Systems & Control Letters 108 (2017) 23–32

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Distributed control of nonstationary LPV systems over arbitrary graphs

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ARTICLE INFO

Article history:
Received 4 November 2016
Received in revised form 20 May 2017
Accepted 26 July 2017
Available online 6 September 2017

Keywords:
Nonstationary linear parameter-varying (NSLPV) systems
Distributed control
Arbitrary graphs
Communication latency
Eventually time-periodic systems

ABSTRACT

This paper deals with the ℓ_2-induced norm control of discrete-time, nonstationary linear parameter-varying (NSLPV) subsystems, represented in a linear fractional transformation (LFT) framework and interconnected over arbitrary directed graphs. Communication between the subsystems is subjected to a one-step time-delay. NSLPV models have state-space matrix-valued functions with explicit dependence on time-varying terms that are known a priori, as well as parameters that are not known a priori but are available for measurement at each discrete time-step. The sought controller has the same interconnection and LFT structures as the plant. Convex analysis and synthesis results are derived using a parameter-independent Lyapunov function. These conditions are infinite dimensional in general, but become finite dimensional in the case of eventually time-periodic subsystems interconnected over finite graphs. The method is applied to an illustrative example.

1. Introduction

This paper is on the distributed control of subsystems interconnected over arbitrary directed graphs. Each subsystem has its own sensing and actuating capabilities, and is modeled using a discrete-time, nonstationary linear parameter-varying (NSLPV) model [1,2], formulated in a linear fractional transformation (LFT) framework. NSLPV models generalize stationary/standard linear parameter-varying (LPV) models in the sense that the state-space matrices can have an explicit dependence on time, in addition to their dependence on the scheduling parameters. We account for a communication latency of one sampling period between the subsystems, i.e., a one-step time-delay on the information transfer between the subsystems. Our aim is to construct a distributed controller which renders the closed-loop system asymptotically stable, and further guarantees some ℓ_2-gain performance level, i.e., an upper bound on the ℓ_2-induced norm of the closed-loop input–output map, for all permissible parameter trajectories. The sought controller inherits the topological structure of the plant. The controller subsystems have NSLPV models, formulated in an LFT framework, and scheduled by the same parameters as their corresponding plant subsystems.

In general, distributed controllers are of interest for the class of interconnected subsystems because of the advantages they present over centralized controllers in terms of computational complexity and practicality. Namely, centralized controllers require a heavy computational burden in the case of a large number of high dimensional subsystems, and also require high connectivity to receive measurements from and send commands to all the subsystems [3]. Distributed controllers may also be more desirable than decentralized controllers in applications with stringent stability and performance requirements for the global system. Additionally, distributed controllers apply whenever the interconnection of the subsystems is stabilizable; whereas, decentralized controllers require that each subsystem be individually stabilizable.

NSLPV models were introduced and motivated in [1,2]. They are extensions of LPV models in that the state-space matrices depend on a priori known time-varying terms, in addition to their dependence on the parameters, which are not known a priori but are available for measurement at each time-step. Like LPV models, NSLPV models allow for capturing the nonlinearities of the studied system, while being amenable to control using linear techniques. NSLPV models arise, for example, when controlling nonlinear systems about prespecified trajectories. They can also be thought of as linear time-varying (LTV) models with time-varying uncertainties. In general, when it comes to describing time-varying nonlinear systems using parameter-varying models, NSLPV models are far less conservative than stationary LPV models, and in some cases, the only stabilizable models attainable are NSLPV [2]. In the context of interconnected subsystems, a distributed NSLPV system can be formed of NSLPV subsystems and/or mixes of LTV and LPV subsystems.

Several works have appeared that address the problem of distributed control for interconnected LPV subsystems, e.g., [4–8], and interconnected uncertain subsystems [9]. These works can

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http://dx.doi.org/10.1016/j.sysconle.2017.07.017
0167-6911/

be classified based upon various criteria. To start, in [4–7,9], the sought controller inherits the interconnection structure of the plant. The controller subsystems in [9] are assumed to have linear time-invariant models; whereas, in [4–7], the subsystems of the controller are parameter-dependent and are described similarly to the subsystems of the plant. In [7], the controller subsystems depend on their own local parameters as well as parameters received from other subsystems. On the other hand, in [8], the structure and the order of the controller are design inputs. Secondly, in [5,7,8], the synthesis results are derived using a parameter-dependent Lyapunov function; whereas, in [4,6,9], a parameter-independent Lyapunov function is used. The use of various types of Lyapunov functions bares consequences on the convexity and the tractability of the derived synthesis results. Thirdly, classification can be based upon the complexity of the interconnection structure and the heterogeneity of the subsystems. Namely, [4,5,8] consider subsystems having the same model and interconnected over an infinite lattice. [6] considers heterogeneous groups of subsystems. Within each group, the subsystems have identical models and the interconnections between the subsystems are undirected. Among different groups, however, subsystems can have different models, and the interconnections can be directed. Heterogeneous subsystems and arbitrary graphs are considered in [7,9]. [7] further allows for directed interconnections, and accounts for communication latency between the subsystems. A fourth classification criterion covers the modeling of the interconnections between the subsystems. Specifically, [4,5,7–9] use spatial states to model the interconnections between the subsystems, in addition to the states associated with the subsystems; whereas, in [6], the possibly time-varying interconnection topology is modeled using a feedback operator in an LFT framework.

