell(Q)$ denotes its side-length. For $A > 0$, we denote by AQ the cube concentric to Q of side-length $A\ell(Q)$.
- Set $Q_0 = (-\frac{1}{2}, \frac{1}{2})^d$. For a cube Q , we set $\mathcal{L}_Q$ to be the canonical affine map (a composition of a dilation and a translation) satisfying $\mathcal{L}_Q(Q_0) = Q$.
- We define the ratio of two cubes Q and Q' by

$$[Q' : Q] = \left| \log_2 \frac{\ell(Q')}{\ell(Q)} \right|.$$

- For any $x \in \mathbb{R}^d$, $r > 0$, we set

$$\mathcal{I}_\mu(B(x, r)) = \int_{\mathbb{R}^d} \varphi\left(\frac{|x - y|}{r}\right) d\mu(y),$$

$$\text{so } \mu(B(x, r)) \leq \mathcal{I}_\mu(B(x, r)) \leq \mu(B(x, 2r)).$$

2.2. Balls associated to cubes

We associate the ball $B_{Q_0} = B(0, 4\sqrt{d})$ to the cube $Q_0 = (-\frac{1}{2}, \frac{1}{2})^d$. Then for an arbitrary cube Q , we set

$$B_Q = \mathcal{L}_Q(B_{Q_0}).$$

Notice that $B_Q = B(x_Q, 4\sqrt{d}\ell(Q))$, where $x_Q = \mathcal{L}_Q(0)$ is the centre of Q .

We associate to the cube Q_0 the function $\varphi_{Q_0}(x) = \varphi(\frac{|x|}{2\sqrt{d}})$, $x \in \mathbb{R}^d$. For any other cube Q we set $\varphi_Q = \varphi_{Q_0} \circ \mathcal{L}_Q^{-1} = \varphi(\frac{|\cdot - x_Q|}{2\sqrt{d}\ell(Q)})$. The reader may wish to keep in mind the following chain of inclusions:

$$3Q \subset B(x_Q, 2\sqrt{d}\ell(Q)) \subset \{\varphi_Q = 1\} \subset \text{supp}(\varphi_Q) \subset B_Q.$$

We set

$$\mathcal{I}_\mu(Q) = \int_{\mathbb{R}^d} \varphi_Q d\mu \quad \left(= \int_{B_Q} \varphi_Q d\mu \right).$$

In relation to our previous notation, we have $\mathcal{I}_\mu(Q) = \mathcal{I}_\mu(\frac{1}{2}B_Q)$. For $n > 0$, we define the n -density of a cube Q by

$$D_{\mu,n}(Q) = \frac{1}{\ell(Q)^n} \int_{\mathbb{R}^d} \varphi_Q d\mu = \frac{1}{\ell(Q)^n} \mathcal{I}_\mu(Q).$$

Thus

$$\frac{\mu(Q)}{\ell(Q)^n} \leq D_{\mu,n}(Q) \leq \frac{\mu(B_Q)}{\ell(Q)^n} \leq \frac{\mu(8\sqrt{d}Q)}{\ell(Q)^n}. \quad (2.1)$$

If $n = s$, then we just write $D_\mu(Q)$ instead of $D_{\mu,s}(Q)$.

2.3. Flatness and transportation coefficients

For $n \in \mathbb{N}$, the n -dimensional β -coefficient of a measure μ in a cube Q is given by

$$\beta_{\mu,n}(Q) = \left[\frac{1}{\mathcal{I}_\mu(Q)} \inf_{L \in \mathcal{P}_n} \int_{\mathbb{R}^d} \left(\frac{\text{dist}(x, L)}{\ell(Q)} \right)^2 \varphi_Q(x) d\mu(x) \right]^{1/2},$$

where, as before, $\mathcal{P}_n$ denotes the collection of n -planes in $\mathbb{R}^d$. We shall write

$$\beta_\mu(Q) = \beta_{\mu, \lfloor s \rfloor}(Q).$$

It is easy to see that there is an n -plane L_Q such that

$$\beta_{\mu,n}(Q) = \left[\frac{1}{\mathcal{I}_\mu(Q)} \int_{\mathbb{R}^d} \left(\frac{\text{dist}(x, L_Q)}{\ell(Q)} \right)^2 \varphi_Q(x) d\mu(x) \right]^{1/2},$$

and we shall call any plane L_Q satisfying this property an optimal n -plane for $\beta_{\mu,n}(Q)$. The following classical fact will prove very useful for our analysis:

Lemma 2.1. *Suppose ν is a non-zero finite measure. Every n -plane L that minimizes the quantity $\int_{\mathbb{R}^d} \text{dist}(x, L)^2 d\nu(x)$ contains the centre of mass of ν , that is, the point $\frac{1}{\nu(\mathbb{R}^d)} \int_{\mathbb{R}^d} x d\nu(x) \in \mathbb{R}^d$.*

Proof. We may assume that $\int_{\mathbb{R}^d} x d\nu(x) = 0$. For a $(d-n)$ -dimensional orthonormal set $v_{n+1}, \dots, v_d$, consider the function $F: \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$F(b) = \int_{\mathbb{R}^d} \left| \sum_{j=n+1}^d \langle (b-x), v_j \rangle v_j \right|^2 d\nu(x), \quad b \in \mathbb{R}^d.$$

For the n -plane $L = b + \text{span}(v_{n+1}, \dots, v_d)^\perp$ to be a minimizer, we must certainly have that $\nabla F(b) = 0$. But

$$\nabla F(b) = \int_{\mathbb{R}^d} 2 \left(\sum_{j=n+1}^d \langle (b-x), v_j \rangle v_j \right) d\nu(x) = 2\nu(\mathbb{R}^d) \sum_{j=n+1}^d \langle b, v_j \rangle v_j.$$

Thus $\nabla F(b) = 0$ if and only if $b \in \text{span}(v_{n+1}, \dots, v_d)^\perp$. Therefore, should L be optimal, then it is necessarily a linear subspace. $\square$

The n -dimensional transportation (or Wasserstein) coefficient of a measure μ in a cube $Q \subset \mathbb{R}^d$ is given by

$$\alpha_{\mu,n}(Q) = \inf_{\substack{L \in \mathcal{P}_n: \\ L \cap \frac{1}{4}B_Q \neq \emptyset}} \sup_{\substack{f \in \text{Lip}_0(3B_Q), \\ \|f\|_{\text{Lip}} \leq \frac{1}{\ell(Q)}}} \left| \int_{\mathbb{R}^d} \varphi_Q f d(\mu - \vartheta_{\mu,L} \mathcal{H}^n|_L) \right|,$$

where $\vartheta_{\mu,L} = \frac{\mathcal{I}_\mu(Q)}{\mathcal{I}_{\mathcal{H}^n|_L}(Q)}$. In the case when $n = s$ we will write $\alpha_\mu(Q) = \alpha_{\mu,s}(Q)$.

Notice that the β -number is a gauge of how flat the measure is within a given cube, while the α -number tells us how close a measure is to a constant multiple of the Lebesgue

measure of an n -plane. As one might expect, for $n \in \mathbb{N}$, we have

$$\beta_{\mu,n}(Q)^2 \leq C\alpha_{\mu,n}(Q).$$

To see this, take an n -plane L that intersects $\frac{1}{4}B_Q$. Then the function

$$f(x) = \left(\frac{\text{dist}(x, L)}{\ell(Q)} \right)^2 \varphi_{3Q}$$

is supported in $3B_Q$ and has Lipschitz norm bounded by $\frac{C}{\ell(Q)}$. This proves the desired inequality since $\varphi_{3Q}\varphi_Q = \varphi_Q$.

2.4. The dyadic energies

Consider a dyadic lattice $\mathcal{D}$. Then, for any finite measure μ we have the following two inequalities:

$$\mathcal{J}(\mu, \mathbb{R}^d) \leq C \sum_{Q \in \mathcal{D}} \beta_\mu(Q)^2 D_\mu(Q)^2 \mathcal{I}_\mu(Q), \quad (2.2)$$

and

$$\mathcal{W}(\mu, \mathbb{R}^d) \leq C \sum_{Q \in \mathcal{D}} D_\mu(Q)^2 \mathcal{I}_\mu(Q). \quad (2.3)$$

Both of these inequalities follow from integrating with respect to μ the pointwise inequalities, for $s \in \mathbb{Z}$,

$$\int_0^\infty \beta_\mu(B(x, r))^2 \left(\frac{\mu(B(x, r))}{r^s} \right)^2 \frac{dr}{r} \leq C \sum_{Q \in \mathcal{D}} \beta_\mu(Q)^2 D_\mu(Q)^2 \varphi_Q(x),$$

and, for $s \in (0, d)$,

$$\int_0^\infty \left(\frac{\mu(B(x, r))}{r^s} \right)^2 \frac{dr}{r} \leq C \sum_{Q \in \mathcal{D}} D_\mu(Q)^2 \varphi_Q(x).$$

We shall just prove the first pointwise inequality (the second one is easier). Rewrite the left hand side as

$$\sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left[\frac{\mu(B(x, r))}{r^s} \inf_{L \in \mathcal{P}_s} \frac{1}{r^s} \int_{B(x, r)} \left(\frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y) \right] \frac{dr}{r}. \quad (2.4)$$

For each $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$, there is a cube $Q \in \mathcal{D}$ with $\ell(Q) = 2^{k+1}$ and $x \in \overline{Q}$. Then, for $r \in (2^k, 2^{k+1})$, $B(x, r) \subset B(x_Q, 2\sqrt{d}\ell(Q))$ and so, for an s -plane L ,

$$\frac{1}{r^s} \int_{B(x, r)} \left(\frac{\text{dist}(y, L)}{r} \right)^2 d\mu(y) \leq 2^{s+2} \frac{1}{\ell(Q)^s} \int_{\mathbb{R}^d} \varphi_Q(y) \left(\frac{\text{dist}(y, L)}{\ell(Q)} \right)^2 d\mu(y),$$

while also $\frac{\mu(B(x, r))}{r^s} \leq 2^s D_\mu(Q)$, and $\varphi_Q(x) = 1$. Thus the sum (2.4) is dominated by a constant multiple of

$$\sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{D}: \ell(Q) = 2^{k+1}} \beta_\mu(Q)^2 D_\mu(Q)^2 \varphi_Q(x).$$

2.5. Lattice stabilization

We say that a sequence of dyadic lattices $\mathcal{D}^{(k)}$ stabilizes in a dyadic lattice $\mathcal{D}'$ if every $Q' \in \mathcal{D}'$ lies in $\mathcal{D}^{(k)}$ for sufficiently large k .

Lemma 2.2. *Suppose $\mathcal{D}^{(k)}$ is a sequence of dyadic lattices with $Q_0 \in \mathcal{D}^{(k)}$ for all k . Then there exists a subsequence of the dyadic lattices that stabilizes to some dyadic lattice $\mathcal{D}'$.*

The lemma is proved via a diagonal argument: For every $n \geq 0$, there are 2^{nd} ways to choose a dyadic cube of sidelength 2^n so that $(-\frac{1}{2}, \frac{1}{2})^d$ is one of its dyadic descendants.

2.6. A basic density result

For an integer n , a set E is called n -rectifiable if it is contained, up to an exceptional set of $\mathcal{H}^n$ -measure zero, in the union of a countable number of images of Lipschitz mappings $f: \mathbb{R}^n \mapsto \mathbb{R}^d$. We shall require the following elementary density property of measures supported on rectifiable sets, whose proof may be found in Mattila [13].

Lemma 2.3. *Suppose that μ is a measure supported on an n -rectifiable set. Then*

$$\liminf_{Q \ni x, \ell(Q) \rightarrow 0} D_{\mu, n}(Q) > 0 \text{ for } \mu\text{-almost every } x \in \mathbb{R}^d.$$

We shall actually only require this result when the support of μ is locally contained in a finite union of smooth n -surfaces.

2.7. The growth condition

Lemma 2.4. *Fix $s \in (0, d)$. If μ is a non-atomic measure for which the square function operator $\mathcal{S}_\mu$ is bounded in $L^2(\mu)$, then $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$ for any lattice $\mathcal{D}$.*

This lemma is well-known, and is essentially due to G. David. Since we could only locate a proof in the case of non-degenerate Calderón–Zygmund operators rather than the square function, we reproduce a sketch of David’s argument (Proposition 1.4 in Chapter 3 of [2]) in the context of the square function. We shall verify that there is a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^d$, $\mu(Q) \leq C\ell(Q)^s$, from which the lemma certainly follows (see (2.1)).

The first step is to use the pigeonhole principle to verify the following:

Claim. *For every integer $A > 100$, there exists $C_0 > 0$, such that for any cube $Q \subset \mathbb{R}^d$, there exists a sub-cube $Q^* \subset Q$, with $\ell(Q^*) = \ell(Q)/A$, satisfying the property that*

$$\mu(Q^*) \geq \left(1 - \frac{C_0}{\lambda^2}\right)\mu(Q), \quad (2.5)$$

where $\lambda = \frac{\mu(Q)}{\ell(Q)^s}$.

Set $\varkappa = \frac{1}{1000}$. We first locate a cube $Q' \subset Q$ of side-length $\ell(Q') = 2\varkappa A^{-1}\ell(Q)$ satisfying³ $\mu(Q') \geq \varkappa^d A^{-d}\mu(Q)$. If the claim fails to hold for a given $C_0 > 0$, then one can find $Q'' \subset Q$, with $\ell(Q'') = \frac{\varkappa\ell(Q)}{A}$, $d(Q'', Q') \geq \frac{\ell(Q)}{5A}$ and satisfying $\mu(Q'') \geq \frac{C_0}{\lambda^2} \frac{\varkappa^d \mu(Q)}{A^d}$.

Notice that, if $f = \chi_{Q''}$, then have $S_\mu(f)(x) \geq c(A, \varkappa) \frac{\mu(Q'')}{\ell(Q)^s}$ for $x \in Q'$. Squaring this bound and integrating over Q' yields that

$$\frac{\mu(Q')\mu(Q'')^2}{\ell(Q)^{2s}} \leq C(A, \varkappa)\mu(Q''), \text{ and hence } \frac{\mu(Q')\mu(Q'')}{\ell(Q)^{2s}} \leq C(A, \varkappa).$$

Plugging in the lower bounds on the measures of Q' and Q'' gives

$$\frac{C_0\mu(Q)^2}{\lambda^2\ell(Q)^{2s}} \leq C(A, \varkappa), \text{ and hence } C_0 \leq C(A, \varkappa).$$

But this is absurd if C_0 was chosen large enough. The claim is proved.

Starting with any cube $Q^{(0)}$, we iterate the claim to find a sequence of cubes $Q^{(j)}$, $j \geq 0$ with $Q^{(j)} \subset Q^{(j-1)}$, $\ell(Q^{(j)}) = \ell(Q^{(j-1)})/A$, and, with $\lambda^{(j)} = \frac{\mu(Q^{(j)})}{\ell(Q^{(j)})^s}$,

$$\lambda^{(j)} \geq A^s \left(1 - \frac{C_0}{(\lambda^{(j-1)})^2}\right) \lambda^{(j-1)}.$$

Assuming $\lambda^{(0)} \geq 1$ is large enough in terms of C_0 , we infer by induction that $\lambda^{(j)} \geq A^{s/2} \lambda^{(j-1)} \geq \dots \geq A^{sj/2} \lambda^{(0)}$. Plugging this back into (2.5) yields that for every j

$$\mu(Q^{(j)}) \geq \prod_{\ell=1}^{j-1} \left(1 - \frac{C_0}{A^{s\ell/2} (\lambda^{(0)})^2}\right) \mu(Q^{(0)}).$$

³ The factor of 2 in the sidelength here is due to the fact that our cubes are open.

Assuming $\lambda^{(0)}$ is large enough, we have that $\mu(Q^{(j)}) \geq \frac{1}{2}\mu(Q^{(0)})$ for every $j \geq 1$, which implies that the non-atomic measure μ has an atom. Consequently, there is an absolute bound $C > 0$ for which $\lambda^{(0)} \leq C$. Since $Q^{(0)}$ was an arbitrary cube, we have proved the desired growth condition on the measure.

3. The basic scheme

3.1. Localization to square function constituents

Let us now suppose that μ is a measure for which the square function operator $\mathcal{S}_\mu$ is bounded in $L^2(\mu)$. For a dyadic lattice $\mathcal{D}$, notice that for each $k \in \mathbb{Z}$, the balls $\{AB_Q : Q \in \mathcal{D}, \ell(Q) = 2^k\}$ have overlap number at most CA^d . Thus,

$$\begin{aligned} \sum_{Q \in \mathcal{D}} \int_{AB_Q} \int_{\frac{\ell(Q)}{A}}^{A\ell(Q)} \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) f(y) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x) \\ \leq C(A) \|\mathcal{S}_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)}^2 \|f\|_{L^2(\mu)}^2, \end{aligned} \quad (3.1)$$

for every $f \in L^2(\mu)$. Here $C(A) = CA^d \log(A)$, as each $t \in (0, \infty)$ can lie in at most $C \log(A)$ of the intervals $[2^k/A, 2^{k+1}/A]$, $k \in \mathbb{Z}$. The precise form of $C(A)$ is not important.

