



## Brief paper

# Balanced truncation model reduction of nonstationary systems interconnected over arbitrary graphs<sup>☆</sup>

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## ABSTRACT

This paper deals with the balanced truncation of discrete-time, linear time-varying, heterogeneous subsystems interconnected over finite arbitrary directed graphs. The information transfer between the subsystems is subject to a communication latency of one time-step. The method guarantees the preservation of the interconnection structure and further allows for its simplification. In addition to truncating temporal states associated with the subsystems, the method allows for the order reduction of spatial states associated with the interconnections between the subsystems and even the removal of whole interconnections. Upper bounds on the  $\ell_2$ -induced norm of the resulting error system are derived. The method is illustrated through an example.

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## 1. Introduction

Various biological and engineering systems consist of multiple interacting agents. Mathematically describing such systems can lead to models with a very large number of states, especially in the case of “large” models of the agents and complicated interconnection structures. Thus arises the need for model order reduction to simplify the control analysis and synthesis problems. Interconnected systems can be treated as one global system; and standard model reduction tools, like balanced truncation (BT) and coprime factors reduction (CFR), can then be applied to these systems. However, such an approach is not always desirable as it does not guarantee the preservation of the interconnection structure of the system. Various works have addressed the problem of structure-preserving BT and CFR for interconnected systems. See, for example, [Abou Jaoude and Farhood \(2015, 2016\)](#), [Al-Taie and Werner \(2016\)](#), [Li and Paganini \(2005\)](#) and [Sandberg and Murray \(2009\)](#). The methods in these references, like the one in [Beck, Doyle, and Glover \(1996\)](#), are based on the existence of block-diagonally structured solutions to linear matrix inequalities (LMIs). BT applies to stable systems, guarantees the stability of the reduced order system, and comes with a guaranteed upper bound on the

norm of the error system. CFR applies to stabilizable and detectable systems, and guarantees the stabilizability and detectability of the reduced order system. The bound from CFR is not in terms of the norm of the error between the full and reduced order systems, but rather is in terms of the norm of the error between their corresponding coprime factorizations.

The aforementioned works can be classified based on the modeling of the interconnections between the subsystems. Namely, [Sandberg and Murray \(2009\)](#) account for the interconnection structure using a transfer function matrix, whereas, [Al-Taie and Werner \(2016\)](#) and [Abou Jaoude and Farhood \(2015, 2016\)](#) model the interconnections between the subsystems as states, which we refer to as spatial states. The latter methods allow for the order reduction of the spatial states in addition to the standard states associated with the subsystems, which we refer to as the temporal states. That is, in addition to guaranteeing the preservation of the interconnection structure, these methods further allow for its simplification. [Al-Taie and Werner \(2016\)](#) deal with homogeneous, linear time-invariant (LTI) or linear parameter-varying (LPV) subsystems, interconnected over a grid. Model reduction is only applied to one subsystem, and then, all the temporal states, all the forward spatial states, and all the backward spatial states are truncated in a uniform way, respectively. [Abou Jaoude and Farhood \(2015, 2016\)](#) deal with heterogeneous, linear time-varying (LTV) subsystems, interconnected over arbitrary directed graphs, allow for individually truncating each of the temporal and spatial states, and even permit the removal of a whole interconnection if it is deemed negligible. Due to the time-varying nature of the subsystems, these methods usually involve solving infinite sequences

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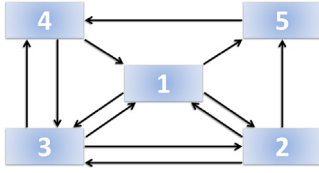


Fig. 1. Example of a directed graph.

of LMIs. Also, if truncation is performed at infinitely many time-steps, the resulting error bound might not be finite. [Abou Jaoude and Farhood \(2015\)](#) show that in the special case of time-periodic subsystems, the sequences of LMIs can be restricted to the first time-period, and the error bound is guaranteed to be finite.

The current work extends the results of [Abou Jaoude and Farhood \(2015\)](#) from time-periodic subsystems to eventually time-periodic subsystems, i.e., subsystems which become time-periodic after some initial finite time-horizon. A tighter expression for the error bound for general LTV subsystems is derived, which applies when the truncated entries of the balanced gramians form monotonic sequences in time. An example which illustrates the application of the method is also provided.

The paper is organized as follows. In Section 2, we define the notation and summarize the adopted state-space framework along with the relevant analysis results. Then, in Section 3, we present the BT method. We derive the error bounds in Section 4 and treat the special class of eventually time-periodic subsystems in Section 5. In Section 6, we apply the method to an example. We conclude the paper with Section 7.

## 2. Preliminaries

### 2.1. Notation

The sets of nonnegative integers, real numbers, and  $n \times n$  symmetric matrices are denoted by  $\mathbb{N}_0$ ,  $\mathbb{R}$ , and  $\mathbb{S}^n$ , respectively. Let  $S$  be an ordered subset of  $\mathbb{N}_0$ . We denote by  $(v_i)_{i \in S}$  the vector-valued sequence associated with  $S$  and by  $\text{vec}(v_i)_{i \in S}$  the vertical concatenation of the elements of  $(v_i)_{i \in S}$ . The elements in  $(v_i)_{i \in S}$  are ordered conformably with the elements in  $S$ . We denote by  $(M_i)_{i \in S}$  the matrix-valued sequence associated with  $S$  and by  $\text{diag}(M_i)_{i \in S}$  the block-diagonal augmentation of the elements of  $(M_i)_{i \in S}$ . For example, if  $S = \{1, 2, 4\}$ , then,  $(v_i)_{i \in S} = (v_1, v_2, v_4)$ ,  $\text{vec}(v_i)_{i \in S} = [v_1^T \ v_2^T \ v_4^T]^T$ ,  $(M_i)_{i \in S} = (M_1, M_2, M_4)$ , and  $\text{diag}(M_i)_{i \in S} = \text{diag}(M_1, M_2, M_4)$ .  $0_{i \times j}$  denotes an  $i \times j$  zero matrix and  $I_i$  denotes an  $i \times i$  identity matrix.

$\mathcal{G}(V, E)$  refers to a directed graph with set of vertices  $V$  and set of directed edges  $E$ . We assume throughout that the directed graph under consideration is finite. That is, both  $V$  and  $E$  are finite sets. We denote by  $N$  the finite number of vertices, and choose  $V = \{1, \dots, N\}$ . The ordered pair  $(i, j)$  is in  $E$  if there exists a directed edge from  $i \in V$  to  $j \in V$ . For each  $k \in V$ , we define the sets  $E_{\text{in}}^{(k)} = \{i \in V \mid (i, k) \in E\}$  and  $E_{\text{out}}^{(k)} = \{j \in V \mid (k, j) \in E\}$  with cardinalities  $m(k)$  and  $p(k)$ , respectively. The elements in these sets are ordered in an increasing fashion. Consider the directed graph in Fig. 1. For  $k = 1$ , we define  $E_{\text{in}}^{(1)} = \{2, 3, 4\}$ ,  $m(1) = 3$ ,  $E_{\text{out}}^{(1)} = \{2, 3, 5\}$ ,  $p(1) = 3$ , and so on.

$X < 0$  (resp.  $> 0$ ) means that the symmetric matrix  $X$  is negative definite (resp. positive definite). Let  $n$  be an integer sequence such that  $n : (t, k) \in \mathbb{N}_0 \times V \rightarrow n^{(k)}(t) \in \mathbb{N}_0$ . We define  $\ell(\{\mathbb{R}^{n^{(k)}(t)}\})$  as the vector space of mappings  $w : (t, k) \in \mathbb{N}_0 \times V \rightarrow w^{(k)}(t) \in \mathbb{R}^{n^{(k)}(t)}$ . The Hilbert space  $\ell_2(\{\mathbb{R}^{n^{(k)}(t)}\})$  is the subspace of  $\ell(\{\mathbb{R}^{n^{(k)}(t)}\})$  consisting of mappings  $w$  with a finite  $\ell_2$ -norm  $\|w\|_2 = (\sum_{(t,k)} w^{(k)}(t)^T w^{(k)}(t))^{\frac{1}{2}}$ . We subsequently use the abbreviated symbols  $\ell$  and  $\ell_2$ .