To the best of our knowledge, the current work is the first on distributed control of interconnected NSLPV subsystems. In this paper, we develop an operator theoretic framework in the context of robust control tools for working with distributed NSLPV systems. This framework builds on previous ones developed for single NSLPV systems [2] and distributed LTV systems [10]. Using this framework, the state-space equations of the complex system under consideration can be represented in a compact operator form that looks formally identical to standard LPV-LFT state-space models, and derivations and proofs of standard analysis and synthesis results [11–13] can be adapted to NSLPV systems interconnected over arbitrary graphs, with many inherently complex manipulations becoming transparent. However, despite the formal analogy between operator-based results and standard ones, there are ensuing intricacies that have to be addressed to make sure that these transparent manipulations go through, which include appropriately characterizing causal and memoryless operators with special structures, and imposing desired structures on analysis solutions with no added conservatism. The standard results also acquire new interpretations and characteristics when extended to the distributed system setting. For instance, there are three types of states associated with distributed NSLPV systems: temporal states which are the standard states of the subsystems, parameter states which are due to formulating the subsystems in an LFT framework, and spatial states which are associated with the interconnections between the subsystems. Additionally, the designed controller inherits both the interconnection structure and the LFT structure of the plant, i.e., the controller is a distributed NSLPV system where the subsystems are formulated in an LFT framework, are affected by the same parameters as the plant, and are interconnected over the same interconnection structure as the plant. In general, the derived analysis and synthesis conditions are given in terms of infinite sequences of linear matrix inequalities (LMIs) due to the explicit dependence on time in the state-space equations of the subsystems. For eventually time-periodic subsystems, where the state-space matrix-valued functions become time-periodic after some initial finite time-horizon, interconnected over finite graphs, i.e., graphs with finite sets of vertices and edges, the LMI sequences become finite dimensional.

The paper is organized as follows. Section 2 introduces the notation and Section 3 gives the operator theoretic framework. Section 4 derives the analysis results and Section 5 is devoted for the synthesis results. Section 6 applies the method to an illustrative example. Conclusions are given in Section 7.

2. Notation

The sets of nonnegative integers, integers, real numbers, and $n \times n$ symmetric matrices are denoted by $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{S}^n$, respectively. $0_{i\times j}$, 0_i, and I_i denote an $i \times j$ zero matrix, an $i \times i$ zero matrix, and an $i \times i$ identity matrix, respectively. diag (M_i) is the block-diagonal augmentation of the elements of the sequence of operators M_i.

Consider a directed graph with a countable set of vertices V and a set of directed edges E. An element of E directed from $i \in V$ to $j \in V$ is denoted by (i, j). The vertex degree $\upsilon(k)$, i.e., the maximum between the indegree and the outdegree of vertex k, is assumed to be uniformly bounded. Without loss of generality, the directed graph under consideration is assumed to be d-regular, i.e., for each $k \in V$, the indegree and the outdegree are equal to d. This is because an arbitrary directed graph can be turned into a d-regular directed graph, where $d = \max_{k\in V} \upsilon(k)$, by the addition of the necessary virtual edges and nodes. With this assumption, d permutations, $\rho_1, \ldots, \rho_d$, of the set of vertices are defined such that if $(i, j) \in E$, then one $e \in \{1, \ldots, d\}$ satisfies $\rho_e(i) = j$ and $\rho_e^{-1}(j) = i$. Fig. 1 shows an example of a directed graph and the same graph rendered 2-regular after the addition of virtual edges. The permutations ρ_1 and ρ_2 are defined as follows: $\rho_1(1) = 2$, $\rho_1(2) = 3$, $\rho_1(3) = 4$, $\rho_1(4) = 1$, $\rho_2(1) = 3$, $\rho_2(3) = 1$, $\rho_2(2) = 4$, and $\rho_2(4) = 2$.

$J_1 \oplus J_2$ denotes the vector space direct sum of the vector spaces J_1 and J_2. Let H and F be Hilbert spaces. The inner product and norm associated with H are denoted by $\langle ., .\rangle_H$ and $\|.\|_H$, respectively. The subscript is dropped when H is clear from context. The spaces of bounded linear operators and bounded linear causal operators mapping H to F are denoted by $\mathcal{L}(H, F)$ and $\mathcal{L}_c(H, F)$, respectively. The symbols simplify to $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$ when $H = F$. For $X \in \mathcal{L}(H, F)$, $\|X\|$ refers to the H to F induced norm of X and X^* denotes the adjoint of X. A self-adjoint operator $X \in \mathcal{L}(H)$ is said to be negative definite, $X \prec 0$, if there exists $\alpha > 0$ such that $\langle x, Xx\rangle < -\alpha\|x\|^2$, for all nonzero $x \in H$.

Given the sequence $n : (t, k) \in \mathbb{Z} \times V \to n(t, k) \in \mathbb{N}_0$, $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is defined as the vector space of mappings $w : (t, k) \in \mathbb{Z} \times V \to w(t, k) \in \mathbb{R}^{n(t,k)}$. The Hilbert space $\ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w with finite norm $\|w\| = \sqrt{\sum_{(t,k)} w^*(t, k)w(t, k)}$. $\ell_{2e}\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ composed of mappings w that satisfy $\sum_k w(t, k)^* w(t, k) < \infty$, for each $t \in \mathbb{Z}$. The symbols ℓ, ℓ_2, and ℓ_{2e} are used when the dimensions are clear from context. Let $\bar{n} = (n_1, \ldots, n_f)$, where $n_1, \ldots, n_f$ are integer sequences similar to n, and define $\ell^{\bar{n}} = \oplus_{i=1}^{f} \ell\left(\{\mathbb{R}^{n_i(t,k)}\}\right)$. Similar definitions apply for $\ell_2^{\bar{n}}$ and $\ell_{2e}^{\bar{n}}$.

