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Model reduction of distributed nonstationary LPV systems

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ABSTRACT

This paper is on the structure-preserving model reduction of distributed systems formed by heterogeneous, discrete-time, nonstationary linear parameter-varying subsystems interconnected over arbitrary directed graphs. The subsystems are formulated in a linear fractional transformation (LFT) framework, and a communication latency of one sampling period is considered. The balanced truncation method is extended to the class of systems of interest, and upper bounds on the ℓ_2-induced norm of the resulting error system are derived. Balanced truncation suffers from conservatism since it only applies to stable systems which possess structured solutions to the generalized Lyapunov inequalities. The coprime factors reduction method is then provided as a partial remedy to this conservatism. An illustrative example is given to demonstrate the efficacy of the proposed approaches.

1. Introduction

This work is on the model reduction of distributed systems formed by heterogeneous, discrete-time, nonstationary linear parameter-varying (NSLPV) subsystems interconnected over arbitrary directed graphs. The subsystems are formulated in a linear fractional transformation (LFT) framework. It is assumed that the information sent by a subsystem at the current time-step is received by the target subsystem at the next time-step. We refer to such systems as distributed NSLPV systems.

NSLPV models [10,11] extend standard/stationary linear parameter-varying (LPV) models in the sense that the state-space matrix-valued functions can have an explicit dependence on a priori known time-varying terms, in addition to their dependence on time-varying scheduling parameters that are not known a priori, but are available for measurement at each time-step. The dependence of the state-space matrices on these parameters is assumed to be rational so as to allow for formulating the subsystems in an LFT framework. This assumption, however, is not generally restrictive as nonlinear functions that are not rational can frequently be approximated by rational ones. An NSLPV model formulated in an LFT framework is basically an interconnection of a nominal linear time-varying (LTV) model and a Δ-operator which consists of all the scheduling parameters. As noted in [14], the analysis results for NSLPV models are effectively tools for robustness analysis of LTV systems against static time-varying uncertainties. NSLPV models arise, for example, when controlling nonlinear systems about prespecified trajectories as a means for capturing the effects of the system nonlinearities while facilitating control design using linear techniques. In such scenarios, and generally whenever a priori known time-varying terms appear in the system equations, NSLPV models usually constitute far less conservative representations of the time-varying nonlinear system dynamics than their stationary LPV counterparts, and in some cases, the only stabilizable parameter-varying models that can be obtained are NSLPV [11].

Distributed NSLPV systems consist of NSLPV subsystems and/or combinations of LTV and LPV subsystems. The control synthesis problem for distributed NSLPV systems is treated in [1], where the systems are compactly described using an operator theoretic framework. This framework models the interconnections between the subsystems as states referred to as spatial states, in addition to the standard states of the subsystems that are referred to as temporal states, and the parameter states which result from the LFT formulation. The size of these systems, and consequently, the sizes of the analysis and synthesis problems, grow with the number and dimension of the temporal, spatial, and parameter states. This calls for the extension of structure-preserving model reduction techniques, namely, balanced truncation (BT) and coprime factors reduction (CFR), to the class of distributed NSLPV systems. Specifically, a reduced-order model is sought that

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approximates the behavior of the full-order system and preserves the interconnection structure and the structure of the Δ-operator.

BT applies to strongly stable systems, i.e., stable systems which possess structured solutions to the generalized Lyapunov inequalities. These solutions, when existent, are called generalized gramians [6]. BT guarantees the strong stability of the reduced-order system as well as an upper bound on the ℓ_2-induced norm of the error system. The proposed method allows for reducing the dimension of each temporal, spatial, and parameter state, individually. If the dimension of a spatial state is reduced to zero for all time-steps, then the corresponding interconnection is removed from the interconnection structure of the reduced-order system. Similarly, if the dimension of a parameter state is reduced to zero for all time-steps, then the corresponding channel is removed from the Δ-operator. However, there exist stable systems which do not possess generalized gramians and cannot be reduced via BT. The reader is referred to [24] and the references therein for a discussion on the existence of structured solutions to Lyapunov inequalities. This conservatism is partially remedied by the CFR method as it applies to systems which, while not necessarily strongly stable, can be represented using a strongly stable coprime factorization. A strongly stabilizable and strongly detectable system possesses the needed factorization and is reducible via CFR. The reduced-order model resulting from CFR is guaranteed to be strongly stabilizable and strongly detectable, and the corresponding error bound can be interpreted in terms of the robust stability of the closed-loop system.

Several works have appeared that treat the problem of structure-preserving BT or CFR for uncertain and LPV systems [7,16], NSLPV systems [10], and interconnected systems [2,3,5,17,21,26]. This paper can be viewed as a generalization of the work in [10] to the class of distributed systems or as a generalization of the works in [2,3] to the case where the subsystems are represented by NSLPV, instead of LTV, models. In particular, while [2,3] deal with systems with an interconnection structure and [10] deals with systems with an uncertainty structure, the present work deals with systems having an interconnection and an uncertainty structure, both of which are to be preserved/simplified during model reduction. The adopted operator theoretic framework gives the equations for distributed NSLPV systems in a compact operator form reminiscent of standard LPV-LFT state-space systems, which allows for a transparent extension of standard model reduction results to the class of systems treated here while addressing intricacies due to more elaborate operator machinery. Since the interconnections between the subsystems are modeled as spatial states, the extended results acquire new characteristics and interpretations. For instance, the generalized gramians are classified into temporal terms, spatial terms, and parameter terms, and the proposed methods allow for the truncation of the three types of states. In other words, the methods allow for reducing the order of the subsystems via truncation of temporal and parameter states, as well as simplifying the interconnection structure via truncation of spatial states. We note that the given results also remain novel when the subsystems have standard LPV models, i.e., when the nominal part of the subsystems is linear time-invariant. The novelty comes from considering heterogeneous subsystems and arbitrary directed graphs and from accounting for communication latency on the information transfer between the subsystems.

An important contribution of this work is Theorem 3, which remains novel when restricted to the class of distributed LTV systems treated in [2,3]. Theorem 3 gives an alternative expression for the BT error bound which can be less conservative than the standard “twice the sum of distinct truncated entries in the balanced generalized gramian” error bound given in Theorem 2. The importance of this theorem is articulated in robust stability analysis [20], where the full-order system can be replaced by the reduced-order system and a bounded perturbation operator whose norm is less than the error bound. Deriving tighter error bounds thus allows for a better quantification of the said operator and consequently for less conservative robustness results. Another contribution resides in the extension of the notion of a coprime factorization to the class of distributed NSLPV systems. This notion is essential for the development of the CFR method and further is of independent interest as it finds applications in robust control theory [27]. Finally, the paper gives an example which illustrates the application and characteristics of the proposed methods. Specifically, the example highlights the flexibility of our methods in truncating various types of states and shows that truncation need not be uniform in time even if the dimensions of the states in the full-order system are constants. The example uses the 1-norm heuristic to improve on the computed error bound by maximizing the number of entries in the balanced generalized gramian which are equal to each other and to some truncation cut-off value [2]. Then, when applying Theorem 2, all the corresponding state variables (regardless of type) are truncated and accounted for only once in the error bound.

The paper is organized as follows. In Section 2, we introduce the notation of the paper. In Section 3, we present the framework and the relevant analysis results of [1]. Section 4 treats the BT method, and Section 5 treats the CFR method. A numerical example is given in Section 6. The paper concludes with Section 7.

2. Notation

$\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of nonnegative integers, integers, and real numbers, respectively. $\mathrm{diag}(M_i)$ denotes the block-diagonal augmentation of the sequence of operators M_i. $0_{i\times j}$ denotes an $i\times j$ zero matrix, and I_i denotes the $i\times i$ identity matrix.

Consider a directed graph with a countable set of vertices V and a set of directed edges E. $(i, j)\in E$ denotes a directed edge from $i\in V$ to $j\in V$. The graph under consideration is assumed to be d-regular, for some integer $d>0$, i.e., for all $k\in V$, both the indegree and outdegree are equal to d. Note that any arbitrary directed graph with a uniformly bounded vertex degree can be turned into a d-regular directed graph, where d is the maximum over all vertex degrees, via the addition of the necessary virtual edges and/or vertices. This assumption allows for the definition of d permutations, $\rho_1,\dots,\rho_d$, of the set of vertices according to the edges. The permutations are chosen such that if $(i, j)\in E$, then one $e\in\{1,\dots,d\}$ satisfies $\rho_e(i)=j$ and $\rho_e^{-1}(j)=i$. See [12] for more details.

Let J_1 and J_2 be vector spaces, and let H and F be Hilbert spaces. $J_1\oplus J_2$ denotes the vector space direct sum of J_1 and J_2. The inner product and the norm associated with H are denoted by $\langle\cdot,\ \cdot\rangle_H$ and $\|\cdot\|_H$, respectively. The subscript is dropped when H is clear from context. $\mathcal{L}(H,F)$ and $\mathcal{L}_c(H,F)$ denote the spaces of bounded linear operators and bounded linear causal operators mapping H to F, respectively. These symbols simplify to $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$ when $H=F$. Let $X\in\mathcal{L}(H,F)$. $\|X\|$ denotes the H to F induced norm of X, and X^* denotes the adjoint of X. A self-adjoint operator $X\in\mathcal{L}(H)$ is said to be negative definite ($X\prec 0$) if there exists $\alpha>0$ such that $\langle x, Xx\rangle<-\alpha\|x\|^2$ for all nonzero $x\in H$.

Given an integer sequence $n:(t,k)\in\mathbb{Z}\times V\to n(t,k)\in\mathbb{N}_0$, $\ell(\{\mathbb{R}^{n(t,k)}\})$ denotes the vector space of mappings $w:(t,k)\in\mathbb{Z}\times V\to w(t,k)\in\mathbb{R}^{n(t,k)}$. The Hilbert space $\ell_2(\{\mathbb{R}^{n(t,k)}\})$ is the subspace of $\ell(\{\mathbb{R}^{n(t,k)}\})$ which consists of mappings w that have a finite norm $\|w\|=\sqrt{\sum_{(t,k)} w(t,k)^* w(t,k)}$. $\ell_{2e}(\{\mathbb{R}^{n(t,k)}\})$ is the subspace of $\ell(\{\mathbb{R}^{n(t,k)}\})$ consisting of mappings w such that $\sum_k w(t,k)^* w(t,k)<\infty$ for each $t\in\mathbb{Z}$. The abbreviated symbols ℓ, ℓ_2, and ℓ_{2e} are frequently used when $n(t, k)$ is clear from context.

We now summarize some of the operator machinery of [12]. An operator Q: $\ell_2\to\ell_2$ is said to be graph-diagonal if there exists a uniformly bounded sequence of matrices $Q(t, k)$ such that

$(Qv)(t,k)=Q(t,k)v(t,k)$ for all $(t,k)\in\mathbb{Z}\times V$. An operator $W=[W_{ij}]$ is said to be partitioned graph-diagonal if each block W_{ij} is a graph-diagonal operator. The mapping $[\![W]\!](t,k)=[W_{ij}(t,k)]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering, i.e., $\|W_1\|=\|[\![W_1]\!]\|$, $[\![W_1W_2]\!]=[\![W_1]\!][\![W_2]\!]$, $[\![W_1+W_2]\!]=[\![W_1]\!]+[\![W_2]\!]$, where W_1 and W_2 are compatible partitioned graph-diagonal operators. If W_1 is self-adjoint, then $W_1\succ 0$ if and only if $[\![W_1]\!]\succ 0$ if and only if $[\![W_1]\!](t,k)\succ\beta I$ for all $(t,k)\in\mathbb{Z}\times V$ and some scalar $\beta>0$. The unitary temporal-shift operator, S_0: $\ell_2\to\ell_2$, and the unitary spatial-shift operators, S_i: $\ell_2\to\ell_2$ for $i=1,\dots,d$, are defined as follows: $(S_0v)(t,k)=v(t-1,k)$, $(S_0^*v)(t,k)=v(t+1,k)$, $(S_iv)(t,k)=v(t,\rho_i^{-1}(k))$, and $(S_i^*v)(t,k)=v(t,\rho_i(k))$. We do not distinguish between the shift operators for different Hilbert spaces ℓ_2. The definitions of graph-diagonal operators and of the shift operators naturally extend to ℓ and ℓ_{2e}.

Let $X\succ 0$ be a graph-diagonal operator where, for all $(t,k)\in\mathbb{Z}\times V$, $X(t,\ k)$ is a diagonal matrix. $\phi(X)$ denotes the sum of distinct diagonal entries of X, i.e., $\phi(X)$ is the sum of the distinct diagonal entries in $\mathrm{diag}(X(t,k))_{(t,k)\in\mathbb{Z}\times V}$. For example, assume that $X(t,k)=0$ for all $(t,\ k)$ except for some $(t_0,\ k_0)$, $(t_0,\ k_1)$, and $(t_1,\ k_1)$, where $X(t_0,k_0)=\mathrm{diag}(w_1,w_1,w_2,w_2)$, $X(t_0,k_1)=\mathrm{diag}(w_1,w_3,w_4)$, and $X(t_1,k_1)=\mathrm{diag}(w_3,w_4)$. Then, $\phi(X)=w_1+w_2+w_3+w_4$. Consider a partitioned graph-diagonal operator $W=\mathrm{diag}(W_i)\succ 0$, where W_i are graph-diagonal operators and, for all $(t,k)\in\mathbb{Z}\times V$, $W_i(t,\ k)$ are diagonal matrices. $\Phi(W)$ denotes the sum of distinct diagonal entries of W, i.e., $\Phi(W)=\phi([\![W]\!])$.

3. Operator theoretic framework

Consider a distributed NSLPV system $\mathcal{G}_\delta$. We represent the interconnection structure of $\mathcal{G}_\delta$ using a d-regular directed graph: each subsystem $G^{(k)}$ in $\mathcal{G}_\delta$ corresponds to a vertex $k\in V$, and the interconnection from $G^{(i)}$ to $G^{(j)}$ corresponds to the directed edge $(i,\ j)\in E$. The dynamics of each $G^{(k)}$ are described by a discrete-time NSLPV model formulated in an LFT framework. The standard states of $G^{(k)}$ are denoted by $x_T(t,\ k)$, where $t\in\mathbb{Z}$ is the discrete time-step. The signals introduced by the LFT formulation are denoted by $\beta(t,\ k)$ and $\alpha(t,\ k)$. $x_T(t,\ k)$ are referred to as the temporal states, and $\beta(t,\ k)$ and $\alpha(t,\ k)$ are referred to as the parameter states for ease of reference. The possibly time-varying dimension of $x_T(t,\ k)$ is denoted by $n_T(t,\ k)$. The parameter states satisfy $\beta(t,k)=\underline{\Delta}(t,k)\alpha(t,k)$, where $\underline{\Delta}(t,k)=\mathrm{diag}(\delta_1(t,k)I_{n_1^P(t,k)},\dots,\delta_r(t,k)I_{n_r^P(t,k)})$. $\delta_j(t,\ k)$, for $j=1,\dots,r$, are time-varying scalar parameters that are not known a priori, but are assumed to be measurable at each discrete time-step t. The parameter states are partitioned into r vector-valued channels conformably with the partitioning of $\underline{\Delta}(t,\ k)$, i.e., $\alpha(t,k)=[\alpha_1^*(t,k)\ \ \alpha_2^*(t,k)\ \ \dots\ \ \alpha_r^*(t,k)]^*$ and $\beta(t,k)=[\beta_1^*(t,k)\ \ \beta_2^*(t,k)\ \ \dots\ \ \beta_r^*(t,k)]^*$, where $\alpha_j(t,\ k)$ and $\beta_j(t,\ k)$ share the dimension $n_j^P(t,k)$. The formulation allows for a local dependence of the state-space matrices on the parameters: different subsystems may depend on different parameters; and if two subsystems are affected by the same parameters, the evolution of the parameters is assumed to be independent in each subsystem. Denote by r_k the number of parameters that affect $G^{(k)}$. Then, $r=\max_{k\in V}r_k$. If $r_{k_0}<r$ for some $k_0\in V$, then $\delta_j(t,k_0)=0$ and $n_j^P(t,k_0)=0$ for all $t\in\mathbb{Z}$ and $j=r_{k_0}+1,\dots,r$. Each subsystem has its own actuating and sensing capabilities. The control inputs and the output measurements of $G^{(k)}$ are denoted by $u(t,\ k)$ and $y(t,\ k)$, respectively, and their corresponding dimensions are given by $n_u(t,\ k)$ and $n_y(t,\ k)$.

