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Coprime factors reduction of distributed nonstationary LPV systems

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ABSTRACT

This paper is on the coprime factors reduction of distributed systems formed by discrete-time, heterogeneous, nonstationary linear parameter-varying subsystems. The subsystems are represented in a linear fractional transformation framework and interconnected over arbitrary directed graphs, and the communication between the subsystems is subjected to a delay of one time-step. Two methods for constructing a contractive coprime factorisation for the full-order system are proposed. This factorisation forms an augmented system which is reducible by the structure-preserving balanced truncation method. A reduced-order contractive coprime factorisation is obtained from which the reduced-order system can be formed. A robustness theorem is also provided to interpret the error bound from coprime factors reduction in terms of robust stability of the closed-loop system. A numerical example is considered at the end of the paper.

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1. Introduction

This paper deals with the coprime factors reduction (CFR) method for distributed systems formed by discrete-time, heterogeneous, nonstationary linear parameter-varying (NSLPV) subsystems. The subsystems are represented in a linear fractional transformation (LFT) framework and interconnected over arbitrary directed graphs, and the communication between the subsystems is subjected to a delay of one time-step. Such systems are referred to as distributed NSLPV systems. The results herein complement the authors' recent work in Abou Jaoude and Farhood (2018), which generalises the structure-preserving balanced truncation (BT) and CFR methods to the class of distributed NSLPV systems. Specifically, this paper provides two methods for constructing contractive coprime factorisations (CCFs) for the class of systems of interest. The CFR algorithm is modified to account for the contractiveness of the factorisations, and the resulting reduced-order factorisation is also proved to be contractive. This paper also derives a result which allows to interpret the error bound from CFR in terms of robust stability of the closed-loop system.

A framework for describing distributed NSLPV systems is presented in Abou Jaoude and Farhood (2017c). The NSLPV models used to describe the subsystems are an extension of standard linear parameter-varying (LPV) models in the sense that the state-space matrices can have an explicit dependence on known time-varying terms, in addition to their dependence on parameters that are not known a priori but are available for measurement at each time-step. A detailed presentation of NSLPV models is given in Farhood and Dullerud (2007). The framework models the interconnections between the subsystems as states. These states are referred to as spatial states, whereas the standard states of the subsystems are referred to as the temporal states. The signals introduced by the LFT formulation are called the parameter states for ease of reference. Distributed NSLPV systems increase in size with the number of temporal, parameter, and spatial states, as well as their corresponding dimensions, which calls for model reduction techniques that reduce the size of the system and the computational complexity of the analysis and synthesis problems. Specifically, structure-preserving techniques are sought that preserve the interpretation of the states in the reduced-order system and further allow for the simplification of the interconnection and uncertainty structures of the system.

Several works have appeared that deal with the problem of structure-preserving model reduction for systems with an uncertainty structure and/or an interconnection structure; see e.g. Abou Jaoude and Farhood (2017a, 2017b, 2018), Al-Taie and Werner (2016), Beck, Doyle, and Glover (1996), Beck (2006), Farhood and Dullerud (2007), Li and Paganini (2005), Li (2014), Sandberg and Murray (2009). These works deal with either BT or CFR, and are based on the existence of structured solutions to linear matrix inequalities (LMIs). This imposed structure ensures that the developed methods are structure-preserving, but also introduces conservatism into the proposed approaches. In this direction, the works of Sootla and Anderson (2016) and Trnka, Sturk, Sandberg, Havlena, and Rehor (2013) identify classes of systems with guaranteed structured solutions to the LMIs therein.

Abou Jaoude and Farhood (2018) treat the problem of model reduction for distributed NSLPV systems. BT is applied for the model reduction of strongly stable systems, i.e. stable systems that possess structured solutions, called generalised gramians, to the generalised Lyapunov inequalities. CFR is then used to reduce strongly stabilisable and strongly detectable systems: a strongly stable coprime factorisation is constructed which is reducible via BT. However, the CFR algorithm therein does not address the issue of contractiveness of the resulting factorisation, which is the main topic of the present work.

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For standard linear time-invariant (LTI) systems, normalised coprime factorisations (NCFs) are employed in the CFR algorithm, which result in the least conservative robustness conditions when the CFR error bound is interpreted in a gap metric sense. Specifically, the CFR bound specifies how far one can proceed with model reduction while guaranteeing that a controller that stabilises the full-order system also stabilises the reduced-order system. The reader is referred to Georgiou and Smith (1990), Meyer (1990), McFarlane and Glover (1990), Vidyasagar (1984), Vinnicombe (1993) for a more detailed treatment of these topics. For systems with structural constraints, however, ensuring normalisation is difficult, and so relaxations of NCFs are pursued instead. Wood, Goddard, and Glover (1996) present CCFs as the natural extension of NCFs for the class of LPV systems, as the normalisation condition may not be satisfied for all permissible parameter trajectories; and for the same class of systems, El-Zobaidi and Jaimoukha (1998) employ CCFs in a unified approach for control synthesis and model reduction. In Beck and Doyle (1993), contractiveness and expansiveness ideas are used to extend the stability margin to behavioural systems with uncertainty. Li and Paganini (2005) treat systems where the state partitioning is to be preserved during model reduction, and propose CFR methods that use expansive or contractive factorisations. Expansive factorisations allow the extension of the robust stability margin, but result in a non-convex optimisation problem as the stability of the factorisation needs to be imposed separately. CCFs are more computationally attractive as they can be constructed from solutions to LMIs and the stability of the factorisation is automatically guaranteed. However, extending the robustness theorem for CCFs requires imposing the difficult condition of some level of expansiveness; and so, a heuristic that makes the factorisation approach normalisation is proposed instead. In a similar direction, Beck and Bendotti (1997) propose a method for finding CCFs for the class of uncertain systems. Moreover, an iterative algorithm is proposed to ensure a level of expansiveness as close to one as possible. An alternative method for computing CCFs for uncertain systems is presented in Li (2014). It is also noted therein that even for discrete-time LTI systems, applying CFR to an NCF only guarantees the contractiveness of the resulting reduced-order factorisation; and so, one may start from CCFs in the first place for discrete-time systems. Finally, CCFs for distributed linear time-varying (LTV) systems were treated in Abou Jaoude and Farhood (2017b).

The present work extends the results in Beck and Bendotti (1997), Li (2014), Abou Jaoude and Farhood (2017b) to the more general class of distributed NSLPV systems. The extension of the results is carried out transparently because of the use of the adopted framework, which models distributed NSLPV systems in a way reminiscent of standard LPV-LFT state-space systems. As compared to the first two works, the results derived here gain novel interpretations, namely, the considered systems possess an interconnection structure in addition to the uncertainty structure, and the nominal part of the system is time-varying. As compared to Abou Jaoude and Farhood (2017b), the differences are threefold: (1) distributed LTV systems are a special case of distributed NSLPV systems where the uncertainty block is fixed and known a priori; (2) the method of Section 5.2 remains novel even when specialised to distributed LTV systems; and (3) Section 6 derives an alternative robustness result to Theorem 5.4 therein. The two CFR methods proposed here are applicable to strongly stabilisable and strongly detectable systems, i.e. systems reducible by the method of Abou Jaoude and Farhood (2018). Thus, the contractiveness requirement does not introduce conservatism into CFR; however, it does induce larger computational costs.

The paper is organised as follows. Section 2 introduces the notation. Sections 3 and 4 summarise the adopted framework and the structure-preserving BT method, respectively. Section 5 details the two CFR methods based on CCFs. Section 6 gives the robustness theorem. Section 7 considers a numerical example, and the paper concludes with Section 8.

2. Notation

We denote the sets of non-negative integers, integers, and real numbers by $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. $\text{diag}(M_i)$ denotes the block-diagonal augmentation of the sequence of operators M_i. The $i \times i$ identity matrix is denoted by I_i.

Consider a directed graph with a countable set of vertices V and a set of directed edges E. $(i, j) \in E$ denotes a directed edge from $i \in V$ to $j \in V$. We assume throughout that the graph under consideration is d-regular, for some integer $d > 0$, i.e. for all $k \in V$, both the indegree and outdegree are equal to d. An arbitrary directed graph with a uniformly bounded vertex degree can be turned into a d-regular directed graph, where d is the maximum over all vertex degrees, via the addition of the necessary virtual edges and/or vertices. For a d-regular directed graph, we define the permutations, $\rho_1, \ldots, \rho_d$, of the set of vertices such that if $(i, j) \in E$, then one $e \in \{1, \ldots, d\}$ satisfies $\rho_e(i) = j$ and $\rho_e^{-1}(j) = i$. See Farhood, Di, and Dullerud (2015) for more details.

$J_1 \oplus J_2$ denotes the vector space direct sum of J_1 and J_2. Let H and F be Hilbert spaces. We denote the inner product and the norm associated with H by $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|_H$, respectively. The subscript is suppressed when H is clear from context. $\mathcal{L}(H, F)$ and $\mathcal{L}_c(H, F)$ denote the spaces of bounded linear operators and bounded linear causal operators mapping H to F, respectively. When $H = F$, we employ the simplified symbols $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$. For $X \in \mathcal{L}(H, F)$, $\|X\|$ denotes the H to F induced norm of X, and X^* denotes the adjoint of X. $X \prec 0$ means that the operator $X = X^* \in \mathcal{L}(H)$ is negative definite, i.e. $\langle x, Xx \rangle < -\mu\|x\|^2$ for some $\mu > 0$ and all non-zero $x \in H$.

Given $n : (t, k) \in \mathbb{Z} \times V \to n(t, k) \in \mathbb{N}_0$, $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ denotes the vector space of mappings $w : (t, k) \in \mathbb{Z} \times V \to w(t, k) \in \mathbb{R}^{n(t,k)}$. The Hilbert space $\ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w with finite norm $\|w\| = \sqrt{\sum_{(t,k)} w(t, k)^* w(t, k)}$. $\ell_{2e}\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w such that $\Sigma_k w(t, k)^* w(t, k) < \infty$ for each $t \in \mathbb{Z}$. We frequently use the abbreviated symbols ℓ, ℓ_2, and ℓ_{2e}.

An operator Q: $\ell_2 \to \ell_2$ is graph-diagonal if there exists a uniformly bounded sequence of matrices $Q(t, k)$ such that $(Qv)(t, k) = Q(t, k)v(t, k)$ for all $(t, k) \in \mathbb{Z} \times V$. An operator $W = [W_{ij}]$ is partitioned graph-diagonal if each constituent block W_{ij} is graph-diagonal. The mapping defined by $[\![W]\!](t, k) = [W_{ij}(t, k)]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering. We define the unitary temporal-shift operator, S_0: $\ell_2 \to \ell_2$, such that $(S_0 v)(t, k) = v(t-1, k)$ and $(S_0^* v)(t, k) = v(t+1, k)$, and the unitary spatial-shift operators, S_i: $\ell_2 \to \ell_2$ for $i = 1, \ldots, d$, such that $(S_i v)(t, k) = v(t, \rho_i^{-1}(k))$ and $(S_i^* v)(t, k) = v(t, \rho_i(k))$. We do not distinguish between the shift operators for different Hilbert spaces ℓ_2. The definitions in this paragraph extend to ℓ and ℓ_{2e}. Further details are found in Farhood et al. (2015).