The following summarizes the operator machinery of [10]. An operator Q on ℓ_2 is said to be graph-diagonal if $(Qw)(t, k) = Q(t, k)w(t, k)$, for all $(t, k) \in \mathbb{Z} \times V$, and the matrix-valued sequence $Q(t, k)$ is uniformly bounded. An operator $W = [W_{ij}]$ is said to be partitioned graph-diagonal if each W_{ij} is a graph-diagonal operator. The mapping $[\![W]\!](t, k) = [W_{ij}(t, k)]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering. The definition of

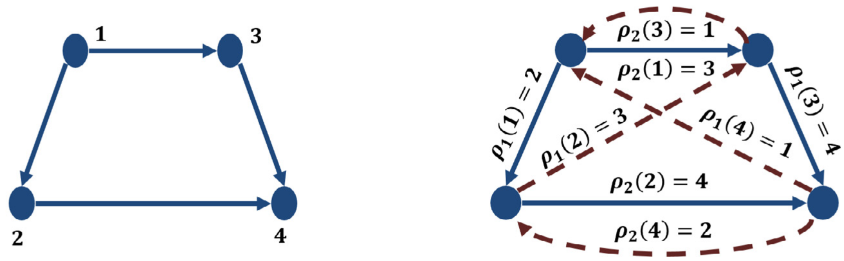

Fig. 1. Directed graph (left) rendered 2-regular (right) via the addition of the necessary virtual edges.

graph-diagonal operators extend to ℓ and ℓ_{2e}. I^q denotes the graph-diagonal identity operator such that $[\![I^q]\!](t,k) = I_{q(t,k)}$, and $0^{e\times h}$ denotes the graph-diagonal zero operator such that $0^{e\times h}(t,k) = 0_{e(t,k)\times h(t,k)}$. We also define the partitioned graph-diagonal operators $I^{(q_1,\ldots,q_m)} = \operatorname{diag}(I^{q_1},\ldots,I^{q_m})$ and $0^{(n_1,\ldots,n_f)\times(m_1,\ldots,m_g)} = [0^{(n_i\times m_j)}]_{i=1,\ldots,f;j=1,\ldots,g}$. If the dimensions are not pertinent to the discussion, the identity and zero operators are simply referred to as I and 0, respectively. The unitary temporal-shift operator S_0 and the unitary spatial-shift operators S_i are defined as follows:

$$
\begin{aligned}
S_0 : \ell_2 \to \ell_2, \quad & (S_0 v)(t,k) = v(t-1,k),\\
& (S_0^* v)(t,k) = v(t+1,k),\\
S_i : \ell_2 \to \ell_2, \quad & (S_i v)(t,k) = v(t,\rho_i^{-1}(k)),\\
& (S_i^* v)(t,k) = v(t,\rho_i(k)), \qquad \text{for } i = 1,\ldots,d.
\end{aligned}
$$

These definitions extend to ℓ and ℓ_{2e}. Subsequently, no distinction is made between the shift operators for vector spaces ℓ with different associated dimensions.

3. Operator theoretic framework

Consider a distributed NSLPV system $\mathcal{G}_\delta$ formed by heterogeneous, discrete-time, NSLPV subsystems $G^{(k)}$ formulated in an LFT framework and subjected to a communication latency. The interconnection structure of $\mathcal{G}_\delta$ is given by a d-regular directed graph, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$, and the interconnections between the subsystems are described by the directed edges. Fig. 2 shows the distributed NSLPV system corresponding to the graph of Fig. 1. For all $(t,k) \in \mathbb{Z}\times V$, the state-space equations of system $\mathcal{G}_\delta$ are as follows:

$$
\begin{bmatrix} x_T(t+1,k) \\ \hline x_1(t+1,\rho_1(k)) \\ \vdots \\ x_d(t+1,\rho_d(k)) \\ \hline \alpha(t,k) \\ \hline z(t,k) \\ \hline y(t,k) \end{bmatrix}
= \left[\begin{array}{c|c|c|c|c}
\bar{A}_{TT}(t,k) & \bar{A}_{TS}(t,k) & \bar{A}_{TP}(t,k) & \bar{B}_{T1}(t,k) & \bar{B}_{T2}(t,k) \\ \hline
\bar{A}_{ST}(t,k) & \bar{A}_{SS}(t,k) & \bar{A}_{SP}(t,k) & \bar{B}_{S1}(t,k) & \bar{B}_{S2}(t,k) \\ \hline
\bar{A}_{PT}(t,k) & \bar{A}_{PS}(t,k) & \bar{A}_{PP}(t,k) & \bar{B}_{P1}(t,k) & \bar{B}_{P2}(t,k) \\ \hline
\bar{C}_{1T}(t,k) & \bar{C}_{1S}(t,k) & \bar{C}_{1P}(t,k) & \bar{D}_{11}(t,k) & \bar{D}_{12}(t,k) \\ \hline
\bar{C}_{2T}(t,k) & \bar{C}_{2S}(t,k) & \bar{C}_{2P}(t,k) & \bar{D}_{21}(t,k) & \bar{D}_{22}(t,k)
\end{array}\right]
\times \begin{bmatrix} x_T(t,k) \\ \hline x_1(t,k) \\ \vdots \\ x_d(t,k) \\ \hline \beta(t,k) \\ \hline w(t,k) \\ \hline u(t,k) \end{bmatrix},
$$

$$
\begin{aligned}
\beta(t,k) &= \operatorname{diag}\left(\delta_1(t,k) I_{n_1^P(t,k)}, \ldots, \delta_r(t,k) I_{n_r^P(t,k)}\right) \alpha(t,k) \\
&= \underline{\Delta}(t,k)\, \alpha(t,k).
\end{aligned} \tag{1}
$$