We shall term the quantity

$$\mathcal{S}_\mu^A(Q) = \int_{AB_Q} \int_{\frac{\ell(Q)}{A}}^{A\ell(Q)} \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x), \quad (3.2)$$

a *square function constituent*. Our aim is to verify the following theorems.

Theorem 3.1. *If $s \notin \mathbb{Z}$, then there are constants $C > 0$ and $A > 0$ such that for any measure μ satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$, we have*

$$\sum_{Q \in \mathcal{D}} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q). \quad (3.3)$$

Theorem 3.2. *If $s \in \mathbb{Z}$, then there are constants $C > 0$ and $A > 0$ such that for any measure μ satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$, we have*

$$\sum_{Q \in \mathcal{D}} \beta_\mu(Q)^2 D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q). \quad (3.4)$$

To see that Theorems 1.1 and 1.2 follow from Theorems 3.2 and 3.1 respectively, let us again assume that μ is a measure for which $\mathcal{S}_\mu$ is bounded on $L^2(\mu)$. Then from

Section 2.7 we see that the condition $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$ holds. Fix a cube $P \in \mathcal{D}$. By testing the inequality (3.1) against the function $f = \chi_P$, we observe that the measure $\mu|_P$ satisfies

$$\sum_{Q \in \mathcal{D}} S_{\mu|_P}^A(Q) \leq C(A) \|S_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)}^2 \mu(P)$$

for every $A > 0$. Now, from Theorems 3.2 and 3.1 applied to $\mu|_P$, we find that if $s \in \mathbb{Z}$, then there is a constant $C > 0$ such that

$$\sum_{Q \in \mathcal{D}} \beta_{\mu|_P}(Q)^2 D_{\mu|_P}(Q)^2 \mathcal{I}_{\mu|_P}(Q) \leq C \|S_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)}^2 \mu(P), \quad (3.5)$$

while, if $s \notin \mathbb{Z}$, then there is a constant $C > 0$ such that

$$\sum_{Q \in \mathcal{D}} D_{\mu|_P}(Q)^2 \mathcal{I}_{\mu|_P}(Q) \leq C \|S_\mu\|_{L^2(\mu) \rightarrow L^2(\mu)}^2 \mu(P). \quad (3.6)$$

Making reference to Section 2.4, we conclude that the energy conditions (1.8) and (1.7) hold.

3.2. The general principle that we will use over and over again

Consider a rule Γ that associates to each measure μ a function

$$\Gamma_\mu : \mathcal{D} \rightarrow [0, \infty).$$

General Principle. Fix $A > 1$ and $\Delta > 0$. If we can verify the following statement:

$$\text{for every measure } \mu \text{ and } Q \in \mathcal{D}, \mathcal{S}_\mu^A(Q) \geq \Delta \Gamma_\mu(Q) \mathcal{I}_\mu(Q), \quad (3.7)$$

then we get that

$$\sum_{Q \in \mathcal{D}} \Gamma_\mu(Q) \mathcal{I}_\mu(Q) \leq \frac{1}{\Delta} \sum_{Q \in \mathcal{D}} S_\mu^A(Q). \quad (3.8)$$

Comparing (3.8) with (3.4) and (3.3), it is natural to attempt to verify (3.7) with the choice

$$\Gamma_\mu(Q) = \begin{cases} \beta_\mu(Q)^2 D_\mu(Q)^2 & \text{for } s \in \mathbb{Z}, \\ D_\mu(Q)^2 & \text{for } s \notin \mathbb{Z}. \end{cases}$$

Unfortunately this is not possible. As such, we shall use the general principle in a more convoluted way.

The key to proving Theorems 3.2 and 3.1 is to first understand the properties of measures for which no non-zero square function constituent can be found in any cube. Following Mattila [13,14], we call such measures φ -symmetric.

4. The structure of φ -symmetric measures

A measure μ is called φ -symmetric if

$$\int_{\mathbb{R}^d} (x-y)\varphi\left(\frac{|x-y|}{t}\right)d\mu(y) = 0 \text{ for every } x \in \text{supp}(\mu) \text{ and } t > 0.$$

We followed Mattila in the nomenclature: A measure is called symmetric if $\int_{B(x,r)} (x-y)d\mu(y) = 0$ for every $x \in \text{supp}(\mu)$ and $r > 0$. Of course this is a closely related object to the φ -symmetric measure, and we will lean heavily on the theory of symmetric measures developed by Mattila [14] and Mattila–Preiss [15].

The reader may want to keep in mind the following example of a φ -symmetric measure: For a linear subspace V of dimension $k \in \{0, \dots, d\}$, a uniformly discrete set E with $E \cap V = \{0\}$ that is symmetric about each of its points (that is, if $x \in E$, and $y \in E$, then $2y - x \in E$), and a non-negative symmetric function f on E (symmetry here means that if $x, y \in E$, then $f(x) = f(2y - x)$), form the measure

$$\mu = \sum_{x \in E} f(x) \mathcal{H}^k|_{V+x}.$$

Then μ is φ -symmetric. Provided that φ is reasonably ‘non-degenerate’, we expect that every φ -symmetric measure (with $0 \in \text{supp}(\mu)$) takes the above form, but we do not explore this too much here.

4.1. Doubling scales

Fix $\tau = 1000\sqrt{d}$ and a constant $C_\tau > \tau^d$ to be chosen later. We shall call $R > 0$ a doubling scale, or doubling radius, if

$$\mathcal{I}_\mu(B(0, \tau R)) \leq C_\tau \mathcal{I}_\mu(B(0, R)).$$

For $\lambda \in (0, \infty)$, we say that a measure has λ -power growth if

$$\limsup_{R \rightarrow \infty} \frac{\mu(B(0, R))}{R^\lambda} < \infty. \quad (4.1)$$

Lemma 4.1. *Suppose that μ is a measure with λ -power growth for some $\lambda \in (0, \infty)$. If $C_\tau > \tau^\lambda$, then for every $R > 0$, there is a doubling scale $R' > R$.*

Proof. Since the statement is trivial if μ is the zero measure, we may assume that $\mathcal{I}_\mu(B(0, R)) > 0$. We consider radii of the form $\tau^k R$, $k \in \mathbb{N}$. If none of these radii are doubling, then for every $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathcal{I}_\mu(B(0, \tau^{k+1} R)) &\geq C_\tau \mathcal{I}_\mu(B(0, \tau^k R)) \geq C_\tau^k \mathcal{I}_\mu(B(0, \tau R)) \\ &\geq C_\tau^k \mathcal{I}_\mu(B(0, R)).\end{aligned}$$

But then as $C_\tau > \tau^\lambda$, we infer that

$$\lim_{k \rightarrow \infty} \frac{\mathcal{I}_\mu(B(0, \tau^k R))}{\tau^{k\lambda}} = \infty,$$

which violates the growth condition (4.1). Thus, under this condition on C_τ , there exists some doubling scale $R' = \tau^k R$ with $k \geq 1$. $\square$

4.2. Behaviour at infinity

We next prove a variation of a powerful perturbation result used by Mattila–Preiss [15].

Lemma 4.2 (*The Mattila–Preiss formula*). *Let μ be a φ -symmetric measure. Suppose that $0 \in \text{supp}(\mu)$ and $x \in \text{supp}(\mu)$. Then, whenever R is a doubling radius with $R > |x|$,*

$$\sup_{r \in [R, 2R]} \left| x + \frac{1}{\mathcal{I}_\mu(B(0, r))} \int_{\mathbb{R}^d} \frac{y}{r} \varphi' \left(\frac{|y|}{r} \right) \left\langle \frac{y}{|y|}, x \right\rangle d\mu(y) \right| \leq \frac{C_\tau C |x|^2}{R}.$$

This formula does not appear precisely as stated in [15]. The formulation is rather close to that of Lemma 8.2 in [23], which in turn was strongly influenced by the techniques in [15].

Proof. Since $\varphi \equiv 1$ on $[0, 1]$, the function $\psi(x) = \varphi(|x|)$ lies in $C_0^\infty(B(0, 2))$. Taylor's theorem ensures that for each $y \in \mathbb{R}^d$,

$$\varphi \left(\frac{|x - y|}{r} \right) = \varphi \left(\frac{|y|}{r} \right) - \frac{1}{r} \left\langle x, \frac{y}{|y|} \right\rangle \varphi' \left(\frac{|y|}{r} \right) + \frac{E_{x,r}(y)}{2}, \quad (4.2)$$

where, for some z on the line segment between 0 and x ,

$$E_{x,r}(y) = \frac{1}{r^2} \left\langle x, D^2 \psi \left(\frac{z - y}{r} \right) x \right\rangle.$$

Therefore, if $r > |x|$, then

$$|E_{x,r}(y)| \leq C \frac{|x|^2}{r^2} \chi_{B(0, 3r)}(y) \leq C \frac{|x|^2}{r^2} \varphi \left(\frac{|y|}{3r} \right).$$

Now, since both x and 0 lie in $\text{supp}(\mu)$, we have

$$\int_{\mathbb{R}^d} (x - y) \varphi\left(\frac{|x - y|}{r}\right) d\mu(y) = 0,$$

and also $\int_{\mathbb{R}^d} y \varphi\left(\frac{|y|}{r}\right) d\mu(y) = 0$. Whence, for $r > |x|$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} x \varphi\left(\frac{|y|}{r}\right) d\mu(y) - \int_{\mathbb{R}^d} (x - y) \frac{1}{r} \left\langle \frac{y}{|y|}, x \right\rangle \varphi'\left(\frac{|y|}{r}\right) d\mu(y) \right| \\ & \leq \int_{\mathbb{R}^d} |x - y| |E_{x,r}(y)| d\mu(y) \leq C \frac{|x|^2}{r} \mathcal{I}_\mu(B(0, 3r)). \end{aligned}$$

In conjunction with the straightforward estimate

$$\int_{\mathbb{R}^d} \frac{|x|}{r} \left| \left\langle \frac{y}{|y|}, x \right\rangle \right| \left| \varphi'\left(\frac{|y|}{r}\right) \right| d\mu(y) \leq \frac{C|x|^2}{r} \mu(B(0, 2r)) \leq \frac{C|x|^2}{r} \mathcal{I}_\mu(B(0, 3r)),$$

we infer that

$$\left| x \mathcal{I}_\mu(B(0, r)) + \int_{\mathbb{R}^d} \frac{y}{r} \varphi'\left(\frac{|y|}{r}\right) \left\langle \frac{y}{|y|}, x \right\rangle d\mu(y) \right| \leq \frac{C|x|^2}{r} \mathcal{I}_\mu(B(0, 3r)). \quad (4.3)$$

Finally, suppose $r \in [R, 2R]$, with $R > |x|$ a doubling radius. Then $\mathcal{I}_\mu(B(0, 3r)) \leq \mathcal{I}_\mu(B(0, \tau R)) \leq C_\tau \mathcal{I}_\mu(B(0, r))$. Thus, after dividing both sides of (4.3) by $\mathcal{I}_\mu(B(0, r))$, we arrive at the desired inequality. $\square$

A variant of this formula was used in [15] to derive a growth rate at infinity of a symmetric measure. We repeat their argument in the form of the following lemma, as we are working under different assumptions on the measure.

Lemma 4.3 (*The growth lemma*). *Let μ be a φ -symmetric measure with $0 \in \text{supp}(\mu)$. If $x_1, \dots, x_k$ is a maximal linearly independent set in $\text{supp}(\mu)$, and R is a doubling radius with $R > \max(|x_1|, \dots, |x_k|)$, then*

$$\sup_{r \in [R, 2R]} \left| k - r \frac{\frac{d}{dr} \mathcal{I}_\mu(B(0, r))}{\mathcal{I}_\mu(B(0, r))} \right| \leq \frac{C(C_\tau, x_1, \dots, x_k)}{R}.$$

Proof. Consider the orthonormal basis $v_1, \dots, v_k$ of $V = \text{span}(\text{supp}(\mu))$ obtained via the Gram–Schmidt algorithm from $x_1, \dots, x_k$. By applying Lemma 4.2 to each element x_j , and using the triangle inequality, we infer that, for every $j = 1, \dots, k$,

$$\sup_{r \in [R, 2R]} \left| v_j + \frac{1}{\mathcal{I}_\mu(B(0, r))} \int_{\mathbb{R}^d} \frac{y}{r} \varphi' \left(\frac{|y|}{r} \right) \left\langle \frac{y}{|y|}, v_j \right\rangle d\mu(y) \right| \leq \frac{C(C_\tau, x_1, \dots, x_k)}{R}. \quad (4.4)$$

But now observe that

$$\begin{aligned} \sup_{r \in [R, 2R]} \left| k + \sum_{j=1}^k \frac{1}{\mathcal{I}_\mu(B(0, r))} \int_{\mathbb{R}^d} \frac{\langle y, v_j \rangle}{r} \varphi' \left(\frac{|y|}{r} \right) \left\langle \frac{y}{|y|}, v_j \right\rangle d\mu(y) \right| \\ \leq \sum_{j=1}^k \sup_{r \in [R, 2R]} \left| v_j + \frac{1}{\mathcal{I}_\mu(B(0, r))} \int_{\mathbb{R}^d} \frac{y}{r} \varphi' \left(\frac{|y|}{r} \right) \left\langle \frac{y}{|y|}, v_j \right\rangle d\mu(y) \right| \\ \leq \frac{C(C_\tau, x_1, \dots, x_k)}{R}. \end{aligned} \quad (4.5)$$

Finally, notice that since $v_1, \dots, v_k$ form an orthonormal basis of V , we have

$$\begin{aligned} \sum_{j=1}^k \int_{\mathbb{R}^d} \frac{\langle y, v_j \rangle}{r} \varphi' \left(\frac{|y|}{r} \right) \left\langle \frac{y}{|y|}, v_j \right\rangle d\mu(y) &= \int_{\mathbb{R}^d} \frac{|y|}{r} \varphi' \left(\frac{|y|}{r} \right) d\mu(y) \\ &= -r \frac{d}{dr} \mathcal{I}_\mu(B(0, r)), \end{aligned}$$

and the lemma follows by inserting this identity into the left hand side of (4.5). $\square$

Lemma 4.4 (*Maximal growth at infinity*). *Let μ be a φ -symmetric measure with $0 \in \text{supp}(\mu)$. Let V denote the linear span of $\text{supp}(\mu)$, and $k = \dim(V)$. Then for any $\varepsilon > 0$,*

$$\liminf_{R \rightarrow \infty} \frac{\mathcal{I}_\mu(B(0, R))}{R^{k-\varepsilon}} = +\infty.$$

Proof. From Lemma 4.3, we may fix $R_0 > 0$ such that if $R \geq R_0$ is a doubling scale, then

$$\sup_{r \in [R, 2R]} \left| k - r \frac{\frac{d}{dr} \mathcal{I}_\mu(B(0, r))}{\mathcal{I}_\mu(B(0, r))} \right| \leq \frac{\varepsilon}{2}, \quad (4.6)$$

and so

$$\frac{\frac{d}{dr} \mathcal{I}_\mu(B(0, r))}{\mathcal{I}_\mu(B(0, r))} \geq \frac{k - \frac{\varepsilon}{2}}{r} \text{ for every } r \in [R, 2R].$$

Integrating this inequality between R and $2R$ yields that

$$\mathcal{I}_\mu(B(0, 2R)) \geq 2^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R)).$$

We therefore infer the following alternative for *any* $R \geq R_0$: Either R is a non-doubling radius, in which case, since $C_\tau \geq \tau^d$,

$$\mathcal{I}_\mu(B(0, \tau R)) \geq C_\tau \mathcal{I}_\mu(B(0, R)) \geq \tau^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R)),$$

or, R is a doubling radius, in which case

$$\mathcal{I}_\mu(B(0, 2R)) \geq 2^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R)).$$

Starting with R_0 , we repeatedly apply the alternative to obtain a sequence of radii $R_j \rightarrow \infty$ with R_j equal to either $2R_{j-1}$ or τR_{j-1} , such that

$$\mathcal{I}_\mu(B(0, R_j)) \geq \left(\frac{R_j}{R_0}\right)^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R_0)).$$

Finally notice that for any $R \geq R_0$, there exists some R_j with $\frac{R}{\tau} \leq R_j \leq R$, so

$$\begin{aligned} \mathcal{I}_\mu(B(0, R)) &\geq \mathcal{I}_\mu(B(0, R_j)) \geq \left(\frac{R_j}{R_0}\right)^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R_0)) \\ &\geq c \left(\frac{R}{R_0}\right)^{k-\frac{\varepsilon}{2}} \mathcal{I}_\mu(B(0, R_0)). \end{aligned} \quad (4.7)$$

The lemma is proved. $\square$

We shall need one additional corollary of the Mattila–Preiss formula. It is a direct analogue for symmetric measures of an influential result of Preiss (see Proposition 6.19 in [4]), which states that if a uniform measure is sufficiently flat at arbitrarily large scales (has small enough coefficient $\beta_{\mu,n}(Q)$ for all cubes Q of sufficiently large side-length), then the measure is flat (supported in an n -plane).

In the case of symmetric measures, this statement is much easier to achieve than for uniform measures due to the strength of the Mattila–Preiss formula.⁴ We give the statement in the contrapositive form as it will be convenient for our purposes.