Let $X \succ 0$ be a graph-diagonal operator such that $X(t, k)$ is a diagonal matrix for all $(t, k) \in \mathbb{Z} \times V$. $\phi(X)$ denotes the sum of distinct diagonal entries of X, i.e. $\phi(X)$ is the sum of the distinct diagonal entries in diag $(X(t,k))_{(t,k)\in\mathbb{Z}\times V}$. For instance, let $X(t, k)$ be equal to diag(w_1, w_1, w_2, w_2) for $(t, k) = (t_0, k_0)$, diag(w_1, w_3, w_4) for $(t, k) = (t_0, k_1)$, diag(w_3, w_4) for $(t, k) = (t_1, k_1)$, and 0 otherwise. Then, $\phi(X) = w_1 + w_2 + w_3 + w_4$. For a partitioned graph-diagonal operator $W = \text{diag}(W_i) \succ 0$, where $W_i(t, k)$ are diagonal matrices for all $(t, k) \in \mathbb{Z} \times V$, we define $\Phi(W)$ as the sum of distinct diagonal entries of W, i.e. $\Phi(W) = \phi([\![W]\!])$.

3. Operator theoretic framework

This section summarises the framework and the analysis results of Abou Jaoude and Farhood (2017c). Consider a distributed NSLPV system $\mathcal{G}_\delta$. The interconnection structure of $\mathcal{G}_\delta$ is represented using a d-regular directed graph, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$, and the interconnection from $G^{(i)}$ to $G^{(j)}$ corresponds to the directed edge $(i, j) \in E$. The subsystems are described using discrete-time NSLPV models formulated in an LFT framework. Let $t \in \mathbb{Z}$ denote the discrete time-step. Then, for all $(t, k) \in \mathbb{Z} \times V$, the state-space equations of system $\mathcal{G}_\delta$ are given by

$$
\begin{bmatrix} x_T(t+1,k) \\ \hline x_1(t+1,\rho_1(k)) \\ \vdots \\ x_d(t+1,\rho_d(k)) \\ \hline \alpha(t,k) \\ \hline y(t,k) \end{bmatrix}
= \begin{bmatrix} \bar{A}_{TT}(t,k) & \bar{A}_{TS}(t,k) & \bar{A}_{TP}(t,k) & \bar{B}_T(t,k) \\ \bar{A}_{ST}(t,k) & \bar{A}_{SS}(t,k) & \bar{A}_{SP}(t,k) & \bar{B}_S(t,k) \\ \bar{A}_{PT}(t,k) & \bar{A}_{PS}(t,k) & \bar{A}_{PP}(t,k) & \bar{B}_P(t,k) \\ \bar{C}_T(t,k) & \bar{C}_S(t,k) & \bar{C}_P(t,k) & \bar{D}(t,k) \end{bmatrix}
\times \begin{bmatrix} x_T(t,k) \\ \hline x_1(t,k) \\ \vdots \\ x_d(t,k) \\ \hline \beta(t,k) \\ \hline u(t,k) \end{bmatrix},
$$

$$
\begin{aligned}
\beta(t,k) &= \text{diag}\left(\delta_1(t,k) I_{n_1^P(t,k)}, \ldots, \delta_r(t,k) I_{n_r^P(t,k)}\right) \alpha(t,k) \\
&= \underline{\Delta}(t,k)\, \alpha(t,k).
\end{aligned} \tag{1}
$$

In (1), the state corresponding to subsystem $G^{(k)}$ is denoted by $x_T(t, k)$ and has a possibly time-varying dimension $n_T(t, k)$. Such states are referred to as the temporal states. The signals introduced by the LFT formulation are denoted by $\beta(t, k)$ and $\alpha(t, k)$. $\alpha(t, k)$ and $\beta(t, k)$ are partitioned into r vector-valued channels conformably with the partitioning of $\underline{\Delta}(t, k)$, e.g. $\alpha(t, k) = [\,\alpha_1^*(t, k)\ \alpha_2^*(t, k)\ \ldots\ \alpha_r^*(t, k)\,]^*$, where $\alpha_j(t, k)$ and $\beta_j(t, k)$ have a dimension $n_j^P(t, k)$. For simplicity, $\beta(t, k)$ and $\alpha(t, k)$ are referred to as the parameter states. For each $j = 1, \ldots, r$, $\delta_j(t, k)$ is a time-varying scalar parameter that is not known a priori, but is measurable at each time-step t. The framework allows for heterogeneous subsystems and for a local dependence of the state-space matrices on the parameters. Different subsystems can depend on different parameters, and if two subsystems are affected by the same parameters, the evolution of the parameters is assumed to be independent in each subsystem. Let r_k be the number of parameters affecting $G^{(k)}$. Then, $r = \max_{k\in V} r_k$. If $r_{k_0} < r$ for some $k_0 \in V$, then the corresponding $\delta_j(t, k_0)$ and $n_j^P(t, k_0)$ are equal to 0 for all $t \in \mathbb{Z}$ and $j = r_{k_0} + 1, \ldots, r$. The interconnections between the subsystems are modelled as spatial states. The spatial state $x_i(t, \rho_i(k))$, with dimension $n_i^S(t, \rho_i(k))$, corresponds to the edge $(k, \rho_i(k))$, i.e. the outgoing edge from vertex k along the permutation ρ_i, and the spatial state $x_i(t, k)$, with dimension $n_i^S(t, k)$, corresponds to the edge $(\rho_i^{-1}(k), k)$, i.e. the incoming edge to vertex k along the permutation ρ_i. The virtual edges are not present in the actual interconnection structure and are only added to render the directed graph d-regular, and so the corresponding spatial states are of zero dimensions for all time-steps. Due to the communication latency, the data sent by a subsystem at the current time-step reaches the target subsystem at the next time-step. Finally, the control inputs and the output measurements corresponding to subsystem $G^{(k)}$ are denoted by $u(t, k)$ and $y(t, k)$ with dimensions $n_u(t, k)$ and $n_y(t, k)$, respectively.

Figure 1 shows a distributed NSLPV system and the graph defining its interconnection structure. The dashed arrows correspond to the virtual edges. The permutations ρ_1 and ρ_2 and the spatial states are also specified. S_0 marks the communication latency between the subsystems.

The state-space matrices are known a priori and are assumed to be uniformly bounded. These matrices are partitioned conformably with the permutations $\rho_1, \ldots, \rho_d$ and the blocks of $\underline{\Delta}(t, k)$. For instance,

$$
\begin{aligned}
\bar{A}_{ST}(t,k) &= \left[A_1^{ST}(t,k)^* \quad \ldots \quad A_d^{ST}(t,k)^* \right]^*, \\
\bar{A}_{TP}(t,k) &= \left[A_1^{TP}(t,k) \quad \ldots \quad A_r^{TP}(t,k) \right], \\
\bar{A}_{SS}(t,k) &= \left[A_{ie}^{SS}(t,k) \right]_{i=1,\ldots,d;\, e=1,\ldots,d}, \\
\bar{A}_{PP}(t,k) &= \left[A_{jf}^{PP}(t,k) \right]_{j=1,\ldots,r;\, f=1,\ldots,r}.
\end{aligned}
$$

The partitions $\bar{A}_{TT}(t, k)$, $A_1^{TS}(t, k)$, and so on define graph-diagonal operators, e.g. A_{TT}, A_1^{TS}, which in turn are augmented to form partitioned graph-diagonal operators A, B, and C such that $[\![C]\!](t, k) = [\,\bar{C}_T(t, k)\ \bar{C}_S(t, k)\ \bar{C}_P(t, k)\,]$,

$$
[\![A]\!](t,k) = \begin{bmatrix} \bar{A}_{TT}(t,k) & \bar{A}_{TS}(t,k) & \bar{A}_{TP}(t,k) \\ \bar{A}_{ST}(t,k) & \bar{A}_{SS}(t,k) & \bar{A}_{SP}(t,k) \\ \bar{A}_{PT}(t,k) & \bar{A}_{PS}(t,k) & \bar{A}_{PP}(t,k) \end{bmatrix}, \text{ and}
$$

$$
[\![B]\!](t,k) = \begin{bmatrix} \bar{B}_T(t,k) \\ \bar{B}_S(t,k) \\ \bar{B}_P(t,k) \end{bmatrix}.
$$

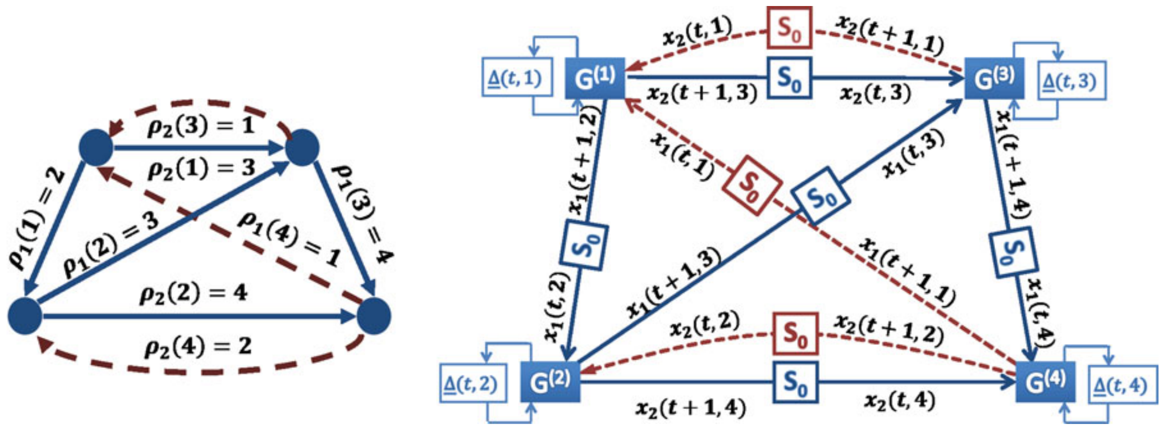

Figure 1. A distributed NSLPV system (right) and the graph defining its interconnection structure (left).

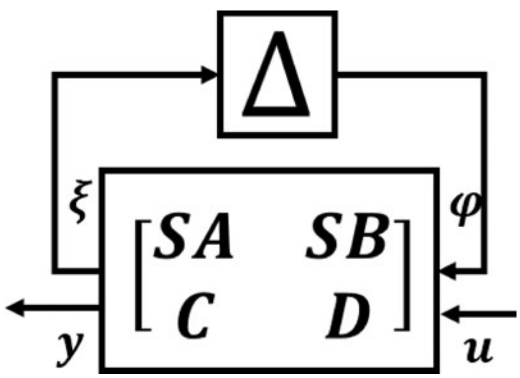

Figure 2. LFT interpretation of a distributed NSLPV system $\mathcal{G}_\delta$ with realisation $(A, B, C, D, \mathbf{\Delta})$.

The matrices $\bar{D}(t, k)$ define the graph-diagonal operator D such that $[\![D]\!](t, k) = \bar{D}(t, k)$. It is useful to construct the partitioned graph-diagonal operator A_{PP} from the terms A_{jf}^{PP} such that $[\![A_{PP}]\!](t, k) = \bar{A}_{PP}(t, k)$ for all $(t, k) \in \mathbb{Z} \times V$.