### 2.2. State-space representation

We now give the state-space equations for a distributed system  $G$ , formed by discrete-time, heterogeneous, LTV subsystems interconnected over finite arbitrary directed graphs and subjected to a communication latency. We represent the interconnection structure of  $G$  using a directed graph  $\mathcal{G}(V, E)$ , where each subsystem  $G^{(k)}$  corresponds to a vertex  $k \in V$ , and the interconnections between the subsystems are described by the directed edges. Each subsystem  $G^{(k)}$  has a discrete-time LTV model, with states  $x^{(k)}(t)$ , inputs  $u^{(k)}(t)$ , and outputs  $y^{(k)}(t)$ . We refer to the states associated with the subsystems as temporal states or vertex/node states. The interconnections between the subsystems are also modeled using states, which we refer to as spatial states or interconnection/edge states. Namely, we associate a state  $x^{(ij)}(t)$  with each edge  $(i, j) \in E$ . Due to the communication latency, the information sent from  $G^{(i)}$  at time  $t$  reaches  $G^{(j)}$  at  $t + 1$ . For each subsystem  $G^{(k)}$ , we define vectors  $x_{\text{in}}^{(k)}(t) = \text{vec}(x^{(ik)}(t))_{i \in E_{\text{in}}^{(k)}}$  and  $x_{\text{out}}^{(k)}(t) = \text{vec}(x^{(kj)}(t))_{j \in E_{\text{out}}^{(k)}}$ , which are partitioned into  $m(k)$  and  $p(k)$  vector-valued channels, respectively. These vectors represent the total information received and sent by  $G^{(k)}$  at time  $t$ . When all subsystems are considered, and since the interconnection input to a subsystem is an output to another subsystem, both  $x_{\text{out}}^{(k)}$  and  $x_{\text{in}}^{(k)}$  contain all spatial states  $x^{(ij)}$ . We assume zero initial conditions for the temporal and the spatial states. Then, for all  $(t, k) \in \mathbb{N}_0 \times V$ , we have  $x^{(k)}(0) = 0$ ,  $x_{\text{in}}^{(k)}(0) = 0$ ,

$$\begin{aligned} \begin{bmatrix} x^{(k)}(t+1) \\ x_{\text{out}}^{(k)}(t+1) \end{bmatrix} &= A^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + B^{(k)}(t) u^{(k)}(t), \\ y^{(k)}(t) &= C^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + D^{(k)}(t) u^{(k)}(t). \end{aligned} \quad (1)$$

The matrix-valued sequences of state-space matrices, e.g.,  $A^{(k)}(t)$ , are known a priori and assumed to be uniformly bounded. The dimensions of signals  $x^{(k)}(t)$ ,  $u^{(k)}(t)$ ,  $y^{(k)}(t)$ , and  $x^{(ij)}(t)$  can vary with  $t$ , and  $k$  or  $(i, j)$ , and are denoted by  $n^{(k)}(t)$ ,  $n_u^{(k)}(t)$ ,  $n_y^{(k)}(t)$ , and  $n^{(ij)}(t)$ , respectively, for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ . We denote the realization of system  $G$  by the quadruple  $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ .

For each  $(t, k)$ , the state-space matrices are naturally partitioned conformably with the partitioning of  $[x^{(k)}(t+1) \ x_{\text{out}}^{(k)}(t+1)]^T$  and  $[x^{(k)}(t) \ x_{\text{in}}^{(k)}(t)]^T$ . For example, consider the distributed system in Fig. 1. The state-space matrices of subsystem  $G^{(1)}$  are partitioned as  $C^{(1)}(t) = [C_0^{(1)}(t) \ C_2^{(1)}(t) \ C_3^{(1)}(t) \ C_4^{(1)}(t)]$ , where  $C_0^{(1)}(t)$  is an  $n_y^{(1)}(t) \times n^{(1)}(t)$  matrix, etc. Similarly,

$$\begin{aligned} A^{(1)}(t) &= \begin{bmatrix} A_{00}^{(1)}(t) & A_{02}^{(1)}(t) & A_{03}^{(1)}(t) & A_{04}^{(1)}(t) \\ A_{20}^{(1)}(t) & A_{22}^{(1)}(t) & A_{23}^{(1)}(t) & A_{24}^{(1)}(t) \\ A_{30}^{(1)}(t) & A_{32}^{(1)}(t) & A_{33}^{(1)}(t) & A_{34}^{(1)}(t) \\ A_{40}^{(1)}(t) & A_{42}^{(1)}(t) & A_{43}^{(1)}(t) & A_{44}^{(1)}(t) \end{bmatrix}, \\ B^{(1)}(t) &= [B_0^{(1)}(t)^T \ B_2^{(1)}(t)^T \ B_3^{(1)}(t)^T \ B_5^{(1)}(t)^T]^T, \end{aligned}$$

where  $A_{00}^{(1)}(t)$  is an  $n^{(1)}(t+1) \times n^{(1)}(t)$  matrix,  $B_0^{(1)}(t)$  is an  $n^{(1)}(t+1) \times n_u^{(1)}(t)$  matrix, etc.

### 2.3. Analysis results

We now summarize the relevant analysis results of [Farhood, Di, and Dullerud \(2015\)](#). Consider a system  $G$  with realization  $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ . Since the state-space equations (1) have zero initial conditions and the state-space matrices are defined for  $t \in \mathbb{N}_0$ , we can equivalently assume that the state-space matrices are zeros for  $t < 0$ . Then, from [Dullerud and D'Andrea \(2004\)](#), it can be shown that  $G$  is well-posed, i.e., given inputs in  $\ell$ , the state-space equations admit unique solutions in  $\ell$ , and further

define a linear causal mapping on  $\ell$ . A well-posed system  $G$  is said to be stable if, given inputs in  $\ell_2$ , the state-space equations admit unique solutions in  $\ell_2$ , and further define a linear causal mapping on  $\ell_2$ . Next, we give a Lyapunov-based test to check if  $G$  is stable. This result constitutes the basis of the proposed BT scheme.

**Lemma 1.** *System  $G$  is stable if there exist uniformly bounded, positive definite, matrix-valued functions  $X^{(k)}(t) \in \mathbb{S}^{n^{(k)}(t)}$  and  $X^{(ij)}(t) \in \mathbb{S}^{n^{(ij)}(t)}$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ , such that, for some  $\beta > 0$ ,  $X^{(k)}(t) \succ \beta I$ ,  $X^{(ij)}(t) \succ \beta I$ , and*

$$A^{(k)}(t)^T \text{diag}(X^{(k)}(t+1), X_{\text{out}}^{(k)}(t+1))A^{(k)}(t) - \text{diag}(X^{(k)}(t), X_{\text{in}}^{(k)}(t)) \prec -\beta I, \quad (2)$$

$$X_{\text{in}}^{(k)}(t) = \text{diag}(X^{(ik)}(t))_{i \in E_{\text{in}}^{(k)}}, X_{\text{out}}^{(k)}(t) = \text{diag}(X^{(kj)}(t))_{j \in E_{\text{out}}^{(k)}}.$$

The solutions to (2) can be classified into temporal terms  $X^{(k)}(t)$  and spatial terms  $X^{(ij)}(t)$ . Due to the time-varying nature of the subsystems, there is an infinite sequence of LMIs associated with each subsystem. Moreover, the LMI sequence associated with a given subsystem is coupled with the LMI sequences of the other subsystems through the spatial terms. The “ $\beta I$ ” terms in (2) are small quantities added to ensure that the matrix sequences on the left-hand side do not converge to singular matrices as  $t$  approaches infinity. Subsequently, we no longer specify the dimensions of  $X^{(k)}(t)$  and  $X^{(ij)}(t)$ .

Systems that satisfy the conditions in Lemma 1 are called strongly stable. Strong stability implies stability, but the converse is not always true. Specifically, strongly stable systems are stable systems which have the required structured solutions to (2). The proposed BT scheme suffers from conservatism as it only applies to strongly stable systems. However, this imposed structure on the solutions of (2) allows for the preservation and the simplification of the interconnection structure during the model reduction process. Trnka, Sturk, Sandberg, Havlena, and Rehor (2013) and Sootla and Anderson (2016) identify classes of systems with guaranteed structured solutions to LMIs.