Lemma 4.5 (*Propagation of non-flatness to infinity*). *Let μ be a φ -symmetric measure with $0 \in \text{supp}(\mu)$. Suppose that $\text{supp}(\mu)$ is not contained in an n -plane. There exists $R_\mu > 0$ such that if $R \geq R_\mu$ is a doubling scale, then*

$$\frac{1}{\mathcal{I}_\mu(B(0, R))} \inf_{L \in \mathcal{P}_n} \int_{\mathbb{R}^d} \left(\frac{\text{dist}(x, L)}{R}\right)^2 \varphi\left(\frac{|x|}{2R}\right) d\mu(x) > \frac{1}{4C_\tau \|\varphi'\|_\infty^2}. \quad (4.8)$$

⁴ It is not true, though, that every symmetric measure is a uniform measure.

Proof of Lemma 4.5. Since 0 is the centre of mass of the measure $\varphi(\frac{\cdot}{2R})d\mu$ (μ is symmetric and $0 \in \text{supp}(\mu)$), we infer from Lemma 2.1 that it suffices to only consider linear subspaces L (instead of all affine subspaces) in the infimum appearing on the left hand side of (4.8).

Set $V = \text{span}(\text{supp}(\mu))$. Then V has dimension $k > n$ by the assumption of the lemma. Notice that if L is an n -dimensional linear subspace, then we have for every $y \in V$,

$$\text{dist}(y, L) \geq \text{dist}(y, L_V),$$

where L_V denotes the orthogonal projection of L onto V .

Let $v_1, \dots, v_k$ be an orthonormal basis of V . Then, using Lemma 4.2 in precisely the same manner as in the first paragraph of the proof of Lemma 4.3, we find $R_\mu > 0$ large enough so that for each $j = 1, \dots, k$, and any doubling scale $R \geq R_\mu$,

$$\left| v_j + \frac{1}{R\mathcal{I}_\mu(B(0, R))} \int_{\mathbb{R}^d} \frac{y}{|y|} \varphi'\left(\frac{|y|}{R}\right) \langle y, v_j \rangle d\mu(y) \right| < \frac{1}{2k}. \quad (4.9)$$

We can find a non-zero vector $x = \sum_{j=1}^k d_j v_j$ so that $x \perp L_V$. Of course, $|x|^2 = \sum_{j=1}^k |d_j|^2$ and so $|d_j| \leq |x|$ for every j . Thus,

$$\begin{aligned} & \left| x + \frac{1}{R\mathcal{I}_\mu(B(0, R))} \int_{\mathbb{R}^d} \frac{y}{|y|} \varphi'\left(\frac{|y|}{R}\right) \langle y, x \rangle d\mu(y) \right| \\ & \leq \sum_{j=1}^k |d_j| \left| v_j + \frac{1}{R\mathcal{I}_\mu(B(0, R))} \int_{\mathbb{R}^d} \frac{y}{|y|} \varphi'\left(\frac{|y|}{R}\right) \langle y, v_j \rangle d\mu(y) \right| \\ & \leq \sum_{j=1}^k |d_j| \frac{1}{2k} \leq \frac{|x|}{2}. \end{aligned}$$

Consequently, we see that for any doubling scale $R > R_0$,

$$\frac{1}{2} < \left| \frac{1}{\mathcal{I}_\mu(B(0, R))} \int_{B(0, 2R)} \frac{y}{|y|} \varphi'\left(\frac{|y|}{R}\right) \frac{\langle y, \frac{x}{|x|} \rangle}{R} d\mu(y) \right|, \quad (4.10)$$

(here we have just used that φ is supported in $B(0, 2R)$). Now notice that, since $x \perp L_V$, $|\langle y, \frac{x}{|x|} \rangle| \leq \text{dist}(y, L_V)$. Therefore, applying the Cauchy–Schwarz inequality to the right hand side of (4.10), we get that

$$\frac{1}{2} \leq \|\varphi'\|_\infty \frac{\sqrt{\mu(B(0, 2R))}}{\mathcal{I}_\mu(B(0, R))} \left(\int_{B(0, 2R)} \left(\frac{\text{dist}(y, L_V)}{R} \right)^2 d\mu(y) \right)^{1/2}.$$

The lemma now follows from the facts that R is a doubling radius, and $\varphi\left(\frac{|y|}{2R}\right) = 1$ for $y \in B(0, 2R)$ so $\mathcal{I}_\mu(B(0, R)) \geq \mu(B(0, 2R))$. $\square$

4.3. Flat φ -symmetric measures

We now look at the behaviour of a φ -symmetric measure that is supported in an n -plane. Suppose that μ is a φ -symmetric measure with power growth (i.e. satisfies (4.1) for some $\lambda > 0$). Then, of course,

$$\int_{\mathbb{R}^d} (x - y) \left[\varphi\left(\frac{|x - y|}{t}\right) - \varphi\left(\frac{2|x - y|}{t}\right) \right] d\mu(y) = 0 \text{ for } x \in \text{supp}(\mu), t > 0.$$

Notice that the function $t \mapsto [\varphi(\frac{1}{t}) - \varphi(\frac{2}{t})]$ is supported in $[1/2, 2]$. Consequently, if we take any bounded function $g : (0, \infty) \rightarrow \mathbb{R}$ that decays faster than any power at infinity, then for $x \in \text{supp}(\mu)$,

$$\begin{aligned} 0 &= \int_0^\infty g(t) \int_{\mathbb{R}^d} (x - y) \left[\varphi\left(\frac{|x - y|}{t}\right) - \varphi\left(\frac{2|x - y|}{t}\right) \right] d\mu(y) \frac{dt}{t} \\ &= \int_{\mathbb{R}^d} (x - y) \int_0^\infty g(|x - y|t) \left[\varphi\left(\frac{1}{t}\right) - \varphi\left(\frac{2}{t}\right) \right] \frac{dt}{t} d\mu(y). \end{aligned}$$

We shall use this idea to show that the support of a φ -symmetric measure is contained in the zero set of a real analytic function. As usual, this idea goes back to Mattila [14].

Lemma 4.6. *Suppose that μ is a φ -symmetric measure with power growth, and $\text{supp}(\mu) \subset L$ for some n -plane L . Then either $\mu = c_L \mathcal{H}^n|_L$ for some $c_L > 0$, or $\text{supp}(\mu)$ is $(n - 1)$ -rectifiable.*

Proof. After applying a suitable affine transformation, we may assume that $0 \in \text{supp}(\mu)$ and $L = \mathbb{R}^n \times \{0\}$, $0 \in \mathbb{R}^{d-n}$.

For $z \in \mathbb{C}^n$, set

$$w(z) = \int_0^\infty e^{-\pi z^2 t^2} \left[\varphi\left(\frac{1}{t}\right) - \varphi\left(\frac{2}{t}\right) \right] \frac{dt}{t},$$

where $z^2 = z_1^2 + \dots + z_n^2$. Since the domain of integration may be restricted to $[1/2, 2]$, we see that w is an entire function on $\mathbb{C}^n$. Consider the function $v : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $v(z) = zw(z)$. Then v is an entire vector field. Notice that, with $\widehat{v}(\xi) = \int_{\mathbb{R}^n} v(x) e^{-2\pi i \langle x, \xi \rangle} dm_n(\xi)$, $\xi \in \mathbb{R}^n$, the Fourier transform of v in $\mathbb{R}^n$, we have

$$\begin{aligned}
 \widehat{v}(\xi) &= c\nabla\widehat{w}(\xi) = c\nabla\int_0^\infty \frac{1}{t^n}e^{-\pi|\xi|^2/t^2}\left[\varphi\left(\frac{1}{t}\right) - \varphi\left(\frac{2}{t}\right)\right]\frac{dt}{t} \\
 &= c\xi\int_0^\infty \frac{1}{t^{n+2}}e^{-\pi|\xi|^2/t^2}\left[\varphi\left(\frac{1}{t}\right) - \varphi\left(\frac{2}{t}\right)\right]\frac{dt}{t}.
 \end{aligned}
 \tag{4.11}$$

The only thing we need from this formula is that $\widehat{v}$ is only zero when $\xi = 0$.

Since μ has power growth, and the entire function v satisfies a straightforward decay estimate $|v(x + iy)| \leq (1 + |y|)e^{\pi|y|^2}(1 + |x|)e^{-\pi|x|^2}$ for $x, y \in \mathbb{R}^n$, we infer that the function

$$u(x) = \int_{\mathbb{R}^n} v(x - y)d\mu(y), \quad x \in \mathbb{R}^n,$$

is a real analytic function on $\mathbb{R}^n$, and $\text{supp}(\mu) \subset u^{-1}(\{0\})$ (recall that μ is symmetric). First suppose that u is identically zero in $\mathbb{R}^n$. Then since μ is a tempered distribution,⁵ we have that

$$\widehat{u} = c\widehat{v} \cdot \widehat{\mu} \equiv 0.$$

Since $\widehat{v}$ is only zero at the origin, we see that $\text{supp}(\widehat{\mu}) \subset \{0\}$. This can only happen if μ has a polynomial density with respect to m_n , $\mu = Pm_n$. Since the function $\int_{\mathbb{R}^n} P(\cdot - y)v(y)dm_n(y)$ is identically zero on $\mathbb{R}^n$, we have that $\int_{\mathbb{R}^n} D^\alpha P(x - y)v(y)dm_n(y) = 0$ for any $x \in \mathbb{R}^n$ and multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. But then if the polynomial P is non-constant, we can, with a suitable differentiation, find a non-constant affine polynomial $\langle a, x \rangle + b$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} [\langle a, (x - y) \rangle + b]v(y)dm_n(y) = 0 \text{ for every } x \in \mathbb{R}^n.$$

Since $\int_{\mathbb{R}^n} v(y)dm_n(y) = \int_{\mathbb{R}^n} yw(y)dm_n(y) = 0$, evaluating this expression at $x = 0$, and taking the scalar product with a yields

$$\int_{\mathbb{R}^n} \langle a, y \rangle^2 w(y)dm_n(y) = 0,$$

which is preposterous. Consequently, μ is equal to a constant multiple of the Lebesgue measure m_n .

If u is not identically zero, then since u is analytic,

$$\mathbb{R}^n = \bigcup_{\alpha \text{ multi-index}} \{x \in \mathbb{R}^n : D^\alpha u(x) \neq 0\},$$

⁵ The power growth assumption is again used here.

and therefore

$$\begin{aligned} \operatorname{supp}(\mu) &\subset u^{-1}(\{0\}) \cap \bigcup_{\alpha \text{ multi-index}} \{x \in \mathbb{R}^n : D^\alpha u(x) \neq 0\} \\ &= \bigcup_{\alpha \text{ multi-index}} \{x \in \mathbb{R}^n : D^\alpha u(x) \neq 0, D^\beta u(x) = 0 \text{ for every } \beta < \alpha\}. \end{aligned}$$

The implicit function theorem ensures that each set in the union on the right hand side is locally contained in a smooth $(n-1)$ -surface. $\square$

4.4. A structure theorem

Here we summarize the results of this section in a form useful for what follows.

Proposition 4.7. *Suppose that μ is a φ -symmetric measure satisfying $B_{Q_0} \cap \operatorname{supp}(\mu) \neq \emptyset$, and such that*

$$\limsup_{R \rightarrow \infty} \frac{\mu(B(0, R))}{R^\lambda} < \infty \text{ for some } \lambda > 0.$$

Then

- (1) *If $\operatorname{supp}(\mu)$ is not contained in any n -plane, then*
- *for any $\varepsilon > 0$ and every $T > 1$, there exists $\ell > 0$ such that if Q is a cube satisfying $\ell(Q) \geq \ell$ and $\frac{1}{2}B_Q \supset B_{Q_0}$, then $D_{\mu, n+1-\varepsilon}(Q) > T$.*
 - *there exists a constant $c^* > 0$, depending on s, d, λ , and $\|\varphi'\|_\infty$, such that whenever $\mathcal{D}$ is a dyadic lattice and $\ell > 0$, there exists $Q' \in \mathcal{D}$ with $\ell(Q') \geq \ell$ satisfying $\frac{1}{2}B_{Q'} \supset B_{Q_0}$ and*

$$\beta_{\mu, n}(Q') \geq c^*.$$

- (2) *If $\operatorname{supp}(\mu) \subset L$ for some n -plane L , then either $\mu = c\mathcal{H}^n|_L$ or $\operatorname{supp}(\mu)$ is $(n-1)$ -rectifiable.*

Proof. First assume that $\operatorname{supp}(\mu)$ is not contained in any n -plane. Fix some point $x_0 \in \operatorname{supp}(\mu) \cap Q_0$. To prove the first property listed in item (1), observe that from Lemma 4.4 applied to the φ -symmetric measure $\mu_{x_0} = \mu(\cdot + x_0)$ it follows that $\lim_{R \rightarrow \infty} \frac{\mathcal{I}_\mu(B(x_0, R))}{R^{n+1-\varepsilon}} = \infty$. But if $\frac{1}{2}B_Q \supset B_{Q_0}$, then $B(x_Q, 2\sqrt{d}\ell(Q))$ contains a ball $B(x_0, R)$ with R comparable to $\ell(Q)$. Then $D_{\mu, n+1-\varepsilon}(Q) \geq c \frac{\mathcal{I}_\mu(B(x_0, R))}{R^{n+1-\varepsilon}}$, and the first statement listed in item (1) follows.

To derive the second property listed in item (1), apply Lemma 4.1 to the φ -symmetric measure $\mu_{x_0} = \mu(\cdot + x_0)$ to infer that, provided $C_\tau > \tau^\lambda$ (we fix C_τ to be of this order of magnitude), the measure μ_{x_0} has a sequence of doubling scales R_j with $R_j \rightarrow \infty$. Lemma 4.5 yields that if j is large enough, then

$$\frac{1}{\mathcal{I}_\mu(B(x_0, R_j))} \inf_{L \in \mathcal{P}_n} \int_{B(x_0, 4R_j)} \left(\frac{\text{dist}(x, L)}{R_j} \right)^2 \varphi\left(\frac{|x - x_0|}{2R_j}\right) d\mu(x) \geq c,$$

for some constant $c > 0$ depending on s , d , λ , and $\|\varphi'\|_\infty$.

Now, for any given lattice $\mathcal{D}$, choose a cube Q intersecting $B(x_0, R_j)$ of side-length between $4R_j$ and $8R_j$. For large enough j , we certainly have that $\frac{1}{2}B_Q \supset B_{Q_0}$. Also notice that $B(x_0, 4R_j) \subset B(x_Q, 2\sqrt{d}\ell(Q)) \subset \{\varphi_Q = 1\} \subset \text{supp}(\varphi_Q) \subset B(x_Q, 4\sqrt{d}\ell(Q)) \subset B(x_0, \tau R_j)$. Consequently, for any n -plane L ,

$$\begin{aligned} & \int_{B(x_0, 4R_j)} \left(\frac{\text{dist}(x, L)}{R_j} \right)^2 \varphi\left(\frac{|x - x_0|}{2R_j}\right) d\mu(x) \\ & \leq C \int_{B_Q} \left(\frac{\text{dist}(x, L)}{\ell(Q)} \right)^2 \varphi_Q(x) d\mu(x), \end{aligned}$$

while $\mathcal{I}_\mu(Q) \leq C_\tau \mathcal{I}_\mu(B(x_0, R_j))$. Bringing these observations together proves the second statement listed in item (1).

Item (2) is merely a restatement of Lemma 4.6. $\square$

5. The rudiments of weak convergence

We say that a sequence of measures μ_k converges weakly to a measure μ , written $\mu_k \rightharpoonup \mu$, if

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f d\mu_k = \int_{\mathbb{R}^d} f d\mu,$$

for every $f \in C_0(\mathbb{R}^d)$ (the space of continuous functions on $\mathbb{R}^d$ with compact support).

5.1. A general convergence result

Our first result is a simple convergence lemma that we shall use in blow-up arguments.

Lemma 5.1. *Suppose that $\nu_k \rightharpoonup \nu$. Fix $\psi \in \text{Lip}(\mathbb{R}^d \times \mathbb{R}^d)$, and a sequence of functions $\psi_k \in \text{Lip}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

- ψ_k converge uniformly to ψ ,
- there exists $R > 0$ such that $\text{supp}(\psi_k(x, \cdot)) \subset B(x, R)$ for every $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, and
- $\sup_k \|\psi_k\|_{\text{Lip}} < \infty$.