The interconnections between the subsystems are modeled as spatial states. The spatial state $x_i(t,\ \rho_i(k))$ is associated with the edge $(k,\ \rho_i(k))$, i.e., the outgoing edge from vertex k along permutation ρ_i. The dimension of $x_i(t,\ \rho_i(k))$ is denoted by $n_i^S(t,\rho_i(k))$. Similarly, the spatial state $x_i(t,\ k)$ with dimension $n_i^S(t,k)$ corresponds to the edge $(\rho_i^{-1}(k),k)$, i.e., the incoming edge to vertex k along permutation ρ_i. The spatial states associated with the virtual edges are of zero dimensions for all time-steps since the virtual edges are not present in the actual interconnection structure and are only added to render the directed graph d-regular. Due to the communication latency, the data sent by a subsystem at the current time-step reaches the target subsystem at the next time-step. Then, for all $(t,k)\in\mathbb{Z}\times V$, the state-space equations of system $\mathcal{G}_\delta$ are given by

$$\begin{bmatrix} x_T(t+1,k) \\ \hline x_1(t+1,\rho_1(k)) \\ \vdots \\ x_d(t+1,\rho_d(k)) \\ \hline \alpha(t,k) \\ \hline y(t,k) \end{bmatrix} = \begin{bmatrix} \bar{A}_{TT}(t,k) & \bar{A}_{TS}(t,k) & \bar{A}_{TP}(t,k) & \bar{B}_T(t,k) \\ \bar{A}_{ST}(t,k) & \bar{A}_{SS}(t,k) & \bar{A}_{SP}(t,k) & \bar{B}_S(t,k) \\ \bar{A}_{PT}(t,k) & \bar{A}_{PS}(t,k) & \bar{A}_{PP}(t,k) & \bar{B}_P(t,k) \\ \bar{C}_T(t,k) & \bar{C}_S(t,k) & \bar{C}_P(t,k) & \bar{D}(t,k) \end{bmatrix} \begin{bmatrix} x_T(t,k) \\ \hline x_1(t,k) \\ \vdots \\ x_d(t,k) \\ \hline \beta(t,k) \\ \hline u(t,k) \end{bmatrix},$$

$$\beta(t,k)=\mathrm{diag}\big(\delta_1(t,k)I_{n_1^P(t,k)},\dots,\delta_r(t,k)I_{n_r^P(t,k)}\big)\alpha(t,k) = \underline{\Delta}(t,k)\ \alpha(t,k). \tag{1}$$

Fig. 1 shows a distributed NSLPV system and the graph defining its interconnection structure. The dashed red arrows correspond to the virtual edges added to render the graph 2-regular. The permutations and the spatial states are specified in the figure. The operator S_0 marks the communication latency.

The state-space matrices are known a priori, are assumed to be uniformly bounded, and are partitioned conformably with the permutations and the blocks of $\underline{\Delta}(t,k)$:

$$\bar{A}_{ST}(t,k)=\begin{bmatrix} A_1^{ST}(t,k) \\ \vdots \\ A_d^{ST}(t,k) \end{bmatrix},\quad \bar{A}_{PT}(t,k)=\begin{bmatrix} A_1^{PT}(t,k) \\ \vdots \\ A_r^{PT}(t,k) \end{bmatrix},$$

$$\bar{B}_S(t,k)=\begin{bmatrix} B_1^S(t,k) \\ \vdots \\ B_d^S(t,k) \end{bmatrix},\quad \bar{B}_P(t,k)=\begin{bmatrix} B_1^P(t,k) \\ \vdots \\ B_r^P(t,k) \end{bmatrix},$$

$$\bar{C}_S(t,k)=\big[C_1^S(t,k)\quad\cdots\quad C_d^S(t,k)\big],$$

$$\bar{C}_P(t,k)=\big[C_1^P(t,k)\quad\cdots\quad C_r^P(t,k)\big],$$

$$\bar{A}_{TS}(t,k)=\big[A_1^{TS}(t,k)\quad\cdots\quad A_d^{TS}(t,k)\big],$$

$$\bar{A}_{TP}(t,k)=\big[A_1^{TP}(t,k)\quad\cdots\quad A_r^{TP}(t,k)\big],$$

$$\bar{A}_{SS}(t,k)=\big[A_{ie}^{SS}(t,k)\big]_{i=1,\dots,d;e=1,\dots,d},$$

$$\bar{A}_{SP}(t,k)=\big[A_{ij}^{SP}(t,k)\big]_{i=1,\dots,d;j=1,\dots,r},$$

$$\bar{A}_{PS}(t,k)=\big[A_{ji}^{PS}(t,k)\big]_{j=1,\dots,r;i=1,\dots,d},$$

$$\bar{A}_{PP}(t,k)=\big[A_{jf}^{PP}(t,k)\big]_{j=1,\dots,r;f=1,\dots,r}.$$

The partitions $\bar{A}_{TT}(t,k)$, $A_1^{TS}(t,k)$, and so on define graph-diagonal operators, e.g., A_{TT}, A_1^{TS}, which in turn, when augmented in the obvious way, form partitioned graph-diagonal operators A, B, and C

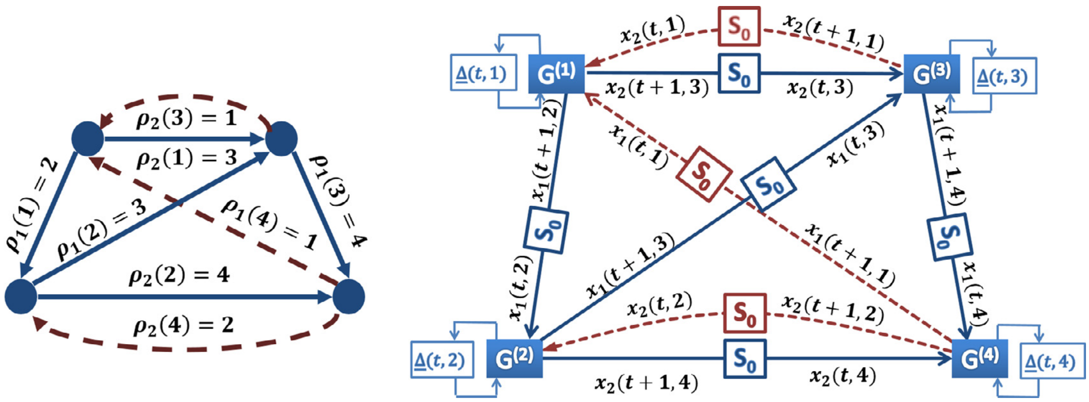

Fig. 1. A distributed NSLPV system (right) and the graph defining its interconnection structure (left).

such that

$$[\![A]\!](t,k) = \begin{bmatrix} \bar{A}_{TT}(t,k) & \bar{A}_{TS}(t,k) & \bar{A}_{TP}(t,k) \\ \bar{A}_{ST}(t,k) & \bar{A}_{SS}(t,k) & \bar{A}_{SP}(t,k) \\ \bar{A}_{PT}(t,k) & \bar{A}_{PS}(t,k) & \bar{A}_{PP}(t,k) \end{bmatrix},$$

$$[\![B]\!](t,k) = \begin{bmatrix} \bar{B}_T(t,k) \\ \bar{B}_S(t,k) \\ \bar{B}_P(t,k) \end{bmatrix},$$

$$[\![C]\!](t,k) = \begin{bmatrix} \bar{C}_T(t,k) & \bar{C}_S(t,k) & \bar{C}_P(t,k) \end{bmatrix}.$$

The matrices $\bar{D}(t,k)$ define the graph-diagonal operator D such that $[\![D]\!](t,k) = \bar{D}(t,k)$. For $j = 1, \ldots, r$, define the graph-diagonal operators Δ_j, where $\Delta_j(t,k) = \delta_j(t,k) I_{n_j^P(t,k)}$, and construct $\Delta_P = \mathrm{diag}(\Delta_1, \ldots, \Delta_r)$ such that $[\![\Delta_P]\!](t,k) = \underline{\Delta}(t,k)$. Let $S = \mathrm{diag}(S_0, S_0 S_1, \ldots, S_0 S_d, I^{(n_1^P, \ldots, n_r^P)})$ be the composite-shift operator and define the partitioned graph-diagonal operator $\Delta = \mathrm{diag}(I^{n_T}, I^{(n_1^S, \ldots, n_d^S)}, \Delta_P)$. The graph-diagonal operator I^q satisfies $[\![I^q]\!](t,k) = I_{q(t,k)}$, and the partitioned graph-diagonal operator $I^{(q_1,\ldots,q_m)}$ is defined as $I^{(q_1,\ldots,q_m)} = \mathrm{diag}(I^{q_1}, \ldots, I^{q_m})$. Eq. (1) are rewritten in compact operator form as

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta S A \begin{bmatrix} x \\ \beta \end{bmatrix} + \Delta S B u, \quad y = C \begin{bmatrix} x \\ \beta \end{bmatrix} + D u,$$
$$x = \begin{bmatrix} x_T^* & x_1^* & \cdots & x_d^* \end{bmatrix}^* \text{ and } \beta = \begin{bmatrix} \beta_1^* & \cdots & \beta_r^* \end{bmatrix}^*. \qquad (2)$$

Δ is restricted to $\mathbf{\Delta} = \{\Delta : \|\Delta\| \le 1\}$. We use the quintuple ($A$, B, C, D, $\mathbf{\Delta}$) to denote the realization of system $\mathcal{G}_\delta$ described by (2). For a fixed $\Delta \in \mathbf{\Delta}$, the input-output map of system $\mathcal{G}_\delta$ can be written as $G_\delta = \Delta \star \left[\begin{array}{c|c} SA & SB \\ \hline C & D \end{array}\right] = C(I - \Delta SA)^{-1} \Delta SB + D$, assuming that the inverse exists. The distributed NSLPV system $\mathcal{G}_\delta$ is then defined as $\mathcal{G}_\delta = \{G_\delta : \Delta \in \mathbf{\Delta}\}$.

Remark 1 [1, Lemma 1]. $\mathcal{G}_\delta$ is well-posed if $I - \Delta SA$ has a causal inverse on $\ell_{2e}(\{\mathbb{R}^{n_T(t,k)}\}) \oplus (\oplus_{i=1}^d \ell_{2e}(\{\mathbb{R}^{n_i^S(t,k)}\})) \oplus (\oplus_{j=1}^r \ell_{2e}$ $(\{\mathbb{R}^{n_j^P(t,k)}\}))$, hereafter simply ℓ_{2e}, for all $\Delta \in \mathbf{\Delta}$. $\mathcal{G}_\delta$ is well-posed if $[\![A]\!](t,k) = 0$ for all $k \in V$ and $t < 0$ and if $I - \Delta_P A_{PP}$ has a causal inverse on $\oplus_{j=1}^r \ell_{2e}(\{\mathbb{R}^{n_j^P(t,k)}\})$ for all $\Delta \in \mathbf{\Delta}$, where the partitioned graph-diagonal operator A_{PP} is partitioned conformably with Δ_P and satisfies $[\![A_{PP}]\!](t,k) = \bar{A}_{PP}(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. Hereafter, all the state-space matrices are assumed to be zeros for $t < 0$.

Definition 1. System $\mathcal{G}_\delta$ is said to be ℓ_2-stable if $I - \Delta SA$ has a bounded causal inverse for all $\Delta \in \mathbf{\Delta}$.

Next, we give a sufficient condition for the ℓ_2-stability of system $\mathcal{G}_\delta$. Let

$\mathcal{T} = \big\{X : X = \mathrm{diag}\big(X_T, X_1^S, \ldots, X_d^S, X_1^P, \ldots, X_r^P\big),$
where X_T, X_i^S, X_j^P are bounded graph-diagonal operators
for $i = 1, \ldots, d$ and $j = 1, \ldots, r$,
$X^{-1} \in \mathcal{L}\big(\ell_2(\{\mathbb{R}^{n_T(t,k)}\}) \oplus \big(\oplus_i \ell_2(\{\mathbb{R}^{n_i^S(t,k)}\})\big) \oplus \big(\oplus_j \ell_2(\{\mathbb{R}^{n_j^P(t,k)}\})\big)\big)\big\}$,
and let $\mathcal{X} = \{X : X = X^* \in \mathcal{T},\ X \succ 0\}$. The sets $\mathcal{T}$ and $\mathcal{X}$ are commutants of $\mathbf{\Delta}$.

Lemma 1 [1, Lemma 2 and Theorem 1]. *$\mathcal{G}_\delta$ is ℓ_2-stable if there exists $X \in \mathcal{X}$, or equivalently $X \succ 0$ in the commutant of $\mathbf{\Delta}$, such that*

$$A^* S^* X S A - X \prec 0. \qquad (3)$$

Let X be in $\mathcal{X}$. (3) can be written in terms of the following equivalent sequences of linear matrix inequalities (LMIs). For all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, r$, and some scalar $\beta > 0$:

$$X_T(t,k) \succ \beta I, \quad X_i^S(t,k) \succ \beta I, \quad X_j^P(t,k) \succ \beta I,$$
$$[\![A^*]\!](t,k)[\![S^*XS]\!](t,k)[\![A]\!](t,k) - [\![X]\!](t,k) \prec -\beta I,$$
$$[\![X]\!](t,k) = \mathrm{diag}(X_T(t,k), X_1^S(t,k), \ldots, X_d^S(t,k),$$
$$X_1^P(t,k), \ldots, X_r^P(t,k)),$$
$$[\![S^*XS]\!](t,k) = \mathrm{diag}(X_T(t+1,k), X_1^S(t+1, \rho_1(k)), \ldots,$$
$$X_d^S(t+1, \rho_d(k)), X_1^P(t,k), \ldots, X_r^P(t,k)).$$

The βI terms ensure that the sequences on the left-hand side of the inequalities do not converge to singular matrices as $t \to \infty$. Due to the explicit dependence on time in the state-space equations of the subsystems, there is an infinite sequence of LMIs associated with each $G^{(k)}$. The sequences corresponding to various subsystems are coupled through the spatial terms X_i^S. The parameter terms $X_j^P(t, k_0)$ only appear in the LMI sequence associated with $G^{(k_0)}$ due to the local dependence of the state-space matrices on the parameters. Since the state-space matrices are assumed to be zeros for negative time-steps, then the sequences of LMIs are trivial for $t < 0$, and t can be restricted to $\mathbb{N}_0$. Moreover, if the subsystems are (h, q)-eventually time-periodic for some integers $h \ge 0$ and $q > 0$, i.e., for all $t, z \in \mathbb{N}_0$ and $k \in V$, the state-space matrices satisfy $[\![Z]\!](t + h + zq, k) = [\![Z]\!](t + h, k)$, $Z \in \{A, B, C, D\}$, then using the averaging techniques of [8,13], we can show that a solution $X \in \mathcal{X}$ to (3) exists if and only if an (h, q)-eventually time-periodic solution X_{eper} exists. Thus, in the case of (h, q)-eventually time-periodic subsystems, we restrict t to the finite time-horizon h and the first time-period, i.e., $0 \le t \le h + q - 1$, when evaluating the sequences of LMIs equivalent to (3).