$x_T(t,k)$ denotes the state associated with subsystem $G^{(k)}$, and has a possibly time-varying dimension $n_T(t,k)$. Such states are referred to as the temporal states. The interconnections between the subsystems are also modeled as states, called spatial states. The spatial state $x_i(t,k)$, with dimension $n_i^S(t,k)$, is associated with the edge $(\rho_i^{-1}(k), k)$, and the spatial state $x_i(t,\rho_i(k))$, with dimension $n_i^S(t,\rho_i(k))$, is associated with the edge $(k,\rho_i(k))$. Due to the communication latency, the information sent by a subsystem at time-step t reaches the target subsystem at $t+1$. The spatial states corresponding to the virtual interconnections and their corresponding blocks in the state-space matrices are of zero dimensions, i.e., nonexistent. $\beta(t,k)$ and $\alpha(t,k)$ are the parameter states due to the LFT formulation, and evolve according to the feedback channel $\beta(t,k) = \underline{\Delta}(t,k)\alpha(t,k)$, where $\delta_j(t,k)$ are scalar functions, for $j = 1,\ldots,r$. The parameters $\delta_j(t,k)$ are not known a priori, but are assumed to be measurable at each t. The vectors $\beta(t,k)$ and $\alpha(t,k)$ are partitioned into r vector-valued channels conformably with the partitioning of $\underline{\Delta}(t,k)$, e.g., $\alpha(t,k) = \left[\alpha_1^*(t,k),\ \alpha_2^*(t,k),\ \ldots,\ \alpha_r^*(t,k)\right]^*$, where the dimension of $\alpha_j(t,k)$ and $\beta_j(t,k)$ is $n_j^P(t,k)$. The dependence of the subsystems on the parameters is local, i.e., different subsystems may depend on different parameters. Even if two subsystems are affected by the same parameters, the evolution of the parameters is assumed to be independent in each subsystem. Let r_k be the number of parameters affecting subsystem $G^{(k)}$. Then, $r = \max_{k\in V} r_k$. If $r_{k_0} < r$, for some $k_0 \in V$, then $\delta_j(t,k_0) = 0$ and $n_j^P(t,k_0) = 0$, for all $t\in\mathbb{Z}$ and $j = r_{k_0}+1,\ldots,r$. $w(t,k)$, $z(t,k)$, $u(t,k)$, and $y(t,k)$ are the exogenous disturbances, the performance outputs, the control inputs, and the measurement outputs associated with subsystem $G^{(k)}$, respectively. Their corresponding dimensions are given by $n_w(t,k)$, $n_z(t,k)$, $n_u(t,k)$, and $n_y(t,k)$. Unless otherwise stated, we assume hereafter that $w \in \ell_2$.

The state-space matrices are known a priori, are assumed to be uniformly bounded, and are partitioned according to the permutations $\rho_1,\ldots,\rho_d$ and the channels in $\underline{\Delta}(t,k)$, e.g., $\bar{A}_{TS}(t,k) = \left[A_1^{TS}(t,k)\ \ldots\ A_d^{TS}(t,k)\right]$; see [14] for more details. These partitions, e.g., $A_i^{TS}(t,k)$, define graph-diagonal operators, e.g., A_i^{TS}, which are augmented to form partitioned graph-diagonal operators, e.g., $A_{TS} = \left[A_1^{TS}\ \ldots\ A_d^{TS}\right]$, such that $[\![A_{TS}]\!](t,k) = \bar{A}_{TS}(t,k)$. The operators A, B_1, B_2, B, C_1, C_2, C, and D are then defined as follows:

$$
A = \begin{bmatrix} A_{TT} & A_{TS} & A_{TP} \\ A_{ST} & A_{SS} & A_{SP} \\ A_{PT} & A_{PS} & A_{PP} \end{bmatrix},
$$

$$
B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} B_{T1} & B_{T2} \\ B_{S1} & B_{S2} \\ B_{P1} & B_{P2} \end{bmatrix},
$$

$$
C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_{1T} & C_{1S} & C_{1P} \\ C_{2T} & C_{2S} & C_{2P} \end{bmatrix}, D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.
$$

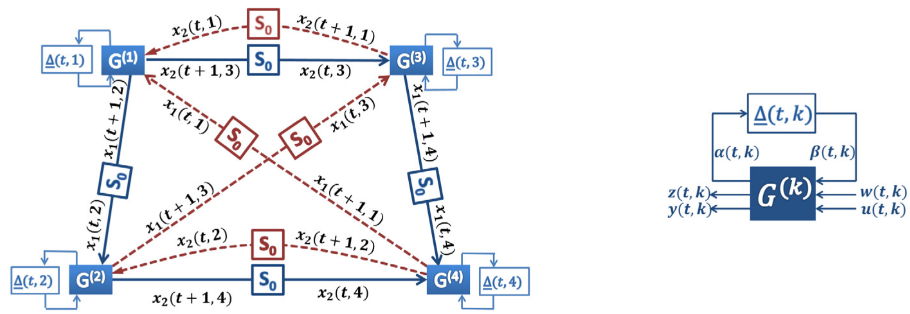

Fig. 2. Left: Distributed system with interconnection structure defined in Fig. 1 and consisting of NSLPV subsystems formulated in an LFT framework. Right: Close up view of the LFT formulation of a subsystem $G^{(k)}$.