Then, for any bounded open set $U \subset \mathbb{R}^d$,

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_U \left| \int_{\mathbb{R}^d} \psi_k(x, y) d\nu_k(y) \right|^2 d\nu_k(x) \\ \geq \int_U \left| \int_{\mathbb{R}^d} \psi(x, y) d\nu(y) \right|^2 d\nu(x). \end{aligned}$$

Proof. Choose M such that $U \subset B(0, M)$. Notice that the function

$$f_k(x) = \int_{\mathbb{R}^d} \psi_k(x, y) d\nu_k(y)$$

has both its modulus of continuity and supremum norm on the set $B(0, M)$ bounded in terms of M , R , $\sup_k \|\psi_k\|_{\text{Lip}}$ and $\sup_k \nu_k(B(0, R + M))$. Consequently, the functions f_k converge uniformly to the function $f(x) = \int_{\mathbb{R}^d} \psi(x, y) d\nu(y)$ on $B(0, M)$. But now, for $g \in C_0(B(0, M))$, the sequence $g|f_k|^2$ converges to $g|f|^2$ uniformly, and so from the weak convergence of ν_k to ν we conclude that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} g|f_k|^2 d\nu_k = \int_{\mathbb{R}^d} g|f|^2 d\nu.$$

The desired lower semi-continuity property readily follows by choosing for g an increasing sequence of functions in $\text{Lip}_0(B(0, M))$ that converges to χ_U pointwise. $\square$

Lemma 5.2. Suppose that μ_k is a sequence of measures satisfying

- (1) $\mathcal{I}_{\mu_k}(Q_0) \geq 1$,
- (2) $\sup_k \mu_k(B(0, R)) < \infty$ for every $R > 0$,
- (3) $\mathcal{S}_{\mu_k}^k(Q_0) \leq \frac{1}{k}$.

Then there is a subsequence of the measures that converges weakly to a φ -symmetric measure μ satisfying $\mathcal{I}_{\mu}(Q_0) \geq 1$.

The reader should compare item (3) in the assumptions of the lemma with the display (3.7). This lemma will be used to argue by contradiction that (3.7) holds for certain choices of function Γ .

Proof. Using the condition (2) we pass to a subsequence of the measures that converges weakly to a measure μ . It is immediate from (1) that $\mathcal{I}_{\mu}(Q_0) \geq 1$. To complete the proof it remains to demonstrate that μ is φ -symmetric, that this,

$$\int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) = 0 \text{ for every } x \in \text{supp}(\mu) \text{ and } t > 0. \quad (5.1)$$

To this end, fix $M > 0$ and $t > 0$. We apply Lemma 5.1 with $\nu_k = \mu_k$, $\nu = \mu$, and $\psi_k(x, y) = (x-y) \varphi\left(\frac{|x-y|}{t}\right)$. This yields that

$$\begin{aligned} & \int_{B(0,M)} \left| \int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right|^2 d\mu(x) \\ & \leq \liminf_{k \rightarrow \infty} \int_{B(0,M)} \left| \int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{t}\right) d\mu_k(y) \right|^2 d\mu_k(x). \end{aligned}$$

After dividing both sides by $\frac{1}{t^{2(s+1)}}$, integrating this inequality over $(\frac{1}{M}, M)$ with respect to $\frac{dt}{t}$ and applying Fatou's lemma we get

$$\int_{B(0,M)} \int_{\frac{1}{M}}^M \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x) \leq \liminf_k \mathcal{S}_{\mu_k, \varphi}^M(Q_0),$$

and the right hand side is equal to 0 because of the condition (3) (just note that $\mathcal{S}_{\mu_k, \varphi}^M(Q_0) \leq \mathcal{S}_{\mu_k, \varphi}^k(Q_0)$ for $k > M$). Since M was chosen arbitrarily, and certainly the function $x \mapsto \int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{t}\right) d\mu(y)$ is continuous, we conclude that (5.1) holds. $\square$

5.2. Geometric properties of measures and weak convergence

In blow-up arguments, we shall often consider a sequence of measures with a weak limit that is φ -symmetric. The lemmas of this section will allow us to extract information about the eventual behaviour of the sequence of measures from our knowledge of the limit measure.

Lemma 5.3. *Suppose $\mu_k \rightharpoonup \mu$, and Q is a cube with $\mathcal{I}_\mu(Q) > 0$. Then for any $n > 0$,*

- $\lim_{k \rightarrow \infty} D_{\mu_k, n}(Q) = D_{\mu, n}(Q)$, while, for $n \in \mathbb{Z}$,
- $\beta_{\mu, n}(Q) = \lim_{k \rightarrow \infty} \beta_{\mu_k, n}(Q)$, and,
- $\alpha_{\mu, n}(Q) = \lim_{k \rightarrow \infty} \alpha_{\mu_k, n}(Q)$.

Proof. The first item of course follows immediately from the definition of weak convergence. For the convergence of the β -coefficients, observe that for any finite subset $\tilde{\mathcal{P}}'$ of the family $\tilde{\mathcal{P}}$ of n -planes that intersect B_Q , we have

$$\lim_{k \rightarrow \infty} \min_{L \in \tilde{\mathcal{P}}'} \int_{B_Q} \varphi_Q \text{dist}(x, L)^2 d\mu_k(x) = \min_{L \in \tilde{\mathcal{P}}'} \int_{B_Q} \varphi_Q \text{dist}(x, L)^2 d\mu(x).$$

From this, the convergence of the β -coefficients follows from observing that the collection of functions $\varphi_Q \operatorname{dist}(\cdot, L)^2$, $L \in \tilde{\mathcal{P}}$, is a relatively compact set in $C(\overline{B_Q})$; and every plane which contains the centre of mass of any of the measures $\varphi_Q \mu_k$ or $\varphi_Q \mu$ must also intersect B_Q (since B_Q is a convex set containing $\operatorname{supp}(\varphi_Q)$).

We argue similarly in the case of the α numbers: In this case we observe that

- $\mathcal{P}_Q = \{L \cap \overline{B_Q} : L \in \mathcal{P}_n, L \cap \frac{1}{4}B_Q \neq \emptyset\}$ is relatively compact in the Hausdorff metric, while, for any constant $K > 0$,
- $\mathcal{F} = \{f \in \operatorname{Lip}_0(3B_Q) : \|f\|_{\operatorname{Lip}} \leq \frac{1}{\ell(Q)}\}$ is a relatively compact subset in $C_0(\mathbb{R}^d)$ equipped with uniform norm.

For any finite subsets $\mathcal{P}'_Q \subset \mathcal{P}_Q$, $\mathcal{F}' \subset \mathcal{F}$, we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \max_{f \in \mathcal{F}'} \min_{L \in \mathcal{P}'_Q} \int_{B_Q} \varphi_Q f(x) d(\mu_k - \vartheta_{\mu_k, L} \mathcal{H}^n|_L)(x) \\ &= \max_{f \in \mathcal{F}'} \min_{L \in \mathcal{P}'_Q} \int_{B_Q} \varphi_Q f(x) d(\mu - \vartheta_{\mu, L} \mathcal{H}^n|_L)(x). \end{aligned}$$

To complete the proof, just notice that for every $f \in \mathcal{F}$, the function

$$L \cap \overline{B_Q} \mapsto \int_{\mathbb{R}^d} f \varphi_Q d\mathcal{H}^n|_L$$

is continuous in the Hausdorff metric with a modulus of continuity bounded in terms of $\|\varphi_Q\|_{\operatorname{Lip}}$, and $\ell(Q)$, while the functionals

$$f \mapsto \int_{\mathbb{R}^d} \varphi_Q f d\mu_k, \text{ and } f \mapsto \int_{\mathbb{R}^d} \varphi_Q f d\mu$$

are continuous in the uniform norm with moduli of continuity bounded independently of k . Since the numbers $\vartheta_{\mu_k, L}$ are uniformly bounded over k and $L \cap \overline{B_Q} \in \mathcal{P}_Q$, the convergence of the α -coefficients follows. $\square$

The next result is a clear consequence of Lemma 5.3 (and also Section 2.5), but it will be useful to state it explicitly.

Corollary 5.4. *Suppose that $\mu_k \rightharpoonup \mu$. Fix a sequence of lattices $\mathcal{D}^{(k)}$ that stabilize in a lattice $\mathcal{D}'$, $n \in \mathbb{Z} \cap (0, d)$, and $m \in (0, d)$. If, for a cube $Q' \in \mathcal{D}'$, we have $\beta_{\mu, n}(Q') > \beta$ and $D_{\mu, m}(Q') > T$, then for all sufficiently large k , we have*

$$Q' \in \mathcal{D}^{(k)}, \beta_{\mu_k, n}(Q') > \beta \text{ and } D_{\mu_k, m}(Q') > T.$$

Lemma 5.5. Fix $n \in \mathbb{N}$, $n < d$. Suppose that $\mu_k \rightarrow \mu$, for some measure μ with $\mathcal{I}_\mu(Q_0) = 1$ for which $\text{supp}(\mu)$ is n -rectifiable. Fix a sequence of lattices $\mathcal{D}^{(k)}$, all containing Q_0 , that stabilize in a lattice $\mathcal{D}'$. For any $\delta \in (0, 1)$ and $\varkappa > 0$, we can find a finite collection of cubes Q_j such that

- (1) $\ell(Q_j) \leq \varkappa$,
- (2) $3B_{Q_j}$ are disjoint, and $3B_{Q_j} \subset 3B_{Q_0}$,

and, for all sufficiently large⁶ k ,

- (3) $Q_j \in \mathcal{D}^{(k)}$,
- (4) $D_{\mu_k, n+\delta}(Q') \leq \left(\frac{\ell(Q_j)}{\ell(Q')}\right)^{\delta/2} D_{\mu_k, n+\delta}(Q_j)$ for every $Q' \in \mathcal{D}^{(k)}$ satisfying $B_{Q_j} \subset B_{Q'} \subset 300B_{Q_0}$,
- (5) $\sum_j \mathcal{I}_{\mu_k}(Q_j) \geq \frac{1}{C} \mathcal{I}_{\mu_k}(Q_0)$.

Proof. From Lemma 2.3 we infer that, for any $\delta' \in (0, \delta/2)$

$$\lim_{\substack{Q' \in \mathcal{D}', x \in Q' \\ \ell(Q') \rightarrow 0}} D_{\mu, n+\delta'}(Q') = \infty \text{ for } \mu\text{-a.e. } x \in \text{supp}(\mu).$$

Fix $T > 0$. Consider the maximal (by inclusion of the associated balls $B_{Q'}$) cubes $Q' \in \mathcal{D}'$ with $B_{Q'} \subset 300B_{Q_0}$ that intersect B_{Q_0} and satisfy $D_{\mu, n+\delta'}(Q') > T$. If T is sufficiently large then $\ell(Q') \leq \varkappa$, and certainly $3B_{Q'} \subset 3B_{Q_0}$, and so property (1), along with the second assertion in property (2), hold for the maximal cubes Q' .

For each maximal cube Q' we have that

$$D_{\mu, n+\delta}(Q'') \leq 2^{-(\delta-\delta')[Q'':Q']} D_{\mu, n+\delta}(Q') \quad (5.2)$$

for every $Q'' \in \mathcal{D}'$ satisfying

$$B_{Q'} \subset B_{Q''} \subset 300B_{Q_0} \text{ and } B_{Q''} \cap B_{Q_0} \neq \emptyset. \quad (5.3)$$

As there are only finitely many Q'' satisfying (5.3), we have that for large enough k (possibly depending on Q')

$$\begin{aligned} D_{\mu_k, n+\delta}(Q'') &\leq 2^{-\frac{\delta}{2}[Q'':Q']} D_{\mu_k, n+\delta}(Q') \\ &\text{for every } Q'' \in \mathcal{D}' \text{ satisfying (5.3).} \end{aligned} \quad (5.4)$$

⁶ This largeness threshold is purely qualitative. It may depend on $\varkappa$, but also on the density properties of μ , and the rate at which the lattices $\mathcal{D}^{(k)}$ stabilize.

Now take a finite subcollection $\mathcal{G}$ of the maximal cubes with the property that $\sum_{Q' \in \mathcal{G}} \mathcal{I}_\mu(Q') > \frac{1}{2} \mathcal{I}_\mu(Q_0) = \frac{1}{2}$ (μ -almost every point in B_{Q_0} is contained in a maximal cube).

If k is sufficiently large, then every cube Q' in the finite collection $\mathcal{G}$ satisfies (5.4). Moreover, since the lattices $\mathcal{D}^{(k)}$ stabilize, we infer that if k is large enough, then every $Q' \in \mathcal{G}$ and every $Q'' \in \mathcal{D}'$ satisfying (5.3) lies in $\mathcal{D}^{(k)}$. It follows that properties (3) and (4) hold for every cube in $\mathcal{G}$ if k is large enough.

Finally, using the Vitali covering lemma we choose a pairwise disjoint sub-collection $\{3B_{Q'_j}\}_j$ of the collection of balls $\{3B_{Q'} : Q' \in \mathcal{G}\}$ such that $\bigcup_j 15B_{Q'_j} \supset \bigcup_{Q' \in \mathcal{G}} 3B_{Q'}$. From (5.2) we derive that $\mu(15B_{Q'_j}) \leq C\mathcal{I}_\mu(Q'_j)$. Whence

$$\frac{1}{2} < \sum_j \mu(15B_{Q'_j}) \leq C \sum_j \mathcal{I}_\mu(Q'_j).$$

Thus, as long as k is large enough, we have that $\sum_j \mathcal{I}_{\mu_k}(Q'_j) \geq \frac{1}{C}$. Consequently, the collection of cubes $(Q'_j)_j$ satisfies all of the desired properties. $\square$

6. Domination from below

Fix $n = \lceil s \rceil - 1$.

We introduce two parameters, $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$, satisfying

$$n + 2\delta + \varepsilon < s, \text{ and } s + 2\varepsilon < \lfloor s \rfloor + 1.$$

6.1. Domination from below

We introduce a filter on a dyadic lattice $\mathcal{D}$ from [9] called domination from below. Fix a measure μ , and subsets $\mathcal{G}, \mathcal{G}' \subset \mathcal{D}$.

Definition 6.1. We say that $Q \in \mathcal{G}$ is dominated from below by a (finite) bunch of cubes $Q_j \in \mathcal{G}'$ if the following conditions hold:

- (1) $D_\mu(Q_j) > 2^{\varepsilon[Q:Q_j]} D_\mu(Q)$,
- (2) $3B_{Q_j}$ are disjoint,
- (3) $3B_{Q_j} \subset 3B_Q$,
- (4) $\sum_j D_\mu(Q_j)^2 2^{-2\varepsilon[Q:Q_j]} \mathcal{I}_\mu(Q_j) > D_\mu(Q)^2 \mathcal{I}_\mu(Q)$.

We set $\mathcal{G}_{\text{down}}(\mathcal{G}')$ to be the set of all cubes Q in $\mathcal{G}$ that cannot be dominated from below by a bunch of cubes in $\mathcal{G}'$ (except for the trivial bunch consisting of Q itself in the case when $Q \in \mathcal{G} \cap \mathcal{G}'$). If $\mathcal{G}' = \mathcal{G}$, then we just write $\mathcal{G}_{\text{down}}$ instead of $\mathcal{G}_{\text{down}}(\mathcal{G})$.

Lemma 6.2. *Suppose that $\sup_{Q \in \mathcal{G}} D_\mu(Q) < \infty$. Then there exists $c(\varepsilon) > 0$ such that*

$$\sum_{Q \in \mathcal{G}_{\text{down}}} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \geq c(\varepsilon) \sum_{Q \in \mathcal{G}} D_\mu(Q)^2 \mathcal{I}_\mu(Q).$$

Proof. We start with a simple claim.

Claim. *Every $Q \in \mathcal{G}$ with $\mathcal{I}_\mu(Q) > 0$ is dominated from below by a bunch of cubes $P_{Q,j}$ in $\mathcal{G}_{\text{down}}$.*

To prove the claim we make two observations. The first is transitivity: if the bunch $Q_1, \dots, Q_N$ dominates $Q' \in \mathcal{G}$ from below, and if (say) Q_1 is itself dominated from below by a bunch $P_1, \dots, P_{N'}$, then the bunch $P_1, \dots, P_{N'}, Q_2, \dots, Q_N$ dominates Q' . The second observation is that there are only finitely many cubes Q' that can participate in a dominating bunch for Q : Indeed, each such cube Q' satisfies $D_\mu(Q') \geq 2^{\varepsilon[Q:Q']} D_\mu(Q)$, and so

$$[Q : Q'] \leq \frac{1}{\varepsilon} \log_2 \left(\frac{\sup_{Q'' \in \mathcal{G}} D_\mu(Q'')}{D_\mu(Q)} \right).$$

With these two observations in hand, we define a partial ordering on the finite bunches of cubes $(Q_j)_j$ that dominate Q from below: For two different dominating bunches $(Q_j^{(1)})_j$ and $(Q_j^{(2)})_j$, we say that $(Q_j^{(1)})_j \prec (Q_j^{(2)})_j$ if for each ball $3B_{Q_j^{(1)}}$, we have $3B_{Q_j^{(1)}} \subset 3B_{Q_k^{(2)}}$ for some k . Since there are only finitely many cubes that can participate in a dominating bunch, there may be only finitely many different dominating bunches of Q , and hence there is a minimal (according to the partial order $\prec$) dominating bunch $(P_{Q,j})_j$. Each cube $P_{Q,j}$ must lie in $\mathcal{G}_{\text{down}}$.