Since (3) is only a sufficient condition for ℓ_2-stability, there exist ℓ_2-stable systems for which a solution in $\mathcal{X}$ to (3) does not exist. Systems which possess structured solutions to (3) are said to be strongly stable.

Lemma 2 [1, Lemma 3 and Theorem 1]**.** *System $\mathcal{G}_\delta$ is strongly stable and satisfies $\|G_\delta\| < \gamma$ for all $\Delta \in \mathbf{\Delta}$ if there exists $X \in \mathcal{X}$, or equivalently, $X \succ 0$ in the commutant of $\mathbf{\Delta}$, such that*

$$\begin{bmatrix} -\begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} & \begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \\ \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} & -\begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \prec 0. \tag{4}$$

Clearly, the size of $\mathcal{G}_\delta$ and the size of the analysis problems increase with the number of subsystems, interconnections, and parameters, as well as the dimensions of the corresponding temporal, spatial, and parameter states. This makes model reduction very useful, sometimes even necessary, for reducing the computational complexity of the problems at hand. Specifically, a reduced-order system $\mathcal{G}_{\text{red},\delta}$ is sought which approximates the behavior of $\mathcal{G}_\delta$ and preserves the interconnection and uncertainty structures of $\mathcal{G}_\delta$, i.e., $\mathcal{G}_{\text{red},\delta}$ is a distributed NSLPV system whose interconnection structure is described using the same graph as $\mathcal{G}_\delta$ and where the LFT formulation of the subsystems retains the same partitioning of the Δ-operator.

4. Balanced truncation

This section treats the BT method. The notion of a balanced realization for a distributed NSLPV system is defined, and strongly stable systems are shown to admit a balanced realization. The reduced-order system resulting from BT is proved to be strongly stable with a balanced realization, and upper bounds on the ℓ_2-induced norm of the error system are derived. The results of this section generalize their counterparts for single NSLPV systems in [10, Lemmas 9 and 10 and Theorems 12, 13, and 17] and for distributed LTV systems in [2, Algorithm 1, Lemma 3, and Theorems 1 and 2]. If only one subsystem is considered, the results of [10] are recovered; and if the operator Δ is fixed and known a priori, the distributed NSLPV system reduces to a distributed LTV system and the results of [2] are recovered. Theorem 3 does not have a counterpart in [2] and remains novel when specialized to the class of systems therein; [2, Theorem 3] only applies when the truncation sequences are monotonic in time, whereas Theorem 3 applies for general truncation sequences.

4.1. Balanced realization

Definition 2. A realization $(A, B, C, D, \mathbf{\Delta})$ of $\mathcal{G}_\delta$ is said to be balanced with balanced generalized gramian Σ if there exists $\Sigma = X = Y \in \mathcal{X}$ that satisfies

$$AXA^* - S^*XS + BB^* \prec 0, \tag{5}$$

$$A^*S^*YSA - Y + C^*C \prec 0, \tag{6}$$

and for each $(t, k) \in \mathbb{Z} \times V$, $[\![\Sigma]\!](t, k)$ is a diagonal matrix.

(5) and (6) are called the generalized Lyapunov inequalities and can be solved separately in order to obtain generalized gramians X and Y in $\mathcal{X}$. $[\![X]\!](t, k)$ and $[\![Y]\!](t, k)$ need not be diagonal matrices.

Lemma 3. *A strongly stable system $\mathcal{G}_\delta$ admits a balanced realization $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \mathbf{\Delta})$.*

Proof. For a strongly stable system, there exists a solution $P \in \mathcal{X}$ to (3), and equivalently, due to the homogeneity and scalability of (3), there exist solutions $X \in \mathcal{X}$ and $Y \in \mathcal{X}$ to (5) and (6), respectively. X and Y are used to construct a balanced realization for $\mathcal{G}_\delta$ and a balanced generalized gramian Σ as follows. First, we perform the Cholesky factorizations $X = R^*R$ and $Y = H^*H$. Then, we perform the singular value decomposition $HR^* = U\Sigma V^*$, where U and V are in $\mathcal{T}$. We define the balancing transformation $T = \Sigma^{-1/2}U^*H \in \mathcal{T}$ and its inverse $T^{-1} = R^*V\Sigma^{-1/2} \in \mathcal{T}$. Σ can then be expressed as $\Sigma = TXT^* = (T^*)^{-1}YT^{-1}$. Σ simultaneously satisfies (5) and (6) for the realization $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \mathbf{\Delta})$ of $\mathcal{G}_\delta$, where $A_{\text{bal}} = (S^*TS)AT^{-1}$, $B_{\text{bal}} = (S^*TS)B$, and $C_{\text{bal}} = CT^{-1}$. Because of the structure imposed on $\mathcal{T}$ and $\mathcal{X}$, the previous computations are performed block-wise, e.g., $X_T = (R_T)^*R_T$, $Y_T = (H_T)^*H_T$, and so on. □

Σ, A_{bal}, B_{bal}, and C_{bal} in the previous proof depend on the generalized gramians X and Y used in the balancing procedure. For model reduction purposes, generalized gramians with minimum traces are sought, e.g., the solution $X \in \mathcal{X}$ to (5) which minimizes $\sum_{(t,k)}(\operatorname{tr} X_T(t,k) + \sum_{i=1}^{d} \operatorname{tr} X_i^S(t,k) + \sum_{j=1}^{r} \operatorname{tr} X_j^P(t,k))$. See Section 6 for more details.

4.2. Balanced truncation

Consider a strongly stable system $\mathcal{G}_\delta$ with a balanced realization $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian $\Sigma = \operatorname{diag}(\Sigma_T, \Sigma_1^S, \ldots, \Sigma_d^S, \Sigma_1^P, \ldots, \Sigma_r^P)$, which is to be reduced via BT. To determine which state variables to truncate, one looks at the entries of Σ and their relative order, the value of γ in Lemma 2, and the error bounds from Theorems 2 and 3. Since Σ has a block-diagonal structure and contains temporal terms $\Sigma_T(t, k)$, spatial terms $\Sigma_i^S(t,k)$, and parameter terms $\Sigma_j^P(t,k)$, then the BT method allows for the truncation of each temporal, spatial, and parameter state individually. Truncation need not be uniform in time, i.e., one may truncate different numbers of variables from a particular state at different time-steps.

We focus on Σ_T and repeat similar steps for Σ_i^S and Σ_j^P for $i = 1, \ldots, d$ and $j = 1, \ldots, r$. For each $(t, k) \in \mathbb{Z} \times V$, $\Sigma_T(t, k)$ is an $n_T(t, k) \times n_T(t, k)$ positive definite, diagonal matrix. Without loss of generality, we assume that the entries of $\Sigma_T(t, k)$ are sorted in a decreasing order with the largest value in the first entry. Denote the dimensions of the reduced temporal states by $m_T(t, k)$, where $0 \le m_T(t, k) \le n_T(t, k)$. We partition $\Sigma_T(t, k)$ into two blocks as in $\Sigma_T(t,k) = \operatorname{diag}(\Gamma_T(t,k), \Omega_T(t,k))$, where $\Gamma_T(t, k)$ is an $m_T(t, k) \times m_T(t, k)$ matrix. If $m_T(t_0, k_0) = 0$ for some $(t_0, k_0) \in \mathbb{Z} \times V$, then $\Gamma_T(t_0, k_0)$ is nonexistent. Similarly, if $n_T(t_0, k_0) - m_T(t_0, k_0) = 0$, then $\Omega_T(t_0, k_0)$ is nonexistent. $\Gamma_T(t, k)$ and $\Omega_T(t, k)$ define graph-diagonal operators denoted by Γ_T and Ω_T. The method also allows for the reduction of the dimensions of the spatial and the parameter states. The dimensions of the spatial and the parameter states in the reduced-order system are given by $m_i^S(t,k)$ and $m_j^P(t,k)$, respectively, where $0 \le m_i^S(t,k) \le n_i^S(t,k)$ and $0 \le m_j^P(t,k) \le n_j^P(t,k)$. The case, $m_{i_0}^S(t,k_0) = 0$ for all time-steps $t \in \mathbb{Z}$, corresponds to the removal of the interconnection $(\rho_{i_0}^{-1}(k_0), k_0)$ altogether from the graph of $\mathcal{G}_{\text{red},\delta}$. Similarly, the case, $m_{j_0}^P(t,k_0) = 0$ for all time-steps, corresponds to the removal of the channel $\Delta_{j_0}(t, k_0)$ from $\underline{\Delta}(t, k_0)$ in $\mathcal{G}_{\text{red},\delta}$. We define the operators $\Gamma = \operatorname{diag}(\Gamma_T, \Gamma_1^S, \ldots, \Gamma_d^S, \Gamma_1^P, \ldots, \Gamma_r^P)$ and $\Omega = \operatorname{diag}(\Omega_T, \Omega_1^S, \ldots, \Omega_d^S, \Omega_1^P, \ldots, \Omega_r^P)$. Γ and Ω are associated with the non-truncated and truncated blocks of Σ, respectively. We now partition the state-space matrices into non-truncated portions, which we mark with a hat, and truncated portions, conformably with the partitioning of $\Sigma_T = [\![\operatorname{diag}(\Gamma_T, \Omega_T)]\!]$, $\Sigma_i^S = [\![\operatorname{diag}(\Gamma_i^S, \Omega_i^S)]\!]$, and $\Sigma_j^P = [\![\operatorname{diag}(\Gamma_j^P, \Omega_j^P)]\!]$. For instance,

$$\bar{A}_{TS}(t,k) = \begin{bmatrix} A_1^{TS}(t,k) & \cdots & A_d^{TS}(t,k) \end{bmatrix}$$

$$= \left[\begin{bmatrix} \hat{A}_1^{TS}(t,k) & A_{1,12}^{TS}(t,k) \\ A_{1,21}^{TS}(t,k) & A_{1,22}^{TS}(t,k) \end{bmatrix} \cdots \begin{bmatrix} \hat{A}_d^{TS}(t,k) & A_{d,12}^{TS}(t,k) \\ A_{d,21}^{TS}(t,k) & A_{d,22}^{TS}(t,k) \end{bmatrix} \right],$$

$$\bar{A}_{SS}(t,k) = \left[A_{ie}^{SS}(t,k) \right]_{i=1,\ldots,d;e=1,\ldots,d}$$

$$= \left[\begin{bmatrix} \hat{A}_{ie}^{SS}(t,k) & A_{ie,12}^{SS}(t,k) \\ A_{ie,21}^{SS}(t,k) & A_{ie,22}^{SS}(t,k) \end{bmatrix} \right]_{i=1,\ldots,d;e=1,\ldots,d},$$

$$\bar{A}_{TP}(t,k) = \left[A_1^{TP}(t,k) \quad \cdots \quad A_r^{TP}(t,k) \right]$$

$$= \left[\begin{bmatrix} \hat{A}_1^{TP}(t,k) & A_{1,12}^{TP}(t,k) \\ A_{1,21}^{TP}(t,k) & A_{1,22}^{TP}(t,k) \end{bmatrix} \cdots \begin{bmatrix} \hat{A}_r^{TP}(t,k) & A_{r,12}^{TP}(t,k) \\ A_{r,21}^{TP}(t,k) & A_{r,22}^{TP}(t,k) \end{bmatrix} \right],$$

$$\bar{A}_{PP}(t,k) = \left[A_{jf}^{PP}(t,k) \right]_{j=1,\ldots,r;f=1,\ldots,r}$$

$$= \left[\begin{bmatrix} \hat{A}_{jf}^{PP}(t,k) & A_{jf,12}^{PP}(t,k) \\ A_{jf,21}^{PP}(t,k) & A_{jf,22}^{PP}(t,k) \end{bmatrix} \right]_{j=1,\ldots,r;f=1,\ldots,r},$$

where $\hat{A}_i^{TS}(t,k)$ is an $m_T(t+1,k) \times m_i^S(t,k)$ matrix, $\hat{A}_{ie}^{SS}(t,k)$ is an $m_i^S(t+1,\rho_i(k)) \times m_e^S(t,k)$ matrix, $\hat{A}_j^{TP}(t,k)$ is an $m_T(t+1,k) \times m_j^P(t,k)$ matrix, and $\hat{A}_{jf}^{PP}(t,k)$ is an $m_j^P(t,k) \times m_f^P(t,k)$ matrix. The partitioning of the state-space matrices is performed at the level of the most elemental blocks, e.g., we do not partition $\bar{A}_{TS}(t,k)$, but rather, we partition $A_i^{TS}(t,k)$ into a non-truncated block $\hat{A}_i^{TS}(t,k)$ and other truncated blocks.

The non-truncated blocks, e.g., $\hat{A}_{TT}(t,k)$, $\hat{A}_i^{TS}(t,k)$, define graph-diagonal operators, e.g., $\hat{A}_{TT}$, $\hat{A}_i^{TS}$, which, when augmented in the obvious way, form the reduced-order system operators A_{red}, B_{red}, and C_{red}. D_{red} is set equal to D. We also define

$$\Delta_{\text{red}} = \operatorname{diag}\left(I^{m_T}, I^{(m_1^S,\ldots,m_d^S)}, \hat{\Delta}_P\right), \quad \text{where } \hat{\Delta}_P = \operatorname{diag}\left(\hat{\Delta}_1,\ldots,\hat{\Delta}_r\right), \tag{7}$$

and $\hat{\Delta}_j$ are graph-diagonal operators that satisfy $\hat{\Delta}_j(t,k) = \delta_j(t,k) I_{m_j^P(t,k)}$ for $j = 1,\ldots,r$. The realization of the reduced-order system $\mathcal{G}_{\text{red},\delta}$ is thus given by $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \boldsymbol{\Delta}_{\text{red}})$, where $\boldsymbol{\Delta}_{\text{red}} = \{\Delta_{\text{red}} = \mathcal{P}\Delta\mathcal{P}^* : \Delta \in \boldsymbol{\Delta}\}$ and $\mathcal{P}$ is an appropriately defined operator that results in (7). BT is said to be structure-preserving since the interconnection structure of $\mathcal{G}_{\text{red},\delta}$ is the same as the interconnection structure of $\mathcal{G}_\delta$, with the spatial states having smaller or equal dimensions, and the structure of the Δ_{red}-operator in $\mathcal{G}_{\text{red},\delta}$ is the same as the structure of the Δ-operator in $\mathcal{G}_\delta$, with the parameter states having smaller or equal dimensions. The method is also structure-simplifying because the spatial states and the parameter states in $\mathcal{G}_{\text{red},\delta}$ are allowed to have zero dimensions.