These operators satisfy relationships of the type $[\![B_g]\!](t,k) = \left[\bar{B}^*_{Tg}(t,k) \;\; \bar{B}^*_{Sg}(t,k) \;\; \bar{B}^*_{Pg}(t,k)\right]^*$, $[\![C_g]\!](t,k) = \left[\bar{C}_{gT}(t,k) \;\; \bar{C}_{gS}(t,k) \;\; \bar{C}_{gP}(t,k)\right]$, for $g \in \{1,2\}$. For $j = 1,\ldots,r$, the graph-diagonal operators Δ_j are defined such that $\Delta_j(t,k) = \delta_j(t,k) I_{n^P_j(t,k)}$. These operators are block-diagonally augmented to construct the partitioned graph-diagonal operator $\Delta_P = \text{diag}\,(\Delta_1,\ldots,\Delta_r)$, where $[\![\Delta_P]\!](t,k) = \underline{\Delta}(t,k)$. Let $\bar{n}_S = (n^S_1,\ldots,n^S_d)$, $\bar{n}_P = (n^P_1,\ldots,n^P_r)$, $n^+_T = S^*_0 n_T S_0$, and $\bar{n}^+_S = (S^*_1 S^*_0 n^S_1 S_0 S_1,\ldots,S^*_d S^*_0 n^S_d S_0 S_d)$, where $(S^*_0 n_T S_0)(t,k) = n_T(t+1,k)$ and $(S^*_i S^*_0 n^S_i S_0 S_i)(t,k) = n^S_i(t+1,\rho_i(k))$, for $i = 1,\ldots,d$. Define the composite-shift operator as $S = \text{diag}\left(S_0, S_0S_1,\ldots,S_0S_d, I^{\bar{n}_P}\right)$ and the operator $\Delta = \text{diag}\left(I^{(n_T,\bar{n}_S)}, \Delta_P\right)$. Then, the equations in (1) can be rewritten in compact operator form as

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta S A \begin{bmatrix} x \\ \beta \end{bmatrix} + \Delta S B \begin{bmatrix} w \\ u \end{bmatrix}, \qquad \begin{bmatrix} z \\ y \end{bmatrix} = C \begin{bmatrix} x \\ \beta \end{bmatrix} + D \begin{bmatrix} w \\ u \end{bmatrix}, \tag{2}$$

where $x = \left[x^*_T, x^*_1,\ldots,x^*_d\right]^*$, $\beta = \left[\beta^*_1,\ldots,\beta^*_r\right]^*$, and Δ is restricted to $\mathbf{\Delta} = \{\Delta : \|\Delta\| \le 1\}$. For every $\Delta \in \mathbf{\Delta}$, and assuming that the relevant inverse exists, the input–output map of system $\mathcal{G}_\delta$ can be expressed as $G_\delta = C(I - \Delta SA)^{-1}\Delta SB + D$. The distributed NSLPV system $\mathcal{G}_\delta$ is then defined as $\mathcal{G}_\delta = \{G_\delta : \Delta \in \mathbf{\Delta}\}$. For an a priori known and fixed Δ, system $\mathcal{G}_\delta$ reduces to a distributed LTV system as in [10]. If only one subsystem is considered, then system $\mathcal{G}_\delta$ reduces to a single NSLPV system as described in [1,2].

To simplify the presentation of subsequent results, the following operators are defined, which group the temporal and spatial blocks of A, B, and C:

$$\hat{A}_{11} = \begin{bmatrix} A_{TT} & A_{TS} \\ A_{ST} & A_{SS} \end{bmatrix}, \hat{A}_{12} = \begin{bmatrix} A_{TP} \\ A_{SP} \end{bmatrix}, \hat{A}_{21} = \begin{bmatrix} A_{PT} & A_{PS} \end{bmatrix}, \qquad \hat{B}_g = \begin{bmatrix} B_{Tg} \\ B_{Sg} \end{bmatrix}, \hat{C}_g = \begin{bmatrix} C_{gT} & C_{gS} \end{bmatrix}, \quad g \in \{1,2\}. \tag{3}$$

4. Analysis results

This section gives the analysis results for a distributed NSLPV system $\mathcal{G}_\delta$. For simplicity, the control inputs $u(t,k)$ and the measurement outputs $y(t,k)$ are neglected. If we rewrite the system equations in (1) so as to eliminate $\alpha(t,k)$ and $\beta(t,k)$, then it is obvious that, for the state-space equations to be well-defined, $I - \bar{A}_{PP}(t,k)\underline{\Delta}(t,k)$ must be invertible, for all $(t,k) \in \mathbb{Z} \times V$.

Definition 1. System $\mathcal{G}_\delta$ is said to be well-posed if (i) given an input $w \in \ell_{2e}$, the Eqs. (2) admit a unique solution $(x,\beta) \in \ell^{(n_T,\bar{n}_S,\bar{n}_P)}_{2e}$, and (ii) G_δ defines a linear causal mapping on ℓ_{2e}, for all $\Delta \in \mathbf{\Delta}$. System $\mathcal{G}_\delta$ is said to be ℓ_2-stable if (i) it is well-posed, (ii) given an input $w \in \ell_2$, the Eqs. (2) admit a unique solution $(x,\beta) \in \ell^{(n_T,\bar{n}_S,\bar{n}_P)}_2$, and (iii) G_δ defines a bounded linear causal mapping on ℓ_2, for all $\Delta \in \mathbf{\Delta}$, i.e., $G_\delta \in \mathcal{L}_c\left(\ell_2\left(\{\mathbb{R}^{n_w(t,k)}\}\right), \ell_2\left(\{\mathbb{R}^{n_z(t,k)}\}\right)\right)$, for all permissible parameter trajectories.

Lemma 1. *If $[\![A]\!](t,k) = 0$, for $t < 0$, and $I - \Delta_P A_{PP}$ has a causal inverse on $\ell^{\bar{n}_P}_{2e}$, for all $\Delta \in \mathbf{\Delta}$, then $I - \Delta SA$ has a causal inverse on $\ell^{(n_T,\bar{n}_S,\bar{n}_P)}_{2e}$, for all $\Delta \in \mathbf{\Delta}$, and system $\mathcal{G}_\delta$ is well-posed.*