Now write

$$\begin{aligned} \sum_{Q \in \mathcal{G}} D_\mu(Q)^2 \mathcal{I}_\mu(Q) &\leq \sum_{Q \in \mathcal{G}} \sum_j D_\mu(P_{Q,j})^2 \mathcal{I}_\mu(P_{Q,j}) 2^{-2\varepsilon[Q:P_{Q,j}]} \\ &\leq \sum_{P \in \mathcal{G}_{\text{down}}} D_\mu(P)^2 \mathcal{I}_\mu(P) \left[\sum_{Q: 3B_Q \supset 3B_P} 2^{-2\varepsilon[Q:P]} \right]. \end{aligned}$$

The inner sum does not exceed $\frac{C}{\varepsilon}$, and the lemma follows. $\square$

Lemma 6.3. *There exists $c(\varepsilon) > 0$ such that*

$$\sum_{Q \in \mathcal{G} \setminus \mathcal{G}_{\text{down}}(\mathcal{G}')} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C(\varepsilon) \sum_{Q \in \mathcal{G}'} D_\mu(Q)^2 \mathcal{I}_\mu(Q).$$

Proof. For each $Q \in \mathcal{G} \setminus \mathcal{G}_{\text{down}}(\mathcal{G}')$, just pick a bunch of cubes $P_{Q,j}$ in $\mathcal{G}'$ that dominates Q from below. Then we merely repeat the final calculation of the previous proof:

$$\begin{aligned}
& \sum_{Q \in \mathcal{G} \setminus \mathcal{G}_{\text{down}}(\mathcal{G}')} D_{\mu}(Q)^2 \mathcal{I}_{\mu}(Q) \\
& \leq \sum_{Q \in \mathcal{G} \setminus \mathcal{G}_{\text{down}}(\mathcal{G}')} \sum_j D_{\mu}(P_{Q,j})^2 \mathcal{I}_{\mu}(P_{Q,j}) 2^{-2\varepsilon[Q:P_{Q,j}]} \\
& \leq \sum_{P \in \mathcal{G}'} D_{\mu}(P)^2 \mathcal{I}_{\mu}(P) \left[\sum_{Q \in \mathcal{D}: 3B_Q \supset 3B_P} 2^{-2\varepsilon[Q:P]} \right],
\end{aligned}$$

and the lemma follows. $\square$

The domination from below filter is used in what follows to preclude the possibility that the support of a measure in a cube $Q \in \mathcal{G}_{\text{down}}$ concentrates on a set of dimension smaller than s . In particular, we shall use the following lemma:

Lemma 6.4. *Suppose that $\mu_k \rightharpoonup \mu$, where $\mathcal{I}_{\mu}(Q_0) = 1$ and $\text{supp}(\mu)$ is n -rectifiable (recall that $n = \lceil s \rceil - 1$). Fix a sequence of lattices $\mathcal{D}^{(k)}$, all containing Q_0 , that stabilize in a lattice $\mathcal{D}'$. Provided that $\varkappa > 0$ is chosen sufficiently small, for all sufficiently large k , the bunch of cubes Q_j constructed in Lemma 5.5 dominates Q_0 from below in the sense of properties (1)–(4) of Definition 6.1.*

Proof. First notice that, by property (4) of the conclusion of Lemma 5.5, we have that $D_{\mu_k, n+\delta}(Q_j) \geq D_{\mu_k, n+\delta}(Q_0)$, and so

$$\begin{aligned}
D_{\mu_k}(Q_j) &= 2^{(s-n-\delta)[Q_0:Q_j]} D_{\mu_k, n+\delta}(Q_j) \\
&\geq 2^{(s-n-\delta)[Q_0:Q_j]} D_{\mu_k, n+\delta}(Q_0) \\
&= 2^{(s-n-\delta)[Q_0:Q_j]} D_{\mu_k}(Q_0),
\end{aligned} \tag{6.1}$$

as long as k is large enough. Therefore property (1) of Definition 6.1 is satisfied, as $s - n - \delta > \varepsilon$. Since properties (2) and (3) of Definition 6.1 clearly hold, it remains to verify that

$$\sum_j D_{\mu_k}(Q_j)^2 2^{-2\varepsilon[Q_0:Q_j]} \mathcal{I}_{\mu_k}(Q_j) > D_{\mu_k}(Q_0)^2 \mathcal{I}_{\mu_k}(Q_0).$$

Since $D_{\mu}(Q_0) = 1$, we have $D_{\mu_k}(Q_0) \geq \frac{1}{2}$ for large k and hence from (6.1) we derive that

$$\begin{aligned}
& \sum_j D_{\mu_k}(Q_j)^2 e^{-2\varepsilon[Q_0:Q_j]} \mathcal{I}_{\mu_k}(Q_j) \\
& \geq \frac{1}{4} \min_j 2^{2(s-n-\delta-\varepsilon)[Q_0:Q_j]} \sum_j \mathcal{I}_{\mu_k}(Q_j).
\end{aligned}$$

Using properties (1) and (5) in the conclusion of Lemma 5.5, the right hand side here is clearly at least

$$\frac{1}{C} \min_j 2^{2(s-n-\delta-\varepsilon)[Q_0:Q_j]} \mathcal{I}_{\mu_k}(Q_0) \geq \frac{1}{C} \varkappa^{-2(s-n-\delta-\varepsilon)} \mathcal{I}_{\mu_k}(Q_0)$$

since $\delta < s - n - \varepsilon$. But the right hand side here is larger than $\mathcal{I}_{\mu_k}(Q_0)$ provided that $\varkappa$ is small enough. $\square$

7. Cubes with lower-dimensional density control are sparse

Recall that $n = \lceil s \rceil - 1$. Fix a measure μ . For $M \in \mathbb{N}$, consider the set $\mathcal{D}_M(\mu)$ of cubes $Q \in \mathcal{D}$ such that

$$D_{\mu, n+\delta}(Q') \leq D_{\mu, n+\delta}(Q)$$

whenever $Q' \in \mathcal{D}$ satisfies $B_{Q'} \supset B_Q$ and $[Q' : Q] \leq M$.

The aim of this section is to prove the following result:

Proposition 7.1. *There exist $M \in \mathbb{N}$, $A > 0$ and $C > 0$ such that if μ is a finite measure satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$, then*

$$\sum_{Q \in \mathcal{D}_M(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

To prove Proposition 7.1, we shall use the domination from below filter with the subsets $\mathcal{G} = \mathcal{G}' = \mathcal{D}_M(\mu)$. Write $\mathcal{D}_{M, \text{down}}(\mu)$ for the set of cubes in $\mathcal{D}_M(\mu)$ that cannot be dominated from below. Lemma 6.2 yields that,

$$\sum_{Q \in \mathcal{D}_{M, \text{down}}(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \geq c(\varepsilon) \sum_{Q \in \mathcal{D}_M(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q).$$

Consequently, referring to the general principle of Section 3.2, we find that in order to prove Proposition 7.1, we need to verify (3.7) with

$$\Gamma_\mu(Q) = \begin{cases} D_\mu(Q)^2 & \text{if } Q \in \mathcal{D}_{M, \text{down}}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

We formulate this precisely as the following lemma:

Lemma 7.2. *There exist $A > 0$, $\Delta > 0$ and $M \in \mathbb{N}$ such that for every measure μ and every cube $Q \in \mathcal{D}_{M, \text{down}}(\mu)$, we have*

$$\mathcal{S}_\mu^A(Q) \geq \Delta D_\mu(Q)^2 \mathcal{I}_\mu(Q).$$

Proof. If the result fails to hold, then for every $k \in \mathbb{N}$, we can find a measure $\tilde{\mu}_k$ and a cube $Q_k \in \mathcal{D}_{k, \text{down}}(\tilde{\mu}_k)$ such that

$$\mathcal{S}_{\tilde{\mu}_k}^k(Q_k) \leq \frac{1}{k} D_{\tilde{\mu}_k}(Q_k)^2 \mathcal{I}_{\tilde{\mu}_k}(Q_k).$$

Consider the measure $\mu_k = \frac{\tilde{\mu}_k(\mathcal{L}_{Q_k}(\cdot))}{\mathcal{I}_{\tilde{\mu}_k}(Q_k)}$. Then $D_{\mu_k}(Q_0) = \mathcal{I}_{\mu_k}(Q_0) = 1$. The preimage of the lattice $\mathcal{D}$ under the affine map $\mathcal{L}_{Q_k}$ is some lattice $\mathcal{D}^{(k)}$ with $Q_0 \in \mathcal{D}^{(k)}$. Of course, we have that $Q_0 \in \mathcal{D}^{(k)}_{k,\text{down}}(\mu_k)$. In addition

$$D_{\mu_k, n+\delta}(Q') \leq 1 \text{ if } Q' \in \mathcal{D}^{(k)}, B_{Q'} \supset B_{Q_0}, [Q' : Q_0] \leq k. \quad (7.1)$$

It readily follows from this that for every $R > 0$, $\sup_k \mu_k(B(0, R)) < \infty$. In addition, we have that $\mathcal{S}_{\mu_k}^k(Q_0) \leq \frac{1}{k}$. As such, we may apply Lemma 5.2 and conclude that, passing to a subsequence if necessary, the sequence μ_k converges weakly to a φ -symmetric measure ν with $\mathcal{I}_\nu(Q_0) = 1$. With the passage to a further subsequence, we may assume that the lattices $\mathcal{D}^{(k)}$ (which all contain Q_0) stabilize in a lattice $\mathcal{D}'$ (see Section 2.5). Then from (7.1) we see that

$$D_{\nu, n+\delta}(Q') \leq D_{\nu, n+\delta}(Q_0) \text{ for every } Q' \in \mathcal{D}' \text{ such that } B_{Q'} \supset B_{Q_0}.$$

From this property, we infer from Proposition 4.7 that $\text{supp}(\nu)$ has to be contained in an n -plane and is therefore n -rectifiable (ν has insufficient growth at infinity for the other possibilities in Proposition 4.7 to hold).

Consequently all the hypotheses of Lemma 5.5 are satisfied, and so we may consider the finite collection of cubes Q_j constructed there, which may have side-length smaller than any prescribed threshold $\varkappa > 0$. Lemma 6.4 ensures that the bunch of cubes Q_j dominate Q_0 from below as long as $\varkappa$ is small enough. Therefore, given that $Q_0 \in \mathcal{D}^{(k)}_{k,\text{down}}(\mu_k)$, we will have reached our desired contradiction once we verify the following:

Claim. *Provided that $\varkappa$ is small enough, and k is sufficiently large, each cube Q_j lies in $\mathcal{D}^{(k)}_k(\mu_k)$.*

To see this, notice that, for a cube Q'' with $B_{Q''} \supset B_{Q_j}$ and $\ell(Q'') \leq 2^k \ell(Q_j)$, it can only happen that property (4) of Lemma 5.5 does not immediately show that $D_{\mu_k, n+\delta}(Q'') \leq D_{\mu_k, n+\delta}(Q_j)$ in the case when $B_{Q''}$ is not contained in $300B_{Q_0}$. But then since $B_{Q_j} \cap B_{Q_0} \neq \emptyset$, the cube Q'' has big side-length (certainly at least $30\ell(Q_0)$). It follows that the grandparent of Q'' , say $\tilde{Q}''$, must satisfy $B_{\tilde{Q}''} \supset B_{Q_0}$, while certainly $[\tilde{Q}'' : Q_0] \leq k$ if $\varkappa$ is small enough. But now we derive that

$$\begin{aligned} D_{\mu_k, n+\delta}(Q'') &\leq C D_{\mu_k, n+\delta}(\tilde{Q}'') \stackrel{(7.1)}{\leq} C D_{\mu_k, n+\delta}(Q_0) \\ &\stackrel{\text{Lemma 5.5, (4)}}{\leq} C \ell(Q_j)^{\delta/2} D_{\mu_k, n+\delta}(Q_j) \leq D_{\mu_k, n+\delta}(Q_j) \end{aligned}$$

as long as $C\varkappa^{\delta/2} < 1$. The claim follows. $\square$

8. Domination from above and the proof of Theorem 1.1

Consider a lattice $\mathcal{D}$, and a non-negative function $\Upsilon : \mathcal{D} \rightarrow [0, \infty)$.

8.1. Domination from above

We say that $Q' \in \mathcal{D}$ dominates $Q \in \mathcal{D}$ from above if $\frac{1}{2}B_{Q'} \supset B_Q$ and

$$\Upsilon(Q') > 2^{\varepsilon[Q':Q]}\Upsilon(Q)$$

We let $\mathcal{D}_{\text{up}}$ denote the set of cubes $Q \in \mathcal{D}$ that are not dominated from above by a cube in $\mathcal{D}$.

Lemma 8.1. *If $\sup_{Q \in \mathcal{D}} \Upsilon(Q) < \infty$, then*

$$\sum_{Q \in \mathcal{D}_{\text{up}}} \Upsilon(Q)^2 \mathcal{I}_\mu(Q) \geq c(\varepsilon) \sum_{Q \in \mathcal{D}} \Upsilon(Q)^2 \mathcal{I}_\mu(Q).$$

Proof. We first claim that every $Q \in \mathcal{D} \setminus \mathcal{D}_{\text{up}}$ with $\Upsilon(Q) > 0$ can be dominated from above by a cube $\tilde{Q} \in \mathcal{D}_{\text{up}}$.

Indeed, note that if Q' dominates Q from above, then certainly

$$[Q' : Q] \leq \frac{1}{\varepsilon} \log_2 \left(\frac{\sup_{Q'' \in \mathcal{D}} \Upsilon(Q'')}{\Upsilon(Q)} \right),$$

or else we would have that $\Upsilon(Q') > \sup_{Q'' \in \mathcal{D}} \Upsilon(Q'')$ (which is absurd). Consequently, there are only finitely many candidates for a cube that dominates Q from above. To complete the proof of the claim, choose $\tilde{Q} \in \mathcal{D}$ to be a cube of largest side-length that dominates Q from above. Then $\tilde{Q} \in \mathcal{D}_{\text{up}}$ (domination from above is transitive).

For each fixed $P \in \mathcal{D}_{\text{up}}$, consider those $Q \in \mathcal{D} \setminus \mathcal{D}_{\text{up}}$ with $\mathcal{I}_\mu(Q) > 0$ and $\tilde{Q} = P$. Then

$$\begin{aligned} \sum_{Q \in \mathcal{D} \setminus \mathcal{D}_{\text{up}}: \tilde{Q}=P} \Upsilon(Q)^2 \mathcal{I}_\mu(Q) &= \sum_{m \geq 1} \sum_{\substack{Q \in \mathcal{D} \setminus \mathcal{D}_{\text{up}}: \\ \ell(Q)=2^{-m}\ell(P), \tilde{Q}=P}} \Upsilon(Q)^2 \mathcal{I}_\mu(Q) \\ &\leq \sum_{m \geq 1} 2^{-2\varepsilon m} \Upsilon(P)^2 \left[\sum_{\substack{Q \in \mathcal{D}: \\ \ell(Q)=2^{-m}\ell(P), B_Q \subset \frac{1}{2}B_P}} \mathcal{I}_\mu(Q) \right]. \end{aligned}$$

The sum in square brackets is bounded by $C\mathcal{I}_\mu(P)$ (as $\varphi_P \equiv 1$ on $\frac{1}{2}B_P$), and so by summing over $P \in \mathcal{D}_{\text{up}}$, we see that

$$\sum_{Q \in \mathcal{D} \setminus \mathcal{D}_{\text{up}}} \Upsilon(Q)^2 \mathcal{I}_\mu(Q) \leq C(\varepsilon) \sum_{P \in \mathcal{D}_{\text{up}}} \Upsilon(P)^2 \mathcal{I}_\mu(P),$$

and the lemma is proved. $\square$

We shall make the choice

$$\Upsilon(Q) = \begin{cases} \beta_\mu(Q)D_\mu(Q) & \text{if } s \in \mathbb{Z}, \\ D_\mu(Q) & \text{if } s \notin \mathbb{Z}. \end{cases}$$

With this function, denote by $\mathcal{D}_{\text{up}}(\mu)$ those cubes that cannot be dominated from above.

Notice that if $s \in \mathbb{Z}$, and $Q \in \mathcal{D}_{\text{up}}(\mu)$, then for every $Q' \in \mathcal{D}$ with $\frac{1}{2}B_{Q'} \supset B_Q$,

$$\beta_\mu(Q')D_\mu(Q') \leq 2^{\varepsilon[Q':Q]} \beta_\mu(Q)D_\mu(Q). \quad (8.1)$$

Provided that $\beta_\mu(Q) > 0$, we readily derive from this inequality that whenever $\frac{1}{2}B_{Q'} \supset B_Q$

$$\begin{aligned} \left(\frac{\ell(Q)}{\ell(Q')}\right)^s D_\mu(Q) &\leq D_\mu(Q') \leq \left(\frac{\ell(Q')}{\ell(Q)}\right)^{s+2\varepsilon} D_\mu(Q), \\ \text{and } \beta_\mu(Q') &\leq \left(\frac{\ell(Q')}{\ell(Q)}\right)^{s+\varepsilon} \beta_\mu(Q). \end{aligned} \quad (8.2)$$

The right hand inequality in the first displayed formula perhaps deserves comment. To see it, we plug the obvious inequality

$$\sqrt{\mathcal{I}_\mu(Q')}\beta_\mu(Q') \geq \sqrt{\mathcal{I}_\mu(Q)}\beta_\mu(Q)$$

into (8.1) to find that

$$\frac{\sqrt{\mathcal{I}_\mu(Q')}}{\ell(Q')^s} \leq 2^{\varepsilon[Q':Q]} \frac{\sqrt{\mathcal{I}_\mu(Q)}}{\ell(Q)^s}.$$

Rearranging this yields the desired inequality.