We define the partitioned graph-diagonal operator $\Delta_P = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$ such that $[\![\Delta_P]\!](t, k) = \underline{\Delta}(t, k)$, where $\Delta_j(t, k) = \delta_j(t, k) I_{n_j^P(t,k)}$ for $j = 1, \ldots, r$ and all $(t, k) \in \mathbb{Z} \times V$. We also construct the partitioned graph-diagonal operator $\Delta = \mathrm{diag}(I, \Delta_P)$, where the partitioned graph-diagonal identity operator satisfies $[\![I]\!](t, k) = I_{n_T(t,k)+n_1^S(t,k)+\cdots+n_d^S(t,k)}$. Δ belongs to $\mathbf{\Delta} = \{\Delta : \|\Delta\| \le 1\}$. We also define the composite-shift operator $S = \mathrm{diag}(S_0, S_0S_1, \ldots, S_0S_d, I)$, where the partitioned graph-diagonal identity operator satisfies $[\![I]\!](t, k) = I_{n_1^P(t,k)+\cdots+n_r^P(t,k)}$. For a fixed $\Delta \in \mathbf{\Delta}$, the Equations (1) are rewritten in compact operator form as

$$\varphi = \Delta\xi,\ \xi = SA\varphi + SBu,\ y = C\varphi + Du, \tag{2}$$

where $\varphi = [x^*\ \beta^*]^*$, $x = [x_T^*\ x_1^* \ldots x_d^*]^*$, and $\beta = [\beta_1^* \ldots \beta_r^*]^*$. Assuming that the inverse exists, the input–output map can be expressed as $G_\delta = C(I - \Delta SA)^{-1}\Delta SB + D$ for every $\Delta \in \mathbf{\Delta}$. The distributed NSLPV system is defined as $\mathcal{G}_\delta = \{G_\delta : \Delta \in \mathbf{\Delta}\}$. We denote the realisation of $\mathcal{G}_\delta$ described by (2) using the quintuple $(A, B, C, D, \mathbf{\Delta})$. A distributed NSLPV system can be interpreted as an LFT on Δ, as shown in Figure 2.

Definition 3.1: System $\mathcal{G}_\delta$ is said to be well-posed if $I - \Delta SA$ has a causal inverse on $\ell_{2e}(\{\mathbb{R}^{n_T(t,k)}\}) \oplus (\oplus_{i=1}^d \ell_{2e}(\{\mathbb{R}^{n_i^S(t,k)}\})) \oplus (\oplus_{j=1}^r \ell_{2e}(\{\mathbb{R}^{n_j^P(t,k)}\}))$ for all $\Delta \in \mathbf{\Delta}$.

If $[\![A]\!](t, k) = 0$ for all $k \in V$ and $t < 0$ and if $I - \Delta_P A_{PP}$ has a causal inverse on $\oplus_{j=1}^r \ell_{2e}(\{\mathbb{R}^{n_j^P(t,k)}\})$ for all $\Delta \in \mathbf{\Delta}$, then $\mathcal{G}_\delta$ is well-posed. Hereafter, all state-space matrices are taken as zeros for $t < 0$.

Definition 3.2: System $\mathcal{G}_\delta$ is ℓ_2-stable if $I - \Delta SA$ has a *bounded* causal inverse for all $\Delta \in \mathbf{\Delta}$.

The next result provides a sufficient condition for ℓ_2-stability; systems that satisfy this condition are referred to as strongly stable systems. Strong stability implies ℓ_2-stability, but the converse is not true in general. To give this result, we first need to make the following definitions. Let the set of transformations, $\mathcal{T}$, be defined as

$$\begin{aligned}\mathcal{T} = \Big\{&X : X = \mathrm{diag}\left(X_T, X_1^S, \ldots, X_d^S, X_1^P, \ldots, X_r^P\right), \\ &\text{where } X_T, X_i^S, X_j^P \text{ are bounded graph-diagonal} \\ &\text{operators for } i = 1, \ldots, d \text{ and } j = 1, \ldots, r, \text{ and} \\ &X^{-1} \in \mathcal{L}\big(\ell_2(\{\mathbb{R}^{n_T(t,k)}\}) \oplus \big(\oplus_i \ell_2(\{\mathbb{R}^{n_i^S(t,k)}\})\big) \\ &\oplus \big(\oplus_j \ell_2(\{\mathbb{R}^{n_j^P(t,k)}\})\big)\big)\Big\},\end{aligned}$$

and let $\mathcal{X} = \{X : X = X^* \in \mathcal{T},\ X \succ 0\}$. The sets $\mathcal{T}$ and $\mathcal{X}$ are commutants of $\mathbf{\Delta}$.

Lemma 3.1 (Abou Jaoude & Farhood, 2017c)**:** *System $\mathcal{G}_\delta$ is strongly stable if and only if there exists $X \in \mathcal{X}$ such that*

$$A^*S^*XSA - X \prec 0. \tag{3}$$

Inequality (3) can be equivalently expressed in terms of sequences of LMIs, namely, $X_T(t, k) \succ \mu I$, $X_i^S(t, k) \succ \mu I$, $X_j^P(t, k) \succ \mu I$, and $[\![A^*]\!](t, k)[\![S^*XS]\!](t, k)[\![A]\!](t, k) - [\![X]\!](t, k) \prec -\mu I$ for some scalar $\mu > 0$ and all $(t, k) \in \mathbb{Z} \times V$, where

$$[\![X]\!](t,k) = \operatorname{diag}\big(X_T(t,k), X_1^S(t,k), \ldots, X_d^S(t,k), X_1^P(t,k), \ldots, X_r^P(t,k)\big),$$
$$[\![S^*XS]\!](t,k) = \operatorname{diag}\big(X_T(t+1,k), X_1^S(t+1,\rho_1(k)), \ldots, X_d^S(t+1,\rho_d(k)), X_1^P(t,k), \ldots, X_r^P(t,k)\big).$$

Remark 3.1: Due to the explicit dependence on time in the state-space equations of the subsystems, there is an infinite sequence of LMIs associated with each subsystem $G^{(k)}$. Since the state-space matrices are zeros for $t < 0$, the sequences of LMIs are trivial for $t < 0$, and t can be restricted to $\mathbb{N}_0$. For subsystems that are (h, q)-eventually time-periodic (ETP) for some integers $h \geq 0$ and $q > 0$, i.e. for all $t, z \in \mathbb{N}_0$ and $k \in V$, the state-space matrices satisfy $[\![Z]\!](t+h+zq,k) = [\![Z]\!](t+h,k)$ for $Z = A$, B, C, D, the existence of a solution $X \in \mathcal{X}$ to (3) is equivalent to the existence of an (h, q)-ETP solution X_{eper}; see Farhood and Dullerud (2002) and Dullerud and Lall (1999). When solving for X_{eper}, t can be restricted to the finite time-horizon h and the first time-period truncation, i.e. $0 \leq t \leq h+q-1$.

This section concludes with the following result from Abou Jaoude and Farhood (2017c) and an associated optimisation problem.

Lemma 3.2 (Abou Jaoude & Farhood, 2017c)**:** *System $\mathcal{G}_\delta$ is strongly stable and satisfies $\|G_\delta\| < \gamma$ for all $\Delta \in \mathbf{\Delta}$ if there exists $X \in \mathcal{X}$ such that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} S^*XS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \quad (4)$$

Associated with this result is the following optimisation problem:

($\mathbf{P}_1$): minimise γ^2 subject to $X \in \mathcal{X}$ and (4).

4. Balanced truncation

This section summarises the BT method of Abou Jaoude and Farhood (2018).

Lemma 4.1 (Abou Jaoude & Farhood, 2018)**:** *Consider a distributed NSLPV system $\mathcal{G}_\delta$ with realisation $(A, B, C, D, \mathbf{\Delta})$. Then, the following are equivalent:*

- *System $\mathcal{G}_\delta$ is strongly stable.*
- *There exists a solution $X \in \mathcal{X}$ to*

$$AXA^* - S^*XS + BB^* \prec 0. \quad (5)$$

- *There exists a solution $Y \in \mathcal{X}$ to*

$$A^*S^*YSA - Y + C^*C \prec 0. \quad (6)$$

Inequalities (5) and (6) are called the generalised Lyapunov inequalities, and X and Y are called the generalised gramians; see Hinrichsen and Pritchard (1990) and Beck et al. (1996).

Definition 4.1: The realisation of system $\mathcal{G}_\delta$ is said to be balanced if there exists $\Sigma \in \mathcal{X}$ that simultaneously satisfies (5) and (6), where $[\![\Sigma]\!](t,k)$ is a diagonal matrix for each $(t, k) \in \mathbb{Z} \times V$. Σ is referred to as a balanced generalised gramian.

Lemma 4.2 (Abou Jaoude & Farhood, 2018)**:** *A strongly stable system $\mathcal{G}_\delta$ admits a balanced realisation.*

The proof of this result in Abou Jaoude and Farhood (2018) gives an algorithm that uses any generalised gramians X and Y to construct a balancing transformation $T \in \mathcal{T}$, a balanced generalised gramian $\Sigma = TXT^* = (T^*)^{-1}YT^{-1}$, and a balanced realisation for $\mathcal{G}_\delta$ given by $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \mathbf{\Delta})$, where $A_{\text{bal}} = (S^*TS)AT^{-1}$, $B_{\text{bal}} = (S^*TS)B$, and $C_{\text{bal}} = CT^{-1}$.

We now show how to apply the BT method to a strongly stable system $\mathcal{G}_\delta$ with balanced realisation $(A, B, C, D, \mathbf{\Delta})$ and balanced generalised gramian Σ. The truncation procedure is illustrated for Σ_T, and a similar procedure is repeated for Σ_i^S and Σ_j^P for all $i = 1, \ldots, d$ and $j = 1, \ldots, r$. For each $(t, k) \in \mathbb{Z} \times V$, $\Sigma_T(t, k)$ is an $n_T(t,k) \times n_T(t,k)$ positive definite, diagonal matrix with the entries sorted in a decreasing order. The dimensions of the temporal states are to be reduced from $n_T(t, k)$ to $m_T(t, k)$, where $0 \leq m_T(t,k) \leq n_T(t,k)$. $\Sigma_T(t, k)$ is partitioned as in $\Sigma_T(t,k) = \operatorname{diag}(\Gamma_T(t,k), \Omega_T(t,k))$, where $\Gamma_T(t,k)$ is an $m_T(t,k) \times m_T(t,k)$ matrix. $\Gamma_T(t, k)$ and $\Omega_T(t, k)$ define the graph-diagonal operators denoted by Γ_T and Ω_T, respectively. The dimensions of the spatial states and the parameter states in the reduced-order system are given by $m_i^S(t,k)$ and $m_j^P(t,k)$, respectively, where $0 \leq m_i^S(t,k) \leq n_i^S(t,k)$ and $0 \leq m_j^P(t,k) \leq n_j^P(t,k)$. Γ and Ω are then defined as $\Gamma = \operatorname{diag}\left(\Gamma_T, \Gamma_1^S, \ldots, \Gamma_d^S, \Gamma_1^P, \ldots, \Gamma_r^P\right)$ and $\Omega = \operatorname{diag}\left(\Omega_T, \Omega_1^S, \ldots, \Omega_d^S, \Omega_1^P, \ldots, \Omega_r^P\right)$.