For a stable system  $G$  mapping  $u \in \ell_2$  to  $y \in \ell_2$  and starting from zero initial conditions, the  $\ell_2$ -induced norm is defined as  $\|G\| = \sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2}$ .

**Lemma 2.** *System  $G$  is strongly stable and satisfies  $\|G\| < \gamma$ , for some  $\gamma > 0$ , if there exist  $\beta > 0$  and uniformly bounded, positive definite, matrix-valued functions  $X^{(k)}(t) \succ \beta I$  and  $X^{(ij)}(t) \succ \beta I$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ , such that*

$$F^{(k)}(t)^T \text{diag}(X^{(k)}(t+1), X_{\text{out}}^{(k)}(t+1), I)F^{(k)}(t) - \text{diag}(X^{(k)}(t), X_{\text{in}}^{(k)}(t), \gamma^2 I) \prec -\beta I, \quad (3)$$

$$\text{where } F^{(k)}(t) = \begin{bmatrix} A^{(k)}(t) & B^{(k)}(t) \\ C^{(k)}(t) & D^{(k)}(t) \end{bmatrix}.$$

### 3. Balanced truncation model reduction

#### 3.1. Balanced realization

We now extend the notions of generalized Lyapunov inequalities and generalized gramians, discussed in Beck et al. (1996) and Hinrichsen and Pritchard (1990), to the class of distributed systems. Namely, the controllability and observability generalized gramians are uniformly bounded, positive definite, matrix-valued functions, denoted by  $X^{(k)}(t)$ ,  $X^{(ij)}(t)$  and  $Y^{(k)}(t)$ ,  $Y^{(ij)}(t)$ , respectively, which, for some scalar  $\beta > 0$ , satisfy  $X^{(k)}(t) \succ \beta I$ ,  $X^{(ij)}(t) \succ$

$\beta I$  and  $Y^{(k)}(t) \succ \beta I$ ,  $Y^{(ij)}(t) \succ \beta I$ , in addition to the following generalized Lyapunov inequalities:

$$A^{(k)}(t) \text{diag}(X^{(k)}(t), X_{\text{in}}^{(k)}(t))A^{(k)}(t)^T + B^{(k)}(t)B^{(k)}(t)^T - \text{diag}(X^{(k)}(t+1), X_{\text{out}}^{(k)}(t+1)) \prec -\beta I, \quad (4)$$

$$A^{(k)}(t)^T \text{diag}(Y^{(k)}(t+1), Y_{\text{out}}^{(k)}(t+1))A^{(k)}(t) - \text{diag}(Y^{(k)}(t), Y_{\text{in}}^{(k)}(t)) + C^{(k)}(t)^T C^{(k)}(t) \prec -\beta I, \quad (5)$$

where  $Y_{\text{in}}^{(k)}(t)$  and  $Y_{\text{out}}^{(k)}(t)$  are defined similarly to  $X_{\text{in}}^{(k)}(t)$  and  $X_{\text{out}}^{(k)}(t)$ . The generalized Lyapunov inequalities and the generalized gramians allow for the definition of a balanced realization of a distributed system as given next.

**Definition 1.** The realization of system  $G$  is said to be balanced if there exist  $\beta > 0$  and uniformly bounded, diagonal, positive definite, matrix-valued functions  $\Sigma^{(k)}(t) \succ \beta I$  and  $\Sigma^{(ij)}(t) \succ \beta I$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ , that simultaneously satisfy (4) and (5), i.e.,  $\Sigma^{(k)}(t) = X^{(k)}(t) = Y^{(k)}(t)$ ,  $\Sigma^{(ij)}(t) = X^{(ij)}(t) = Y^{(ij)}(t)$ , and  $\Sigma^{(k)}(t)$  and  $\Sigma^{(ij)}(t)$  are diagonal matrices.

We can show (see the proof of Theorem 4) that the existence of solutions to (2) is equivalent to the existence of solutions to (4) and (5), respectively. Thus, the generalized gramians are only defined for strongly stable systems. These gramians can be used to construct a balanced realization and balanced generalized gramians  $\Sigma^{(k)}(t)$ ,  $\Sigma^{(ij)}(t)$  for a given system  $G$ , as outlined next.

**Algorithm 1.** We construct a balanced realization for a given strongly stable system  $G$  with generalized gramians  $X^{(k)}(t)$ ,  $X^{(ij)}(t)$  and  $Y^{(k)}(t)$ ,  $Y^{(ij)}(t)$  as follows. First, we compute the Cholesky factorizations  $X^{(k)}(t) = R^{(k)}(t)^T R^{(k)}(t)$  and  $Y^{(k)}(t) = H^{(k)}(t)^T H^{(k)}(t)$ , and we perform the singular value decompositions  $H^{(k)}(t)R^{(k)}(t)^T = U^{(k)}(t)\Sigma^{(k)}(t)V^{(k)}(t)^T$ . Then, we define the balancing transformations  $T^{(k)}(t) = \Sigma^{(k)}(t)^{-1/2} U^{(k)}(t)^T H^{(k)}(t)$  and their inverses  $T^{(k)}(t)^{-1} = R^{(k)}(t)^T V^{(k)}(t) \Sigma^{(k)}(t)^{-1/2}$ . Similar steps are repeated for the spatial terms.  $\Sigma^{(k)}(t)$ ,  $\Sigma^{(ij)}(t)$  are the balanced generalized gramians. We augment the obtained transformations as in  $T_{\text{pre}}^{(k)}(t) = \text{diag}(T^{(k)}(t), T_{\text{out}}^{(k)}(t))$ ,  $T_{\text{post}}^{(k)}(t) = \text{diag}(T^{(k)}(t), T_{\text{in}}^{(k)}(t))^{-1}$ . A balanced realization  $(A_{\text{bal}}^{(k)}(t), B_{\text{bal}}^{(k)}(t), C_{\text{bal}}^{(k)}(t), D^{(k)}(t))$  of  $G$  is then given by

$$A_{\text{bal}}^{(k)}(t) = T_{\text{pre}}^{(k)}(t+1)A^{(k)}(t)T_{\text{post}}^{(k)}(t),$$

$$B_{\text{bal}}^{(k)}(t) = T_{\text{pre}}^{(k)}(t+1)B^{(k)}(t), \quad \text{and}$$

$$C_{\text{bal}}^{(k)}(t) = C^{(k)}(t)T_{\text{post}}^{(k)}(t).$$

An alternative balancing algorithm is also given in Abou Jaoude and Farhood (2015). Clearly, the balanced realization of a strongly stable system is not unique as it depends on the followed algorithm as well as the solutions to (4) and (5) used in the algorithm. To obtain useful results for model reduction, namely entries in the balanced generalized gramians that yield reasonable error bounds, we use a trace heuristic, i.e., we find generalized gramians with minimum sum of traces, see e.g., Farhood and Beck (2014) and Farhood and Dullerud (2007).

#### 3.2. Balanced truncation

Let  $G$  be a distributed system with a balanced realization  $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ . We assume, without loss of generality, that the diagonal entries of the balanced generalized gramians are ordered in a decreasing fashion. The essence of BT is to truncate the state variables associated with the negligible entries. We partition the gramians into two blocks: one corresponding to the non-truncated states and the other to the truncated states. We illustrate the partitioning process for the temporal terms. Given

integers  $r^{(k)}(t)$ , such that  $0 \leq r^{(k)}(t) \leq n^{(k)}(t)$ , we partition  $\Sigma^{(k)}(t)$  as  $\Sigma^{(k)}(t) = \text{diag}(\Gamma^{(k)}(t), \Omega^{(k)}(t))$ , where  $\Gamma^{(k)}(t) \in \mathbb{S}^{r^{(k)}(t)}$  are reduced order gramians associated with the non-truncated states and  $\Omega^{(k)}(t)$  correspond to the truncated states. By allowing  $r^{(k)}(t)$  to be equal to 0 or  $n^{(k)}(t)$  for some  $(t, k)$ , we allow that either all or no variables be truncated from the corresponding temporal state  $x^{(k)}(t)$ . This results in either  $\Gamma^{(k)}(t)$  or  $\Omega^{(k)}(t)$  having a zero dimension, which is a slight abuse of notation. The proposed method allows for the evaluation of the importance of a particular interconnection, and accordingly, the reduction of the dimension of the spatial state vector associated with it and even the complete removal of the interconnection. For example, the case where  $r^{(ij)}(t) = 0$ , for all  $t \in \mathbb{N}_0$ , corresponds to the removal of edge  $(i, j)$  altogether from the interconnection structure.