If instead $s \notin \mathbb{Z}$ and $Q \in \mathcal{D}_{\text{up}}(\mu)$ then we have much better density control:

$$D_\mu(Q') \leq \left(\frac{\ell(Q')}{\ell(Q)}\right)^\varepsilon D_\mu(Q) \text{ whenever } Q' \in \mathcal{D}, \frac{1}{2}B_{Q'} \supset B_Q.$$

For the remainder of the paper, let us fix M so that Proposition 7.1 holds. Our goal will be to prove the following alternative.

Alternative 8.2. *For each $\Lambda > 4$ and $\alpha > 0$, there exist $A > 0$ and $\Delta > 0$ such that for every measure μ and cube $Q \in \mathcal{D}_{\text{up}}(\mu)$, with the additional properties that $\beta_\mu(Q) > 0$ and $\alpha_\mu(\Lambda Q) \geq \alpha$ if $s \in \mathbb{Z}$, we have that either*

$$(a) \quad \mathcal{S}_\mu^A(Q) \geq \Delta D_\mu(Q)^2 \mathcal{I}_\mu(Q)$$

or

(b) Q is dominated from below by a bunch of cubes in $\mathcal{D}_M(\mu)$.

Before we prove the alternative, let us see how we shall use it. Fix $\Lambda > 0$ and $\alpha > 0$. For $s \in \mathbb{Z}$, set

$$\mathcal{D}_{\text{up}}^*(\mu) = \{Q \in \mathcal{D}_{\text{up}}(\mu) : \alpha_\mu(\Lambda Q) \geq \alpha \text{ and } \beta_\mu(Q) > 0\},$$

while for $s \notin \mathbb{Z}$, set $\mathcal{D}_{\text{up}}^*(\mu) = \mathcal{D}_{\text{up}}(\mu)$.

Corollary 8.3. *If $s \in \mathbb{Z}$, then there exist $\Delta > 0$ and $A > 0$, depending on M , Λ , α such that for every finite measure μ satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$,*

$$\sum_{Q \in \mathcal{D}_{\text{up}}^*(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq \frac{1}{\Delta} \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

If $s \notin \mathbb{Z}$, then there exist $\Delta > 0$ and $A > 0$, depending on M , such that for every finite measure μ satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$,

$$\sum_{Q \in \mathcal{D}_{\text{up}}(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq \frac{1}{\Delta} \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

Proof of Corollary 8.3. One uses the general principle (3.7) to control the contribution of the sum over the cubes where alternative (a) occurs. Indeed, making the choice

$$\Gamma_\mu(Q) = \begin{cases} D_\mu(Q)^2 & \text{if } Q \in \mathcal{D}_{\text{up}}^*(\mu) \text{ and alternative (a) holds for } Q \\ 0 & \text{otherwise,} \end{cases}$$

we get from (3.8) that

$$\sum_{\substack{Q \in \mathcal{D}_{\text{up}}^*(\mu): \\ \text{alternative (a) holds}}} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq \frac{1}{\Delta} \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

For the cubes where alternative (b) holds, we apply Lemma 6.3 with $\mathcal{G}' = \mathcal{D}_M(\mu)$ and $\mathcal{G} = \mathcal{D}_{\text{up}}^*(\mu)$. Since

$$\{Q \in \mathcal{D}_{\text{up}}^*(\mu) : \text{alternative (b) holds}\} \subset \mathcal{G} \setminus \mathcal{G}_{\text{down}}(\mathcal{G}'),$$

we infer that

$$\sum_{\substack{Q \in \mathcal{D}_{\text{up}}^*(\mu): \\ \text{alternative (b) holds}}} D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C(\varepsilon) \sum_{Q \in \mathcal{D}_M(\mu)} D_\mu(Q)^2 \mathcal{I}_\mu(Q).$$

Proposition 7.1 ensures that the right hand side here is bounded by the sum of square function constituents. $\square$

Notice that, in conjunction with Lemma 8.1, Corollary 8.3 completes the proof of Theorem 3.1, and with it Theorem 1.1.

We now move onto proving the alternative.

Proof of Alternative 8.2. We (rather predictably) proceed by contradiction. If the alternative fails to hold, then for some $\Lambda > 0$ and $\alpha > 0$, and every $k \in \mathbb{N}$, we can find a measure $\tilde{\mu}_k$ and a cube $Q_k \in \mathcal{D}_{\text{up}}^*(\tilde{\mu}_k)$ such that

$$\mathcal{S}_{\tilde{\mu}_k}^k(Q_k) \leq \frac{1}{k} D_{\tilde{\mu}_k}(Q_k)^2 \mathcal{I}_{\tilde{\mu}_k}(Q_k),$$

but also Q_k cannot be dominated from below by a bunch of cubes in $\mathcal{D}_M(\tilde{\mu}_k)$.

We consider the measure $\mu_k = \frac{\tilde{\mu}_k(\mathcal{L}_{Q_k}(\cdot))}{\mathcal{I}_{\tilde{\mu}_k}(Q_k)}$, which satisfies $D_{\mu_k}(Q_0) = 1$. The preimage of the lattice $\mathcal{D}$ under $\mathcal{L}_{Q_k}$ is some lattice $\mathcal{D}^{(k)}$ with $Q_0 \in \mathcal{D}^{(k)}$. Moreover, $Q_0 \in \mathcal{D}_{\text{up}}^{(k)*}(\mu_k)$, and so from (8.2) we have that

$$D_{\mu_k}(Q') \leq C \ell(Q')^{s+1} \text{ whenever } \frac{1}{2} B_{Q'} \supset B_{Q_0}.$$

This polynomial growth bound allows us to apply Lemma 5.2 and pass to a subsequence of the measures that converges weakly to a φ -symmetric measure μ with $D_\mu(Q_0) = \mathcal{I}_\mu(Q_0) = 1$. With the passage to a further subsequence, we assume that the lattices $\mathcal{D}^{(k)}$ stabilize in some lattice $\mathcal{D}'$.

We first suppose that $\text{supp}(\mu)$ is not contained in an $[s]$ -plane. Then, since $\mu(B(0, R)) \leq CR^{2s+2\varepsilon}$ for large $R > 0$, we may apply Proposition 4.7, and find that there exists a cube $Q' \in \mathcal{D}'$ with $\frac{1}{2} B_{Q'} \supset B_{Q_0}$ of arbitrarily large side-length we have that $\beta_\mu(Q') > c^*$ and $D_\mu(Q') > \ell(Q')^{\lfloor s \rfloor + 1 - s - \varepsilon}$, where $c^* > 0$ depends only on d , s , and $\|\varphi'\|_\infty$. But Lemma 5.4 then ensures that for sufficiently large k , we have that $Q' \in \mathcal{D}^{(k)}$, $\beta_{\mu_k}(Q') > c^*$, and $D_{\mu_k}(Q') > \ell(Q')^{\lfloor s \rfloor + 1 - s - \varepsilon} D_{\mu_k}(Q_0)$. Provided that $c^* \ell(Q')^{\lfloor s \rfloor + 1 - s - \varepsilon} > 4\sqrt{d} \cdot \ell(Q')^\varepsilon$ ($\geq \beta_{\mu_k}(Q_0) 2^{\varepsilon[Q':Q_0]}$), this contradicts the fact that $Q_0 \in \mathcal{D}_{\text{up}}^{(k)}(\mu_k)$, and such a contradictory choice of $\ell(Q')$ is possible since $2\varepsilon < 1 + \lfloor s \rfloor - s$. Thus $\text{supp}(\mu) \subset L$ for some $[s]$ -plane L (which must intersect B_{Q_0}).

Our next claim is that $\text{supp}(\mu)$ is n -rectifiable, with $n = \lfloor s \rfloor - 1$. This is already proved in the case when $s \notin \mathbb{Z}$, as μ is supported in an n -plane in this case. If $s \in \mathbb{Z}$, then we notice that Proposition 4.7 guarantees that either $\mu = c\mathcal{H}_L^s$, or that $\text{supp}(\mu)$ is n -rectifiable. But the first case is ruled out since $\alpha_\mu(\Lambda Q_0) > 0$ (note that $\Lambda > 4$), so we must indeed have that $\text{supp}(\mu)$ is n -rectifiable.

Consequently, we may apply Lemma 5.5 with $\varkappa \leq 2^{-M}$ to find a finite collection of cubes Q_j , each of sidelength less than 2^{-M} , such that the balls $3B_{Q_j}$ are disjoint, $3B_{Q_j} \subset 3B_{Q_0}$, and for all sufficiently large k we have, for every j ,

$$D_{\mu_k, n+\delta}(Q') \leq D_{\mu_k, n+\delta}(Q_j)$$

whenever $Q' \in \mathcal{D}^{(k)}$ with $B_{Q_j} \subset B_{Q'} \subset 300B_{Q_0}$ (this property is weaker than property (4) of the conclusion of Lemma 5.5). In particular this ensures that each Q_j lies in $\mathcal{D}^{(k)}_M(\mu_k)$.

On the other hand, by choosing $\varkappa$ smaller if necessary, we conclude from Lemma 6.4 that the bunch Q_j dominates Q_0 from below. This contradicts the fact that Q_0 cannot be dominated from below by a finite bunch of cubes from $\mathcal{D}^{(k)}_M(\mu_k)$, and with this final contradiction we complete the proof of the alternative. $\square$

9. The reduction to one last square function estimate

For the remainder of the paper, we restrict our attention to proving Theorem 3.2, so we shall henceforth assume that $s \in \mathbb{Z}$. It remains to show that there exist constants $\alpha > 0$, $\Lambda > 4$, $A > 1$ and $C > 0$, such that if μ is a finite measure satisfying $\sup_{Q \in \mathcal{D}} D_\mu(Q) < \infty$, then

$$\sum_{Q \in \mathcal{D}_{\text{up}}(\mu), \alpha_\mu(\Lambda Q) \leq \alpha} \beta_\mu(Q)^2 D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

When combined with Corollary 8.3, this would show that (with a possibly larger constant A),

$$\sum_{Q \in \mathcal{D}_{\text{up}}(\mu)} \beta_\mu(Q)^2 D_\mu(Q)^2 \mathcal{I}_\mu(Q) \leq C \sum_{Q \in \mathcal{D}} \mathcal{S}_\mu^A(Q).$$

Then Theorem 3.2 follows from Lemma 8.1. Following the general principle (3.7) with the choice

$$\Gamma_\mu(Q) = \begin{cases} \beta_\mu(Q)^2 D_\mu(Q)^2 & \text{if } Q \in \mathcal{D}_{\text{up}}(\mu) \text{ satisfies } \alpha_\mu(\Lambda Q) \leq \alpha \\ 0 & \text{otherwise,} \end{cases}$$

it will suffice to demonstrate the following proposition:

Proposition 9.1. *There exist $\Lambda > 0$, $\alpha > 0$, $A > 1$, and $\Delta > 0$ such that for every measure μ and $Q \in \mathcal{D}_{\text{up}}(\mu)$ satisfying $\alpha_\mu(\Lambda Q) \leq \alpha$ and $\beta_\mu(Q) > 0$, we have*

$$\mathcal{S}_\mu^A(Q) \geq \Delta \beta_\mu(Q)^2 D_\mu(Q)^2 \mathcal{I}_\mu(Q). \quad (9.1)$$

Notice here that the β -number is present on the right hand side of (9.1). It is not possible to estimate the square function coefficient in terms of the density alone (i.e., (9.1) couldn't possibly be true in general if one removes the $\beta_\mu(Q)^2$ term on the right hand side), as μ may well be the s -dimensional Hausdorff measure associated to an s -plane, in which case the left hand side of (9.1) equals to zero.

10. The pruning lemma

For an n -plane L and $\beta > 0$, $L_\beta = \{x \in \mathbb{R}^d : \text{dist}(x, L) \leq \beta\}$ denotes the closed β -neighbourhood of L .

Lemma 10.1. *Let $R > 0$. Fix a measure μ with $\mu(B(0, R)) > 0$. Suppose that for some hyperplane H and $\beta > 0$, we have*

$$\frac{1}{\mu(B(0, R))} \int_{B(0, 10R)} \left(\frac{\text{dist}(x, H)}{R} \right)^2 d\mu(x) \leq \beta^2.$$

Then

$$\begin{aligned} & \left(\frac{\mu(B(0, R))}{R^s} \right)^2 \int_{B(0, 2R) \setminus H_{3\beta R}} \left(\frac{\text{dist}(x, H)}{R} \right)^2 d\mu(x) \\ & \leq C \int_{B(0, 2R)} \int_{3R}^{4R} \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{x-y}{t}\right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x). \end{aligned} \quad (10.1)$$

Proof. We may assume that $R = 1$ and $\mu(B(0, R)) = 1$. Suppose that $H = b + e^\perp$ for $b \in \mathbb{R}^d$ and $e \in \mathbb{R}^d$ with $|e| = 1$, and for $x \in \mathbb{R}^d$ set $z_x = \langle x - b, e \rangle$. Then

$$\left| \int_{\mathbb{R}^d} (x - y) \varphi\left(\frac{|x - y|}{t}\right) d\mu(y) \right| \geq \left| \int_{\mathbb{R}^d} (z_x - z_y) \varphi\left(\frac{|x - y|}{t}\right) d\mu(y) \right|.$$

Fix $x \in B(0, 2)$ with $|z_x| = \text{dist}(x, H) > 3\beta$. We will assume that $z_x > 3\beta$. Then

$$\begin{aligned} \int_{\mathbb{R}^d} (z_x - z_y) \varphi\left(\frac{|x - y|}{t}\right) d\mu(y) & \geq \int_{\{z_y < 2\beta\}} (z_x - z_y) \varphi\left(\frac{|x - y|}{t}\right) d\mu(y) \\ & \quad - \int_{\{z_y > z_x\}} (z_y - z_x) \varphi\left(\frac{|x - y|}{t}\right) d\mu(y). \end{aligned} \quad (10.2)$$

Notice that $\mu(\{y \in B(0, 1) : z_y \geq 2\beta\}) \leq \frac{1}{4\beta^2} \int_{B(0, 1)} z_y^2 d\mu \leq \frac{1}{4}$. Consequently, if $t \in (3, 4)$, we get that the first integral appearing on the right hand side of (10.2) is at least $\frac{z_x}{3} \mu(B(0, 1) \cap \{z_y < 2\beta\}) \geq \frac{z_x}{4}$. On the other hand, the second integral on the right hand side of (10.2) is at most

$$\int_{B(0, 10) \cap \{z_y > 3\beta\}} |z_y| d\mu(y) \leq \frac{1}{3\beta} \int_{B(0, 10) \cap \{z_y > 3\beta\}} |z_y|^2 d\mu(y) \leq \frac{\beta}{3} \leq \frac{z_x}{9}.$$

Thus

$$\left| \int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right| \geq \frac{|z_x|}{8}.$$

It is easy to see that the conclusion also holds when $z_x < -3\beta$. Squaring this inequality and integrating it yields,

$$\begin{aligned} & \int_{B(0,2) \setminus H_{3\beta}} |z_x|^2 d\mu(x) \\ & \leq C \int_{B(0,2)} \int_3^4 \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x), \end{aligned}$$

as required. $\square$

We shall use this lemma as an alternative:

Corollary 10.2 (*The pruning alternative*). *Fix a measure μ satisfying $\mu(B(0, R)) > 0$. Fix $\Delta > 0$. Suppose that, for some s -plane L , and $R > 0$,*

$$\frac{1}{\mu(B(0, R))} \int_{B(0, 10R)} \left(\frac{\text{dist}(x, L)}{R} \right)^2 d\mu(x) \leq \beta^2.$$

Then, we have that either

$$\begin{aligned} & \int_{B(0, 2R)} \int_3^{4R} \left| \int_{\mathbb{R}^d} \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) d\mu(y) \right|^2 \frac{dt}{t} d\mu(x) \\ & \geq \Delta \beta^2 \left(\frac{\mu(B(0, R))}{R^s} \right)^2 \mu(B(0, R)), \end{aligned}$$

or

$$\int_{B(0, 2R) \setminus L_{3\beta(d-s)R}} \left(\frac{\text{dist}(x, L)}{R} \right)^2 d\mu(x) \leq C \Delta \beta^2 \mu(B(0, R)).$$

Suppose $L = b + \{v_{d-s+1}, \dots, v_d\}^\perp$ for an orthonormal set of vectors $v_{d-s+1}, \dots, v_d$. One derives the corollary by applying Lemma 10.1 to the collection of $d-s$ hyperplanes $H^{(1)} = b + \{v_{d-s+1}\}^\perp, \dots, H^{(d-s)} = b + \{v_d\}^\perp$, whose intersection is L . One merely needs to notice that, on the one hand, for each $j \in \{1, \dots, d-s\}$, we have $\text{dist}(\cdot, H^{(j)}) \leq \text{dist}(\cdot, L)$. But on the other hand $\text{dist}(\cdot, L) \leq \sum_j \text{dist}(\cdot, H^{(j)})$,

and so for each $x \notin L_{3\beta(d-s)R}$ there is some j such that $x \notin H_{3\beta R}^{(j)}$ and moreover $\text{dist}(x, L) \leq (d-s) \text{dist}(x, H^{(j)})$.