The state-space matrices are partitioned conformably with $\Sigma_T = [\![\operatorname{diag}(\Gamma_T, \Omega_T)]\!]$, $\Sigma_i^S = [\![\operatorname{diag}(\Gamma_i^S, \Omega_i^S)]\!]$, and $\Sigma_j^P = [\![\operatorname{diag}(\Gamma_j^P, \Omega_j^P)]\!]$. For instance,

$$\bar{A}_{SS}(t,k) = \left[A_{ie}^{SS}(t,k)\right]_{i=1,\ldots,d;e=1,\ldots,d} = \left[\begin{bmatrix} \hat{A}_{ie}^{SS}(t,k) & A_{ie,12}^{SS}(t,k) \\ A_{ie,21}^{SS}(t,k) & A_{ie,22}^{SS}(t,k) \end{bmatrix}\right]_{i=1,\ldots,d;e=1,\ldots,d},$$

where $\hat{A}_{ie}^{SS}(t,k)$ is an $m_i^S(t+1,\rho_i(k)) \times m_e^S(t,k)$ matrix. The non-truncated blocks (marked with a hat) define the graph-diagonal operators that are augmented to form the reduced-order system operators A_{red}, B_{red}, and C_{red}. Let $\Delta_{\text{red}} = \operatorname{diag}(I, \hat{\Delta}_P)$, where $[\![I]\!](t,k) = I_{m_T(t,k)+m_1^S(t,k)+\cdots+m_d^S(t,k)}$, $\hat{\Delta}_P = \operatorname{diag}(\hat{\Delta}_1, \ldots, \hat{\Delta}_r)$, and the graph-diagonal operators $\hat{\Delta}_j$ satisfy $\hat{\Delta}_j(t,k) = \delta_j(t,k) I_{m_j^P(t,k)}$ for $j = 1, \ldots, r$. The realisation of the reduced-order system $\mathcal{G}_{\text{red},\delta}$ is $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \mathbf{\Delta}_{\text{red}})$, where $\mathbf{\Delta}_{\text{red}}$ is defined similarly to $\mathbf{\Delta}$ but using Δ_{red} instead of Δ.

Theorem 4.1 (Abou Jaoude & Farhood, 2018)**:** *$\mathcal{G}_{\text{red},\delta}$ is strongly stable, the realisation $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \mathbf{\Delta}_{\text{red}})$ is balanced with balanced generalised gramian Γ, and $\|(G_\delta - G_{\text{red},\delta})\| < 2\Phi(\Omega)$ for all $\Delta \in \mathbf{\Delta}$.*

The error bound can become infinitely large as the number of distinct entries in Ω increases. If the state-space matrices of $\mathcal{G}_\delta$ are (h, q)-ETP, then there exists an (h, q)-ETP balanced generalised gramian Σ_{eper}, and the realisation of $\mathcal{G}_{\text{red},\delta}$ is (h, q)-ETP. In this case, when evaluating $\Phi(\Omega_{\text{eper}})$, t is restricted to $0 \leq t \leq h+q-1$. If, in addition, V and E are finite sets, then $\Phi(\Omega_{\text{eper}})$ is guaranteed to be finite.

Remark 4.1: The balanced generalised gramian and the balanced realisation for system $\mathcal{G}_\delta$ depend on the generalised gramians X and Y employed in the balancing algorithm, as well as the balancing algorithm itself. For model reduction purposes, generalised gramians with minimum traces are usually sought; see Sandberg and Murray (2009) and Bendotti and Beck (1999). Namely, one finds the solution $X \in \mathcal{X}$ to (5) that minimises $\sum_{(t,k)}(\text{trace}\, X_T(t,k) + \sum_{i=1}^{d} \text{trace}\, X_i^S(t,k) + \sum_{j=1}^{r} \text{trace}\, X_j^P(t,k))$ and the solution $Y \in \mathcal{X}$ to (6) that minimises $\sum_{(t,k)}(\text{trace}\, Y_T(t,k) + \sum_{i=1}^{d} \text{trace}\, Y_i^S(t,k) + \sum_{j=1}^{r} \text{trace}\, Y_j^P(t,k))$. After obtaining the balanced realisation, Abou Jaoude and Farhood (2017a) propose to solve the following problem, denoted by $(\mathbf{P_2})$:

$(\mathbf{P_2})$: minimise

$$a_1 \times \epsilon + \sum_{(t,k)} \Bigg(\|\text{vec}(\Sigma_T(t,k) - \epsilon I)\|_1 + \sum_{i=1}^{d} \left\|\text{vec}(\Sigma_i^S(t,k) - \epsilon I)\right\|_1 + \sum_{j=1}^{r} \left\|\text{vec}(\Sigma_j^P(t,k) - \epsilon I)\right\|_1 \Bigg)$$

subject to $\epsilon > 0$, $\Sigma \succeq \epsilon I$, (5) and (6) expressed for the balanced realization of the system, and $X = Y = \Sigma \in \mathcal{X}$, with $[\![\Sigma]\!](t,k)$ being a diagonal matrix for all $(t,k) \in \mathbb{Z} \times V$. Here, $\text{vec}(\mathcal{Q})$ denotes the vector formed by the diagonal entries of a square matrix $\mathcal{Q}$, and $\|v\|_1$ is the 1-norm of vector v.

In $(\mathbf{P_2})$, a balanced generalised gramian is sought that simultaneously satisfies both of the generalised Lyapunov inequalities. Such a gramian is guaranteed to exist by the definition of a balanced realisation. ϵ is the truncation cut-off value, i.e. all the state variables corresponding to an entry equal to ϵ in Σ are truncated. By Theorem 4.1, $\|(G_\delta - G_{\text{red},\delta})\| < 2\epsilon$ for all $\Delta \in \mathbf{\Delta}$, and so a small ϵ is desirable as it results in a meaningful error bound. The 1-norm heuristic is used in the second term of the objective function to yield a solution Σ with many entries equal to ϵ as this increases the number of truncated state variables. Thus, the chosen cost function accounts for two competing objectives, a small error bound and a large number of truncated state variables. The design parameter a_1 is varied to determine the optimal trade-off point. An alternative log-determinant heuristic is proposed in Al-Taie and Werner (2016).

5. Coprime factors reduction

This section presents two CFR methods based on CCFs.

5.1 First method

First, the notion of strong stabilisability is defined and an equivalent convex characterisation is provided. Then, it is shown how to construct a strongly stable CCF for a strongly stabilisable and strongly detectable system $\mathcal{G}_\delta$. Using this factorisation, an augmented strongly stable system $\mathcal{H}_\delta^c$ is formed, which is reducible via BT. The reduced-order system $\mathcal{H}_{\text{red},\delta}^c$ results in a CCF from which the reduced-order system $\mathcal{G}_{\text{red},\delta}$ is constructed.

Let $\mathcal{F}$ be the set of operators $F = [\, F_T\ F_1^S\ \ldots\ F_d^S\ F_1^P\ \ldots\ F_r^P \,]$, where the blocks $F_T \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_T(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, $F_i^S \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_i^S(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, for $i = 1, \ldots, d$, and $F_j^P \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_j^P(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, for $j = 1, \ldots, r$, are graph-diagonal operators.

Definition 5.1: A well-posed system $\mathcal{G}_\delta$ is strongly stabilisable if there exists $F \in \mathcal{F}$ such that the resulting closed-loop system is strongly stable, or equivalently, if there exist $F \in \mathcal{F}$ and $P \in \mathcal{X}$ such that $(A + BF)P(A + BF)^* - S^*PS \prec 0$. Strong detectability is defined as the dual notion to strong stabilisability.

Theorem 5.1: *$\mathcal{G}_\delta$ is strongly stabilisable if and only if there exist $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ such that*

$$\begin{bmatrix} -P & (AP+BQ)^* & Q^* & (CP+DQ)^* \\ AP+BQ & -S^*PS & 0 & 0 \\ Q & 0 & -I & 0 \\ CP+DQ & 0 & 0 & -I \end{bmatrix} \prec 0. \quad (7)$$

Then, one choice for a strongly stabilising feedback operator is $F = QP^{-1}$.

Proof: From Lemma 4.1 and Definition 5.1, system $\mathcal{G}_\delta$ is strongly stabilisable if and only if there exist $Y \in \mathcal{X}$ and $F \in \mathcal{F}$ such that

$$(A + BF)^* S^* Y S (A + BF) - Y + \begin{bmatrix} C + DF \\ F \end{bmatrix}^* \begin{bmatrix} C + DF \\ F \end{bmatrix} \prec 0. \quad (8)$$

Assume $\mathcal{G}_\delta$ is strongly stabilisable and (8) holds. Pre- and post-multiplying (8) by Y^{-1}, and letting $P = Y^{-1}$ and $Q = FY^{-1}$, one retrieves

$$(AP + BQ)^* S^* P^{-1} S (AP + BQ) - P + \begin{bmatrix} CP + DQ \\ Q \end{bmatrix}^* \begin{bmatrix} CP + DQ \\ Q \end{bmatrix} \prec 0,$$

which is equivalent by the Schur complement formula to (7). Conversely, assume there exist $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ such that (7) holds. Then, applying the Schur complement formula to (7) and pre- and post-multiplying the resulting inequality by P^{-1} yield

$$-P^{-1} + (A + BQP^{-1})^* S^* P^{-1} S (A + BQP^{-1}) + (QP^{-1})^*(QP^{-1}) + (C + DQP^{-1})^*(C + DQP^{-1}) \prec 0.$$

(8) is retrieved by defining $F = QP^{-1}$ and $Y = P^{-1}$. Thus, $\mathcal{G}_\delta$ is strongly stabilisable, and F is a strongly stabilizing feedback operator. □

Lemma 5.1: *If $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ satisfy (7), then P and $Q^c \in \mathcal{F}$ satisfy (7), where*

$$Z = I + D^*D + B^*S^*P^{-1}SB = Z^*,$$
$$Q^c = -Z^{-1}(B^*S^*P^{-1}SAP + D^*CP). \quad (9)$$

Moreover, $F^c = Q^cP^{-1} = -Z^{-1}(B^*S^*P^{-1}SA + D^*C)$ strongly stabilises system $\mathcal{G}_\delta$.

Proof: By the Schur complement formula and since $P \in \mathcal{X}$, (7) is equivalent to

$$-P + (AP + BQ)^* S^* P^{-1} S(AP + BQ) \\ + Q^* Q + (CP + DQ)^* (CP + DQ) \prec 0.$$

Adding and subtracting $(PC^*D + PA^*S^*P^{-1}SB)Z^{-1}(D^*CP + B^*S^*P^{-1}SAP)$, and after some algebraic manipulations, the previous inequality can be rewritten as

$$\begin{aligned} &- P + PC^*CP + PA^*S^*P^{-1}SAP \\ &- (PC^*D + PA^*S^*P^{-1}SB)Z^{-1}(D^*CP + B^*S^*P^{-1}SAP) \\ &+ \big(Q^* + (PC^*D + PA^*S^*P^{-1}SB)Z^{-1}\big) \\ &\times Z\big(Q + Z^{-1}(D^*CP + B^*S^*P^{-1}SAP)\big) \prec 0. \end{aligned}$$

In other words, P and Q satisfy (7) if and only if they satisfy the above inequality. Since replacing Q with Q^c in the last term on the left-hand-side (LHS) of the above inequality makes that term zero, then it is not difficult to see that P and Q^c also satisfy the above inequality and, hence, satisfy (7). In addition, $F^c = Q^c P^{-1}$ strongly stabilises system $\mathcal{G}_\delta$ as per Theorem 5.1. □

Definition 5.2: Two operators N_δ and M_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ are said to be right coprime if there exist two operators U_δ and V_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ such that $U_\delta N_\delta + V_\delta M_\delta = I$. Two ℓ_2-stable distributed NSLPV systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are said to be right coprime if their input–output maps N_δ and M_δ are right coprime for all $\Delta \in \mathbf{\Delta}$.

Definition 5.3: The pair $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ of ℓ_2-stable distributed NSLPV systems is said to be a right coprime factorisation (RCF) for system $\mathcal{G}_\delta$ if $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are right coprime and, for all $\Delta \in \mathbf{\Delta}$, M_δ has a causal inverse on ℓ_{2e} and $G_\delta = N_\delta M_\delta^{-1}$. Furthermore, the RCF $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is said to be contractive if $(N_\delta^c)^* N_\delta^c + (M_\delta^c)^* M_\delta^c \preceq I$ for all $\Delta \in \mathbf{\Delta}$.