The next step is to partition the blocks of the state-space matrices in accordance with the partitioning of the blocks of  $\text{diag}(\Sigma^{(k)}(t+1), \Sigma^{(k)}_{\text{out}}(t+1))$  and  $\text{diag}(\Sigma^{(k)}(t), \Sigma^{(k)}_{\text{in}}(t))$ . Consider subsystem  $G^{(1)}$  in Fig. 1.  $A^{(1)}_{00}(t)$  is partitioned according to the partitioning of  $\Sigma^{(1)}(t+1)$  and  $\Sigma^{(1)}(t)$  as in  $A^{(1)}_{00}(t) = \begin{bmatrix} \hat{A}^{(1)}_{00}(t) & A^{(1)}_{0012}(t) \\ A^{(1)}_{0021}(t) & A^{(1)}_{0022}(t) \end{bmatrix}$ , where  $\hat{A}^{(1)}_{00}(t)$  is an  $r^{(1)}(t+1) \times r^{(1)}(t)$  matrix.  $B^{(1)}_0(t)$  is partitioned conformably with the partitioning of  $\Sigma^{(1)}(t+1)$  as in  $B^{(1)}_0(t) = \begin{bmatrix} \hat{B}^{(1)}_0(t)^T & B^{(1)}_{02}(t)^T \end{bmatrix}^T$ , where  $\hat{B}^{(1)}_0(t)$  is an  $r^{(1)}(t+1) \times n^{(1)}_u(t)$  matrix. Likewise,  $C^{(1)}_0(t)$  is partitioned according to the partitioning of  $\Sigma^{(1)}(t)$ , namely,  $C^{(1)}_0(t) = \begin{bmatrix} \hat{C}^{(1)}_0(t) & C^{(1)}_{02}(t) \end{bmatrix}$ , where  $\hat{C}^{(1)}_0(t)$  is an  $n^{(1)}_y(t) \times r^{(1)}(t)$  matrix, and so on. Now, we form the realization  $(A^{(k)}_r(t), B^{(k)}_r(t), C^{(k)}_r(t), D^{(k)}(t))$  of the reduced order system  $G_r$ . For  $A^{(k)}_r(t)$ ,  $B^{(k)}_r(t)$ , and  $C^{(k)}_r(t)$ , we keep the blocks that correspond to the non-truncated states, i.e., the partitions marked with a hat, e.g.,  $C^{(1)}_r(t) = [\hat{C}^{(1)}_0(t) \quad \hat{C}^{(1)}_2(t) \quad \hat{C}^{(1)}_3(t) \quad \hat{C}^{(1)}_4(t)]$ .

It will be useful to permute the original state-space matrices and balanced gramians in order to group together the non-truncated blocks. For example,

$$A^{(k)}_b(t) = \begin{bmatrix} A^{(k)}_r(t) & \bar{A}^{(k)}_{12}(t) \\ \bar{A}^{(k)}_{21}(t) & \bar{A}^{(k)}_{22}(t) \end{bmatrix}, \quad \Gamma^{(k)}_{\text{in}}(t) = \text{diag}(\Gamma^{(k)}(t), \Gamma^{(k)}_{\text{in}}(t)),$$

$\Gamma^{(k)}_{\text{out}}(t) = \text{diag}(\Gamma^{(k)}(t), \Gamma^{(k)}_{\text{out}}(t))$ , for appropriately defined  $\bar{A}^{(k)}_{12}(t)$ ,  $\bar{A}^{(k)}_{21}(t)$ , and  $\bar{A}^{(k)}_{22}(t)$ . We define  $B^{(k)}_b(t)$  and  $C^{(k)}_b(t)$  similarly to  $A^{(k)}_b(t)$ , and  $\Omega^{(k)}_{\text{in}}(t)$  and  $\Omega^{(k)}_{\text{out}}(t)$  similarly to  $\Gamma^{(k)}_{\text{in}}(t)$  and  $\Gamma^{(k)}_{\text{out}}(t)$ , respectively.

**Lemma 3.** System  $G_r$  is strongly stable and the given realization is balanced with balanced generalized gramians  $\Gamma^{(k)}(t)$  and  $\Gamma^{(ij)}(t)$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ .

**Proof.** Since the realization of  $G$  is balanced, then there exist balanced generalized gramians simultaneously solving (4) and (5). We focus on (4), then treat (5) similarly. We apply appropriate permutations to get

$$A^{(k)}_b(t) \text{diag}(\Gamma^{(k)}_{\text{in}}(t), \Omega^{(k)}_{\text{in}}(t)) A^{(k)}_b(t)^T + B^{(k)}_b(t) B^{(k)}_b(t)^T - \text{diag}(\Gamma^{(k)}_{\text{out}}(t+1), \Omega^{(k)}_{\text{out}}(t+1)) < -\beta I. \quad (6)$$

From this inequality, we can infer that

$$A^{(k)}_r(t) \text{diag}(\Gamma^{(k)}(t), \Gamma^{(k)}_{\text{in}}(t)) A^{(k)}_r(t)^T + B^{(k)}_r(t) B^{(k)}_r(t)^T - \text{diag}(\Gamma^{(k)}(t+1), \Gamma^{(k)}_{\text{out}}(t+1)) < -\beta I.$$

Thus,  $G_r$  is strongly stable and the given realization is balanced. ■

#### 4. Error bounds

Next, we develop upper bounds on the  $\ell_2$ -induced norm of the error system  $(G - G_r)$ , which generalize their counterparts for single LTV systems in Sandberg and Rantzer (2004) and single nonstationary LPV systems in Farhood and Dullerud (2007). For each  $t \in \mathbb{N}_0$ , we define  $\hat{\Omega}(t) = \text{diag}(\Omega^{(k)}(t))_{k \in V}$ , and let  $\tilde{\Omega}(t)$  be the block-diagonal augmentation of  $\hat{\Omega}^{(ij)}(t)$ , where  $(i, j) \in E$ . The specific ordering of the diagonal blocks in  $\tilde{\Omega}(t)$  is inconsequential for our purposes. Then, we define

$$\tilde{\Omega}(t) = \text{diag}(\hat{\Omega}(t), \tilde{\Omega}(t)) \text{ and } \Omega = \text{diag}(\tilde{\Omega}(t))_{t \in \mathbb{N}_0}. \quad (7)$$

Depending on the states that are to be truncated, some diagonal blocks of  $\Omega$  may have zero dimensions and, hence, are nonexistent. For instance, suppose for some  $(t_0, k_0) \in \mathbb{N}_0 \times V$  and  $(i_0, j_0) \in E$ , we have  $n^{(k_0)}(t_0) = r^{(k_0)}(t_0)$  and  $n^{(i_0 j_0)}(t_0) = r^{(i_0 j_0)}(t_0)$ . Then, the diagonal blocks  $\Omega^{(k_0)}(t_0)$  and  $\Omega^{(i_0 j_0)}(t_0)$  have zero dimensions and do not appear in  $\Omega$ . Alternatively, if  $n^{(k)}(t) \neq r^{(k)}(t)$  and  $n^{(ij)}(t) \neq r^{(ij)}(t)$  only for  $t = t_0$ ,  $k = k_0$ , and  $(i, j) = (i_0, j_0)$ , then  $\Omega = \text{diag}(\Omega^{(k_0)}(t_0), \Omega^{(i_0 j_0)}(t_0))$ .

**Theorem 1.** If  $\Omega^{(k)}(t) = I$  and  $\Omega^{(ij)}(t) = I$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ , then  $\|(G - G_r)\| < 2$ .