11. The cylinder blow-up argument: the conclusion of the proof of Proposition 9.1

We shall work in the following parameter regime: Fix $\Lambda \gg 1$ to be chosen later, and let $\alpha \rightarrow 0$, $\Delta \rightarrow 0$, and $A \rightarrow \infty$.

Suppose that, for each $k \in \mathbb{N}$, there is a measure $\tilde{\mu}_k$, a cube $Q_k \in \mathcal{D}_{\text{up}}(\tilde{\mu}_k)$ such that $\alpha_{\tilde{\mu}_k}(\Lambda Q_k) \leq \frac{1}{k}$, $\beta_{\tilde{\mu}_k}(Q_k) > 0$, and

$$\mathcal{S}_{\tilde{\mu}_k}^k(Q_k) \leq \frac{1}{k} \beta_{\tilde{\mu}_k}(Q_k)^2 D_{\tilde{\mu}_k}(Q_k)^2 \mathcal{I}_{\tilde{\mu}_k}(Q_k).$$

Proposition 9.1 will follow if we deduce a contradiction for some sufficiently large $\Lambda > 0$.

Consider the measure $\mu_k = \frac{\tilde{\mu}_k(\mathcal{L}_k \cdot)}{\mathcal{I}_{\tilde{\mu}_k}(Q_k)}$. Then $D_{\mu_k}(Q_0) = \mathcal{I}_{\mu_k}(Q_0) = 1$. The preimage of $\mathcal{D}$ under $\mathcal{L}_k$ is some lattice $\mathcal{D}^{(k)}$ containing Q_0 . Passing to a subsequence we may assume that the lattices $\mathcal{D}^{(k)}$ stabilize in some lattice $\mathcal{D}'$. Also observe that

$$\mathcal{S}_{\mu_k}^k(Q_0) \leq \frac{1}{k} \beta_{\mu_k}(Q_0)^2. \quad (11.1)$$

Inasmuch as $Q_0 \in \mathcal{D}^{(k)}_{\text{up}}(\mu_k)$ and $\beta_{\mu_k}(Q_0) > 0$, we infer from (8.2) that for any $N \geq 1$

$$D_{\mu_k}(NQ_0) \leq CN^{s+2\varepsilon}. \quad (11.2)$$

Notice that, since $\alpha_{\mu_k}(\Lambda Q_0) \leq \frac{1}{k}$, we have $\beta_{\mu_k}(\Lambda Q_0) \leq \frac{C}{\sqrt{k}}$. From this and (11.2) we find that $\beta_{\mu_k}(Q_0) \rightarrow 0$ as $k \rightarrow \infty$.

11.1. Good density bounds for medium sized cubes containing Q_0

In this section we shall prove the following result.

Lemma 11.1. *There exists $C > 0$ such that if $Q' \in \mathcal{D}^{(k)}$ with $\frac{\Lambda}{2} B_{Q_0} \supset \frac{1}{2} B_{Q'} \supset B_{Q_0}$, then*

$$\frac{1}{C} \leq D_{\mu_k}(Q') \leq C. \quad (11.3)$$

(Here C depends only on d and s .)

Proof. The growth property (11.2) ensures that $\mu_k(B_{\Lambda Q_0}) \leq C(\Lambda)$. Consequently, from the fact that $\alpha_{\mu_k}(\Lambda Q_0) \leq \frac{1}{k}$, we infer that for each k there exists an s -plane V_k that intersects $\frac{1}{4} B_{\Lambda Q_0}$ such that for every $f \in \text{Lip}_0(B_{\Lambda Q_0})$ with $\|f\|_{\text{Lip}} \leq 1$,

$$\left| \int_{\mathbb{R}^d} \varphi_{\Lambda Q_0} f d[\mu_k - \vartheta_k \mathcal{H}^s|_{V_k}] \right| \leq \frac{C(\Lambda)}{k}, \quad (11.4)$$

where $\vartheta_k = \frac{\mathcal{I}_{\mu_k}(\Lambda Q_0)}{\mathcal{I}_{\mathcal{H}^s|_{V_k}}(\Lambda Q_0)}$.

Since $D_{\mu_k}(Q_0) = 1$, we readily see by testing (11.4) with $f = \varphi_{Q_0}$ that $\vartheta_k \mathcal{I}_{\mathcal{H}^s|_{V_k}}(Q_0) \geq \frac{1}{2}$ if k is large enough. Thus the plane V_k intersects B_{Q_0} . Consequently, $\frac{1}{C} \ell(Q')^s \leq \mathcal{I}_{\mathcal{H}^s|_{V_k}}(Q') \leq C \ell(Q')^s$ whenever $\frac{\Lambda}{2} B_{Q_0} \supset \frac{1}{2} B_{Q'} \supset B_{Q_0}$. But also $1 \leq \mathcal{I}_{\mu_k}(3Q_0) \leq C$ from (11.2). Testing (11.4) with $f = \varphi_{3Q_0}$ therefore yields that $\frac{1}{C} \leq \vartheta_k \leq C$ (for large k). Finally, testing (11.4) against $f = \varphi_{Q'}$, with Q' as in the statement of the lemma, we infer that (11.3) holds. $\square$

Fix R to be an integer power of 2 that satisfies $1 \ll R \ll \Lambda$. We choose a dyadic ancestor of Q_0 in $\mathcal{D}'$, say $\widehat{Q}_0$, of sidelength $16R$. Since the lattices $\mathcal{D}^{(k)}$ stabilize, $\widehat{Q}_0$ is a dyadic ancestor of Q_0 in the lattice $\mathcal{D}^{(k)}$ for large enough k . Insofar as $Q_0 \in \mathcal{D}^{(k)}_{\text{up}}(\mu_k)$, from (8.1) and (11.3) we derive that

$$\beta_{\mu_k}(\widehat{Q}_0) \leq CR^\varepsilon \beta_{\mu_k}(Q_0). \quad (11.5)$$

Set $\beta_k = \beta_{\mu_k}(\widehat{Q}_0)$. Note that $\lim_{k \rightarrow \infty} \beta_k = 0$.

11.2. Concentration around the optimal least squares plane

Denote by L_k an optimal s -plane for $\beta_{\mu_k}(\widehat{Q}_0)$. Since $\mathcal{I}_{\mu_k}(Q_0) = 1$, it is easily seen from Chebyshev's inequality that L_k passes through $B_{Q_0} = B(0, 4\sqrt{d})$ for all sufficiently large k , so the closest point x_k in L_k to 0 lies in B_{Q_0} . Then clearly we have that

$$B(x_k, \frac{r}{2}) \subset B(0, r) \subset B(x_k, 2r) \text{ for every } r \geq 8\sqrt{d}. \quad (11.6)$$

In this section our aim is to demonstrate the following lemma:

Lemma 11.2. *There is a constant $C_1 > 0$, depending on d and s , such that if $\widetilde{\beta}_k = C_1 \beta_k$, then*

$$\int_{B(0, 2R) \setminus L_{k, \widetilde{\beta}_k R}} \text{dist}(x, L_k)^2 d\mu_k(x) \leq \frac{C(R) \widetilde{\beta}_k^2}{k}, \quad (11.7)$$

where $L_{k, \widetilde{\beta}_k R} = \{x \in \mathbb{R}^d : \text{dist}(x, L_k) \leq \widetilde{\beta}_k R\}$.

This is a much stronger concentration property around the plane L_k than the one that the β -number alone provides us with. It will play a crucial role in the subsequent argument.

Proof of Lemma 11.2. We look to apply the pruning alternative. Observe that, provided k is large enough

$$\begin{aligned} \int_{B(0,2R)} \int_{3R}^{4R} \int_{\mathbb{R}^d} \left| \frac{x-y}{t^{s+1}} \varphi\left(\frac{|x-y|}{t}\right) d\mu_k(y) \right|^2 \frac{dt}{t} d\mu_k(x) \\ \leq \mathcal{S}_{\mu_k}^k(Q_0) \stackrel{(11.1)}{\leq} \frac{1}{k} \beta_{\mu_k}(Q_0)^2 \stackrel{(11.3)}{\leq} \frac{CR^s}{k} \beta_k^2. \end{aligned} \quad (11.8)$$

On the other hand, using (11.3) once again we derive that $\mathcal{I}_{\mu_k}(\widehat{Q}_0) \leq C\mu_k(B(0,R))$, while $\varphi_{\widehat{Q}_0} \geq 1$ on $B(0,10R)$, so we certainly have that

$$\frac{1}{\mu_k(B(0,R))} \int_{B(0,10R)} \frac{\text{dist}(x, L_k)^2}{R^2} d\mu_k(x) \leq C\beta_k^2. \quad (11.9)$$

Consider the alternative in Corollary 10.2, with $\Delta = \frac{CR^s}{k}$, and $\beta = \widetilde{\beta}_k = C_1\beta_k$. If C_1 is chosen appropriately in terms of d and s , the inequality (11.8) forces us into the first case of Corollary 10.2, which is to say that

$$\int_{B(0,2R) \setminus L_{k, \widetilde{\beta}_k R}} \frac{\text{dist}(x, L_k)^2}{R^2} d\mu_k(x) \leq \frac{CR^s \widetilde{\beta}_k^2}{k} \mu(B(0,R)),$$

as required. $\square$

11.3. Stretching the measure around the least squares plane

Let $\mathcal{A}^{(k)}$ denote a rigid motion that maps the s -plane $\{0\} \times \mathbb{R}^s$ (with $0 \in \mathbb{R}^{d-s}$) to L_k and $0 \in \mathbb{R}^d$ to x_k . We introduce the co-ordinates $x = (x', x'')$, $x' \in \mathbb{R}^{d-s}$, $x'' \in \mathbb{R}^s$. Then from (11.6) and (11.7) we have

$$\int_{B(0,R) \setminus (\{0\} \times \mathbb{R}^s)_{\widetilde{\beta}_k R}} \frac{|x'|^2}{\widetilde{\beta}_k^2} d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') \leq \frac{C(R)}{k}. \quad (11.10)$$

We define the squash mapping $\mathcal{S}_\beta(x) = (\beta x', x'')$ for $\beta > 0$, along with the stretched measure

$$\nu_k(\cdot) = \mu_k(\mathcal{A}^{(k)} \circ \mathcal{S}_{\widetilde{\beta}_k}(\cdot)).$$

Since $\widetilde{\beta}_k < 1$ for large enough k , we have $\nu_k(B(0,N)) \leq (\mu_k \circ \mathcal{A}^{(k)})(B(0,N))$ for $N > 0$. As μ_k satisfies (11.2), we see that we may pass to a subsequence such that ν_k converge weakly to a measure ν .

For $m \in \mathbb{N}$, denote by $B^m(z, r)$ the m -dimensional ball centred at $z \in \mathbb{R}^m$ with radius $r > 0$. Under our change of variables, the inequality (11.10) becomes

$$\int_{[S_{\beta_k}^{-1}(B(0, R))] \setminus (\overline{B^{d-s}(0, R)} \times \mathbb{R}^s)} |x'|^2 d\nu_k(x) \leq \frac{C(R)}{k}. \quad (11.11)$$

Whence,

$$\text{supp}(\nu) \cap [\mathbb{R}^{d-s} \times B^s(0, R)] \subset \overline{B^{d-s}(0, R)} \times B^s(0, R). \quad (11.12)$$

On the other hand, $\mu_k \circ \mathcal{A}^{(k)}(B(0, 8\sqrt{d})) \geq D_{\mu_k}(Q_0) = 1$, and so from (11.11) we derive that $\nu_k(\overline{B^{d-s}(0, R)} \times B^s(0, 8\sqrt{d})) \geq 1 - \frac{C(R)}{k}$. Thus

$$\nu(\mathbb{R}^{d-s} \times \overline{B^s(0, 8\sqrt{d})}) = \nu(\overline{B^{d-s}(0, R)} \times \overline{B^s(0, 8\sqrt{d})}) \geq 1.$$

Lemma 11.3. *The following three properties hold:*

(1) *If $f \in \text{Lip}_0(B(0, R))$, then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f(x', x'') d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') = \int_{\mathbb{R}^d} f(0, x'') d\nu(x', x'').$$

(2) *If $f \in \text{Lip}_0(\mathbb{R}^{d-s} \times B^s(0, R))$, then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^d} f\left(\frac{x'}{\beta_k}, x''\right) d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') = \int_{\mathbb{R}^d} f(x', x'') d\nu(x', x'').$$

(3) *If $t \in (0, \frac{R}{8})$, then*

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_{B(0, R/2)} \left| \int_{\mathbb{R}^d} \frac{x' - y'}{\tilde{\beta}_k} \varphi\left(\frac{|x - y|}{t}\right) d(\mu_k \circ \mathcal{A}^{(k)})(y', y'') \right|^2 d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') \\ \geq \int_{\mathbb{R}^{d-s} \times B^s(0, R/2)} \left| \int_{\mathbb{R}^d} (x' - y') \varphi\left(\frac{|x'' - y''|}{t}\right) d\nu(y', y'') \right|^2 d\nu(x', x''). \end{aligned}$$

The proof is a slightly cumbersome exercise in weak convergence, using the property (11.11). As such, we postpone the proof Section 11.7.

11.4. The limit measure ν is a cylindrically φ -symmetric measure

For $r \in (0, \frac{R}{8})$, let us examine the inequality

$$\int_{B(0,R)} \left| \int_{\mathbb{R}^d} (x-y) \varphi\left(\frac{|x-y|}{r}\right) d\mu_k(y) \right|^2 d\mu_k(x) \leq \frac{C(R)\tilde{\beta}_k^2}{k}$$

(see (11.8)). We would like to see what happens to this inequality under the change of variables that takes μ_k to ν_k . First notice that, because of (11.6) (and the fact that a rigid motion is an isometry)

$$\begin{aligned} \int_{B(0,R/2)} \left| \int_{\mathbb{R}^d} \frac{x'-y'}{\tilde{\beta}_k} \varphi\left(\frac{|x-y|}{r}\right) d(\mu_k \circ \mathcal{A}^{(k)})(y', y'') \right|^2 d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') \\ \leq \frac{C(R)}{k}. \end{aligned}$$

From this, we deduce from item (3) of Lemma 11.3 that

$$\begin{aligned} \int_{\mathbb{R}^d} (x' - y') \varphi\left(\frac{|x'' - y''|}{r}\right) d\nu(y', y'') = 0 \\ \text{for every } (x', x'') \in \text{supp}(\nu) \cap [\mathbb{R}^{d-s} \times B^s(0, R/2)]. \end{aligned} \quad (11.13)$$

We will establish the following lemma:

Lemma 11.4. *There exists a constant $C > 0$ such that for sufficiently large k ,*

$$\beta_{\mu_k}(Q_0) \leq \frac{C}{R} \tilde{\beta}_k.$$

The estimate in this lemma is inconsistent with (11.5) if R is large enough. A contradictory choice of R is possible once Λ is chosen large enough in terms of d and s . As such, we will have completed the proof of Proposition 9.1 once the lemma is established.

The key to proving Lemma 11.4 will be to show that, when restricted to $\mathbb{R}^{d-s} \times B^s(0, R/2)$, the support of ν is the graph of an $\mathbb{R}^{d-s}$ -valued harmonic function on $B^s(0, R/2)$. For this, we shall use the fact that $\alpha_{\mu_k}(\Lambda Q_0)$ tends to zero as $k \rightarrow \infty$ in a more substantial way than we have up to this point.

11.5. Large projections of the limit measure

In this section we shall prove the following result.

Lemma 11.5. *There exists $\vartheta_0 > 0$ such that for every $f : \mathbb{R}^s \rightarrow \mathbb{R}$, $f \in \text{Lip}_0(B^s(0, \frac{3R}{4}))$, we have*

$$\int_{\mathbb{R}^d} f(x'') d(\nu - \vartheta_0 \mathcal{H}^s|_{\{0\} \times \mathbb{R}^s})(x', x'') = 0$$

Proof. Recall (see the proof of Lemma 11.1) that for every k there is an s -plane V_k for which (11.4) holds for every $f \in \text{Lip}_0(B_{\Lambda_{Q_0}})$ with $\|f\|_{\text{Lip}} \leq 1$, and $\frac{1}{C} \leq \vartheta_k \leq C$. Also recall that L_k is an optimal s -plane for $\beta_{\mu_k}(\widehat{Q}_0)$. Both V_k and L_k pass through B_{Q_0} if k is sufficiently large.