Theorem 5.2: *Given a strongly stabilisable and strongly detectable system $\mathcal{G}_\delta$, let $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ satisfy (7), and define Z and $F^c \in \mathcal{F}$ as in Lemma 5.1. Then, the pair $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ of strongly stable systems with the following realisations forms a CCF for system $\mathcal{G}_\delta$:*

$$\begin{aligned} &\mathcal{N}_\delta^c : (A + BF^c, BZ^{-1/2}, C + DF^c, DZ^{-1/2}, \mathbf{\Delta}), \\ &\mathcal{M}_\delta^c : (A + BF^c, BZ^{-1/2}, F^c, Z^{-1/2}, \mathbf{\Delta}). \end{aligned} \tag{10}$$

Proof: The proof of this theorem is twofold. First, $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is shown to be an RCF for $\mathcal{G}_\delta$. Second, this RCF is shown to be contractive. The factorisation $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is defined similarly to the RCF in Abou Jaoude and Farhood (2018) with the additional scaling factor $Z^{-1/2}$. $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are strongly stable as a consequence of Lemma 5.1. For each $\Delta \in \mathbf{\Delta}$, the input–output maps of $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are given by $N_\delta^c = (C + DF^c)(I - \Delta S(A + BF^c))^{-1}\Delta SBZ^{-1/2} + DZ^{-1/2}$ and $M_\delta^c = F^c(I - \Delta S(A + BF^c))^{-1}\Delta SBZ^{-1/2} + Z^{-1/2}$. System $\mathcal{R}_\delta$ with realisation $(A, B, -Z^{1/2}F^c, Z^{1/2}, \mathbf{\Delta})$ is the inverse system of $\mathcal{M}_\delta^c$, i.e. $R_\delta M_\delta^c = M_\delta^c R_\delta = I$ for all $\Delta \in \mathbf{\Delta}$, where $R_\delta = -Z^{1/2}F^c(I - \Delta SA)^{-1}\Delta SB + Z^{1/2}$. $\mathcal{R}_\delta$ is well-posed since $\mathcal{G}_\delta$ is well-posed, i.e. by Definition 3.1, $I - \Delta SA$ has a causal inverse for all $\Delta \in \mathbf{\Delta}$. One can then verify that $G_\delta = N_\delta^c(M_\delta^c)^{-1} = N_\delta^c R_\delta$ for all $\Delta \in \mathbf{\Delta}$. Since system $\mathcal{G}_\delta$ is strongly detectable, there exists an operator K with a structure similar to $(F^c)^*$ and appropriate dimensions such that the resulting closed-loop system is strongly stable. Using K, the strongly stable systems $\mathcal{U}_\delta$ and $\mathcal{V}_\delta$ are defined with realisations $(A + KC, K, Z^{1/2}F^c, 0, \mathbf{\Delta})$ and $(A + KC, B + KD, -Z^{1/2}F^c, Z^{1/2}, \mathbf{\Delta})$, respectively, and input–output maps $U_\delta = Z^{\frac{1}{2}}F^c(I - \Delta S(A + KC))^{-1}\Delta SK$ and $V_\delta = -Z^{\frac{1}{2}}F^c(I - \Delta S(A + KC))^{-1}\Delta S(B + KD) + Z^{\frac{1}{2}}$ for all $\Delta \in \mathbf{\Delta}$. These systems are used to show that $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are right coprime, namely, some algebraic manipulations allow to verify that $U_\delta N_\delta^c + V_\delta M_\delta^c = I$ for all $\Delta \in \mathbf{\Delta}$. Thus, $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is an RCF for system $\mathcal{G}_\delta$.

It remains to show that this RCF is contractive, i.e. $(N_\delta^c)^* N_\delta^c + (M_\delta^c)^* M_\delta^c \preceq I$ or $\|H_\delta^c\| \le 1$ for all $\Delta \in \mathbf{\Delta}$, where H_δ^c is the input–output map of the augmented system $\mathcal{H}_\delta^c = [\,(\mathcal{N}_\delta^c)^*\ (\mathcal{M}_\delta^c)^*\,]^*$. The realisation of $\mathcal{H}_\delta^c$ is denoted by $(A_H^c, B_H^c, C_H^c, D_H^c, \mathbf{\Delta})$, where $A_H^c = A + BF^c$, $B_H^c = BZ^{-\frac{1}{2}}$, $C_H^c = [(C + DF^c)^*\ (F^c)^*]^*$, and $D_H^c = [(DZ^{-\frac{1}{2}})^*\ Z^{-\frac{1}{2}}]^*$. To prove the contractiveness of the RCF, we need to show that there exists a solution in $\mathcal{X}$ to (4) for the realization of $\mathcal{H}_\delta^c$ and all $\gamma > 1$. In the following, we verify that $P^{-1} \in \mathcal{X}$ is one such solution. Specifically, we substitute P^{-1} and the state-space operators of $\mathcal{H}_\delta^c$ into the LHS of (4) and carry out the necessary algebraic operations to arrive at a 2×2 block operator. Then, we show that the diagonal blocks of this operator are negative definite and the off-diagonal ones are zeros, hence proving that P^{-1} satisfies (4) for the realization of $\mathcal{H}_\delta^c$ and all $\gamma > 1$. To start, the $(1, 1)$-block can be expressed as the LHS of (8) with $Y = P^{-1}$ and F replaced with F^c. Then, by Lemma 5.1 and following a similar argument to the one used in the proof of the 'if direction' of Theorem 5.1, we can show that the $(1, 1)$-block is negative definite. The off-diagonal blocks are zeros since

$$\begin{aligned} &\big(A_H^c\big)^* S^* P^{-1} S B_H^c + \big(C_H^c\big)^* D_H^c \\ &= \big((A + BF^c)^* S^* P^{-1} SB + (C + DF^c)^* D + (F^c)^*\big) Z^{-\frac{1}{2}} \\ &= \big(A^* S^* P^{-1} SB + C^* D + (F^c)^* (B^* S^* P^{-1} SB + D^* D + I)\big) Z^{-\frac{1}{2}} \\ &= \big(A^* S^* P^{-1} SB + C^* D + (ZF^c)^*\big) Z^{-\frac{1}{2}} \\ &= \big(A^* S^* P^{-1} SB + C^* D - A^* S^* P^{-1} SB - C^* D\big) Z^{-\frac{1}{2}} = 0. \end{aligned}$$

The $(2, 2)$-block can be written as $(B_H^c)^* S^* P^{-1} S B_H^c + (D_H^c)^* D_H^c - \gamma^2 I = (1 - \gamma^2)I$, since $(B_H^c)^* S^* P^{-1} S B_H^c + (D_H^c)^* D_H^c = Z^{-\frac{1}{2}}(B^* S^* P^{-1} SB + D^* D + I)Z^{-\frac{1}{2}} = Z^{-\frac{1}{2}} Z Z^{-\frac{1}{2}} = I$. Thus, the $(2, 2)$-block is negative definite for all $\gamma > 1$. □

Consider a strongly stabilisable and strongly detectable system $\mathcal{G}_\delta$ with realisation $(A, B, C, D, \mathbf{\Delta})$. The procedure to reduce $\mathcal{G}_\delta$ via CFR is given in the following algorithm, which modifies the algorithm of Abou Jaoude and Farhood (2018) to account for the CCF defined in Theorem 5.2. The resulting reduced-order RCF is then proved to be contractive.

Algorithm 1:

(1) Find solutions $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ to (7).
(2) Define the operators $Z = I + D^*D + B^*S^*P^{-1}SB$ and $F^c = -Z^{-1}(B^*S^*P^{-1}SA + D^*C)$.
(3) Construct a strongly stable CCF $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ for system $\mathcal{G}_\delta$ as in (10).

(4) Form the augmented strongly stable distributed NSLPV system $\mathcal{H}^c_\delta = [(\mathcal{N}^c_\delta)^* \ (\mathcal{M}^c_\delta)^*]^*$ with realisation $(A^c_H, B^c_H, C^c_H, D^c_H, \mathbf{\Delta}) = (A + BF^c, BZ^{-\frac{1}{2}}, [(C + DF^c)^* \ (F^c)^*]^*, [(DZ^{-\frac{1}{2}})^* \ Z^{-\frac{1}{2}}]^*, \mathbf{\Delta})$.

(5) Find a generalised controllability gramian $X \in \mathcal{X}$ that satisfies $A^c_H X (A^c_H)^* - S^* XS + B^c_H (B^c_H)^* = (A + BF^c)X(A + BF^c)^* - S^*XS + BZ^{-1}B^* \prec 0$.
   - Set the generalised observability gramian Y equal to P^{-1}.
   - See Remark 5.1 for choosing the objective functions.

(6) Construct a balanced realisation $(A^c_{H,\mathrm{bal}}, B^c_{H,\mathrm{bal}}, C^c_{H,\mathrm{bal}}, D^c_H, \mathbf{\Delta})$ for system $\mathcal{H}^c_\delta$.
   - Construct a balancing transformation $T \in \mathcal{T}$ and define $\Sigma = TXT^* = (T^{-1})^* P^{-1} T^{-1}$.
   - Define $A_{\mathrm{bal}} = (S^*TS)AT^{-1}$, $B_{\mathrm{bal}} = (S^*TS)B$, $C_{\mathrm{bal}} = CT^{-1}$, and $F^c_{\mathrm{bal}} = F^c T^{-1}$.
   - Define $A^c_{H,\mathrm{bal}} = (S^*TS)A^c_H T^{-1} = A_{\mathrm{bal}} + B_{\mathrm{bal}} F^c_{\mathrm{bal}}$, $B^c_{H,\mathrm{bal}} = (S^*TS)B^c_H = B_{\mathrm{bal}} Z^{-\frac{1}{2}}$, and $C^c_{H,\mathrm{bal}} = C^c_H T^{-1} = [\,(C_{\mathrm{bal}} + DF^c_{\mathrm{bal}})^* \ (F^c_{\mathrm{bal}})^*\,]^*$.

(7) Reduce this balanced realisation via the BT method of Section 4 and obtain a reduced-order system $\mathcal{H}^c_{\mathrm{red},\delta} = [(\mathcal{N}^c_{\mathrm{red},\delta})^* \ (\mathcal{M}^c_{\mathrm{red},\delta})^*]^*$.
   - Denote the realisation of the reduced-order system $\mathcal{H}^c_{\mathrm{red},\delta}$ by $(A^c_{H,\mathrm{red}}, B^c_{H,\mathrm{red}}, C^c_{H,\mathrm{red}}, D^c_H, \mathbf{\Delta}_{\mathrm{red}})$, the input–output map by $H^c_{\mathrm{red},\delta}$, and the reduced-order balanced generalised gramian by Γ.
   - Obtain an upper bound on $\|(H^c_\delta - H^c_{\mathrm{red},\delta})\|$ for all $\Delta \in \mathbf{\Delta}$ from Theorem 4.1.