**Proof.** Only a sketch of the proof is given here due to space considerations. The reader is referred to Abou Jaoude and Farhood (2017) for further details. The proof entails constructing solutions to (3) for a realization of the strongly stable error system  $\frac{1}{2}(G - G_r)$  and  $\gamma = 1$ . To do so, we apply the Schur complement formula twice to (6) and invoke its counterpart for (5) to show that

$$K^{(k)}(t)^T R^{(k)}_2(t+1)^{-1} K^{(k)}(t) - R^{(k)}_1(t) < -\beta I, \quad (8)$$

$$\text{with } K^{(k)}(t) = \begin{bmatrix} 0 & 0 & A^{(k)}_b(t) \\ 0 & 0 & C^{(k)}_b(t) \\ A^{(k)}_b(t) & B^{(k)}_b(t) & 0 \end{bmatrix} \text{ and}$$

$$R^{(k)}_1(t) = \text{diag}(\Gamma^{(k)}_{\text{in}}(t)^{-1}, \Omega^{(k)}_{\text{in}}(t)^{-1}, I_{n^{(k)}_u(t)}, \Gamma^{(k)}_{\text{in}}(t), \Omega^{(k)}_{\text{in}}(t)).$$

$R^{(k)}_2(t+1)$  is constructed similarly to  $R^{(k)}_1(t)$  but using  $\Gamma^{(k)}_{\text{out}}(t+1)$ ,  $\Omega^{(k)}_{\text{out}}(t+1)$ , and  $I_{n^{(k)}_y(t)}$ . Appropriate permutations  $P^{(k)}(t)$  and  $L^{(k)}(t)$  are applied to (8) to yield

$$\begin{aligned} & (L^{(k)}(t) K^{(k)}(t) P^{(k)}(t))^T \left( L^{(k)}(t) R^{(k)}_2(t+1)^{-1} L^{(k)}(t)^T \right) \\ & \times (L^{(k)}(t) K^{(k)}(t) P^{(k)}(t)) - P^{(k)}(t)^T R^{(k)}_1(t) P^{(k)}(t) < -\beta I. \end{aligned}$$

In the above,  $P^{(k)}(t)^T R^{(k)}_1(t) P^{(k)}(t) =$

$$\frac{1}{2} \text{diag} \left( \begin{bmatrix} \Gamma^{(k)}_{\text{in}}(t)^{-1} + \Gamma^{(k)}_{\text{in}}(t) & \Gamma^{(k)}_{\text{in}}(t)^{-1} - \Gamma^{(k)}_{\text{in}}(t) \\ \Gamma^{(k)}_{\text{in}}(t)^{-1} - \Gamma^{(k)}_{\text{in}}(t) & \Gamma^{(k)}_{\text{in}}(t)^{-1} + \Gamma^{(k)}_{\text{in}}(t) \end{bmatrix}, \right. \\ \left. \begin{bmatrix} \Omega^{(k)}_{\text{in}}(t)^{-1} + \Omega^{(k)}_{\text{in}}(t) & 0 & \Omega^{(k)}_{\text{in}}(t)^{-1} - \Omega^{(k)}_{\text{in}}(t) \\ 0 & 2I_{n^{(k)}_u(t)} & 0 \\ \Omega^{(k)}_{\text{in}}(t)^{-1} - \Omega^{(k)}_{\text{in}}(t) & 0 & \Omega^{(k)}_{\text{in}}(t)^{-1} + \Omega^{(k)}_{\text{in}}(t) \end{bmatrix} \right).$$

$L^{(k)}(t) R^{(k)}_2(t+1)^{-1} L^{(k)}(t)^T$  has a similar structure.

Also,  $L^{(k)}(t) K^{(k)}(t) P^{(k)}(t) = \begin{bmatrix} M^{(k)}(t) & N^{(k)}_{12}(t) \\ N^{(k)}_{21}(t) & \bar{A}^{(k)}_{22}(t) \end{bmatrix}$ , where

$$M^{(k)}(t) = \begin{bmatrix} A^{(k)}_r(t) & 0 & \frac{1}{\sqrt{2}} B^{(k)}_r(t) \\ 0 & A^{(k)}_b(t) & \frac{1}{\sqrt{2}} B^{(k)}_b(t) \\ -\frac{1}{\sqrt{2}} C^{(k)}_r(t) & \frac{1}{\sqrt{2}} C^{(k)}_b(t) & 0 \end{bmatrix}, \text{ and } N^{(k)}_{12}(t)$$



and  $N_{21}^{(k)}(t)$  are appropriately defined. The matrices in  $M^{(k)}(t)$  can be used to describe the dynamics of  $\frac{1}{2}(G - G_r)$ . The resulting equations are not in the form of (1), but can be equivalently expressed in that form through the use of appropriate permutations. Since  $\Omega^{(k)}(t) = I$  and  $\Omega^{(ij)}(t) = I$ , for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ , one can conclude from the above that

$$M^{(k)}(t)^T \begin{bmatrix} V_2^{(k)}(t+1) & 0 \\ 0 & I \end{bmatrix} M^{(k)}(t) - \begin{bmatrix} V_1^{(k)}(t) & 0 \\ 0 & I \end{bmatrix} < -\beta I.$$

$V_2^{(k)}(t+1) > \beta I$  and  $V_1^{(k)}(t) > \beta I$  correspond to the upper left corner blocks of  $L^{(k)}(t)R_2^{(k)}(t+1)^{-1}L^{(k)}(t)^T$  and  $P^{(k)}(t)^TR_1^{(k)}(t)P^{(k)}(t)$ , respectively. Applying the appropriate permutations and invoking Lemma 2 with  $\gamma = 1$ , we get  $\|\frac{1}{2}(G - G_r)\| < 1$ . ■

**Theorem 2.** The error system satisfies  $\|(G - G_r)\| < 2\zeta(\Omega)$ , where  $\zeta(X)$  is the sum of distinct diagonal entries of a square, possibly infinite dimensional, matrix  $X$ .

**Proof.** As a truncated realization is itself balanced by Lemma 3, the truncation procedure can be implemented in multiple steps. For each step  $i$ , we find the smallest diagonal entry in  $\Omega$ , which we denote by  $q_i$ , and truncate all the state variables with a corresponding diagonal entry in  $\Omega$  equal to  $q_i$ . We then update  $\Omega$  by removing all the entries equal to  $q_i$ . The resulting reduced order system is denoted by  $G_{r,i}$ . Suppose that, in the first step,  $\Omega^{(k)}(t) = q_1 I$  for some  $(t, k) \in \mathbb{N}_0 \times V$  and/or  $\Omega^{(ij)}(t) = q_1 I$  for some  $(i, j) \in E$  and (perhaps different)  $t \in \mathbb{N}_0$ . We want to show that  $\|(G - G_{r,1})\| < 2q_1$ . To do so, we construct a scaled system  $G_n$  with realization  $(A^{(k)}(t), \frac{1}{\sqrt{q_1}}B^{(k)}(t), \frac{1}{\sqrt{q_1}}C^{(k)}(t), \frac{1}{q_1}D^{(k)}(t))$ . Then,  $\Omega_n^{(k)}(t) = I$  and/or  $\Omega_n^{(ij)}(t) = I$ . Let  $G_{n,r,1}$  be the reduced order system obtained by applying BT to  $G_n$ . By Theorem 1,  $\|(G_n - G_{n,r,1})\| < 2$ . But,  $\|(G_n - G_{n,r,1})\| = \frac{1}{q_1}\|(G - G_{r,1})\|$ . And so,  $\|(G - G_{r,1})\| < 2q_1$ . The same procedure is applied in the following step to obtain  $\|(G_{r,1} - G_{r,2})\| < 2q_2$ . Then, by the triangle inequality,  $\|(G - G_{r,2})\| = \|(G - G_{r,1} + G_{r,1} - G_{r,2})\| \leq \|(G - G_{r,1})\| + \|(G_{r,1} - G_{r,2})\| < 2(q_1 + q_2)$ , and so on. ■

Theorem 2 gives an upper bound on  $\|(G - G_r)\|$  which may not always be finite as there may be infinitely many distinct entries in  $\Omega$ . Next, we derive a tighter expression for the error bound which applies when the diagonal entries of  $\Omega^{(k)}(t)$  and  $\Omega^{(ij)}(t)$  define monotonic sequences in time. We define the subsets of time at which truncation occurs as  $\mathcal{F}_k = \{t \in \mathbb{N}_0 | n^{(k)}(t) \neq r^{(k)}(t)\}$  and  $\mathcal{F}_{(ij)} = \{t \in \mathbb{N}_0 | n^{(ij)}(t) \neq r^{(ij)}(t)\}$ , for all  $k \in V$  and  $(i, j) \in E$ . Definition 2 is from Farhood and Dullerud (2007).