Lemma 4. *The reduced-order system $\mathcal{G}_{\text{red},\delta}$ is strongly stable, and its given realization is balanced.*

Proof. There exists a unique partitioned graph-diagonal operator Q such that $Q^*\Sigma Q = \operatorname{diag}(\Gamma,\Omega)$, $QQ^* = I$, and $Q^*Q = I$. The reader is referred to Appendix A for the detailed structure of the operator Q. It is also not difficult to see that Q satisfies $Q^*\Delta Q = \operatorname{diag}(\Delta_{\text{red}}, \bar{\Delta}_2) = \bar{\bar{\Delta}}$,

$$Q^*SAQ = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A_{\text{red}} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \tilde{S}\bar{\bar{A}},$$

$$Q^*SB = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B_{\text{red}} \\ \bar{B}_2 \end{bmatrix} = \tilde{S}\bar{\bar{B}}, \quad CQ = \begin{bmatrix} C_{\text{red}} & \bar{C}_2 \end{bmatrix} = \bar{\bar{C}},$$

where $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_2$, $\bar{C}_2$, and $\bar{\Delta}_2$ are appropriately defined partitioned graph-diagonal operators.

With Σ satisfying (5) and (6), pre- and post-multiplying (5) by $\tilde{S}^*Q^*S$ and $S^*Q\tilde{S}$, respectively, and (6) by Q^* and Q, respectively, we get

$$\bar{\bar{A}} \operatorname{diag}(\Gamma,\Omega)\bar{\bar{A}}^* - \tilde{S}^*\operatorname{diag}(\Gamma,\Omega)\tilde{S} + \bar{\bar{B}}\bar{\bar{B}}^* \prec 0, \tag{8}$$

$$\bar{\bar{A}}^*\tilde{S}^*\operatorname{diag}(\Gamma,\Omega)\tilde{S}\bar{\bar{A}} - \operatorname{diag}(\Gamma,\Omega) + \bar{\bar{C}}^*\bar{\bar{C}} \prec 0. \tag{9}$$

From (8) and (9), $A_{\text{red}}\Gamma A_{\text{red}}^* - S^*\Gamma S + B_{\text{red}}B_{\text{red}}^* \prec 0$ and $A_{\text{red}}^*S^*\Gamma S A_{\text{red}} - \Gamma + C_{\text{red}}^*C_{\text{red}} \prec 0$. Thus, $\mathcal{G}_{\text{red},\delta}$ is strongly stable, and its given realization is balanced with a balanced generalized gramian Γ. □

4.3. Error bounds

Consider a strongly stable system $\mathcal{G}_\delta$ with a balanced realization $(A, B, C, D, \boldsymbol{\Delta})$ and a balanced generalized gramian Σ, and denote the reduced-order model obtained from BT by $\mathcal{G}_{\text{red},\delta}$ with balanced realization $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \boldsymbol{\Delta}_{\text{red}})$ and balanced generalized gramian Γ. We now derive expressions for the upper bound on the ℓ_2-induced norm of the resulting error system.

Theorem 1. *If $\Omega = I$, i.e., for all $i = 1,\ldots,d$, $j = 1,\ldots,r$, and $(t,k) \in \mathbb{Z} \times V$, $\Omega_T(t,k) = I$, $\Omega_i^S(t,k) = I$, and $\Omega_j^P(t,k) = I$, then $\|(G_\delta - G_{\text{red},\delta})\| < 2$ for all $\Delta \in \boldsymbol{\Delta}$.*

Proof. Since $\mathcal{G}_\delta$ and $\mathcal{G}_{\text{red},\delta}$ are strongly stable systems, then so is the error system $\mathcal{E}_\delta = \{\frac{1}{2}(G_\delta - G_{\text{red},\delta}) : \Delta \in \boldsymbol{\Delta}\}$. Recalling the denotations $\tilde{S}$, $\bar{\bar{A}}$, $\bar{\bar{B}}$, $\bar{\bar{C}}$, and $\bar{\bar{\Delta}}$ defined in Lemma 4, one can see that

$$\frac{1}{2}\left(G_\delta - G_{\text{red},\delta}\right) = \begin{bmatrix} \Delta_{\text{red}} & 0 \\ 0 & \bar{\bar{\Delta}} \end{bmatrix} \star \left[\begin{array}{cc|c} SA_{\text{red}} & 0 & \frac{1}{\sqrt{2}}SB_{\text{red}} \\ 0 & \tilde{S}\bar{\bar{A}} & \frac{1}{\sqrt{2}}\tilde{S}\bar{\bar{B}} \\ \hline -\frac{1}{\sqrt{2}}C_{\text{red}} & \frac{1}{\sqrt{2}}\bar{\bar{C}} & 0 \end{array} \right].$$

As per Lemma 2, we show that $\|\frac{1}{2}(G_\delta - G_{\text{red},\delta})\| < 1$ for all $\Delta \in \boldsymbol{\Delta}$ by constructing an operator $\mathcal{V} \succ 0$ that commutes with $\operatorname{diag}(\Delta_{\text{red}}, \bar{\bar{\Delta}})$ for all $\Delta \in \boldsymbol{\Delta}$ and satisfies (4) for the given realization of system $\mathcal{E}_\delta$. By direct application of the Schur complement formula twice to (8) and (9), we verify that $\begin{bmatrix} -R_1 & K^* \\ K & -S_a^*R_2S_a \end{bmatrix} \prec 0$, where $S_a = \operatorname{diag}(\tilde{S}, I, \tilde{S})$, $R_i = \operatorname{diag}(\Gamma^{-1}, \Omega^{-1}, I^{q_i}, \Gamma, \Omega)$, $q_1 = n_u$, $q_2 = n_y$, and $K = \begin{bmatrix} 0 & 0 & \bar{\bar{A}} \\ 0 & 0 & \bar{\bar{C}} \\ \bar{\bar{A}} & \bar{\bar{B}} & 0 \end{bmatrix}$. Let

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I^{n_y} & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}$$

and

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I^{n_u} & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}.$$

We pre- and post-multiply the previous inequality by $\operatorname{diag}(P^*, L)$ and $\operatorname{diag}(P, L^*)$, respectively, to obtain $\begin{bmatrix} -P^*R_1P & P^*K^*L^* \\ LKP & -LS_a^*R_2S_aL^* \end{bmatrix} \prec 0$, where

$$LKP = \left[\begin{array}{c|c} M & N_{12} \\ \hline N_{21} & N_{22} \end{array} \right] = \left[\begin{array}{cccc|c} A_{\text{red}} & 0 & 0 & \frac{1}{\sqrt{2}}B_{\text{red}} & \bar{A}_{12} \\ 0 & A_{\text{red}} & \bar{A}_{12} & \frac{1}{\sqrt{2}}B_{\text{red}} & 0 \\ 0 & \bar{A}_{21} & \bar{A}_{22} & \frac{1}{\sqrt{2}}\bar{B}_2 & 0 \\ \frac{-1}{\sqrt{2}}C_{\text{red}} & \frac{1}{\sqrt{2}}C_{\text{red}} & \frac{1}{\sqrt{2}}\bar{C}_2 & 0 & \frac{-1}{\sqrt{2}}\bar{C}_2 \\ \hline \bar{A}_{21} & 0 & 0 & \frac{1}{\sqrt{2}}\bar{B}_2 & \bar{A}_{22} \end{array} \right].$$

Since $\Omega = I$ and $S^*S = I$, then

$$P^*R_1P = \mathrm{diag}\left(\frac{1}{2}\begin{bmatrix}\Gamma^{-1}+\Gamma & \Gamma^{-1}-\Gamma\\ \Gamma^{-1}-\Gamma & \Gamma^{-1}+\Gamma\end{bmatrix}, \mathrm{diag}(I, I^{n_u}, I)\right)$$

and

$$LS_a^*R_2S_aL^* = \mathrm{diag}\left(\frac{1}{2}\begin{bmatrix}S^*(\Gamma^{-1}+\Gamma)S & S^*(\Gamma-\Gamma^{-1})S\\ S^*(\Gamma-\Gamma^{-1})S & S^*(\Gamma^{-1}+\Gamma)S\end{bmatrix}, \mathrm{diag}(I, I^{n_y}, I)\right).$$

Let

$$\mathcal{V} = \frac{1}{2}\mathrm{diag}\left(\begin{bmatrix}\Gamma^{-1}+\Gamma & \Gamma^{-1}-\Gamma\\ \Gamma^{-1}-\Gamma & \Gamma^{-1}+\Gamma\end{bmatrix}, 2I\right).$$

$\mathcal{V}$ satisfies

$$\begin{bmatrix} -\begin{bmatrix}\mathcal{V} & 0\\ 0 & I^{n_u}\end{bmatrix} & M^* \\ M & -\begin{bmatrix}\begin{bmatrix}S & 0\\ 0 & \tilde{S}\end{bmatrix}^*\mathcal{V}^{-1}\begin{bmatrix}S & 0\\ 0 & \tilde{S}\end{bmatrix} & 0\\ 0 & I^{n_y}\end{bmatrix}\end{bmatrix} \prec 0$$

and $\mathcal{V} \succ 0$ and commutes with all $\mathrm{diag}(\Delta_{\mathrm{red}}, \bar{\bar{\Delta}})$. By Lemma 2, $\|\frac{1}{2}(G_\delta - G_{\mathrm{red},\delta})\| < 1$ for all $\Delta \in \boldsymbol{\Delta}$. □

The error bound for the case of a general Ω is given next. The theorem follows by scaling and repeated application of Theorem 1 as detailed in [2, Proof of Theorem 2]; and Lemma 4 ensures that BT is applicable to the resulting intermediate reduced-order systems.

Theorem 2. *The reduced-order system $\mathcal{G}_{\mathrm{red},\delta}$ satisfies $\|(G_\delta - G_{\mathrm{red},\delta})\| < 2\Phi(\Omega)$ for all $\Delta \in \boldsymbol{\Delta}$.*

The bound in Theorem 2 can become infinitely large as the number of distinct entries in Ω increases. If the system is strongly stable and the subsystems are (h, q)-eventually time-periodic, then there exists an (h, q)-eventually time-periodic balanced realization for the system and an (h, q)-eventually time-periodic balanced generalized gramian Σ_{eper}. The realization of the reduced-order system obtained from BT is also (h, q)-eventually time-periodic. In this case, when evaluating $\Phi(\Omega_{\mathrm{eper}})$, t is restricted to the finite time-horizon h and the first time-period, i.e., $0 \le t \le h+q-1$. If, in addition, the graph is finite, i.e., V and E are finite sets, then $\Phi(\Omega_{\mathrm{eper}})$ is guaranteed to be finite. As demonstrated in the example, heuristics can be used in order to increase the number of small and equal entries in Ω_{eper}, thereby increasing the effectiveness of Theorem 2. Next, we derive an alternative expression for the error bound which can be less conservative than the expression in Theorem 2 for general NSLPV subsystems. The counterparts of the following result for single LTV systems and single NSLPV systems are found in [22, Theorem 2] and [10, Theorem 17], respectively. For all $k \in V$, $i = 1, \ldots, d$, and $j = 1, \ldots, r$, we define the sets $\mathcal{F}_T(k) = \{t \in \mathbb{N}_0 : m_T(t,k) \ne n_T(t,k)\}$, $\mathcal{F}_i^S(k) = \{t \in \mathbb{N}_0 : m_i^S(t,k) \ne n_i^S(t,k)\}$, and $\mathcal{F}_j^P(k) = \{t \in \mathbb{N}_0 : m_j^P(t,k) \ne n_j^P(t,k)\}$.

Definition 3 [10, Definition 14]. Consider a scalar sequence α_t defined on a subset $\mathcal{W}$ of $\mathbb{N}_0$, and let $t_{\min} = \min\{t : t \in \mathcal{W}\}$. We define the following rule which extends the domain of definition of α_t to all $t \in \mathbb{Z}$:

$$\alpha_t = \begin{cases}\alpha_{t_{\min}} & \text{if } t \le t_{\min},\\ \alpha_e & \text{if } t > t_{\min}, \text{ where } e = \max\{\tau \le t : \tau \in \mathcal{W}\}.\end{cases}$$

Definition 4 [22, Definition 1]. Consider a scalar sequence $\nu = (\nu_1, \nu_2, \ldots, \nu_s)$ for some integer $s \ge 1$, where ν_1 cannot be considered as a local maximum and ν_s cannot be considered as a local minimum. Then, the sequence ν has l local maxima and l local minima for some $l \in \mathbb{N}_0$. If $l \ge 1$, we denote the local maxima and local minima by $\nu_{\max,e}$ and $\nu_{\min,e}$ for $e = 1, \ldots, l$. The min-max ratio S_ν of the sequence ν is defined as follows:

$$S_\nu = \begin{cases}\nu_1 & \text{if } l = 0,\\ \nu_1 \prod_{e=1}^{l}\left(\frac{\nu_{\max,e}}{\nu_{\min,e}}\right) & \text{if } l > 0.\end{cases}$$

Theorem 3. *If $\Omega_T(t,k) = w_T(t,k)I$ for $t \in \mathcal{F}_T(k)$, $\Omega_i^S(t,k) = w_i^S(t,k)I$ for $t \in \mathcal{F}_i^S(k)$, $\Omega_j^P(t,k) = w_j^P(t,k)I$ for $t \in \mathcal{F}_j^P(k)$, and $\mathcal{F}_T(k)$, $\mathcal{F}_i^S(k)$, $\mathcal{F}_j^P(k)$ are finite sets for all $k \in V$, $i = 1, \ldots, d$, and $j = 1, \ldots, r$, then*

$$\|(G_\delta - G_{\mathrm{red},\delta})\| < 2\sum_{k\in V}\left(S_{w_T(t,k)} + \sum_{i=1}^{d} S_{w_i^S(t,k)} + \sum_{j=1}^{r} S_{w_j^P(t,k)}\right)$$

for all $\Delta \in \boldsymbol{\Delta}$. (10)

Proof. The result is proved for the special case where only one state is truncated, e.g., the temporal state $x_T(t, k_0)$. The general case then follows by repeated application of this special result. Without loss of generality, assume that $w_T(t,k_0) \le 1$ for all $t \in \mathcal{F}_T(k_0)$. This assumption can be ensured by scaling system $\mathcal{G}_\delta$. Suppose that the sequence, $w_T(t,k_0) \le 1$ for all $t \in \mathcal{F}_T(k_0)$, has l local maxima and l local minima for some $l \in \mathbb{N}_0$, where the first element in the sequence cannot be considered as a local maximum and the last element cannot be considered as a local minimum, as stipulated in Definition 4. If $l \ge 1$, then the set $\mathcal{F}_T(k_0)$ is of the form $\{t_1, \ldots, t_{\min,1}, \ldots, t_{\max,1}, \ldots, t_{\max,l}, \ldots, t_s\}$. For $e = 1, \ldots, l$, $t_{\min,e}$ and $t_{\max,e}$ denote the time-steps at which $w_T(t,k_0)$ reaches its local minima and local maxima, respectively. The domain of $w_T(t,k_0)$ is extended to all $t \in \mathbb{Z}$ using Definition 3. Note that $w_T(t,k_0) = w_T(t_1,k_0)$ for $t \le t_1$ and $w_T(t,k_0) = w_T(t_s,k_0)$ for $t \ge t_s$. For all $(t,k) \in \mathbb{Z} \times V$, we define the state transformation $T = T^* \in \mathcal{T}$ as

$$[\![T]\!](t,k) = \begin{cases} w_T^{1/2}(t,k_0)\, I & \text{for } t \le t_{\min,1},\\ w_T(t_{\min,1},k_0) \times w_T^{-1/2}(t,k_0)\, I & \text{for } t_{\min,1}+1 \le t \le t_{\max,1},\\ w_T(t_{\min,1},k_0) \times w_T^{-1}(t_{\max,1},k_0)\, w_T^{1/2}(t,k_0)\, I & \text{for } t_{\max,1}+1 \le t \le t_{\min,2},\\ \vdots & \vdots\\ \rho\, w_T^{1/2}(t,k_0)\, I & \text{for } t \ge t_{\max,l}+1,\end{cases}$$

where $\rho = \prod_{e=1}^{l}\left(w_T(t_{\min,e},k_0)\, w_T^{-1}(t_{\max,e},k_0)\right)$. Since $\Sigma_T(t,k_0) \succ \beta I$ for some $\beta > 0$ and all $t \in \mathbb{Z}$, then T and T^{-1} are bounded. We define a new realization $(A_{\mathrm{new}}, B_{\mathrm{new}}, C_{\mathrm{new}}, D, \boldsymbol{\Delta})$ for $\mathcal{G}_\delta$, where $A_{\mathrm{new}} = (S^*TS)AT^{-1}$, $B_{\mathrm{new}} = (S^*TS)B$, and $C_{\mathrm{new}} = CT^{-1}$, and refer to system $\mathcal{G}_\delta$ with this new realization as system $\mathcal{G}_{\mathrm{new},\delta}$. We define operators P, Q, and W in $\mathcal{T}$ such that $[\![P]\!](t,k) = w_T^{1/2}(t,k_0)I$, $[\![Q]\!](t,k) = [\![T]\!](t,k)[\![P]\!](t,k)^{-1}$, and $[\![W]\!](t,k) = [\![T]\!](t,k)[\![P]\!](t,k)$. For each $k \in V$, the sequences $\alpha_1(t,k)$, $\alpha_2(t,k)$, and $\alpha_3(t,k)$, where $[\![T]\!](t,k) = \alpha_1(t,k)I$, $[\![Q]\!](t,k) = \alpha_2(t,k)I$, and $[\![W]\!](t,k) = \alpha_3(t,k)I$, are nonincreasing in t.