(8) Define the operators A_{red}, B_{red}, C_{red}, and F^c_{red} as follows: $C^c_{H,\mathrm{red}} = [(C_{\mathrm{red}} + DF^c_{\mathrm{red}})^* \ (F^c_{\mathrm{red}})^*]^*$, $B_{\mathrm{red}} = B^c_{H,\mathrm{red}} Z^{\frac{1}{2}}$, and $A_{\mathrm{red}} = A^c_{H,\mathrm{red}} - B_{\mathrm{red}} F^c_{\mathrm{red}}$.
   - Systems $\mathcal{N}^c_{\mathrm{red},\delta}$ and $\mathcal{M}^c_{\mathrm{red},\delta}$ with realisations $(A_{\mathrm{red}}+B_{\mathrm{red}}F^c_{\mathrm{red}}, B_{\mathrm{red}}Z^{-\frac{1}{2}}, C_{\mathrm{red}}+DF^c_{\mathrm{red}}, DZ^{-\frac{1}{2}}, \mathbf{\Delta}_{\mathrm{red}})$ and $(A_{\mathrm{red}} + B_{\mathrm{red}}F^c_{\mathrm{red}}, B_{\mathrm{red}}Z^{-\frac{1}{2}}, F^c_{\mathrm{red}}, Z^{-\frac{1}{2}}, \mathbf{\Delta}_{\mathrm{red}})$, respectively, are strongly stable and right coprime.

(9) If $I - \Delta_{\mathrm{red}} S A_{\mathrm{red}}$ has a causal inverse for all $\Delta_{\mathrm{red}} \in \mathbf{\Delta}_{\mathrm{red}}$ (see Remark 5.2), then
   (a) $(A_{\mathrm{red}}, B_{\mathrm{red}}, C_{\mathrm{red}}, D, \mathbf{\Delta}_{\mathrm{red}})$ is a realisation for the reduced-order system $\mathcal{G}_{\mathrm{red},\delta}$;
   (b) $(\mathcal{N}^c_{\mathrm{red},\delta}, \mathcal{M}^c_{\mathrm{red},\delta})$ is an RCF for system $\mathcal{G}_{\mathrm{red},\delta}$; and
   (c) F^c_{red} strongly stabilises system $\mathcal{G}_{\mathrm{red},\delta}$.

Theorem 5.3: *The RCF $(\mathcal{N}^c_{\mathrm{red},\delta}, \mathcal{M}^c_{\mathrm{red},\delta})$ for $\mathcal{G}_{\mathrm{red},\delta}$ obtained in Algorithm 1 is contractive.*

Proof: As shown in the proof of Theorem 5.2, the following inequality holds for all $\gamma > 1$:

$$\begin{bmatrix} A^c_H & B^c_H \\ C^c_H & D^c_H \end{bmatrix}^* \begin{bmatrix} S^*P^{-1}S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^c_H & B^c_H \\ C^c_H & D^c_H \end{bmatrix} - \begin{bmatrix} P^{-1} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \tag{11}$$

From Abou Jaoude and Farhood (2018), there exists a partitioned graph-diagonal operator L such that $L^*L = I$, $LL^* = I$, $L^*\Sigma L = \mathrm{diag}(\Gamma, \Omega)$, $L^*SA_{\mathrm{bal}}L = \mathrm{diag}(S, S)\left[\begin{smallmatrix} A_{\mathrm{red}} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{smallmatrix}\right]$, $L^*SB_{\mathrm{bal}} = \mathrm{diag}(S, S)[\,B^*_{\mathrm{red}} \ \bar{B}^*_2\,]^*$, $C_{\mathrm{bal}}L = [\,C_{\mathrm{red}} \ \bar{C}_2\,]$, $F^c_{\mathrm{bal}}L = [\,F^c_{\mathrm{red}} \ \bar{F}_2\,]$, and $L^*\Delta L = \mathrm{diag}(\Delta_{\mathrm{red}}, \bar{\Delta}_2)$ for all $\Delta \in \mathbf{\Delta}$, where the operators $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_2$, $\bar{C}_2$, $\bar{F}_2$, and $\bar{\Delta}_2$ are appropriately defined. The following relations also hold: $L^*SA^c_{H,\mathrm{bal}}L = \left[\begin{smallmatrix} S & 0 \\ 0 & S \end{smallmatrix}\right]\left[\begin{smallmatrix} A^c_{H,\mathrm{red}} & A^c_{H,\mathrm{rem1}} \\ A^c_{H,\mathrm{rem2}} & A^c_{H,\mathrm{rem3}} \end{smallmatrix}\right]$, $L^*SB^c_{H,\mathrm{bal}} = \left[\begin{smallmatrix} S & 0 \\ 0 & S \end{smallmatrix}\right]\left[\begin{smallmatrix} B^c_{H,\mathrm{red}} \\ B^c_{H,\mathrm{rem}} \end{smallmatrix}\right]$, and $C^c_{H,\mathrm{bal}}L = [\,C^c_{H,\mathrm{red}} \ C^c_{H,\mathrm{rem}}\,]$, where the terms with subscript 'rem' are defined in the obvious way. Pre- and post-multiplying (11) by $\mathrm{diag}((T^{-1})^*, I)$ and its adjoint, respectively, one gets

$$\begin{bmatrix} SA^c_{H,\mathrm{bal}} & SB^c_{H,\mathrm{bal}} \\ C^c_{H,\mathrm{bal}} & D^c_H \end{bmatrix}^* \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA^c_{H,\mathrm{bal}} & SB^c_{H,\mathrm{bal}} \\ C^c_{H,\mathrm{bal}} & D^c_H \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0 \ \text{ for all } \gamma > 1.$$

This inequality is then pre- and post-multiplied by $\mathrm{diag}(L^*, I)$ and $\mathrm{diag}(L, I)$, respectively, to obtain

$$\begin{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} A^c_{H,\mathrm{red}} & A^c_{H,\mathrm{rem1}} \\ A^c_{H,\mathrm{rem2}} & A^c_{H,\mathrm{rem3}} \end{bmatrix} & \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} B^c_{H,\mathrm{red}} \\ B^c_{H,\mathrm{rem}} \end{bmatrix} \\ \begin{bmatrix} C^c_{H,\mathrm{red}} & C^c_{H,\mathrm{rem}} \end{bmatrix} & D^c_H \end{bmatrix}^* \times \begin{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}$$
$$\times \begin{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} A^c_{H,\mathrm{red}} & A^c_{H,\mathrm{rem1}} \\ A^c_{H,\mathrm{rem2}} & A^c_{H,\mathrm{rem3}} \end{bmatrix} & \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} B^c_{H,\mathrm{red}} \\ B^c_{H,\mathrm{rem}} \end{bmatrix} \\ \begin{bmatrix} C^c_{H,\mathrm{red}} & C^c_{H,\mathrm{rem}} \end{bmatrix} & D^c_H \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0 \ \text{ for all } \gamma > 1.$$

Thus, we conclude that

$$\begin{bmatrix} A^c_{H,\mathrm{red}} & B^c_{H,\mathrm{red}} \\ C^c_{H,\mathrm{red}} & D^c_H \end{bmatrix}^* \begin{bmatrix} S^*\Gamma S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^c_{H,\mathrm{red}} & B^c_{H,\mathrm{red}} \\ C^c_{H,\mathrm{red}} & D^c_H \end{bmatrix} - \begin{bmatrix} \Gamma & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0$$

for all $\gamma > 1$, i.e. $\|H^c_{\mathrm{red},\delta}\| \le 1$ for all $\Delta_{\mathrm{red}} \in \mathbf{\Delta}_{\mathrm{red}}$ and the RCF $(\mathcal{N}^c_{\mathrm{red},\delta}, \mathcal{M}^c_{\mathrm{red},\delta})$ is contractive. □

At this point, we briefly compare Algorithm 1 with its counterpart in Abou Jaoude and Farhood (2018). First, the operator F^c in Step 2 is always well-defined, which relaxes the assumption therein that $[\![B]\!](t, k)$ has full column rank for all $(t, k) \in \mathbb{Z} \times V$. Second, one does not need to separately solve for the generalised observability gramian Y. Instead, Y is set equal to P^{-1} in Step 5, since $(A^c_H)^* S^* Y S A^c_H - Y + (C^c_H)^* C^c_H \prec 0$ corresponds to (8) with $Y = P^{-1}$ and F replaced with F^c. Finally, when defining the operators of the reduced-order system $\mathcal{G}_{\mathrm{red},\delta}$ in Step 8, we must account for the scaling factor in our case, which is used to ensure contractiveness.

Remark 5.1: To make the derived CCF approach normalisation, the problems $(\mathbf{P_3})$ and $(\mathbf{P_4})$ are solved; see Li and Paganini (2005), Li (2014).

$(\mathbf{P_3})$: minimise $\sum_{(t,k)}(\mathrm{trace}\, U_T(t, k) + \sum_i \mathrm{trace}\, U^S_i(t, k) + \sum_j \mathrm{trace}\, U^P_j(t, k))$ subject to $P \in \mathcal{X}$, $Q \in \mathcal{F}$, $U \in \mathcal{X}$, (7), and $\left[\begin{smallmatrix} U & I \\ I & P \end{smallmatrix}\right] \succ 0$, i.e. $\left[\begin{smallmatrix} U_T(t,k) & I \\ I & P_T(t,k) \end{smallmatrix}\right] \succ \mu I$, $\left[\begin{smallmatrix} U^S_i(t,k) & I \\ I & P^S_i(t,k) \end{smallmatrix}\right] \succ \mu I$, and $\left[\begin{smallmatrix} U^P_j(t,k) & I \\ I & P^P_j(t,k) \end{smallmatrix}\right] \succ \mu I$ for some $\mu > 0$ and all $(t, k) \in \mathbb{Z} \times V$, $i = 1, ..., d$, and $j = 1, ..., r$.

($\mathbf{P_4}$): minimise $\sum_{(t,k)}(\operatorname{trace} X_T(t,k) + \sum_i \operatorname{trace} X_i^S(t,k) + \sum_j \operatorname{trace} X_j^P(t,k))$ subject to $X \in \mathcal{X}$ and $(A+BF^c)X(A+BF^c)^* - S^*XS + BZ^{-1}B^* \prec 0$.

The feasibility version of ($\mathbf{P_3}$) is defined as follows:

($\mathbf{P_5}$): Find $P \in \mathcal{X}$ and $Q \in \mathcal{F}$ subject to (7).

Problem ($\mathbf{P_3}$) introduces the largest computational burden in Algorithm 1. In the algorithm of Abou Jaoude and Farhood (2018), all the problems are of a comparable size to ($\mathbf{P_4}$). For ETP subsystems interconnected over a finite graph, ($\mathbf{P_3}$) and ($\mathbf{P_4}$) reduce to finite dimensional semi-definite programs (SDPs), and exact expressions for the computational complexity measures can be obtained by formulating the corresponding dual problems; see Abou Jaoude and Farhood (2017b). In Section 7, an example-specific comparison is given for the computational complexity of both methods of this paper and the method of Abou Jaoude and Farhood (2018). The higher computational cost of ($\mathbf{P_3}$) may become prohibitive for some systems when the SDPs from the other methods are still computationally tractable. The computational cost of ($\mathbf{P_3}$) is also significantly higher than that of its feasibility version ($\mathbf{P_5}$). This increase in computational complexity is partly due to the addition of the variable U needed to render the optimisation problem convex: U is introduced since the constraint (7) is in terms of P and the desired objective function is in terms of P^{-1}.

Remark 5.2: $I - \Delta_{\text{red}} S A_{\text{red}}$ has a causal inverse for all $\Delta_{\text{red}} \in \mathbf{\Delta}_{\text{red}}$ if the generalised gramian X found in Step 5 of Algorithm 1 also satisfies $A_{PP} \operatorname{diag}\left(X_1^P, \dots, X_r^P\right) A_{PP}^* - \operatorname{diag}\left(X_1^P, \dots, X_r^P\right) \prec 0$; see Abou Jaoude and Farhood (2018) for the details.