**Definition 2.** Consider a scalar sequence  $\alpha_t$  which is defined on some subset  $\mathcal{W}$  of  $\mathbb{N}_0$ , and let  $t_{\min} = \min\{t | t \in \mathcal{W}\}$ . We extend the domain of definition of  $\alpha_t$  to all  $t \in \mathbb{N}_0$  by defining the following rule:

$$\alpha_t = \begin{cases} \alpha_{t_{\min}} & \text{if } 0 \leq t \leq t_{\min}, \\ \alpha_d & \text{if } t_{\min} < t, \text{ where } d = \max\{\tau \leq t | \tau \in \mathcal{W}\}. \end{cases}$$

**Theorem 3.** For all  $k \in V$  and  $(i, j) \in E$ , if  $\Omega^{(k)}(t) = w^{(k)}(t)I$  for  $t \in \mathcal{F}_k$  and  $\Omega^{(ij)}(t) = w^{(ij)}(t)I$  for  $t \in \mathcal{F}_{(ij)}$ , where the sequences  $w^{(k)}(t)$  and  $w^{(ij)}(t)$  are monotonic in time, then

$$\|(G - G_r)\| < 2 \left( \sum_k \sup_{t \in \mathcal{F}_k} w^{(k)}(t) + \sum_{(i,j)} \sup_{t \in \mathcal{F}_{(ij)}} w^{(ij)}(t) \right).$$

**Proof.** We prove the result for the case where only one state (temporal or spatial) is truncated. The theorem then follows by repeated application of this result. Namely, we fix  $k = k_0$ , and

assume that the corresponding temporal state is the only truncated state. Without loss of generality, we assume that  $w^{(k_0)}(t) \leq 1$  for all  $t \in \mathcal{F}_{k_0}$ , as this can be always achieved by scaling. We extend the domain of  $w^{(k_0)}(t)$  to all  $t \in \mathbb{N}_0$  using Definition 2, and note that the extended sequence is still monotonic in time. We first consider the case where  $w^{(k_0)}(t)$  is monotone nondecreasing. For all  $(t, k) \in \mathbb{N}_0 \times V$ , we define the state-space transformations  $T_{\text{pre}}^{(k)}(t) = w^{(k_0)}(t)^{-1/2}I$  and  $T_{\text{post}}^{(k)}(t) = w^{(k_0)}(t)^{1/2}I$ . Note that  $T_{\text{pre}}^{(k)}(t)$  is bounded since  $\Sigma^{(k_0)}(t) > \beta I$ , for some  $\beta > 0$  and all  $t \in \mathbb{N}_0$ . Then, we define a new realization  $(A_n^{(k)}(t), B_n^{(k)}(t), C_n^{(k)}(t), D^{(k)}(t))$  for system  $G$ , where

$$\begin{aligned} A_n^{(k)}(t) &= T_{\text{pre}}^{(k)}(t+1)A^{(k)}(t)T_{\text{post}}^{(k)}(t), \\ B_n^{(k)}(t) &= T_{\text{pre}}^{(k)}(t+1)B^{(k)}(t), \quad C_n^{(k)}(t) = C^{(k)}(t)T_{\text{post}}^{(k)}(t). \end{aligned}$$

For simplicity, we refer to system  $G$  with the new realization as  $G_n$ . We now show that the realization of  $G_n$  is balanced. Recall that  $\Sigma^{(k)}(t)$  and  $\Sigma^{(ij)}(t)$  satisfy (4) and (5). We pre- and post-multiply (4) by  $T_{\text{pre}}^{(k)}(t+1)$ , insert  $T_{\text{post}}^{(k)}(t)T_{\text{post}}^{(k)}(t)^{-1} = T_{\text{post}}^{(k)}(t)^{-1}T_{\text{post}}^{(k)}(t) = I$  as needed, and define  $\Sigma_n^{(k)}(t) = w^{(k_0)}(t)^{-1}\Sigma^{(k)}(t)$  and  $\Sigma_n^{(ij)}(t) = w^{(k_0)}(t)^{-1}\Sigma^{(ij)}(t)$ , to get

$$\begin{aligned} A_n^{(k)}(t) \text{diag}(\Sigma_n^{(k)}(t), \Sigma_{n,\text{in}}^{(k)}(t)) A_n^{(k)}(t)^T + B_n^{(k)}(t) B_n^{(k)}(t)^T \\ - \text{diag}(\Sigma_n^{(k)}(t+1), \Sigma_{n,\text{out}}^{(k)}(t+1)) < -\beta I. \end{aligned}$$

We also pre- and post-multiply (5) by  $T_{\text{pre}}^{(k)}(t)$  and insert  $T_{\text{pre}}^{(k)}(t+1)^{-1}T_{\text{pre}}^{(k)}(t+1) = T_{\text{pre}}^{(k)}(t+1)T_{\text{pre}}^{(k)}(t+1)^{-1} = I$  to get

$$\begin{aligned} w^{(k_0)}(t+1) A_n^{(k)}(t)^T \text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1)) A_n^{(k)}(t) \\ - w^{(k_0)}(t) \text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t)) + C_n^{(k)}(t)^T C_n^{(k)}(t) < -\beta I. \end{aligned}$$

Since  $0 < w^{(k_0)}(t) \leq 1$ , then  $w^{(k_0)}(t)^{-1} \geq w^{(k_0)}(t)$ . Also, since  $w^{(k_0)}(t)$  is monotone nondecreasing, then  $w^{(k_0)}(t) \leq w^{(k_0)}(t+1)$  and  $w^{(k_0)}(t)^{-1} \geq w^{(k_0)}(t+1)^{-1}$ . With this in mind, it is not difficult to verify that

$$\begin{aligned} w^{(k_0)}(t+1)^{-1} A_n^{(k)}(t)^T \text{diag}(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1)) A_n^{(k)}(t) \\ - w^{(k_0)}(t)^{-1} \text{diag}(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t)) + C_n^{(k)}(t)^T C_n^{(k)}(t) < -\beta I. \end{aligned}$$

Thus,  $\Sigma_n^{(k)}(t)$  and  $\Sigma_n^{(ij)}(t)$  satisfy (4) and (5) for the realization of  $G_n$ , i.e., the realization of  $G_n$  is balanced and can be reduced by BT. We denote by  $G_{n,r}$  the reduced order system obtained by truncating the temporal state of subsystem  $G_n^{(k_0)}$ . Since  $\Omega^{(k_0)}(t) = w^{(k_0)}(t)I$ , then  $\Omega_n^{(k_0)}(t) = I$  for  $t \in \mathcal{F}_{k_0}$ . By Theorem 1,  $\|(G_n - G_{n,r})\| < 2$ . But, because of the special structure of  $T_{\text{post}}^{(k)}(t)$  and  $T_{\text{pre}}^{(k)}(t)$ , system  $(G_n - G_{n,r})$  is equivalent to  $(G - G_r)$ , and so,  $\|(G - G_r)\| < 2$ . The proof is similar for the case of a monotone nonincreasing  $w^{(k_0)}(t)$ , with the state-space transformations defined as  $T_{\text{pre}}^{(k)}(t) = w^{(k_0)}(t)^{1/2}I$  and  $T_{\text{post}}^{(k)}(t) = w^{(k_0)}(t)^{-1/2}I$ , for all  $(t, k) \in \mathbb{N}_0 \times V$ . ■

For example, suppose that the only truncated state is  $x^{(k_0)}$ , the truncation only occurs at three time-steps  $t_1, t_2, t_3$ , and  $\Omega^{(k_0)}(t_1) = \text{diag}(7, 3, 2)$ ,  $\Omega^{(k_0)}(t_2) = \text{diag}(4, 2, 2)$ ,  $\Omega^{(k_0)}(t_3) = \text{diag}(6, 5, 2)$ . The bound from Theorem 2 gives  $2 \times (7 + 6 + 5 + 4 + 3 + 2) = 54$ , whereas the bound from Theorem 3 gives  $2 \times (2 + 5 + 7) = 28$ . In the latter case, we consider the truncation sequences as  $\{2, \text{diag}(2, 2, 2)\}$ ,  $\{3, 4, 5\}$ , and  $\{7, 6\}$ .