Consider a cut-off function $h \in \text{Lip}_0(B(0, R))$, with $h \equiv 1$ on $B(0, 3R/4)$ and $\|h\|_{\text{Lip}} \leq 1$. Then the function $x \mapsto h(x)(\text{dist}(x, L_k))^2$ is $C(R)$ -Lipschitz, and so, by (11.4) and the definition of the β -coefficient, we infer that

$$\int_{B(0, 3R/4)} \text{dist}(x, L_k)^2 d\mathcal{H}^k|_{V_k}(x) \leq \frac{C(R)}{k} + C(R)\tilde{\beta}_k^2.$$

Given that the planes L_k and V_k both pass through B_{Q_0} , this implies that the intersection of the plane $[\mathcal{A}^{(k)}]^{-1}(V_k)$ with the ball $B(0, \frac{3R}{4})$ lies within a $C(R)\omega_k$ neighbourhood of $[\{0\} \times \mathbb{R}^s] \cap B(0, \frac{3R}{4})$, where $\omega_k \rightarrow 0$ as $k \rightarrow \infty$. Consequently, if $F \in \text{Lip}_0(B(0, \frac{3R}{4}))$, $\|F\|_{\text{Lip}} \leq 1$, then

$$\left| \int_{\mathbb{R}^d} F(x', x'') d(\mu_k \circ \mathcal{A}^{(k)} - \vartheta_k \mathcal{H}^s|_{\{0\} \times \mathbb{R}^s})(x', x'') \right| \leq C(R)\omega_k. \quad (11.14)$$

Passing to a subsequence so that ϑ_k converges to ϑ_0 , we get from item (1) of Lemma 11.3 that

$$\int_{\mathbb{R}^d} F(0, x'') d(\nu - \vartheta_0 \mathcal{H}^s|_{\{0\} \times \mathbb{R}^s})(x', x'') = 0.$$

The lemma follows immediately from this statement. $\square$

As a consequence of the lemma, note that whenever $x'' \in B^s(0, \frac{R}{2})$ and $t < R/8$, we have

$$\begin{aligned} \int_{\mathbb{R}^{d-s} \times B^s(x'', t)} \varphi\left(\frac{|x'' - y''|}{t}\right) d\nu(y', y'') &= \vartheta_0 \mathcal{I}_{\mathcal{H}^s}(B^s(x'', t)) \\ &= \vartheta_0 \mathcal{I}_{\mathcal{H}^s}(B^s(0, t)). \end{aligned}$$

11.6. The final contradiction: the proof of Lemma 11.4

From the observations of the previous section along with the property (11.13), we find if $(x', x'') \in \text{supp}(\nu) \cap [\mathbb{R}^{d-s} \times B^s(0, \frac{R}{2})]$ and $r \in (0, R/8)$, then

$$x' = \frac{1}{\vartheta_0 \mathcal{I}_{\mathcal{H}^s}(B^s(0, r))} \int_{\mathbb{R}^d} y' \varphi\left(\frac{|x'' - y''|}{r}\right) d\nu(y', y'').$$

This formula determines x' in terms of x'' . From this, we derive that $\text{supp}(\nu) \cap (\mathbb{R}^{d-s} \times B^s(0, \frac{R}{2}))$ is a graph given by $\{(u(x''), x'') : x'' \in B^s(0, \frac{R}{2})\}$ for some $u : B^s(0, \frac{R}{2}) \rightarrow \overline{B^{d-s}(0, R)}$. As, for each Borel set $E \subset B^s(0, \frac{R}{2})$,

$$\nu(\mathbb{R}^{d-s} \times E) = \nu(\overline{B^{d-s}(0, R)} \times E) = \vartheta_0 \mathcal{H}^s(E), \quad (11.15)$$

we have that whenever $B^s(x'', 2r) \subset B^s(0, R/2)$,

$$u(x') = \frac{1}{\mathcal{I}_{\mathcal{H}^s}(B(x'', r))} \int_{\mathbb{R}^s} u(y'') \varphi\left(\frac{|x'' - y''|}{r}\right) d\mathcal{H}^s(y'').$$

This certainly ensures that u is a smooth function, but moreover it is harmonic. Indeed, for each $x'' \in B(0, R/2)$ we have that for small enough r ,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^s} \varphi\left(\frac{|x'' - y''|}{r}\right) [u(y'') - u(x'')] d\mathcal{H}^s(y'') \\ &= c \int_0^{2r} \int_{\mathbb{S}^{s-1}} [u(x'' + t\omega) - u(x'')] d\sigma(\omega) \varphi\left(\frac{t}{r}\right) t^s \frac{dt}{t}, \end{aligned} \quad (11.16)$$

where $d\sigma$ denotes the surface area measure on the unit s -sphere $\mathbb{S}^{s-1}$. With Δ_s denoting the Laplacian in $\mathbb{R}^s$, we infer from Taylor's formula (or the divergence theorem) that

$$\int_{\mathbb{S}^{s-1}} [u(x'' + t\omega) - u(x'')] d\sigma(\omega) = ct^2 \Delta_s u(x'') + O(t^3) \text{ as } t \rightarrow 0$$

for some constant $c > 0$. Plugging the preceding identity into (11.16) yields that $r^{s+2} |\Delta u(x'')| \leq Cr^{s+3}$ for all small r . Hence $\Delta u(x'') = 0$ for $x'' \in B(0, \frac{R}{2})$.

Since $|u(x'')| \leq R$ for every $x'' \in B^s(0, \frac{R}{2})$ (see (11.12)), standard gradient estimates yield that $|\nabla u(x'')| \leq C$ if $x'' \in B^s(0, \frac{R}{4})$. In order to prove Lemma 11.4, we shall employ the following simple estimate for harmonic functions. We introduce the notation $f_E d\mathcal{H}^s := \frac{1}{\mathcal{H}^s(E)} \int_E f d\mathcal{H}^s$.

Lemma 11.6. *If $B^s(x'', r) \subset B(0, \frac{R}{16})$, then*

$$\begin{aligned} \int_{B^s(x'', r)} |u(y'') - u(x'') - Du(x'')(y'' - x'')|^2 d\mathcal{H}^s(y'') \\ \leq C \left(\frac{r}{R}\right)^4 \int_{B^s(0, \frac{R}{2})} |u|^2 d\mathcal{H}^s. \end{aligned}$$

Proof. Note that if $y'' \in B^s(x'', r)$, then Taylor's theorem ensures that for some $z'' \in B^s(x'', r)$,

$$|u(y'') - u(x'') - Du(x'')(y'' - x'')| \leq Cr^2 |D^2u(z'')|.$$

But now since u is harmonic, from standard gradient estimates and the mean value property we obtain that

$$|D^2u(z'')| \leq \frac{C}{R^2} \sup_{B(z'', \frac{R}{4})} |u| \leq \frac{C}{R^2} \int_{B^s(x'', R/2)} |u| d\mathcal{H}^s.$$

Squaring both sides of the resulting inequality, taking the integral average over $B(x'', r)$, and using the Cauchy–Schwarz inequality, we arrive at the desired statement. $\square$

Written in terms of ν , the previous estimate, along with the property (11.15), ensures that there exist a $(d-s) \times s$ matrix A and a vector $b \in \mathbb{R}^s$ such that

$$\begin{aligned} \int_{\mathbb{R}^{d-s} \times B^s(0, 300\sqrt{d}\ell(Q_0))} |x' - Ax'' - b|^2 d\nu(x', x'') \\ \leq \left(\frac{C}{R}\right)^2 \int_{\mathbb{R}^{d-s} \times B^s(0, \frac{R}{2})} \left(\frac{|x'|}{R}\right)^2 d\nu(x', x''). \end{aligned} \quad (11.17)$$

Furthermore we have $A = \nabla u(0)$, and $b = u(0)$, and so $|b| \leq R$ and $|A| \leq C$.

Consider the function $f : \mathbb{R}^s \rightarrow \mathbb{R}$ given by $f(x'') = \varphi\left(\frac{|x''|}{100\sqrt{d}}\right)$ and fix a non-negative function $g \in \text{Lip}_0(B^{d-s}(0, 2R))$ with $g \equiv 1$ on $\overline{B^{d-s}(0, R)}$. Then from statement (2) of Lemma 11.3 we get that

$$\begin{aligned} \int_{\mathbb{R}^d} g(x') f(x'') |x' - Ax'' - b|^2 d\nu(x', x'') \\ = \lim_{k \rightarrow \infty} \frac{1}{\tilde{\beta}_k^2} \int_{\mathbb{R}^d} g\left(\frac{x'}{\tilde{\beta}_k}\right) f(x'') |x' - \tilde{\beta}_k Ax'' - \tilde{\beta}_k b|^2 d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') \end{aligned}$$

$$\geq \limsup_{k \rightarrow \infty} \frac{1}{\widetilde{\beta}_k^2} \int_{\{|x'| \leq \widetilde{\beta}_k R\}} \varphi_{25Q_0}(x) |x' - \widetilde{\beta}_k A x'' - \widetilde{\beta}_k b|^2 d(\mu_k \circ \mathcal{A}^{(k)})(x', x'').$$

(In the final line we have used the trivial observation that $f(x'') \geq \varphi_{25Q_0}(x)$ for $x = (x', x'') \in \mathbb{R}^d$.) On the other hand, using (11.10) and (11.3), statement (2) of Lemma 11.3 ensures that

$$\begin{aligned} & \int_{\mathbb{R}^{d-s} \times B^s(0, \frac{R}{2})} \left(\frac{|x'|}{R} \right)^2 d\nu(x', x'') \\ & \leq \liminf_{k \rightarrow \infty} \frac{C}{\mathcal{I}_{\mu_k}(\widetilde{Q}_0)} \int_{B(0, \frac{R}{2})} \left(\frac{|x'|}{R \widetilde{\beta}_k} \right)^2 d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') \\ & \leq \liminf_{k \rightarrow \infty} \frac{C \beta_k^2}{\widetilde{\beta}_k^2} \leq C. \end{aligned}$$

Comparing the previous two observations with (11.17), and using our bounds for A and b , we find for all sufficiently large k some s -plane $\widetilde{L}_k$ with $B(0, \frac{R}{2}) \cap \widetilde{L}_k \subset \{\text{dist}(x, L_k) \leq C \widetilde{\beta}_k R\}$, such that

$$\frac{1}{\widetilde{\beta}_k^2} \int_{\{\text{dist}(x, L_k) \leq \widetilde{\beta}_k R\}} \varphi_{25Q_0}(\mathcal{A}^{(k)}(x)) \text{dist}(x, \widetilde{L}_k)^2 d\mu_k(x) \leq \frac{C}{R^2}.$$

On the other hand, if $x \in B(0, \frac{R}{2})$ satisfies $\text{dist}(x, L_k) > \widetilde{\beta}_k R$, then certainly $\text{dist}(x, \widetilde{L}_k) \leq C \text{dist}(x, L_k)$. Whence, from (11.7), we infer that all for large enough k ,

$$\begin{aligned} & \int_{\{\text{dist}(x, L_k) > \widetilde{\beta}_k R\}} \varphi_{25Q_0}(\mathcal{A}^{(k)}(x)) \text{dist}(x, \widetilde{L}_k)^2 d\mu_k(x) \\ & \leq C \int_{B(0, \frac{R}{2}) \cap \{\text{dist}(x, L_k) > \widetilde{\beta}_k R\}} \text{dist}(x, L_k)^2 d\mu_k(x) \leq \frac{C(R)}{k} \widetilde{\beta}_k^2 \leq \frac{1}{R^2} \widetilde{\beta}_k^2. \end{aligned}$$

Notice that (11.6) ensures that $\varphi_{25Q_0}(\mathcal{A}^{(k)} \cdot) \geq \varphi_{Q_0}$. Consequently, by combining our observations, we see that for sufficiently large k ,

$$\beta_{\mu_k}(Q_0) \leq \frac{C}{R} \widetilde{\beta}_k, \quad (11.18)$$

and so Lemma 11.4 is proved.

11.7. The proof of Lemma 11.3

We now turn to proving Lemma 11.3.

Proof of Lemma 11.3. Note the identity

$$\int_{\mathbb{R}^d} f(x', x'') d(\mu_k \circ \mathcal{A}^{(k)})(x', x'') = \int_{\mathbb{R}^d} f(\tilde{\beta}_k x', x'') d\nu_k(x', x'').$$

By replacing f in this identity with $(x', x'') \mapsto f(\frac{x'}{\beta_k}, x'')$, we see that item (2) of the Lemma follows directly from the weak convergence of ν_k to ν . Fix $g \in \text{Lip}_0(B^{d-s}(0, 2R))$ satisfying $g \equiv 1$ on $\overline{B^{d-s}(0, R)}$. Because of (11.11), if $f \in \text{Lip}_0(B(0, R))$, $\|f\|_{\text{Lip}} \leq 1$, then

$$\left| \int_{\mathbb{R}^d} f(\tilde{\beta}_k x', x'') d\nu_k(x', x'') - \int_{\mathbb{R}^d} g(x', x'') f(\tilde{\beta}_k x', x'') d\nu_k(x', x'') \right| \leq \frac{C(R)}{k}.$$

But the function $(x', x'') \mapsto g(x', x'') f(\tilde{\beta}_k x', x'')$ converges to the function $(x', x'') \mapsto g(x', x'') f(0, x'')$ uniformly on $\overline{B^{d-s}(0, 2R)} \times B^s(0, R)$, and $\int_{\mathbb{R}^d} f(0, x'') d\nu(x', x'') = \int_{\mathbb{R}^d} g(x', x'') f(0, x'') d\nu(x', x'')$. Item (1) follows immediately from these two observations.

To prove item (3), we shall look to apply Lemma 5.1. For $t \in (0, \frac{R}{8})$, consider the integral I_k given by

$$\int_{S_{\tilde{\beta}_k}^{-1}(B(0, \frac{R}{2}))} \left| \int_{\mathbb{R}^d} (x' - y') \varphi\left(\frac{|\tilde{\beta}_k[x' - y'], [x'' - y'']|}{t}\right) d\nu_k(y', y'') \right|^2 d\nu_k(x', x'').$$

Notice that if we choose $f \in \text{Lip}_0(B^s(0, R))$ with $f \equiv 1$ on $B^s(0, \frac{3R}{4})$, then inserting a factor of $f(y'')f(x'')$ in the inner integral does not affect the value of the double integral. Consider the measure $d\tilde{\nu}_k(x', x'') = f(x'')d\nu_k(x', x'')$. The error introduced by replacing I_k with the integral $\tilde{I}_k$, defined by

$$\begin{aligned} \int_{S_{\tilde{\beta}_k}^{-1}(B(0, \frac{R}{2}))} \left| \int_{\mathbb{R}^d} \left[(x' - y') \varphi\left(\frac{|\tilde{\beta}_k[x' - y'], [x'' - y'']|}{t}\right) \right. \right. \\ \left. \left. \cdot g(x')g(y') \right] d\tilde{\nu}_k(y', y'') \right|^2 d\tilde{\nu}_k(x', x''), \end{aligned} \quad (11.19)$$

is bounded by a constant multiple of

$$\begin{aligned} & \int_{[S_{\tilde{\beta}_k}^{-1}(B(0, \frac{R}{2}))] \setminus [\overline{B^{d-s}(0, R)} \times B^s(0, \frac{R}{2})]} |x'|^2 d\nu_k(x', x'') \nu_k(S_{\tilde{\beta}_k}^{-1}(B(0, R)))^2 \\ & + \nu_k(S_{\tilde{\beta}_k}^{-1}(B(0, \frac{R}{2}))) \left(\int_{[S_{\tilde{\beta}_k}^{-1}(B(0, R))] \setminus [\overline{B^{d-s}(0, R)} \times B^s(0, R)]} |y'| d\nu_k(y', y'') \right)^2. \end{aligned}$$

From (11.11) we therefore infer that $\lim_{k \rightarrow \infty} |I_k - \tilde{I}_k| \leq \lim_{k \rightarrow \infty} \frac{C(R)}{k} = 0$. (Note that, from (11.6), $\nu_k(\mathcal{S}_{\tilde{\beta}_k}^{-1}(B(0, \frac{R}{2}))) \leq \mu_k(B(0, R)) \leq CR^s$.)

Observe that the function

$$\psi_k(x, y) = (x' - y')\varphi\left(\frac{|\tilde{\beta}_k(x' - y'), x'' - y''|}{t}\right)g(x')g(y')$$

converges uniformly as $k \rightarrow \infty$ to

$$\psi(x, y) = (x' - y')\varphi\left(\frac{|x'' - y''|}{t}\right)g(x')g(y'),$$

and for each $x \in \mathbb{R}^d$, $\text{supp}(\psi_k(x, \cdot)) \subset B(x, 2\sqrt{d}R)$. Clearly $\sup_k \|\psi_k\|_{\text{Lip}} < \infty$, as the $\tilde{\beta}_k$ factor can only decrease the Lipschitz norm of φ . Appealing to Lemma 5.1 with the sequence of measures $\tilde{\nu}_k$, which converge weakly to the measure $d\tilde{\nu}(x', x'') = f(x'')d\nu(x', x'')$, and $U = B^{d-s}(0, 2R) \times B^s(0, \frac{R}{2})$, we infer that $\liminf_k I_k$ is at least

$$\int_{B^{d-s}(0, 2R) \times B^s(0, \frac{R}{2})} \left| \int_{\mathbb{R}^d} (x' - y')\varphi\left(\frac{|x'' - y''|}{t}\right)g(x')g(y')d\tilde{\nu}(y', y'') \right|^2 d\tilde{\nu}(x', x''),$$

and, after recalling the basic properties of g and f , this proves (3). $\square$

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