5.2 Second method

The method presented next builds on the CFR method of Abou Jaoude and Farhood (2018). Namely, after obtaining the RCF $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ of system $\mathcal{G}_\delta$, we solve for a scaling factor that ensures contractiveness. A slightly modified version of Algorithm 1 can then be applied to construct the reduced-order system. The desired scaling factor is shown to always exist. Beck and Bendotti (1997) discuss the methodology adopted in this section for uncertain systems.

Lemma 5.2: *If system $\mathcal{G}_\delta$ with realization $(A, B, C, D, \mathbf{\Delta})$ is strongly stable, then there exist $\gamma > 0$ and $X \in \mathcal{X}$ such that (4) holds.*

Proof: By Lemma 4.1, since $\mathcal{G}_\delta$ is strongly stable, there exists $X \in \mathcal{X}$ satisfying (6). Then, by the Schur complement formula, X satisfies (4) for some $\gamma > 0$ if and only if X satisfies

$$B^*S^*XSB + D^*D - (A^*S^*XSB + C^*D)^* \times (A^*S^*XSA + C^*C - X)^{-1}(A^*S^*XSB + C^*D) \prec \gamma^2 I. \tag{12}$$

Clearly, if the LHS of (12) is bounded, then it is possible to find a sufficiently large $\gamma > 0$ such that (12) holds. Given that the solution $X \in \mathcal{X}$ and the state-space operators are bounded, and since the sum and product of bounded operators are bounded, then it only remains to show that the inverse of $Y = -(A^*S^*XSA + C^*C - X)$ is bounded to prove the boundedness of the LHS. Since $Y \succ 0$, the perturbed version $Y \succ \epsilon I$ holds for a sufficiently small $\epsilon > 0$. By applying the Schur complement formula twice, the latter inequality is equivalent to $0 \prec Y^{-1} \prec (1/\epsilon) I$. □

The following result gives an alternative procedure for constructing a CCF for a given system $\mathcal{G}_\delta$.

Theorem 5.4: *Consider a strongly stabilisable and strongly detectable system $\mathcal{G}_\delta$ with realisation $(A, B, C, D, \mathbf{\Delta})$. Suppose that $F \in \mathcal{F}$ is a strongly stabilising feedback operator for $\mathcal{G}_\delta$. Then, there exist $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ such that*

$$\begin{bmatrix} A+BF \\ C+DF \\ F \end{bmatrix} \tilde{T} \begin{bmatrix} (A+BF)^* & (C+DF)^* & F^* \end{bmatrix} - \begin{bmatrix} S^*\tilde{T}S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} B \\ D \\ I \end{bmatrix} Z^{-1} \begin{bmatrix} B^* & D^* & I \end{bmatrix} \prec 0. \tag{13}$$

Furthermore, the pair of strongly stable systems $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ with realisations $(A+BF, BZ^{-1/2}, C+DF, DZ^{-1/2}, \mathbf{\Delta})$ and $(A+BF, BZ^{-1/2}, F, Z^{-1/2}, \mathbf{\Delta})$, respectively, defines a CCF for $\mathcal{G}_\delta$.

Proof: Since F is a strongly stabilising feedback operator for $\mathcal{G}_\delta$, the system defined by the realisation $(A+BF, B, [(C+DF)^*\ F^*]^*, [D^*\ I]^*, \mathbf{\Delta})$ is strongly stable. Then, by Lemma 5.2, there exist $X \in \mathcal{X}$ and some sufficiently large $\gamma > 0$ such that (4) holds for this realisation. Applying the Schur complement formula twice to (4), one retrieves (13) with $\tilde{T} = X^{-1} \in \mathcal{X}$ and $Z^{-1} = (1/\gamma^2) I \succ 0$. Thus, we have shown that there exist $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ satisfying (13).

The systems $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$, as defined in the theorem statement, are strongly stable. An argument similar to the one in the proof of Theorem 5.2 can then be used to show that $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is an RCF for system $\mathcal{G}_\delta$. We now prove that this RCF is contractive. Let $\mathcal{H}_\delta^c = [(\mathcal{N}_\delta^c)^*\ (\mathcal{M}_\delta^c)^*]^*$. This system is strongly stable by construction and has a realisation $(A+BF, BZ^{-\frac{1}{2}}, \left[(C+DF)^*\ F^*\right]^*, \left[(DZ^{-\frac{1}{2}})^*\ Z^{-\frac{1}{2}}\right]^*, \mathbf{\Delta})$. Then, the RCF is contractive if and only if $\|H_\delta^c\| \le 1$ for all $\Delta \in \mathbf{\Delta}$. In this case, it is possible to show that $\|H_\delta^c\| < 1$ (strict inequality) for all $\Delta \in \mathbf{\Delta}$, and hence the RCF can be shown to be strictly contractive. We will prove this claim by showing that there exists a solution in $\mathcal{X}$ to inequality (4), expressed for the realisation of $\mathcal{H}_\delta^c$, with $\gamma = 1$. Applying the Schur complement formula twice to (13), we observe that $\tilde{T}^{-1} \in \mathcal{X}$ is one such solution. □

Inequality (13) is linear in Z^{-1}; the inverse sign is retained for notational consistency. The changes in Algorithm 1 required to apply the method of this section are now outlined. In Step 1, solve for $P \in \mathcal{X}$ such that $APA^* - S^*PS - BB^* \prec 0$, and define $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$; see Abou Jaoude and Farhood (2018). This feasibility problem involves variables and constraints of the same size and structure as those of ($\mathbf{P_4}$). In Step 2, solve the problem ($\mathbf{P_6}$) defined as follows:

($\mathbf{P_6}$): Find $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ subject to (13).

In Step 3, construct the CCF according to Theorem 5.4. In Step 5, solve for the generalised observability gramian, i.e. find $Y \in \mathcal{X}$ with the minimum trace such that $(A_H^c)^*S^*YSA_H^c - Y + (C_H^c)^*C_H^c \prec 0$, and correspondingly in Step 6, define $\Sigma = TXT^* = (T^{-1})^*YT^{-1}$.

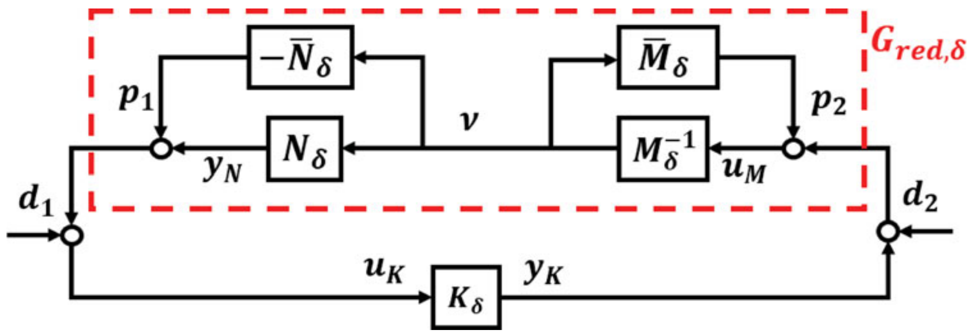

Figure 3. Standard feedback configuration with the reduced-order system represented using its RCF.

Remark 5.3: To make the CCF approach normalisation, one should seek F, $\tilde{T}$, and Z^{-1} such that the LHS of (13) is as close to zero as possible. However, as discussed before, one typical choice of F is given by $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$, and so finding the optimal solutions requires solving non-convex coupled inequalities. To circumvent this difficulty, suboptimal solutions are sought instead. Beck and Bendotti (1997) propose to iterate over the solutions until some pre-specified distance from zero is achieved. Two problems are solved per iteration, and at the end of each iteration, the system operators are updated by applying a transformation that is constructed from $\tilde{T} \in \mathcal{X}$. Such an algorithm needs to be further studied and thoroughly tested; however, this issue is not pursued here.

6. Robustness analysis

In this section, we derive a bound on the CFR error such that a controller that stabilises the full-order system also stabilises the reduced-order system obtained via CFR. Similar discussions for uncertain systems are found in Beck (2006), Li (2014). Suppose the CFR method is applied to system $\mathcal{G}_\delta$, which has a strongly stable coprime factorisation $(\mathcal{N}_\delta, \mathcal{M}_\delta)$, and a reduced-order system $\mathcal{G}_{\text{red},\delta}$ is obtained, which has a strongly stable coprime factorisation $(\mathcal{N}_{\text{red},\delta}, \mathcal{M}_{\text{red},\delta})$. Let ϵ be the error bound obtained from CFR, i.e.

$$\left\| \begin{bmatrix} N_\delta \\ M_\delta \end{bmatrix} - \begin{bmatrix} N_{\text{red},\delta} \\ M_{\text{red},\delta} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \bar{N}_\delta \\ \bar{M}_\delta \end{bmatrix} \right\| < \epsilon \quad \text{for all } \Delta \in \mathbf{\Delta}, \tag{14}$$

where $\bar{N}_\delta = N_\delta - N_{\text{red},\delta}$ and $\bar{M}_\delta = M_\delta - M_{\text{red},\delta}$ for all $\Delta \in \mathbf{\Delta}$. We seek a bound on ϵ such that if a distributed NSLPV controller $\mathcal{K}_\delta$ stabilises $\mathcal{G}_\delta$, it also stabilises the reduced-order system $\mathcal{G}_{\text{red},\delta}$.

Let $\mathcal{K}_\delta$ be a distributed NSLPV controller with realisation $(A_K, B_K, C_K, D_K, \mathbf{\Delta}_K)$ that stabilises system $\mathcal{G}_\delta$ and inherits the interconnection and uncertainty structures of $\mathcal{G}_\delta$. Such a controller can be designed using the method in Abou Jaoude and Farhood (2017c).