## 5. Eventually time-periodic systems

We now show that for eventually time-periodic (ETP) subsystems, i.e., subsystems with state-space matrices which become time-periodic after some initial amount of time, the bound in Theorem 2 reduces to a finite sum. System  $G$  is said to be  $(h, q)$ -ETP, for some integers  $h \geq 0$  and  $q > 0$ , if the state-space matrices of the subsystems are  $(h, q)$ -ETP, e.g.,  $A^{(k)}(t + h + zq) =$

$A^{(k)}(t+h)$ , for all  $t, z \in \mathbb{N}_0$  and  $k \in V$ . The class of ETP subsystems includes as special cases time-periodic subsystems  $((0, q)$ -ETP) and finite time-horizon subsystems  $((h, 1)$ -ETP with zero state-space matrices for  $t \geq h$ ). The following result is from [Abou Jaoude and Farhood \(2015\)](#).

**Lemma 4.** For a strongly stable,  $q$  time-periodic system, there exist  $q$  time-periodic solutions to (2).

**Theorem 4.** For a strongly stable,  $(h, q)$ -ETP system  $G$ , there exist  $(h, q)$ -ETP solutions  $X_{\text{eper}}^{(k)}(t)$ ,  $X_{\text{eper}}^{(ij)}(t)$  and  $Y_{\text{eper}}^{(k)}(t)$ ,  $Y_{\text{eper}}^{(ij)}(t)$  to (4) and (5), respectively, for all  $(t, k) \in \mathbb{N}_0 \times V$  and  $(i, j) \in E$ .

**Proof.** Since  $G$  is strongly stable, then there exist solutions  $W^{(k)}(t)$  and  $W^{(ij)}(t)$  to (2). From these solutions, we construct  $(h, q)$ -ETP solutions to (2) as shown next. We only need the LMIs that correspond to  $t \geq h-1$ , and so we assume  $h = 1$ . For all  $k \in V$ , we have

$$\begin{aligned} & A^{(k)}(0)^T \text{diag}(W^{(k)}(1), W_{\text{out}}^{(k)}(1)) A^{(k)}(0) \\ & - \text{diag}(W^{(k)}(0), W_{\text{in}}^{(k)}(0)) < -\beta I, \\ & A^{(k)}(t)^T \text{diag}(W^{(k)}(t+1), W_{\text{out}}^{(k)}(t+1)) A^{(k)}(t) \\ & - \text{diag}(W^{(k)}(t), W_{\text{in}}^{(k)}(t)) < -\beta I, \end{aligned} \quad (9)$$

for  $t \geq 1$ . By [Lemma 4](#), there exist  $q$  time-periodic solutions  $W_{\text{per}}^{(k)}(t)$  and  $W_{\text{per}}^{(ij)}(t)$  to (9). We choose  $\alpha > 0$  such that  $\alpha W_{\text{per}}^{(k)}(1) < W^{(k)}(1)$  and  $\alpha W_{\text{per}}^{(ij)}(1) < W^{(ij)}(1)$ .

$$\begin{aligned} & \text{Then, } A^{(k)}(0)^T \text{diag}(\alpha W_{\text{per}}^{(k)}(1), \alpha W_{\text{out, per}}^{(k)}(1)) A^{(k)}(0) \\ & - \text{diag}(W^{(k)}(0), W_{\text{in}}^{(k)}(0)) < -\beta I. \end{aligned}$$

We set  $W_{\text{eper}}^{(k)}(0) = W^{(k)}(0)$  and  $W_{\text{eper}}^{(k)}(t) = \alpha W_{\text{per}}^{(k)}(t)$  for  $t \geq 1$ .  $W_{\text{eper}}^{(ij)}(t)$  are defined similarly. Since  $W_{\text{eper}}^{(k)}(t)$  and  $W_{\text{eper}}^{(ij)}(t)$  are  $(h, q)$ -ETP solutions to (2), we can choose  $\mu > 0$  such that  $Y_{\text{eper}}^{(k)}(t) = \mu W_{\text{eper}}^{(k)}(t)$  and  $Y_{\text{eper}}^{(ij)}(t) = \mu W_{\text{eper}}^{(ij)}(t)$  are solutions to (5). Also, by applying the Schur complement formula twice to (2), we see that we can choose  $\xi > 0$  such that  $X_{\text{eper}}^{(k)}(t) = \xi W_{\text{eper}}^{(k)}(t)^{-1}$  and  $X_{\text{eper}}^{(ij)}(t) = \xi W_{\text{eper}}^{(ij)}(t)^{-1}$  are solutions to (4). ■

**Corollary 1.** A strongly stable,  $(h, q)$ -ETP system  $G$  has an  $(h, q)$ -ETP balanced realization and  $(h, q)$ -ETP generalized gramians  $\Sigma_{\text{eper}}^{(k)}(t) = \text{diag}(\Gamma_{\text{eper}}^{(k)}(t), \Omega_{\text{eper}}^{(k)}(t))$  and  $\Sigma_{\text{eper}}^{(ij)}(t) = \text{diag}(\Gamma_{\text{eper}}^{(ij)}(t), \Omega_{\text{eper}}^{(ij)}(t))$ . Moreover, system  $G_r$ , resulting from BT, has an  $(h, q)$ -ETP balanced realization and satisfies  $\|(G - G_r)\| < 2\zeta(\Omega_{\text{eper}})$ , where  $\Omega_{\text{eper}}$  is the  $(h, q)$ -truncation of  $\Omega$  defined in (7), i.e.,  $\Omega_{\text{eper}} = \text{diag}(\bar{\Omega}_{\text{eper}}(t))_{t \in \{0, 1, \dots, h+q-1\}}$ .

The bound in [Corollary 1](#) may be further improved by using [Theorem 3](#) to compute the bound due to the states truncated over the finite time-horizon, i.e., for  $0 \leq t < h$ .

## 6. Illustrative example

In this section, we apply BT to a distributed system  $G$  with  $N = 5$  agents interconnected as in [Fig. 1](#). The temporal and spatial states are of constant dimensions  $n_T = 6$  and  $n_S = 3$ , respectively. There are two sets of building blocks for the state-space matrices: one for odd-numbered subsystems and one for even-numbered subsystems. The blocks of the state-space matrices are constants, except for the  $A_{00}^{(k)}(t)$  terms which are  $(h = 0, q = 28)$ -ETP, i.e., satisfy  $A_{00}^{(k)}(t + 28z) = A_{00}^{(k)}(t)$ , for all  $t, z \in \mathbb{N}_0$  and  $k \in V$ .  $A_{00}^{(k)}(t) = A_{TT}$  for  $t = 0, \dots, 6$ ,  $A_{00}^{(k)}(t) = \mathcal{M}A_{TT}\mathcal{M}^T$  for  $t = 7, \dots, 13$ ,  $A_{00}^{(k)}(t) = \mathcal{M}^2A_{TT}(\mathcal{M}^T)^2$  for  $t = 14, \dots, 20$ , and  $A_{00}^{(k)}(t) =$

$\mathcal{M}^3A_{TT}(\mathcal{M}^T)^3$  for  $t = 21, \dots, 27$ , where  $A_{TT}$  and  $\mathcal{M}$  are building blocks. For odd-numbered subsystems,

$$\begin{aligned} A_{ST} &= 0.1 [\text{diag}(1, -2, 0.1) \quad \text{diag}(0.5, 0.4, 0.2)], \quad A_{SS} = 0_{3 \times 3}, \\ B_S &= 0.1 [I_2 \quad 0_{2 \times 1}]^T, \quad C_T = [I_2 \quad 0_{2 \times 4}], \quad C_S = 0_{2 \times 3}, \\ A_{TT} &= 0.1 \begin{bmatrix} \begin{bmatrix} -9 & -7 \\ -5 & -5 \end{bmatrix} & \begin{bmatrix} 0.1 & 0.3 & -0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 & -0.2 \end{bmatrix} \\ \text{diag}(1, -2) & 0.01 \text{diag}(-5, 1, -3, 2) \\ -I_2 & \end{bmatrix}, \\ A_{TS} &= 0.1 \begin{bmatrix} \begin{bmatrix} -0.5 & 0.5 & 0.01 \\ 0.5 & -0.5 & -0.02 \end{bmatrix} \\ 0_{4 \times 3} \end{bmatrix}, \quad B_T = 0.2 \begin{bmatrix} I_2 \\ 0_{4 \times 2} \end{bmatrix}. \end{aligned}$$