The realization of $\mathcal{G}_\delta$ is balanced with the balanced generalized gramian Σ satisfying both (5) and (6). Pre- and post-multiplying (5) by S^*TS and making use of the fact that $TT^{-1} = T^{-1}T = I$, we get $A_{\mathrm{new}}\,T\,\Sigma\,T\,A_{\mathrm{new}}^* - (S^*TS)\,(S^*\Sigma S)\,(S^*TS) + B_{\mathrm{new}}B_{\mathrm{new}}^* \prec 0$, i.e.,

$$\begin{aligned}&[\![A_{\mathrm{new}}]\!](t,k)[\![T]\!](t,k)[\![\Sigma]\!](t,k)[\![T]\!](t,k)[\![A_{\mathrm{new}}^*]\!](t,k)\\ &- [\![S^*TS]\!](t,k)[\![S^*\Sigma S]\!](t,k)[\![S^*TS]\!](t,k)\\ &+ [\![B_{\mathrm{new}}]\!](t,k)[\![B_{\mathrm{new}}^*]\!](t,k) \prec -\beta I\end{aligned}$$

for all $(t,k) \in \mathbb{Z} \times V$ and some $\beta > 0$. Pre- and post-multiplying each inequality by $[\![S^*WS]\!](t,k)^{-1} = \alpha_3^{-1}(t+1,k)I$, where $\alpha_3(t,k) = \alpha_1(t,k)w_T^{1/2}(t,k_0)$, or equivalently, $w_T^{-1/2}(t,k_0) =$

$\alpha_3^{-1}(t,k)\alpha_1(t,k)$, we get

$$\begin{aligned}&[\![A_{\text{new}}]\!](t,k)\,\alpha_1(t,k)\alpha_3^{-1}(t+1,k)\;[\![\Sigma]\!](t,k)\\&\times\alpha_1(t,k)\alpha_3^{-1}(t+1,k)\;[\![A_{\text{new}}^*]\!](t,k)\\&-w_T^{-1}(t+1,k_0)[\![S^*\Sigma S]\!](t,k)+\alpha_3^{-1}(t+1,k)\;[\![B_{\text{new}}]\!](t,k)\\&\times[\![B_{\text{new}}^*]\!](t,k)\alpha_3^{-1}(t+1,k)\prec-\beta I.\end{aligned}$$

Since, for each $k\in V$, the sequence $\alpha_3(t,\,k)$ is nonincreasing in t, i.e., $0<\alpha_3(t+1,k)\le\alpha_3(t,k)\le w_T(t_1,k_0)$ and $\alpha_3^{-1}(t+1,k)\ge\alpha_3^{-1}(t,k)\ge w_T^{-1}(t_1,k_0)$, one can verify that the following holds:

$$\begin{aligned}&[\![A_{\text{new}}]\!](t,k)\,w_T^{-1}(t,k_0)\;[\![\Sigma]\!](t,k)\;[\![A_{\text{new}}^*]\!](t,k)\\&-w_T^{-1}(t+1,k_0)\;[\![S^*\Sigma S]\!](t,k)\\&+w_T^{-1}(t_1,k_0)\,[\![B_{\text{new}}]\!](t,k)[\![B_{\text{new}}^*]\!](t,k)w_T^{-1}(t_1,k_0)\prec-\beta I.\end{aligned}\tag{11}$$

Similarly, we pre- and post-multiply (6) by T^{-1} and insert $(S^*T^{-1}S)(S^*TS)=(S^*TS)\big(S^*T^{-1}S\big)=I$ as needed to get

$$A_{\text{new}}^*(S^*T^{-1}S)(S^*\Sigma S)(S^*T^{-1}S)A_{\text{new}}-T^{-1}\Sigma T^{-1}+C_{\text{new}}^*C_{\text{new}}\prec0,$$

i.e.,

$$\begin{aligned}&[\![A_{\text{new}}^*]\!](t,k)[\![S^*T^{-1}S]\!](t,k)[\![S^*\Sigma S]\!](t,k)[\![S^*T^{-1}S]\!](t,k)\\&\times[\![A_{\text{new}}]\!](t,k)-[\![T]\!](t,k)^{-1}[\![\Sigma]\!](t,k)[\![T]\!](t,k)^{-1}\\&+[\![C_{\text{new}}^*]\!](t,k)[\![C_{\text{new}}]\!](t,k)\prec-\beta I\end{aligned}$$

for all $(t,k)\in\mathbb{Z}\times V$ and some $\beta>0$. Pre- and post-multiplying each inequality by $[\![Q]\!](t,k)=\alpha_2(t,k)I$, and using the fact that $\alpha_2(t,k)=\alpha_1(t,k)w_T^{-1/2}(t,k_0)$, i.e., $\alpha_2(t,k)\alpha_1^{-1}(t,k)=w_T^{-1/2}(t,k_0)$, we get

$$\begin{aligned}&[\![A_{\text{new}}^*]\!](t,k)\,\alpha_2(t,k)\,\alpha_1^{-1}(t+1,k)\\&\times[\![S^*\Sigma S]\!](t,k)\,\alpha_1^{-1}(t+1,k)\,\alpha_2(t,k)\;[\![A_{\text{new}}]\!](t,k)\\&-w_T^{-1}(t,k_0)\;[\![\Sigma]\!](t,k)+\alpha_2(t,k)\\&\times[\![C_{\text{new}}^*]\!](t,k)[\![C_{\text{new}}]\!](t,k)\,\alpha_2(t,k)\prec-\beta I.\end{aligned}$$

Since, for each $k\in V$, the sequence $\alpha_2(t,\,k)$ is nonincreasing in t, i.e., $\rho\le\alpha_2(t+1,k)\le\alpha_2(t,k)$, we obtain

$$\begin{aligned}&[\![A_{\text{new}}^*]\!](t,k)w_T^{-1}(t+1,k_0)[\![S^*\Sigma S]\!](t,k)[\![A_{\text{new}}]\!](t,k)\\&-w_T^{-1}(t,k_0)[\![\Sigma]\!](t,k)+\rho[\![C_{\text{new}}^*]\!](t,k)\\&\times[\![C_{\text{new}}]\!](t,k)\rho\prec-\beta I.\end{aligned}\tag{12}$$

From (11) and (12), one sees that the scaled realization $(A_{\text{new}},w_T^{-1}(t_1,k_0)B_{\text{new}},\rho C_{\text{new}},w_T^{-1}(t_1,k_0)\rho D,\mathbf{\Delta})$ of $\mathcal{G}_{\text{new},\delta}$ is balanced with balanced generalized gramian Σ_{new}, where $[\![\Sigma_{\text{new}}]\!](t,k)=w_T^{-1}(t,k_0)[\![\Sigma]\!](t,k)$. In particular, $\Omega_{\text{new},T}(t,k_0)=w_T^{-1}(t,k_0)\Omega_T(t,k_0)=I$ for all $t\in\mathcal{F}_T(k_0)$. We reduce system $\mathcal{G}_{\text{new},\delta}$ via BT by only truncating the temporal state of subsystem $G_{\text{new}}^{(k_0)}$ at time-steps $t\in\mathcal{F}_T(k_0)$ and denote the resulting reduced-order system by $\mathcal{G}_{\text{new,red},\delta}$. From Theorem 1, we have $w_T^{-1}(t_1,k_0)\times\rho\times\|(\mathcal{G}_{\text{new},\delta}-\mathcal{G}_{\text{new,red},\delta})\|<2$, i.e., $\|(\mathcal{G}_{\text{new},\delta}-\mathcal{G}_{\text{new,red},\delta})\|<2\,S_{w_T(t,k_0)}$ for all $\Delta\in\mathbf{\Delta}$. Finally, because of the special structure of T, $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|=\|(\mathcal{G}_{\text{new},\delta}-\mathcal{G}_{\text{new,red},\delta})\|$ for all $\Delta\in\mathbf{\Delta}$.

We now give the proof for the case $l=0$, i.e., when the sequence $w_T(t,k_0)$ is monotone nonincreasing in t; see also [2, Proof of Theorem 3]. The balancing transformation T is defined as $[\![T]\!](t,k)=w_T^{1/2}(t,k_0)I$ for all $(t,k)\in\mathbb{Z}\times V$. The realization $(A_{\text{new}}$, B_{new}, C_{new}, D, $\mathbf{\Delta})$ of $\mathcal{G}_{\text{new},\delta}$ satisfies

$$\begin{aligned}&[\![A_{\text{new}}]\!](t,k)w_T(t,k_0)[\![\Sigma]\!](t,k)[\![A_{\text{new}}^*]\!](t,k)\\&-w_T(t+1,k_0)[\![S^*\Sigma S]\!](t,k)+[\![B_{\text{new}}]\!](t,k)[\![B_{\text{new}}^*]\!](t,k)\prec-\beta I,\\&[\![A_{\text{new}}^*]\!](t,k)w_T^{-1}(t+1,k_0)[\![S^*\Sigma S]\!](t,k)[\![A_{\text{new}}]\!](t,k)\\&-w_T^{-1}(t,k_0)[\![\Sigma]\!](t,k)+[\![C_{\text{new}}^*]\!](t,k)[\![C_{\text{new}}]\!](t,k)\prec-\beta I.\end{aligned}$$

Since $w_T(t,k_0)$ is nonincreasing in t, then $0<w_T(t+1,k_0)\le w_T(t,k_0)$ and $w_T^{-1}(t,k_0)\le w_T^{-1}(t+1,k_0)$; and since $w_T(t,k_0)\le1$ for all $t\in\mathbb{Z}$, then $w_T(t,k_0)\le w_T^{-1}(t,k_0)$. Thus, one can verify that

$$\begin{aligned}&[\![A_{\text{new}}]\!](t,k)w_T^{-1}(t,k_0)[\![\Sigma]\!](t,k)[\![A_{\text{new}}^*]\!](t,k)-w_T^{-1}(t+1,k_0)\\&\times[\![S^*\Sigma S]\!](t,k)+[\![B_{\text{new}}]\!](t,k)[\![B_{\text{new}}^*]\!](t,k)\prec-\beta I.\end{aligned}$$

That is, the realization (A_{new}, B_{new}, C_{new}, D, $\mathbf{\Delta}$) of $\mathcal{G}_{\text{new},\delta}$ is balanced with balanced generalized gramian Σ_{new} such that $[\![\Sigma_{\text{new}}]\!](t,k)=w_T^{-1}(t,k_0)[\![\Sigma]\!](t,k)$. In particular, $\Omega_{\text{new},T}(t,k_0)=w_T^{-1}(t,k_0)\Omega_T(t,k_0)=I$ for all $t\in\mathcal{F}_T(k_0)$. We reduce the temporal state corresponding to subsystem $G_{\text{new}}^{(k_0)}$. Then, from Theorem 1 and due to the special structure of T, we have $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|=\|(\mathcal{G}_{\text{new},\delta}-\mathcal{G}_{\text{new,red},\delta})\|<2=2w_T(t_1,k_0)$ for all $\Delta\in\mathbf{\Delta}$. The final equality follows from the fact that $w_T(t_1,k_0)=1$; this is true since the system is scaled to ensure that the monotone nonincreasing sequence $w_T(t,k_0)$ satisfies $w_T(t,k_0)\le1$ for all $t\in\mathcal{F}_T(k_0)$. □

We now illustrate the application of Theorems 2 and 3. Suppose that we are only to truncate the temporal state $x_T(t,k_0)$ and that $\Omega_T(t,k_0)=\frac{1}{t}I$, where $t\in\mathcal{F}_T(k_0)=\{1,2,3,4,5\}$. Using Theorem 2, $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|<2\times(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})\approx4.57$ for all $\Delta\in\mathbf{\Delta}$. But, since $w_T(t,k_0)$ is a monotone decreasing sequence, then from Theorem 3, $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|<2\times1=2$ for all $\Delta\in\mathbf{\Delta}$. This bound represents a 56% improvement over the bound of Theorem 2. For the same set $\mathcal{F}_T(k_0)$, assume that $\Omega_T(1,k_0)=\text{diag}(2,1)$, $\Omega_T(2,k_0)=\text{diag}(17,0.5)$, $\Omega_T(3,k_0)=\text{diag}(4,2)$, $\Omega_T(4,k_0)=\text{diag}(6,1)$, $\Omega_T(5,k_0)=\text{diag}(16,4)$. From Theorem 2, $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|<2\times(17+16+6+4+2+1+0.5)=93$ for all $\Delta\in\mathbf{\Delta}$, whereas from Theorem 3, $\|(\mathcal{G}_\delta-\mathcal{G}_{\text{red},\delta})\|<2\times(2\times\frac{17}{2}\times\frac{16}{4}+1\times\frac{2}{0.5}\times\frac{4}{1})=168$. As illustrated below, this bound is obtained by first truncating the boxed sequence (1, 0.5, 2, 1, 4) and then truncating the circled sequence (2, 17, 4, 6, 16).