Figure 3 shows the standard feedback interconnection formed by the reduced-order system $\mathcal{G}_{\text{red},\delta}$, represented using its coprime factorisation, and the controller $\mathcal{K}_\delta$. The derivation of the robust stability margin relies on showing that this interconnection is equivalent to the interconnection shown in Figure 4 and then applying the small gain theorem. It is assumed that the interconnection is well-defined. We define $d_{in} = (d_1, d_2)$ and $p = (p_1, p_2)$, where d_1, d_2 are the exogenous signals and p_1, p_2 are internal signals as shown in Figure 3. The output of $\mathcal{N}_\delta$ is denoted by y_N, the input to $\mathcal{M}_\delta^{-1}$ is denoted by u_M, and the input and output to $\mathcal{K}_\delta$ are denoted by u_K and y_K, respectively. We define the signal ν as shown in Figure 3. $p = \left[\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -\bar{N}_\delta \\ \bar{M}_\delta \end{smallmatrix}\right]\nu$, and since $y_N = N_\delta\, \nu$ and $\nu = M_\delta^{-1} u_M$, i.e. $u_M = M_\delta \nu$, then, for all $\Delta \in \mathbf{\Delta}$,

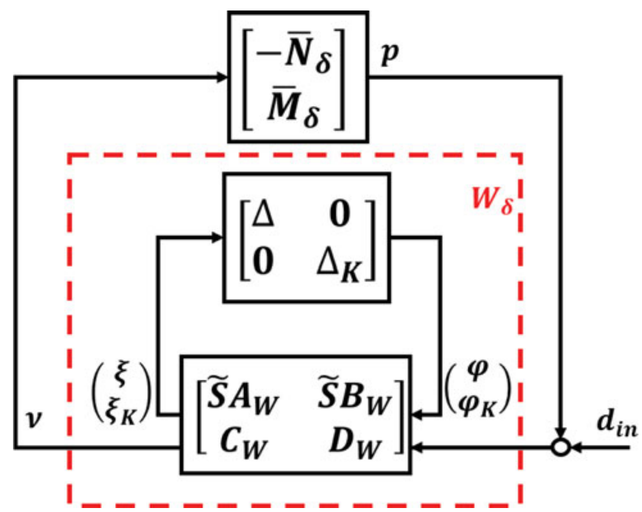

Figure 4. Equivalent interconnection to the interconnection in Figure 3.

$$\begin{aligned} &\varphi = \Delta\xi, \quad \xi = SA_N\varphi + SB_N\nu, \\ &y_N = C_N\varphi + D_N\nu, \quad u_M = C_M\varphi + D_M\nu. \end{aligned} \tag{15}$$

The subscripts in these equations are used to indicate the systems with which the operators or signals are associated. In (15), $\varphi_N = \varphi_M = \varphi$ and $\xi_N = \xi_M = \xi$ since $A_N = A_M$ and $B_N = B_M$ for the coprime factorisations used in the paper. Similarly, the controller equations are given by

$$\begin{aligned} &\varphi_K = \Delta_K\xi_K, \quad \xi_K = SA_K\varphi_K + SB_K u_K, \\ &y_K = C_K\varphi_K + D_K u_K \quad \text{for all } \Delta_K \in \mathbf{\Delta}_K. \end{aligned} \tag{16}$$

Combining (15), (16), $u_K = d_1 + y_N + p_1$, and $u_M = p_2 + d_2 + y_K$ yields

$$\begin{aligned}\begin{bmatrix}\xi\\ \xi_K\end{bmatrix} &= \left(\begin{bmatrix}SA_N & 0\\ SB_KC_N & SA_K\end{bmatrix} + \begin{bmatrix}SB_N\\ SB_KD_N\end{bmatrix}\right.\\ &\quad \left.\times (D_M - D_KD_N)^{-1}\begin{bmatrix}D_KC_N - C_M & C_K\end{bmatrix}\right)\begin{bmatrix}\varphi\\ \varphi_K\end{bmatrix}\\ &\quad + \left(\begin{bmatrix}0 & 0\\ SB_K & 0\end{bmatrix} + \begin{bmatrix}SB_N\\ SB_KD_N\end{bmatrix}(D_M - D_KD_N)^{-1}\right.\\ &\quad \left.\times \begin{bmatrix}D_K & I\end{bmatrix}\right)\begin{bmatrix}p_1 + d_1\\ p_2 + d_2\end{bmatrix},\\ \nu &= (D_M - D_KD_N)^{-1}\begin{bmatrix}D_KC_N - C_M & C_K\end{bmatrix}\begin{bmatrix}\varphi\\ \varphi_K\end{bmatrix}\\ &\quad + (D_M - D_KD_N)^{-1}\begin{bmatrix}D_K & I\end{bmatrix}(p + d_{in}),\end{aligned}$$

i.e. $\begin{bmatrix}\xi\\ \xi_K\end{bmatrix} = \tilde{S}A_W\begin{bmatrix}\varphi\\ \varphi_K\end{bmatrix} + \tilde{S}B_W(p + d_{in})$ and $\nu = C_W\begin{bmatrix}\varphi\\ \varphi_K\end{bmatrix} + D_W(p + d_{in})$, where $\tilde{S} = \mathrm{diag}(S, S)$, and A_W, B_W, C_W, and D_W are defined in the obvious way. Moreover, $\begin{bmatrix}\varphi\\ \varphi_K\end{bmatrix} = \begin{bmatrix}\Delta & 0\\ 0 & \Delta_K\end{bmatrix}\begin{bmatrix}\xi\\ \xi_K\end{bmatrix}$. Denote the distributed NSLPV system thus constructed that maps $(p + d_{in})$ to ν by $\mathcal{W}_\delta$, with input–output map W_δ for all $\Delta \in \mathbf{\Delta}$ and $\Delta_K \in \mathbf{\Delta}_K$. The preceding discussion establishes that the interconnections in Figures 3 and 4 are equivalent.

The given equations of $\mathcal{W}_\delta$ are not in the standard form of (2) and need to be rearranged to become in this form. This rearrangement is made to group together the temporal, spatial, and parameter states of the plant with their controller counterparts, since the uncertainty structures Δ_K and Δ are not independent from each other and neither are the interconnection structures. Define $\gamma_{\min}$ as the square root of the optimal value of the following optimisation problem:

$$\begin{aligned}&\text{minimise } \gamma^2 \text{ subject to } X \in \mathcal{X} \text{ and inequality (4) expressed}\\ &\text{for the realisation of } \mathcal{W}_\delta \text{ (standard form equations).}\end{aligned} \tag{17}$$

The existence of $\gamma_{\min}$ ensures that system $\mathcal{W}_\delta$ is strongly stable and $\|W_\delta\| < \gamma_{\min}$ for all permissible parameter trajectories. We are now ready to state the robustness theorem.

Theorem 6.1: *Consider a distributed NSLPV system $\mathcal{G}_\delta$, which has a strongly stable coprime factorisation $(\mathcal{N}_\delta, \mathcal{M}_\delta)$. Suppose that this system is reduced via CFR, and denote the reduced-order system by $\mathcal{G}_{\mathrm{red},\delta}$ and its strongly stable coprime factorisation by $(\mathcal{N}_{\mathrm{red},\delta}, \mathcal{M}_{\mathrm{red},\delta})$. Let ϵ be the error bound obtained from CFR and defined as in (14). In addition, suppose that $\mathcal{K}_\delta$ is a distributed NSLPV controller that renders the closed-loop system $\mathcal{W}_\delta$ defined in Figure 4 strongly stable, and define $\gamma_{\min}$ as the square root of the optimal value of the optimisation problem in (17). Then, $\mathcal{K}_\delta$ stabilises system $\mathcal{G}_{\mathrm{red},\delta}$ if $\epsilon \le \frac{1}{\gamma_{\min}}$.*

This result follows by application of the small gain theorem to the interconnected systems in Figure 4; see for instance Dullerud and Paganini (2000).

7. Numerical example

Various features of the proposed model reduction methods have been illustrated in the examples of Abou Jaoude and Farhood (2017a, 2017b, 2018), e.g. the simplification of the interconnection structure through the removal of whole interconnections in the reduced-order system (and similarly for the uncertainty structure), the truncation of different numbers of state variables at different time instants, the truncation of different types of state variables, and for distributed NSLPV systems, the need to separately impose/verify the well-posedness of the reduced-order system obtained from CFR as per Remark 5.2. Such features will thus not be stressed here. The focus is placed instead on computational complexity issues; and areas that need further investigation and testing before making final/decisive conclusions are pointed out.

Consider a distributed NSLPV system $\mathcal{G}_\delta$ formed by four subsystems interconnected as in Figure 1, where $G^{(1)}$ is an $(h = 0,\ q = 28)$-ETP, discrete-time LTV subsystem, and $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$ are discrete-time LPV subsystems with $r_k = r = 1$ for $k = 2, 3, 4$. The dimensions of the states are constant for all $t = 0, \ldots, h + q - 1$ and are given by $n_T(t, k) = 6$, $n_i^S(t, k) = 3$ for $k = 1, \ldots, 4$ and $i = 1, 2$, and $n_j^P(t, k) = 4$ for $k = 2, 3, 4$ and $j = 1$. The detailed construction of the state-space matrices is given in Abou Jaoude and Farhood (2018) and is omitted here for space considerations.

System $\mathcal{G}_\delta$ is not strongly stable and is not reducible via BT, but it is strongly stabilisable and strongly detectable and so is reducible via CFR. We apply both Methods 1 and 2 of this paper and the method of Abou Jaoude and Farhood (2018) for comparison. In all methods, after solving for the balanced realisation of the augmented system $\mathcal{H}_\delta$ ($\mathcal{H}_\delta^c$), we re-solve the generalised Lyapunov inequalities for a balanced generalised gramian Σ as per problem ($\mathbf{P_2}$). We also solve ($\mathbf{P_1}$) to find the minimum upper bound $\gamma_{\min}$ on $\|H_\delta\|$ ($\|H_\delta^c\|$) for all $\Delta \in \mathbf{\Delta}$. This bound helps, although not solely conclusive, in deciding on how many state variables to truncate. In general, guidelines need to be developed for determining what constitutes a good reduced-order model. Specifically, for open-loop systems, e.g. ones that cannot be reduced via BT, and since the CFR error bound has closed-loop robust stability interpretations, looking at the CFR error bound alone is not sufficient for determining the quality of the reduced-order model. This observation applies to various classes of systems and not only to distributed NSLPV systems. In a similar direction, it is observed that, for comparable reduction and reduced-order system behaviour, the error bounds obtained by applying Method 1 are generally more conservative than those obtained from using the other methods. For instance, we obtain $\gamma_{\min} = 0.99$ and a truncation cut-off value $\epsilon = 0.019$ (setting $a_1 = 740$ in the objective function of ($\mathbf{P_2}$)) using Method 1, i.e. the CFR error bound is $3.78\%\gamma_{\min}$. For a comparable open-loop behaviour of the reduced-order system and a similar total number of truncated state variables, Method 2 gives $\epsilon = 0.006$, i.e. an error bound equal to $2.3424\%\gamma_{\min}$ with $\gamma_{\min} = 0.5317$. Further testing is needed to substantiate and interpret these observations.

Since the state-space matrices are (h, q)-ETP, then (h, q)-ETP solutions are sought for all the SDPs solved in this example. The SDPs in question are modelled using Yalmip

Table 1 Computational complexity measures and solution times for the various SDPs considered in the example.

Problem number	Dimension of SDP variable	Dimension of linear variable	Number of constraints	Number of SDP blocks	CPU time (s)	Wall clock time (s)
($\mathbf{P_1}$)	3080	1	4033	448	8	12
($\mathbf{P_2}$)	4284	3193	3193	560	3.5	18
($\mathbf{P_3}$)	6160	0	10,920	448	973	978
($\mathbf{P_4}$)	2856	0	4032	448	3	6
($\mathbf{P_5}$)	4732	0	6888	448	206	210
($\mathbf{P_6}$)	3528	0	4368	560	5.5	12

and are solved using SDPT3; see Lofberg (2004), Tutuncu, Toh, and Todd (2003). The computations are carried out in Matlab (R2017*b*) 9.3.0.713579 (The MathWorks Inc., Natick, Massachusetts, U.S.A.) on a Dell desktop with i7 − 7700 Intel Quad Core, 3.60 GHz processors, and 16 GB of RAM running Windows 10 Pro. The measures of computational complexity and the time needed to solve the SDPs are given in Table 1. ($\mathbf{P_5}$) is also solved for comparison with ($\mathbf{P_3}$). Note that the method of Abou Jaoude and Farhood (2018) only involves solving problems of comparable sizes to ($\mathbf{P_4}$).

8. Conclusion

This paper gives two CFR methods for distributed NSLPV systems based on CCFs. A robustness theorem is provided, which indicates how far one can proceed with truncation using CFR while ensuring that a stabilizing controller for the full-order system is also stabilizing for the reduced-order system. The paper reveals various issues that provide directions for future research, e.g. the connections of the robust stability margin to the gap and graph metrics, the properties of the iterative algorithm for approximate normalisation using Method 2, the conservatism of the error bound obtained by applying Method 1, and the need for general CFR truncation guidelines.

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