Proof. The proof parallels [1,15]. The well-posedness of $\mathcal{G}_\delta$ is equivalent to the existence of a causal inverse for $I - \Delta SA$ on $\ell^{(n_T,\bar{n}_S,\bar{n}_P)}_{2e}$, for all $\Delta \in \mathbf{\Delta}$. Using (3), and for each $\Delta \in \mathbf{\Delta}$, $I - \Delta SA$ can be factored as

$$I - \Delta SA = \begin{bmatrix} I - \hat{S}_0\hat{S}\hat{A}_{11} & -\hat{S}_0\hat{S}\hat{A}_{12} \\ -\Delta_P\hat{A}_{21} & I - \Delta_P A_{PP} \end{bmatrix} = \begin{bmatrix} I & -\hat{S}_0\hat{S}\hat{A}_{12}(I - \Delta_P A_{PP})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \hat{S}_0\hat{S}R & 0 \\ -\Delta_P\hat{A}_{21} & I - \Delta_P A_{PP} \end{bmatrix}$$

since $I - \Delta_P A_{PP}$ has a causal inverse on $\ell^{\bar{n}_P}_{2e}$, where $\hat{S}_0 = \text{diag}(S_0,\ldots,S_0)$ ($d+1$ blocks), $\hat{S} = \text{diag}\,(I^{n_T}, S_1,\ldots,S_d)$, and $R = \hat{A}_{11} + \hat{A}_{12}(I - \Delta_P A_{PP})^{-1}\Delta_P\hat{A}_{21}$. So, if $I - \hat{S}_0\hat{S}R$ has a causal inverse on $\ell^{(n_T,\bar{n}_S)}_{2e}$, then $I - \Delta SA$ has a causal inverse on $\ell^{(n_T,\bar{n}_S,\bar{n}_P)}_{2e}$. By generalizing the characterizations of causality and memorylessness to partitioned operators mapping multiple inputs to multiple outputs, i.e., operators mapping $\oplus^{N_1}_{i=1}\ell_{2e}$ to $\oplus^{N_2}_{i=1}\ell_{2e}$, for some N_1, N_2, and by extending [15, Lemma 6] to this class of operators, we show that since R is a linear memoryless operator and $[\![A]\!](t,k) = 0$, for $t < 0$, then $(I - \hat{S}_0\hat{S}R)^{-1}$ exists and is causal. □

Hereafter, we assume $[\![A]\!](t,k) = 0$, $[\![B]\!](t,k) = 0$, $[\![C]\!](t,k) = 0$, and $[\![D]\!](t,k) = 0$, for all $t < 0$ and $k \in V$. To check for the well-posedness of $\mathcal{G}_\delta$, it becomes sufficient to check that $I - \Delta_P A_{PP}$ is invertible on $\ell^{\bar{n}_P}_{2e}$, for all $\Delta \in \mathbf{\Delta}$. The inverse, when existent, is memoryless (and causal) since $I - \Delta_P A_{PP}$ is memoryless. The following result gives a sufficient condition for the ℓ_2-stability of system $\mathcal{G}_\delta$, i.e., the validity of this condition implies that $I - \Delta SA$ has a bounded causal inverse, for all $\Delta \in \mathbf{\Delta}$. The proof of the result parallels its counterpart for single LPV and uncertain systems found in [16], and so is omitted. Let

$$\mathcal{X} = \Big\{X : X = \text{diag}\Big(X_T, X^S_1,\ldots,X^S_d, X^P_1,\ldots,X^P_r\Big) = X^* \in \mathcal{L}_c\Big(\ell^{(n_T,\bar{n}_S,\bar{n}_P)}_2\Big),\; X \succ 0,\; X^{-1} \in \mathcal{L}\Big(\ell^{(n_T,\bar{n}_S,\bar{n}_P)}_2\Big)\Big\},$$

where X_T, X^S_i, X^P_j, for $i = 1,\ldots,d$ and $j = 1,\ldots,r$, are graph-diagonal operators. Clearly, $\mathcal{X}$ is a commutant of $\mathbf{\Delta}$. The symbol $\mathcal{X}$ is used for similarly defined sets regardless of the associated dimensions.

Lemma 2. *If there exists $X \in \mathcal{X}$ such that $A^*S^*XSA - X \prec 0$, then system $\mathcal{G}_\delta$ is ℓ_2-stable.*

S^*XS is a partitioned graph-diagonal operator with a block-diagonal structure similar to the structure of X, i.e., $S^*XS = \text{diag}(S_0^*X_TS_0, S_1^*S_0^*X_1^SS_0S_1, \ldots, S_d^*S_0^*X_d^SS_0S_d, X_1^P, \ldots, X_r^P)$. The condition $X \in \mathcal{X}$ and the inequality in Lemma 2 can be expressed in terms of equivalent sequences of LMIs, i.e., for some scalar $\beta > 0$ and all $(t, k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, r$, $X_T(t, k) \succ \beta I$, $X_i^S(t, k) \succ \beta I$, $X_j^P(t, k) \succ \beta I$, and

$$\begin{aligned}&[\![A^*]\!](t, k)\,\text{diag}\left(X_T(t+1, k), X_1^S(t+1, \rho_1(k)), \ldots,\right.\\&\left.X_d^S(t+1, \rho_d(k)), X_1^P(t, k), \ldots, X_r^P(t, k)\right)[\![A]\!](t, k)\\&-\text{diag}\left(X_T(t, k), X_1^S(t, k), \ldots, X_d^S(t, k),\right.\\&\left.X_1^P(t, k), \ldots, X_r^P(t, k)\right) \prec -\beta I.\end{aligned}$$

One sequence of LMIs is associated with every subsystem $G^{(k)}$. These sequences are in general infinite dimensional because of the explicit dependence on time in the state-space equations of the subsystems. The sequences associated with the various subsystems are coupled through the spatial terms $X_i^S(t, k)$, but not through the parameter terms $X_j^P(t, k)$. This highlights the local dependence of the state-space matrices on the parameters. The spatial terms associated with the virtual interconnections have zero dimensions and do not appear in the LMIs. Similarly, if $r_{k_0} < r$, for some $k_0 \in V$, then $X_j^P(t, k_0)$ has zero dimensions, for $j = r_{k_0} + 1, \ldots, r$ and all $t \in \mathbb{Z}$. The βI terms ensure that the matrix sequences on the left-hand side do not converge to singular matrices as t approaches infinity. Since the state-space matrices are zeros for $t < 0$, the sequences are trivial for $t < 0$, and so t is restricted to $\mathbb{N}_0$.