$$\Omega_T(1,k_0)=\begin{bmatrix}\textcircled{2}&0\\0&\boxed{1}\end{bmatrix},\quad\Omega_T(2,k_0)=\begin{bmatrix}\textcircled{17}&0\\0&\boxed{0.5}\end{bmatrix},$$

$$\Omega_T(3,k_0)=\begin{bmatrix}\textcircled{4}&0\\0&\boxed{2}\end{bmatrix},\quad\Omega_T(4,k_0)=\begin{bmatrix}\textcircled{6}&0\\0&\boxed{1}\end{bmatrix},$$

$$\Omega_T(5,k_0)=\begin{bmatrix}\textcircled{16}&0\\0&\boxed{4}\end{bmatrix}.$$

However, we can split the truncation sequences into more than two sequences as illustrated below:

$$\Omega_T(1,k_0)=\begin{bmatrix}\triangle\,2&0\\0&\textcircled{1}\end{bmatrix},\quad\Omega_T(2,k_0)=\begin{bmatrix}\textcircled{17}&0\\0&\boxed{0.5}\end{bmatrix},$$

$$\Omega_T(3,k_0)=\begin{bmatrix}\triangle\,4&0\\0&\boxed{2}\end{bmatrix},\quad\Omega_T(4,k_0)=\begin{bmatrix}\triangle\,6&0\\0&\underline{1}\end{bmatrix},$$

$$\Omega_T(5,k_0)=\begin{bmatrix}\textcircled{16}&0\\0&\boxed{4}\end{bmatrix}.$$

These sequences are truncated in the following order: the underlined sequence (1), the boxed sequence (0.5, 2, 4), the circled sequence (1, 17, 16), and the sequence (2, 4, 6) inside the triangles. In this case, the bound from Theorem 3 becomes $2\times(1+0.5\times$

$\frac{4}{0.5} + 1 \times \frac{17}{1} + 2 \times \frac{6}{2}) = 56$, which represents a 40% improvement over the bound from Theorem 2. The same bound can be retrieved by considering the truncation sequences (0.5, 1), (1, 2, 6, 4), (2, 4), and (17, 16), respectively. This raises the question of how to best apply Theorem 3 to obtain the least conservative error bound. Answering this question is indeed a nontrivial task, and future work will focus on developing a fast computational algorithm which effectively applies Theorem 3 to compute a useful bound for a given truncation sequence. One possible approach consists of modeling the truncation sequence as a directed graph, where the vertices correspond to the truncated values and the directed edges are obtained from the allowable truncation sequences. The problem then becomes one of finding the graph partition which minimizes the upper bound expression from Theorem 3. The precise details of this modeling approach are still not fully developed, and other possibilities still need to be explored. Solutions to the formulated problem can then be computed using heuristics and approximation algorithms [15]. Adopting ideas from Dijkstra's shortest path algorithm [4] and set packing [23], among others, may prove useful in this direction. Finally, Theorem 3 can also be used to improve on the error bound due to truncation over the finite time-horizon in the case of eventually time-periodic subsystems.

5. Coprime factors reduction

Section 4 applies the BT method to strongly stable systems. However, there exist ℓ_2-stable systems with no solutions in $\mathcal{X}$ to (3), i.e., ℓ_2-stable systems that are not strongly stable, which introduces conservatism into BT. [9] points out the difficulty in quantitatively and sharply assessing this type of conservatism. [24] gives sufficient conditions for the existence of structured solutions to the LMI therein, and [26] identifies a class of systems with guaranteed structured solutions to the generalized Lyapunov inequalities. This section presents the CFR method as a means of extending the range of applicability of BT. CFR consists of finding a strongly stable coprime factorization for system $\mathcal{G}_\delta$ which is not necessarily strongly stable. This factorization forms an augmented system $\mathcal{H}_\delta$ which is reducible via BT. The reduced-order system $\mathcal{H}_{\text{red},\delta}$ results in a strongly stable coprime factorization for the reduced-order system $\mathcal{G}_{\text{red},\delta}$. The results in this section extend their counterparts in [10, Propositions 26 and 31 and Theorem 27], [3, Theorem 1, Lemma 3, and Algorithm 2], and [19, Section III] to the class of distributed NSLPV systems.

5.1. Strong stabilizability

Let $\mathcal{F}$ be the set of partitioned graph-diagonal operators $F = [F_T \;\; F_1^S \;\; \cdots \;\; F_d^S \;\; F_1^P \;\; \cdots \;\; F_r^P]$, where for $i = 1, \ldots, d$ and $j = 1, \ldots, r$, $F_T \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_T(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, $F_i^S \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_i^S(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, and $F_j^P \in \mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_j^P(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$.

Definition 5. A well-posed distributed NSLPV system $\mathcal{G}_\delta$ is said to be strongly stabilizable if there exists a feedback operator $F \in \mathcal{F}$ such that the resulting closed-loop system is strongly stable, i.e., if and only if there exists $F \in \mathcal{F}$ and $P \in \mathcal{X}$ such that

$$(A + BF)P(A + BF)^* - S^*PS \prec 0. \qquad (13)$$

The notion of strong detectability is defined as the dual notion of strong stabilizability.

Theorem 4. *There exist $F \in \mathcal{F}$ and $P \in \mathcal{X}$ that satisfy (13) if and only if there exists $Q \in \mathcal{X}$ that satisfies $AQA^* - S^*QS - BB^* \prec 0$. Furthermore, when the relevant inverse exists, a strongly stabilizing choice of F is given by $F = -(B^*S^*Q^{-1}SB)^{-1}B^*S^*Q^{-1}SA$.*

The proof of this theorem is similar to that of [3, Theorem 1] and so is omitted.

5.2. Right coprime factorization

We now define the notion of a right coprime factorization (RCF) for a distributed NSLPV system $\mathcal{G}_\delta$ with realization $(A, B, C, D, \mathbf{\Delta})$. We show that if $\mathcal{G}_\delta$ is a strongly stabilizable and strongly detectable system, then it admits a strongly stable RCF.

Definition 6. Two operators N_δ and M_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ are said to be right coprime if there exist two operators U_δ and V_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ such that $U_\delta N_\delta + V_\delta M_\delta = I$. Two ℓ_2-stable distributed NSLPV systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are said to be right coprime if their input-output maps N_δ and M_δ are right coprime for all $\Delta \in \mathbf{\Delta}$.

Definition 7. The pair $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ of ℓ_2-stable distributed NSLPV systems is said to be an RCF for $\mathcal{G}_\delta$ if $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are right coprime and, for all $\Delta \in \mathbf{\Delta}$, M_δ has a causal inverse on ℓ_{2e} and $G_\delta = N_\delta M_\delta^{-1}$.

Theorem 5. *A strongly stabilizable and strongly detectable system $\mathcal{G}_\delta$ admits a strongly stable RCF $(\mathcal{N}_\delta, \mathcal{M}_\delta)$, where the realizations of $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are given by $(A + BF, B, C + DF, D, \mathbf{\Delta})$ and $(A + BF, B, F, I, \mathbf{\Delta})$, respectively, and F is any strongly stabilizing feedback operator.*

Proof. $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are strongly stable systems since F is a strongly stabilizing feedback operator. We verify that the pair $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ satisfies the conditions in Definition 7. First, we show that M_δ has a causal inverse on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$. Let $R_\delta = \Delta \star \left[\begin{array}{c|c} SA & SB \\ \hline -F & I \end{array}\right]$. R_δ is well-defined and causal on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$ since system $\mathcal{G}_\delta$ is well-posed, i.e., by Remark 1, $I - \Delta SA$ has a causal inverse on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$. We verify that $M_\delta R_\delta = R_\delta M_\delta = I$ for all $\Delta \in \mathbf{\Delta}$. For compactness, we introduce the symbol $\zeta = [x^* \;\; \beta^*]^*$. For a fixed $\Delta \in \mathbf{\Delta}$, $\zeta_M = \Delta S(A + BF)\zeta_M + \Delta SBu_M$, $y_M = F\zeta_M + u_M$, $\zeta_R = \Delta SA\zeta_R + \Delta SBu_R$, and $y_R = -F\zeta_R + u_R$. If $y_M = u_R$, then $(I - \Delta SA)(\zeta_M - \zeta_R) = 0$. But, since $I - \Delta SA$ has a causal inverse on ℓ_{2e}, then $\zeta_M - \zeta_R = 0$ and $y_R = u_M$, i.e., $R_\delta M_\delta = I$. We conclude similarly that $M_\delta R_\delta = I$. Thus, R_δ is the inverse of M_δ for all $\Delta \in \mathbf{\Delta}$. The second step is to show that $G_\delta = N_\delta M_\delta^{-1} = N_\delta R_\delta$ for all $\Delta \in \mathbf{\Delta}$. We have

$$N_\delta R_\delta = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} S(A+BF) & -SBF & SB \\ 0 & SA & SB \\ \hline C+DF & -DF & D \end{array}\right] = \bar{\Delta} \star \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right].$$

Let $Q = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, where I has a compatible structure with Δ. Then,

$$N_\delta R_\delta = \left(Q^{-1}\bar{\Delta}Q\right) \star \left[\begin{array}{c|c} Q^{-1}\bar{A}Q & Q^{-1}\bar{B} \\ \hline \bar{C}Q & \bar{D} \end{array}\right] = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star$$

$$\left[\begin{array}{cc|c} S(A+BF) & 0 & 0 \\ 0 & SA & SB \\ \hline C+DF & C & D \end{array}\right] = C(I - \Delta SA)^{-1}\Delta SB + D.$$

That is, $N_\delta R_\delta = G_\delta$. Finally, we show that $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are right coprime. Since $\mathcal{G}_\delta$ is strongly detectable, there exists a bounded, partitioned graph-diagonal operator K, with a structure similar to F^* and appropriate dimensions, that renders the resulting closed-loop system strongly stable. For each $\Delta \in \mathbf{\Delta}$, consider the operators U_δ and V_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ defined as

$$U_\delta = \Delta \star \left[\begin{array}{c|c} S(A+KC) & SK \\ \hline F & 0 \end{array}\right] \quad \text{and}$$

$$V_\delta = \Delta \star \left[\begin{array}{c|c} S(A+KC) & S(B+KD) \\ \hline -F & I \end{array}\right].$$

To prove that $U_\delta N_\delta + V_\delta M_\delta = I$, we let $u_N = u_M$, $u_U = y_N$, and $u_V = y_M$ and show that $y_U + y_V = u_M$. We write the equations for U_δ, N_δ, and V_δ similarly to the equations of M_δ and R_δ. Then, using the above relations, we get $(I - \Delta S(A + KC))(\zeta_U - \zeta_V + \zeta_M) =$

$\Delta SK(C+DF)(\zeta_N-\zeta_M)$, $y_U+y_V=F(\zeta_U-\zeta_V+\zeta_M)+u_M$, and $(I-\Delta S(A+BF))(\zeta_N-\zeta_M)=0$. Since $(I-\Delta S(A+BF))$ and $(I-\Delta S(A+KC))$ have bounded causal inverses, then $\zeta_N-\zeta_M=0$, $\zeta_U-\zeta_V+\zeta_M=0$, and $y_U+y_V=u_M$. □

5.3. Coprime factors reduction method

Algorithm 1. Given a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ with realization $(A, B, C, D, \mathbf{\Delta})$, we obtain a reduced-order system $\mathcal{G}_{\text{red},\delta}$ via the CFR method as follows:

1. Find $P\in\mathcal{X}$ such that $APA^*-S^*PS-BB^*\prec 0$.
2. Define $F=-(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$. Ensure that F is well-defined by removing any control redundancies so that $[\![B]\!](t,k)$ has full column rank for all $(t,k)\in\mathbb{Z}\times V$; see [3, Theorem 1].
3. Construct a strongly stable RCF $(\mathcal{N}_\delta,\mathcal{M}_\delta)$ for system $\mathcal{G}_\delta$ as in the proof of Theorem 5.
   - $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ have realizations $(A+BF, B, C+DF, D, \mathbf{\Delta})$ and $(A+BF, B, F, I, \mathbf{\Delta})$, respectively.
4. Form an augmented strongly stable system $\mathcal{H}_\delta=[\mathcal{N}_\delta^*\ \ \mathcal{M}_\delta^*]^*$ with realization $(A_H, B_H, C_H, D_H, \mathbf{\Delta})=(A+BF, B, [(C+DF)^*\ \ F^*]^*, [D^*\ \ I]^*, \mathbf{\Delta})$.
5. Find generalized gramians X and Y for system $\mathcal{H}_\delta$.
   - Find $X\in\mathcal{X}$ with minimum trace such that $A_HXA_H^*-S^*XS+B_HB_H^*\prec 0$.
   - Find $Y\in\mathcal{X}$ with minimum trace such that $A_H^*S^*YSA_H-Y+C_H^*C_H\prec 0$.
6. Construct a balanced realization $(A_{H,\text{ bal}}, B_{H,\text{ bal}}, C_{H,\text{ bal}}, D_H, \mathbf{\Delta})$ for $\mathcal{H}_\delta$ as in the proof of Lemma 3.
   - Find a balancing transformation T and express the balanced generalized gramian as $\Sigma=TXT^*$.
   - Define $A_{\text{bal}}=(S^*TS)AT^{-1}$, $B_{\text{bal}}=(S^*TS)B$, $C_{\text{bal}}=CT^{-1}$, and $F_{\text{bal}}=FT^{-1}$.
   - Define $A_{H,\text{bal}}=A_{\text{bal}}+B_{\text{bal}}F_{\text{bal}}$, $B_{H,\text{bal}}=B_{\text{bal}}$, and $C_{H,\text{bal}}=[(C_{\text{bal}}+DF_{\text{bal}})^*\ \ F_{\text{bal}}^*]^*$.
7. Apply the BT method to this balanced realization and obtain a reduced-order system $\mathcal{H}_{\text{red},\delta}=[\mathcal{N}_{\text{red},\delta}^*\ \ \mathcal{M}_{\text{red},\delta}^*]^*$.
   - Determine the dimensions of the reduced-order system based on the following:
     (a) The upper bound on $\|(H_\delta-H_{\text{red},\delta})\|$, for all $\Delta\in\mathbf{\Delta}$, obtained by judiciously applying Theorems 2 and 3.
     (b) The upper bound γ on $\|H_\delta\|$, for all $\Delta\in\mathbf{\Delta}$, obtained by applying Lemma 2.
     (c) The absolute and relative orders of the diagonal entries in Σ.
   - Denote the realization of $\mathcal{H}_{\text{red},\delta}$ by $(A_{H,\text{ red}}, B_{H,\text{ red}}, C_{H,\text{ red}}, D_H, \mathbf{\Delta}_{\text{red}})$.
   - Denote the reduced-order balanced generalized gramian by $\Gamma=\text{diag}(\Gamma_T,\Gamma_1^S,\ldots,\Gamma_d^S,\Gamma_1^P,\ldots,\Gamma_r^P)$.
8. Define the operators A_{red}, B_{red}, C_{red}, and F_{red} as follows: $B_{\text{red}}=B_{H,\text{red}}$, $[(C_{\text{red}}+DF_{\text{red}})^*\ \ F_{\text{red}}^*]^*=C_{H,\text{red}}$, and $A_{\text{red}}=A_{H,\text{red}}-B_{\text{red}}F_{\text{red}}$.
   - With Q defined as in the proof of Lemma 4, notice that the previous operators also satisfy $Q^*SA_{\text{bal}}Q=\begin{bmatrix}S&0\\0&S\end{bmatrix}\begin{bmatrix}A_{\text{red}}&\tilde{A}_{12}\\\tilde{A}_{21}&\tilde{A}_{22}\end{bmatrix}$, $\begin{bmatrix}C_{\text{bal}}\\F_{\text{bal}}\end{bmatrix}Q=\begin{bmatrix}C_{\text{red}}&\tilde{C}_2\\F_{\text{red}}&\tilde{F}_2\end{bmatrix}$, and $Q^*SB_{\text{bal}}=\begin{bmatrix}S&0\\0&S\end{bmatrix}\begin{bmatrix}B_{\text{red}}\\\tilde{B}_2\end{bmatrix}$.
9. Systems $\mathcal{N}_{\text{red},\delta}$ and $\mathcal{M}_{\text{red},\delta}$ with realizations $(A_{\text{red}}+B_{\text{red}}F_{\text{red}}, B_{\text{red}}, C_{\text{red}}+DF_{\text{red}}, D, \mathbf{\Delta}_{\text{red}})$ and $(A_{\text{red}}+B_{\text{red}}F_{\text{red}}, B_{\text{red}}, F_{\text{red}}, I, \mathbf{\Delta}_{\text{red}})$ are strongly stable and right coprime.
10. If $I-\Delta_{\text{red}}SA_{\text{red}}$ has a causal inverse on ℓ_{2e} (see Remark 2) for all $\Delta_{\text{red}}\in\mathbf{\Delta}_{\text{red}}$, then
    (a) $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \mathbf{\Delta}_{\text{red}})$ is a realization for the reduced-order system $\mathcal{G}_{\text{red},\delta}$;
    (b) $(\mathcal{N}_{\text{red},\delta},\mathcal{M}_{\text{red},\delta})$ is an RCF for system $\mathcal{G}_{\text{red},\delta}$, i.e., $G_{\text{red},\delta}=N_{\text{red},\delta}M_{\text{red},\delta}^{-1}$ for all $\Delta_{\text{red}}\in\mathbf{\Delta}_{\text{red}}$;
    (c) F_{red} strongly stabilizes $\mathcal{G}_{\text{red},\delta}$.