Subscripts  $T$  and  $S$  refer to temporal and spatial terms, respectively. E.g., for  $k = 1$  and all  $t \in \mathbb{N}_0$ ,  $A_{02}^{(1)}(t) = A_{03}^{(1)}(t) = A_{04}^{(1)}(t) = A_{TS}$ ,  $B_0^{(1)}(t) = B_T$ ,  $C_0^{(1)}(t) = C_T$ ,  $B_2^{(1)}(t) = B_3^{(1)}(t) = B_5^{(1)}(t) = B_S$ ,  $C_2^{(1)}(t) = C_3^{(1)}(t) = C_4^{(1)}(t) = C_S$ ,  $A_{20}^{(1)}(t) = A_{30}^{(1)}(t) = A_{50}^{(1)}(t) = A_{ST}$ , etc. The permutation matrix  $\mathcal{M} = [e_5 \ e_3 \ e_1 \ e_6 \ e_4 \ e_2]^T$ , where  $\{e_d | d = 1, \dots, 6\}$  is the standard basis for the six-dimensional Euclidean space. For even-numbered subsystems,

$$\begin{aligned} A_{TS} &= 0.1 \begin{bmatrix} \text{diag}(-0.5, 0.1, -0.1) \\ \text{diag}(-0.5, -0.2, 0.3) \end{bmatrix}, \quad C_T = \begin{bmatrix} e_2^T \\ e_4^T \end{bmatrix}, \\ A_{ST} &= [0.2I_3 \quad -0.03I_3], \quad B_T = 0.1 [e_1 \ e_3], \quad C_S = [-I_2 \quad 0_{2 \times 1}], \\ A_{TT} &= 0.1 \begin{bmatrix} \begin{bmatrix} 1 & -4 & -0.3 & 0.1 & 0.5 & 0.3 \\ 3 & -5 & 0.2 & 0.1 & -0.2 & 0.3 \\ 0.1 & -0.3 & -0.5 & 0.2 & 0.1 & 0.1 \end{bmatrix} \\ \text{diag}(-0.2, 0.1, 0.3) \quad \text{diag}(-0.1, 0.15, -0.1) \end{bmatrix}, \end{aligned}$$

$$A_{SS} = 0_{3 \times 3}, \quad B_S = 0_{3 \times 2}, \quad \mathcal{M} = [e_4 \ -e_5 \ e_2 \ -e_1 \ e_6 \ e_3]^T.$$

For all  $(t, k) \in \mathbb{N}_0 \times V$ ,  $D^{(k)}(t) = 0_{2 \times 2}$ . Using [Lemma 2](#), we show that system  $G$  is strongly stable and find an upper bound  $\gamma$  on  $\|G\|$ . Namely, we find  $(0, 28)$ -ETP solutions to (3) while minimizing  $\gamma$ . We denote the resulting semi-definite programming (SDP) problem by P1. The result,  $\gamma = 3.47$ , helps in assessing the upper bound on  $\|(G - G_r)\|$  and in choosing how many temporal and spatial state variables to truncate. Then, we find  $(0, 28)$ -ETP solutions to (4) which minimize  $\sum_{t=0}^{27} (\sum_k \text{trace } X^{(k)}(t) + \sum_{(i,j)} \text{trace } X^{(ij)}(t))$ , and  $(0, 28)$ -ETP solutions  $Y^{(k)}(t)$ ,  $Y^{(ij)}(t)$  to (5) which minimize a similar objective function. We denote the resulting SDP problems by P2 and P3. We determine the computational complexity of these problems by formulating the corresponding dual problems. Let  $N_l = 12$  be the number of interconnections, and  $n_u = 2$  be the number of inputs to each subsystem. For P1, the dimension of the SDP variable is  $(h+q)(N(2n_T + n_u) + 2N_l n_S) = 3976$ , the dimension of the linear variable is 1, the number of SDP blocks is  $(h+q)(2N + N_l) = 616$ , and the number of constraints is  $\frac{1}{2}(h+q)(Nn_T(n_T+1) + N_l n_S(n_S+1)) + 1 = 4957$ . As for P2 and P3, these numbers are, respectively,  $2(h+q)(Nn_T + N_l n_S) = 3696$ ,  $0$ ,  $(h+q)(2N + N_l) = 616$ , and  $\frac{1}{2}(h+q)(Nn_T(n_T+1) + N_l n_S(n_S+1)) = 4956$ . We use Yalmip to model these problems and SDPT3 to solve them; see [Lofberg \(2004\)](#) and [Toh, Todd, and Tutuncu \(1999\)](#). We carry out the computations in Matlab 7.10.0.499 (The MathWorks Inc., Natick, Massachusetts, USA) on a Hewlett-Packard laptop with 2 Intel Cores, 2.30 GHz processors, and 4 GB of RAM running Windows 7. The most time consuming problem is P1. The corresponding elapsed time is about 30 s (CPU time 24 s).

We use [Algorithm 1](#) to construct a  $(0, 28)$ -ETP balanced realization for  $G$ . To obtain useful error bounds for BT, we resolve the Lyapunov inequalities for the balanced realization of  $G$ . Namely, we find  $(0, 28)$ -ETP diagonal solutions  $\Sigma^{(k)}(t) \geq$

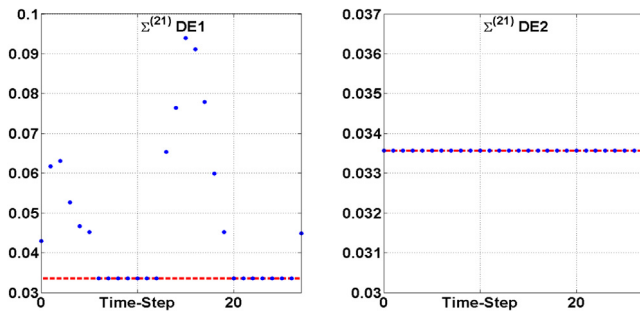


Fig. 2. First and second diagonal entries (DE) of  $\Sigma^{(21)}(t)$ .

$\epsilon I$  and  $\Sigma^{(ij)}(t) \succeq \epsilon I$  that satisfy (4) and (5), and minimize  $\sum_t (\sum_k \|\text{vect}(\Sigma^{(k)}(t) - \epsilon I)\|_1 + \sum_{(i,j)} \|\text{vect}(\Sigma^{(ij)}(t) - \epsilon I)\|_1) + a_1 \times \epsilon$ , where  $\text{vect}(M)$  is the vector formed by the diagonal entries of the matrix  $M$  and  $\|v\|_1$  is the 1-norm of vector  $v$ .  $a_1 = 750$  is the weight given to  $\epsilon$  in the cost function. This value of  $a_1$  gives the best trade-off between the two following objectives. The first objective is to minimize the first term in the cost function, where the 1-norm is used as a heuristic for finding diagonal gramians with many entries equal to  $\epsilon$ . All the temporal and spatial state variables whose corresponding entries in the gramians are equal to  $\epsilon$  are then truncated. The second objective is to minimize  $\epsilon$  since, by Corollary 1,  $\|(G - G_r)\| < 2\epsilon$ . We get  $\epsilon = 0.034$ , i.e.,  $\|(G - G_r)\| < 2\% \gamma$ . Fig. 2 shows the first and second diagonal entries of  $\Sigma^{(21)}(t)$ , for  $0 \leq t < 28$ . The red dashed line corresponds to  $\epsilon$ . The dimension of  $x^{(21)}$  varies between 0 and 1 in  $G_r$  in contrast to 3 in  $G$ , i.e., in  $G_r$ , the interconnection (2, 1) disappears at certain time-steps from the interconnection structure. In total, between 14 and 18 temporal variables and between 20 and 25 spatial variables are truncated at each time-step. Simulations of systems  $G$  and  $G_r$  are carried out using the same sets of applied inputs. As expected from the error bound, the responses are very close. A typical run is shown in Abou Jaoude and Farhood (2017).

## 7. Conclusion

This work applies BT for the model reduction of distributed LTV systems. The method provides a priori error bounds, preserves the interconnection structure, and allows for its simplification. While BT is only applicable to strongly stable systems, CFR extends the applicability of BT to strongly stabilizable and strongly detectable systems.

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