Systems that satisfy the previous condition are said to be strongly stable. Strong stability implies ℓ_2-stability, but the converse is not necessarily true: A strongly stable system is an ℓ_2-stable system with a solution $X \in \mathcal{X}$ to $A^*S^*XSA - X \prec 0$; but there exist ℓ_2-stable systems which are not strongly stable.

Lemma 3. *$\mathcal{G}_\delta$ is strongly stable and $\|G_\delta\| < \gamma$, for all $\Delta \in \mathbf{\Delta}$, if there exists $X \in \mathcal{X}$ satisfying*

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \quad (4)$$

The proof resembles that of a counterpart result for interconnected LTV subsystems in [15], and so is omitted. Lemmas 2 and 3 require $X \in \mathcal{X}$. In fact, X only needs to be positive definite and in the commutant of $\mathbf{\Delta}$. We show next, however, that the imposed structure does not introduce conservatism. A similar result for single NSLPV systems is given in [2].

Theorem 1. *A solution $\bar{X} \succ 0$ to (4) in the commutant of $\mathbf{\Delta}$ exists if and only if a solution $X \in \mathcal{X}$ exists.*

Proof. The 'if' direction is trivial because $\mathcal{X}$ is a commutant of $\mathbf{\Delta}$. The 'only if' direction is proved next. We construct a solution $X \in \mathcal{X}$ from the solution $\bar{X}$. Define the operator $E_{(\tau,\zeta)} : \mathbb{R}^g \to \ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$, for some mapping n that satisfies $n(\tau, \zeta) = g$. If $E_{(\tau,\zeta)}e = v \in \ell_2$, then $v(t, k) = e$ if $(t, k) = (\tau, \zeta)$ and $v(t, k) = 0$ otherwise. The adjoint operator $E_{(\tau,\zeta)}^* : \ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right) \to \mathbb{R}^{n(\tau,\zeta)}$ satisfies $E_{(\tau,\zeta)}^*v = v(\tau, \zeta)$. Any $v \in \ell_2$ can be written as $v = \sum_{(t,k)} E_{(t,k)}v(t, k)$. Thus, for any operator Q on ℓ_2, $Qv = \sum_{(t,k)} QE_{(t,k)}v(t, k)$. As $\bar{X}$ is in the commutant of $\mathbf{\Delta}$, then $\bar{X} = \text{diag}\left(\bar{X}_T, \bar{X}_1^S, \ldots, \bar{X}_d^S, \bar{X}_1^P, \ldots, \bar{X}_r^P\right)$. Since operators Δ_j are graph-diagonal, then $\bar{X}_j^P$ must also be graph-diagonal as they must satisfy $\bar{X}_j^P\Delta_jv = \Delta_j\bar{X}_j^Pv$, for all $v \in \ell_2$ and $j = 1, \ldots, r$. To see this, both sides of the previous equation are evaluated at (τ, ζ), i.e., pre-multiplied by $E_{(\tau,\zeta)}^*$. We get

$$\begin{aligned}&\sum_{(t,k)} \delta_j(t, k)E_{(\tau,\zeta)}^*\bar{X}_j^PE_{(t,k)}v(t, k)\\&\quad= \delta_j(\tau, \zeta)\sum_{(t,k)} E_{(\tau,\zeta)}^*\bar{X}_j^PE_{(t,k)}v(t, k).\end{aligned}$$

For this equality to hold for an arbitrary $v \in \ell_2$, $\bar{X}_j^P$ must be graph-diagonal. That is, $E_{(\tau,\zeta)}^*\bar{X}_j^PE_{(t,k)} = \bar{X}_j^P(\tau, \zeta)$, for $(t, k) = (\tau, \zeta)$, and $E_{(\tau,\zeta)}^*\bar{X}_j^PE_{(t,k)} = 0$, otherwise. The same, however, cannot be concluded about $\bar{X}_T$ and $\bar{X}_i^S$, for $i = 1, \ldots, d$. Since $\bar{X}$ satisfies (4), then, for some $\beta > 0$, the following holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} S^*\bar{X}S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} \bar{X} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec -\beta I.$$

Define the operator $\hat{E}_{(\tau,\zeta)} = \text{diag}(E_{(\tau,\zeta)}, E_{(\tau,\zeta)}, \ldots, E_{(\tau,\zeta)})$ ($1+d+r$ blocks), which satisfies

$$\begin{aligned}&\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & E_{(\tau,\zeta)} \end{bmatrix}\\&\quad= \begin{bmatrix} \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & E_{(\tau,\zeta)} \end{bmatrix} \left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\!\right](\tau, \zeta).\end{aligned}$$

The previous inequality is pre- and post-multiplied by $\text{diag}(\hat{E}_{(\tau,\zeta)}, E_{(\tau,\zeta)})^*$ and its adjoint, which results in

$$\begin{aligned}&\left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\!\right]^*(\tau, \zeta) \begin{bmatrix} \hat{E}_{(\tau,\zeta)}^*S^*\bar{X}S\hat{E}_{(\tau,\zeta)} & 0 \\ 0 & I \end{bmatrix}\\&\quad\times \left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\!\right](\tau, \zeta) - \begin{bmatrix} \hat{E}_{(\tau,\zeta)}^*\bar{X}\hat{E}_{(\tau,\zeta)} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec -\beta I,\end{aligned}$$

with $E_{(\tau,\zeta)}^*S_0^*\bar{X}_TS_0E_{(\tau,\zeta)} = E_{(\tau+1,\zeta)}^*\bar{X}_TE_{(\tau+1,\zeta)}$ and