Remark 2. Let $A_{\text{red},\ PP}$ and $A_{\text{bal},\ PP}$ be defined similarly to A_{PP} in Remark 1 but with the blocks of A_{red} and A_{bal} used instead of A, respectively. To ensure that $I-\Delta_{\text{red}}SA_{\text{red}}$ has a causal inverse on ℓ_{2e} for all $\Delta_{\text{red}}\in\mathbf{\Delta}_{\text{red}}$, and since the state-space matrices are zeros for $t<0$, we only need to ensure that $I-\hat{\Delta}_PA_{\text{red},PP}$ has a causal inverse on $\oplus_{j=1}^r\ell_{2e}(\{\mathbb{R}^{m_j^P(t,k)}\})$ for all $\Delta_{\text{red}}\in\mathbf{\Delta}_{\text{red}}$, where $\hat{\Delta}_P$ is defined in (7). This is guaranteed if the generalized gramian X in Algorithm 1 additionally satisfies $A_{PP}\,\text{diag}(X_1^P,\ldots,X_r^P)A_{PP}^*-\text{diag}(X_1^P,\ldots,X_r^P)\prec 0$. Namely, since $\Sigma=TXT^*$ and Σ and T have block-diagonal structures, the previous inequality is equivalent to $A_{\text{bal},PP}\,\text{diag}(\Sigma_1^P,\ldots,\Sigma_r^P)A_{\text{bal},PP}^*-\text{diag}(\Sigma_1^P,\ldots,\Sigma_r^P)\prec 0$. Then, similarly to Lemma 4, we can show that $A_{\text{red},PP}\,\text{diag}(\Gamma_1^P,\ldots,\Gamma_r^P)A_{\text{red},PP}^*-\text{diag}(\Gamma_1^P,\ldots,\Gamma_r^P)\prec 0$, i.e., $\|\text{diag}(\Gamma_1^P,\ldots,\Gamma_r^P)^{-\frac{1}{2}}A_{\text{red},PP}\,\text{diag}(\Gamma_1^P,\ldots,\Gamma_r^P)^{\frac{1}{2}}\|<1$. But, $\text{diag}(\Gamma_1^P,\ldots,\Gamma_r^P)$ commutes with every permissible $\hat{\Delta}_P$, and so, using the sub-multiplicative property, we see that $\hat{\Delta}_PA_{\text{red},PP}$ has a spectral radius less than 1 and $I-\hat{\Delta}_PA_{\text{red},PP}$ has a bounded causal inverse for all $\Delta_{\text{red}}\in\mathbf{\Delta}_{\text{red}}$. An alternative similar condition can be derived based on the fact that $\Sigma=(T^*)^{-1}YT^{-1}$.

Remark 3. The bound from BT explicitly relates to the ℓ_2-induced norm of the error system and can be used in robustness analysis [20], wherein the full-order system is replaced by the reduced-order system and a perturbation operator whose norm is less than the bound. Namely, for all $\Delta\in\mathbf{\Delta}$, G_δ can be expressed as $G_\delta=G_{\text{red},\delta}+(G_\delta-G_{\text{red},\delta})$, where $\|(G_\delta-G_{\text{red},\delta})\|$ is less than the BT error bound. Thus, the possibly tighter bound from Theorem 3 helps in better quantifying the perturbation operator and yielding less conservative robustness results. Similarly, the bound from CFR on $\|(H_\delta-H_{\text{red},\delta})\|$, for all $\Delta\in\mathbf{\Delta}$, can be interpreted in terms of the robust stability of the closed-loop system as in [3, Theorem 5]. Namely, the CFR bound can be related to the maximum number of state variables that one can truncate from $\mathcal{G}_\delta$ while ensuring that a controller $\mathcal{K}_\delta$ which stabilizes $\mathcal{G}_\delta$ also stabilizes $\mathcal{G}_{\text{red},\delta}$. Let $\mathcal{K}_\delta$ be a distributed NSLPV controller that inherits the structures of $\mathcal{G}_\delta$ and renders the closed-loop system strongly stable [1]. That is, $\mathcal{K}_\delta$ strongly stabilizes $\mathcal{G}_\delta$, or equivalently, $\mathcal{G}_\delta$ strongly stabilizes $\mathcal{K}_\delta$. In other words, the controller is strongly stabilizable and strongly detectable, has right and left coprime factorizations, and is reducible via CFR. If CFR is applied to $\mathcal{K}_\delta$, then the CFR bound indicates how far one can proceed with the truncation while ensuring that the reduced-order controller still stabilizes $\mathcal{G}_\delta$.

Remark 4. Various distributed control techniques, e.g., [1,9,12], apply to strongly stabilizable and strongly detectable systems and guarantee that the resulting closed-loop system is strongly stable. Thus, the systems to which these synthesis techniques apply are reducible via CFR. Since the synthesis problems involve solving sequences of LMIs of a larger size than the generalized Lyapunov inequalities, model reduction can be used to render the control synthesis problems computationally feasible. Moreover, since the distributed control techniques usually yield distributed NSLPV controllers that are of a comparable size to the plant and that inherit both the interconnection and the uncertainty structures, then model reduction can also be used to construct reduced-order controllers. Namely, applying model reduction prior to control synthesis ensures that the least controllable and least observable modes of the plant are not reflected in the designed controller. Thus, in addition to simplifying the computational complexity of the analysis and synthesis problems, model reduction is also beneficial from an implementation point of view in scenarios where resources are limited. Specifically, since the proposed BT and CFR methods allow for the evaluation of the importance of a particular interconnection and the possible truncation of the corresponding spatial state, then the methods can be used to simplify the communication network in a rigorous manner by removing inconsequential communication links.

6. Numerical example

We now apply CFR to a distributed NSLPV system $\mathcal{G}_\delta$ formed by 4 subsystems interconnected as in Fig. 1. The leader $G^{(1)}$ has discrete-time LTV dynamics and the followers have discrete-time LPV dynamics. There is only one parameter affecting each of the followers, i.e., $r = 1$, and so the parameter subscript is dropped. The example illustrates the characteristics of the proposed methods. For instance, the example demonstrates the truncation of the various types of states and further shows that the truncation need not be uniform in time, even if the dimensions of the states in the full-order system are constants. As per Remark 2, the example shows that, when applying Algorithm 1, one needs to impose/verify the well-posedness of the resulting reduced-order system. The example also illustrates the use of the trace minimization heuristic discussed after Lemma 3 and the 1-norm heuristic [2] for improving on the computed error bound.

For the leader, all the state-space matrices are constants, except for $\bar{A}_{TT}(t,1)$ which is $(h = 0, q = 28)$-eventually time-periodic, i.e., $\bar{A}_{TT}(t+q,1) = \bar{A}_{TT}(t,1)$ for all $t \in \mathbb{N}_0$. Specifically, $\bar{A}_{TT}(t,1) = \mathcal{A}$ for $t = 0,\dots,6$; $\bar{A}_{TT}(t,1) = Q\mathcal{A}Q^*$ for $t = 7,\dots,13$; $\bar{A}_{TT}(t,1) = Q^2\mathcal{A}(Q^*)^2$ for $t = 14,\dots,20$; $\bar{A}_{TT}(t,1) = Q^3\mathcal{A}(Q^*)^3$ for $t = 21,\dots,27$, where

$$\mathcal{A} = 0.15\begin{bmatrix} 9 & 5 & 1 & 0.3 & -0.2 & 0.2 \\ -2 & 7 & 1 & 0.1 & 0.1 & -0.3 \\ 1 & -1 & -1 & 0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & -0.3 & 0 & 0 & 0 \\ -0.3 & 0.1 & -0.1 & 0 & 0 & 0 \\ -0.2 & 0.3 & 0.2 & 0 & 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are no incoming edges to vertex $k = 1$ and $G^{(1)}$ is not affected by any parameter, the state-space matrices $A_e^{TS}(t,1)$, $\bar{A}_{TP}(t,1)$, $A_{ie}^{SS}(t,1)$, $A_i^{SP}(t,1)$, $\bar{A}_{PT}(t,1)$, $A_e^{PS}(t,1)$, $\bar{A}_{PP}(t,1)$, $\bar{B}_P(t,1)$, $C_e^S(t,1)$, and $\bar{C}_P(t,1)$, for $i, e \in \{1, 2\}$, have at least one dimension equal to zero, i.e., are non-existent. The remaining state-space matrices are defined as follows for all $t \in \mathbb{N}_0$: $\bar{C}_T(t,1) = [I_2 \;\; 0_{2\times4}]$,

$$A_i^{ST}(t,1) = 0.1\begin{bmatrix} 2 & 2 & -1 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 & 0 \\ 0.1 & -0.1 & 0.2 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B}_T(t,1) = 0.1\begin{bmatrix} I_2 \\ 0_{4\times2} \end{bmatrix}, \quad B_i^S(t,1) = 0.1\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The state-space matrices of the followers are constants for all $t \in \mathbb{N}_0$, $i, e \in \{1, 2\}$, and $k = 2, 3, 4$; these matrices are given by

$$\bar{A}_{TT}(t,k) = 0.15\begin{bmatrix} 7 & 4 & 1 & 0.2 & -0.1 & 0.2 \\ -3 & 5 & 1 & 0.2 & -0.2 & -0.1 \\ 1 & -3 & -2 & 0.1 & 0.1 & 0.2 \\ -0.1 & 0.3 & 0.1 & 0 & 0 & 0 \\ 0.2 & 0.1 & -0.2 & 0 & 0 & 0 \\ 0.1 & -0.2 & 0.1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_{TP}(t,k) = 0.01\begin{bmatrix} 1 & 3 & 2 & 0.1 \\ 2 & 1 & 1 & 0.1 \\ -3 & 4 & -3 & -0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_{PT}(t,k) = \begin{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} & 0_{2\times4} \\ \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix} & 0_{2\times4} \end{bmatrix},$$

$$A_e^{PS}(t,k) = \begin{bmatrix} 0.1 & 0.2 & 0 \\ -0.2 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_e^{TS}(t,k) = 0.05\begin{bmatrix} -3 & 2 & 0.1 \\ 4 & 4 & 0.2 \\ 2 & -3 & -0.2 \\ -0.2 & -0.1 & 0 \\ 0.1 & 0.3 & 0 \\ 0.3 & -0.2 & 0 \end{bmatrix},$$

$\bar{A}_{PP}(t,k) = 0.1I_4$, $\bar{B}_T(t,k) = 0.1\begin{bmatrix} I_2 \\ 0_{4\times2} \end{bmatrix}$, $\bar{B}_P(t,k) = 0_{4\times2}$, $\bar{C}_T(t,k) = [I_2 \;\; 0_{2\times4}]$, $C_e^S(t,k) = 0_{2\times3}$, $\bar{C}_P(t,k) = 0_{2\times4}$, $A_i^{ST}(t,k) = W\bar{A}_{TT}(t,k)$, $A_{ie}^{SS}(t,k) = WA_e^{TS}(t,k)$, $A_i^{SP}(t,k) = W\bar{A}_{TP}(t,k)$, $B_i^S(t,k) = W\bar{B}_T(t,k)$, where $W = 0.25[I_3 \;\; 0_3]$. Finally, $\bar{D}(t,k) = 0$ for all $(t,k) \in \mathbb{N}_0 \times \{1,2,3,4\}$.

System $\mathcal{G}_\delta$ is not strongly stable and cannot be reduced via BT. However, $\mathcal{G}_\delta$ is strongly stabilizable and strongly detectable and is reducible via CFR. Since the subsystems are (0, 28)-eventually time-periodic, the sought solutions to the subsequent semi-definite programming (SDP) problems are all (0, 28)-eventually time-periodic. These problems are modeled using Yalmip [18] and are solved using SDPT3 [25]. The computations are carried out in Matlab 7.10.0.499 (The MathWorks Inc., Natick, Massachusetts, USA) on a Hewlett-Packard laptop with 2 Intel Cores, 2.30 GHz processors, and 4 GB of RAM running Windows 7. First, we find $P \in \mathcal{X}$ such that $APA^* - S^*PS - BB^* \prec 0$ and define the strongly stabilizing feedback operator $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$. Then, we form the strongly stable augmented system $\mathcal{H}_\delta$ with realization (A_H, B_H, C_H, D_H, $\mathbf{\Delta}$), where $A_H = A + BF$, $B_H = B$, $C_H = [(C+DF)^* \;\; F^*]^*$, and $D_H = [D^* \;\; I]^*$. Using Lemma 2, we find an upper bound $\gamma = 2.23$ on $\|H_\delta\|$ for all $\Delta \in \mathbf{\Delta}$. This SDP is the most computationally expensive: the total number of constraints is 4033, the dimension of the SDP variable is 3080, the number of SDP blocks is 448, and the dimension of the linear variable is 1. The corresponding wall-clock time is about 29.6 s (CPU time 25.8 s). Then, we find generalized gramians for system $\mathcal{H}_\delta$. We solve for $X \in \mathcal{X}$ that satisfies $A_HXA_H^* - S^*XS + B_HB_H^* \prec 0$ and minimizes $\sum_{t=0}^{27}\left(\sum_{k=1}^{4}\left(\operatorname{tr} X_T(t,k) + \sum_{i=1}^{2}\operatorname{tr} X_i^S(t,k)\right) + \sum_{k=2}^{4}\operatorname{tr} X_P(t,k)\right)$. We also solve for $Y \in \mathcal{X}$ that satisfies $A_H^*S^*YSA_H - Y + C_H^*C_H \prec 0$ and

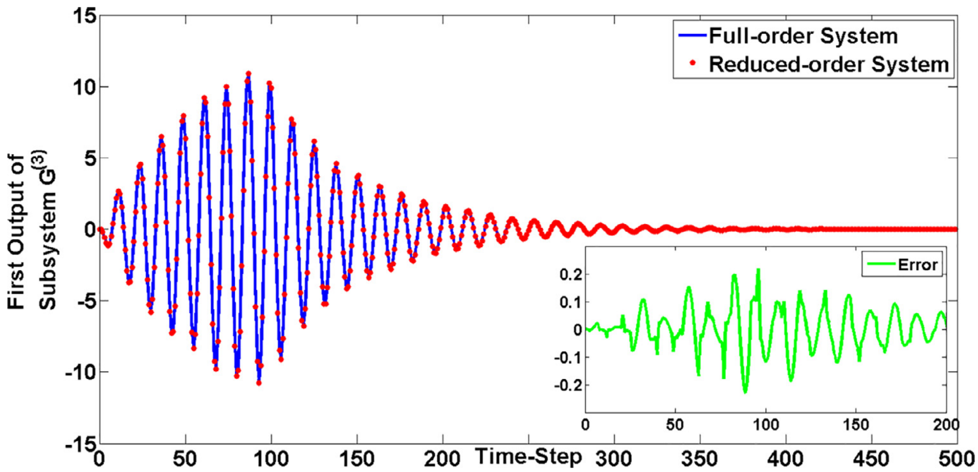

Fig. 2. Responses of $\mathcal{G}_\delta$ and $\mathcal{G}_{\text{red},\delta}$ for the same set of applied inputs.

minimizes a similar objective function. We construct a balanced realization for $\mathcal{H}_\delta$ as in Step 6 of Algorithm 1.

To obtain a useful error bound, we re-solve the generalized Lyapunov inequalities for the obtained balanced realization. We now seek a balanced generalized gramian $\Sigma \succ \epsilon I$ and a scalar $\epsilon > 0$ that simultaneously satisfy both generalized Lyapunov inequalities and minimize the following objective function:

$$a_1 \times \epsilon + \sum_{t=0}^{27}\left(\sum_{k=1}^{4}\left(\|\text{vec}(\Sigma_T(t,k)-\epsilon I)\|_1 + \sum_{i=1}^{2}\left\|\text{vec}(\Sigma_i^S(t,k)-\epsilon I)\right\|_1\right)\right.$$
$$\left. + \sum_{k=2}^{4}\|\text{vec}(\Sigma_P(t,k)-\epsilon I)\|_1\right),$$

where $\text{vec}(\mathcal{Q})$ denotes the vector formed by the diagonal entries of a square matrix $\mathcal{Q}$ and $\|v\|_1$ is the 1-norm of vector v. The sought $[\![\Sigma]\!](t,k)$ are diagonal matrices, which are guaranteed to exist because a balanced realization of $\mathcal{H}_\delta$ is used. The objective function is chosen to be the sum of two cost functions. The first, $a_1\epsilon$, ensures that the optimal value of ϵ is small. In the second cost function, the 1-norm is used as a heuristic for finding a solution Σ with many entries equal to ϵ. Thus, ϵ can be regarded as the truncation cut-off value, i.e., all the state variables corresponding to an entry equal to ϵ in Σ are truncated. Clearly, we want ϵ to be as small as possible because the error bound obtained from Theorem 2 is 2ϵ, i.e., $\|(H_\delta - H_{\text{red},\delta})\| < 2\epsilon$ for all $\Delta \in \mathbf{\Delta}$. We would also like many entries in Σ to be equal to ϵ in order to truncate many state variables, regardless of their type. The weight given to ϵ is $a_1 = 850$. This value gives the best trade-off between the competing objectives of a small error bound and a large number of truncated state variables. The result is $\epsilon = 0.0233$, i.e., $\|(H_\delta - H_{\text{red},\delta})\| < 2.11\%\gamma$. Thus, BT is applied to $\mathcal{H}_\delta$ and each state variable with a corresponding entry in Σ equal to ϵ is truncated. The reduced-order system is denoted by $\mathcal{H}_{\text{red},\delta}$ and its balanced realization is given by $(A_{H,\text{ red}}, B_{H,\text{ red}}, C_{H,\text{ red}}, D_H, \mathbf{\Delta}_{\text{red}})$. At each time-step, the total numbers of truncated temporal, spatial, and parameter state variables range from 9 to 13, 3 to 5, and 5 to 6, respectively.

The realization $(A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D, \mathbf{\Delta}_{\text{red}})$ of system $\mathcal{G}_{\text{red},\delta}$ is formed from the realization of $\mathcal{H}_{\text{red},\delta}$ as follows: define $B_{\text{red}} = B_{H,\text{red}}$, deduce C_{red} and F_{red} from $[(C_{\text{red}} + DF_{\text{red}})^* \; F_{\text{red}}^*]^* = C_{H,\text{red}}$, and compute $A_{\text{red}} = A_{H,\text{red}} - B_{\text{red}}F_{\text{red}}$. The condition in Remark 2 is satisfied, which guarantees that $\mathcal{G}_{\text{red},\delta}$ is well-posed. As an example, systems $\mathcal{G}_\delta$ and $\mathcal{G}_{\text{red},\delta}$ are subjected to the same set of sinusoidal inputs of various amplitudes and frequencies for the first 100 time-steps and are left to evolve on their own afterwards. The parameter values $\delta(t,k)$, for $k = 2, 3, 4$, are varied randomly from -1 to 1. As expected from the small error bound, the responses of the full-order and the reduced-order systems, which are plotted in Fig. 2, are very close.

7. Conclusion

BT and CFR are extended to the class of distributed NSLPV systems. BT applies to strongly stable systems, and CFR applies to strongly stabilizable and strongly detectable systems. The methods are structure-preserving since the interpretation of the temporal, spatial, and parameter states is retained in the reduced-order system. The methods are also structure-simplifying because whole interconnections and whole channels from the Δ-operator can be removed during model reduction. In general, the methods involve solving infinite sequences of LMIs. However, for the class of eventually time-periodic subsystems interconnected over finite graphs, the required computations become finite dimensional with no added conservatism.

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Appendix A

This appendix details the structure of the operator Q needed in the proof of Lemma 4. To simplify the presentation, we assume that the balanced generalized gramian Σ is partitioned into two blocks as in $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$. The given structure of Q immediately generalizes to arbitrary partitions of Σ. Note that the particular type of Σ_1 and Σ_2, i.e., whether they correspond to the temporal, spatial, or parameter blocks of Σ, does not affect the discussion. For this reason, we simply use the subscript $i = 1, 2$. From Section 4.2, recall that $\Sigma_i = [\![\text{diag}(\Gamma_i, \Omega_i)]\!]$, where both sides of the equality correspond to graph-diagonal operators. The problem is to find a partitioned graph-diagonal operator Q such that $Q^*\Sigma Q = \text{diag}(\Gamma, \Omega)$, $QQ^* = I$, and $Q^*Q = I$, where $\Gamma = \text{diag}(\Gamma_1, \Gamma_2)$ and $\Omega = \text{diag}(\Omega_1, \Omega_2)$. To this end, we define the graph-diagonal operators $Q_{\Gamma_i} = [\![\begin{smallmatrix} I \\ 0 \end{smallmatrix}]\!]$ and $Q_{\Omega_i} = [\![\begin{smallmatrix} 0 \\ I \end{smallmatrix}]\!]$, i.e., $Q_{\Gamma_i}(t,k) = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $Q_{\Omega_i}(t,k) = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ for all $(t,k) \in \mathbb{Z} \times V$. In particular, $\Sigma_i Q_{\Gamma_i} = [\![\begin{smallmatrix} \Gamma_i & 0 \\ 0 & \Omega_i \end{smallmatrix}]\!][\![\begin{smallmatrix} I \\ 0 \end{smallmatrix}]\!] = [\![[\begin{smallmatrix} \Gamma_i & 0 \\ 0 & \Omega_i \end{smallmatrix}][\begin{smallmatrix} I \\ 0 \end{smallmatrix}]]\!] = [\![\begin{smallmatrix} \Gamma_i \\ 0 \end{smallmatrix}]\!]$ and $\Sigma_i Q_{\Omega_i} = [\![\begin{smallmatrix} 0 \\ \Omega_i \end{smallmatrix}]\!]$. The operator Q_{Γ_i} further satisfies

$Q^*_{\Gamma_i}\left[\!\left[\begin{smallmatrix}\Gamma_i\\0\end{smallmatrix}\right]\!\right] = \left[\!\left[\begin{smallmatrix}I & 0\end{smallmatrix}\right]\!\right]\left[\!\left[\begin{smallmatrix}\Gamma_i\\0\end{smallmatrix}\right]\!\right] = \Gamma_i$. Similarly, $Q^*_{\Gamma_i}\left[\!\left[\begin{smallmatrix}0\\\Omega_i\end{smallmatrix}\right]\!\right] = 0$, $Q^*_{\Omega_i}\left[\!\left[\begin{smallmatrix}\Gamma_i\\0\end{smallmatrix}\right]\!\right] = 0$, $Q^*_{\Omega_i}\left[\!\left[\begin{smallmatrix}0\\\Omega_i\end{smallmatrix}\right]\!\right] = \Omega_i$, $Q^*_{\Gamma_i}Q_{\Gamma_i} = I$, $Q^*_{\Gamma_i}Q_{\Omega_i} = 0$, $Q^*_{\Omega_i}Q_{\Gamma_i} = 0$, and $Q^*_{\Omega_i}Q_{\Omega_i} = I$. Finally, $Q_{\Gamma_i}Q^*_{\Gamma_i} + Q_{\Omega_i}Q^*_{\Omega_i} = \left[\!\left[\begin{smallmatrix}I\\0\end{smallmatrix}\right]\!\right]\left[\!\left[\begin{smallmatrix}I & 0\end{smallmatrix}\right]\!\right] + \left[\!\left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]\!\right]\left[\!\left[\begin{smallmatrix}0 & I\end{smallmatrix}\right]\!\right] = \left[\!\left[\begin{smallmatrix}I & 0\\0 & I\end{smallmatrix}\right]\!\right] = I$.

Then, the operator Q can be defined as $Q = \left[\begin{smallmatrix}Q_{\Gamma_1} & 0 & Q_{\Omega_1} & 0\\ 0 & Q_{\Gamma_2} & 0 & Q_{\Omega_2}\end{smallmatrix}\right]$; and the relations $Q^*\Sigma Q = Q^*\mathrm{diag}(\Sigma_1, \Sigma_2)Q = \mathrm{diag}(\Gamma, \Omega) = \mathrm{diag}(\Gamma_1, \Gamma_2, \Omega_1, \Omega_2)$, $QQ^* = I$, and $Q^*Q = I$ can be readily verified. Namely,

$$Q^*\Sigma Q = \begin{bmatrix} Q^*_{\Gamma_1} & 0\\ 0 & Q^*_{\Gamma_2}\\ Q^*_{\Omega_1} & 0\\ 0 & Q^*_{\Omega_2}\end{bmatrix}\begin{bmatrix}\Sigma_1 Q_{\Gamma_1} & 0 & \Sigma_1 Q_{\Omega_1} & 0\\ 0 & \Sigma_2 Q_{\Gamma_2} & 0 & \Sigma_2 Q_{\Omega_2}\end{bmatrix}$$

$$= \begin{bmatrix} Q^*_{\Gamma_1} & 0\\ 0 & Q^*_{\Gamma_2}\\ Q^*_{\Omega_1} & 0\\ 0 & Q^*_{\Omega_2}\end{bmatrix}\begin{bmatrix}\left[\!\left[\begin{matrix}\Gamma_1\\0\end{matrix}\right]\!\right] & 0 & \left[\!\left[\begin{matrix}0\\\Omega_1\end{matrix}\right]\!\right] & 0\\ 0 & \left[\!\left[\begin{matrix}\Gamma_2\\0\end{matrix}\right]\!\right] & 0 & \left[\!\left[\begin{matrix}0\\\Omega_2\end{matrix}\right]\!\right]\end{bmatrix}$$

$$= \begin{bmatrix} Q^*_{\Gamma_1}\left[\!\left[\begin{matrix}\Gamma_1\\0\end{matrix}\right]\!\right] & 0 & Q^*_{\Gamma_1}\left[\!\left[\begin{matrix}0\\\Omega_1\end{matrix}\right]\!\right] & 0\\ 0 & Q^*_{\Gamma_2}\left[\!\left[\begin{matrix}\Gamma_2\\0\end{matrix}\right]\!\right] & 0 & Q^*_{\Gamma_2}\left[\!\left[\begin{matrix}0\\\Omega_2\end{matrix}\right]\!\right]\\ Q^*_{\Omega_1}\left[\!\left[\begin{matrix}\Gamma_1\\0\end{matrix}\right]\!\right] & 0 & Q^*_{\Omega_1}\left[\!\left[\begin{matrix}0\\\Omega_1\end{matrix}\right]\!\right] & 0\\ 0 & Q^*_{\Omega_2}\left[\!\left[\begin{matrix}\Gamma_2\\0\end{matrix}\right]\!\right] & 0 & Q^*_{\Omega_2}\left[\!\left[\begin{matrix}0\\\Omega_2\end{matrix}\right]\!\right]\end{bmatrix}$$

$$= \begin{bmatrix}\Gamma_1 & 0 & 0 & 0\\ 0 & \Gamma_2 & 0 & 0\\ 0 & 0 & \Omega_1 & 0\\ 0 & 0 & 0 & \Omega_2\end{bmatrix},$$

$$Q^*Q = \begin{bmatrix} Q^*_{\Gamma_1} & 0\\ 0 & Q^*_{\Gamma_2}\\ Q^*_{\Omega_1} & 0\\ 0 & Q^*_{\Omega_2}\end{bmatrix}\begin{bmatrix}Q_{\Gamma_1} & 0 & Q_{\Omega_1} & 0\\ 0 & Q_{\Gamma_2} & 0 & Q_{\Omega_2}\end{bmatrix}$$

$$= \begin{bmatrix} Q^*_{\Gamma_1}Q_{\Gamma_1} & 0 & Q^*_{\Gamma_1}Q_{\Omega_1} & 0\\ 0 & Q^*_{\Gamma_2}Q_{\Gamma_2} & 0 & Q^*_{\Gamma_2}Q_{\Omega_2}\\ Q^*_{\Omega_1}Q_{\Gamma_1} & 0 & Q^*_{\Omega_1}Q_{\Omega_1} & 0\\ 0 & Q^*_{\Omega_2}Q_{\Gamma_2} & 0 & Q^*_{\Omega_2}Q_{\Omega_2}\end{bmatrix}$$

$$= \begin{bmatrix}I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix},$$

$$QQ^* = \begin{bmatrix}Q_{\Gamma_1} & 0 & Q_{\Omega_1} & 0\\ 0 & Q_{\Gamma_2} & 0 & Q_{\Omega_2}\end{bmatrix}\begin{bmatrix} Q^*_{\Gamma_1} & 0\\ 0 & Q^*_{\Gamma_2}\\ Q^*_{\Omega_1} & 0\\ 0 & Q^*_{\Omega_2}\end{bmatrix}$$

$$= \begin{bmatrix}Q_{\Gamma_1}Q^*_{\Gamma_1} + Q_{\Omega_1}Q^*_{\Omega_1} & 0\\ 0 & Q_{\Gamma_2}Q^*_{\Gamma_2} + Q_{\Omega_2}Q^*_{\Omega_2}\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}.$$

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