$$E_{(\tau,\zeta)}^*S_i^*S_0^*\bar{X}_i^SS_0S_iE_{(\tau,\zeta)} = E_{(\tau+1,\rho_i(\zeta))}^*\bar{X}_i^SE_{(\tau+1,\rho_i(\zeta))},$$

for $i = 1, \ldots, d$. In the above inequality, note that $E_{(\tau,\zeta)}^*E_{(\tau,\zeta)} = I$. We define graph-diagonal operators X_T and X_i^S such that $X_T(t, k) = E_{(t,k)}^*\bar{X}_TE_{(t,k)}$ and $X_i^S(t, k) = E_{(t,k)}^*\bar{X}_i^SE_{(t,k)}$. The previous inequality holds for all $(\tau, \zeta) \in \mathbb{Z} \times V$, and so one can verify that $X = \text{diag}\left(X_T, X_1^S, \ldots, X_d^S, \bar{X}_1^P, \ldots, \bar{X}_r^P\right) \in \mathcal{X}$ satisfies the sequences of LMIs which are equivalent to (4). Thus, we have found $X \in \mathcal{X}$ that satisfies (4). □

5. Synthesis results

This section addresses the control synthesis problem. The plant $\mathcal{G}_\delta$ is assumed to be well-posed, and the state-space matrices are taken as zeros, for $t < 0$. We assume that $D_{22}(t, k) = 0$, for all $(t, k) \in \mathbb{Z} \times V$. The sought controller $\mathcal{K}_\delta$ is a distributed system with the same interconnection and LFT structures as the plant. That is, the controller consists of NSLPV subsystems, formulated in an LFT framework, and interconnected over the same interconnection structure as the plant. The information transfer between the controller subsystems is subjected to a one-step time-delay. For all $k \in V$, the controller subsystem $K^{(k)}$ is affected by the same parameters as the plant subsystem $G^{(k)}$. Thus, the controller equations are in the form of (1), with inputs y and outputs u. The controller operators are denoted using the same symbols as the plant operators with the additional superscript K. The controller dimensions are denoted by $m_T(t, k)$, $m_i^S(t, k)$, $m_j^P(t, k)$, for $i = 1, \ldots, d$ and $j = 1, \ldots, r$. These dimensions are obtained from the synthesis solutions, as will be seen later. $\bar{m}_S$ is defined similarly to $\bar{n}_S$, and so on. Fig. 3 shows the closed-loop system formed by

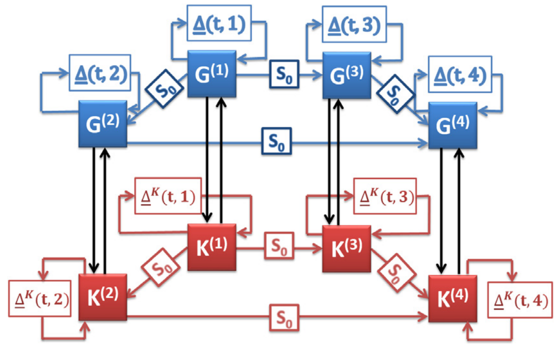

Fig. 3. Controller inheriting the interconnection and LFT structures of the plant. The information exchange between a controller subsystem and its corresponding plant subsystem is depicted using black arrows.

the plant of Fig. 2 and the corresponding controller. Using (3), the controller equations are written as

$$\begin{bmatrix} x_K \\ \alpha_K \\ u \end{bmatrix} = \begin{bmatrix} \hat{S}_0\hat{S}\hat{A}^K_{11} & \hat{S}_0\hat{S}\hat{A}^K_{12} & \hat{S}_0\hat{S}\hat{B}^K \\ \hat{A}^K_{21} & A^K_{PP} & B^K_P \\ \hat{C}^K & C^K_P & D^K \end{bmatrix} \begin{bmatrix} x_K \\ \beta_K \\ y \end{bmatrix},$$

$$\begin{bmatrix} x_K \\ \beta_K \end{bmatrix} = \begin{bmatrix} I^{(m_T,\bar{m}_S)} & 0 \\ 0 & \Delta^K_P \end{bmatrix} \begin{bmatrix} x_K \\ \alpha_K \end{bmatrix} = \Delta^K \begin{bmatrix} x_K \\ \alpha_K \end{bmatrix}, \tag{5}$$

where $\Delta^K_P = \operatorname{diag}\left(\Delta^K_1, \ldots, \Delta^K_r\right)$. For $j = 1, \ldots, r$, the graph-diagonal operators Δ^K_j satisfy $\Delta^K_j(t,k) = \delta_j(t,k) I_{m^P_j(t,k)}$. The parameters $\delta_j(t,k)$ are the same as those of the plant. The controller state-space matrices are zeros, for $t < 0$. Then, by Lemma 1, the controller is well-posed if $I - \Delta^K_P A^K_{PP}$ has a causal inverse on $\ell_{2e}^{\bar{m}_P}$, for all $\Delta^K \in \mathbf{\Delta}^K$. The closed-loop equations are obtained by combining the plant equations and (5). Let $\tilde{S} = \operatorname{diag}(S, S) = \operatorname{diag}\left(\hat{S}_0\hat{S}, I^{\bar{n}_P}, \hat{S}_0\hat{S}, I^{\bar{m}_P}\right)$, $\tilde{\Delta} = \operatorname{diag}(\Delta, \Delta^K) = \operatorname{diag}(I^{(n_T,\bar{n}_S)}, \Delta_P, I^{(m_T,\bar{m}_S)}, \Delta^K_P)$, and $x_{\text{cl}} = \begin{bmatrix} x^* & \beta^* & x_K^* & \beta_K^* \end{bmatrix}^*$. Then, the closed-loop equations are $x_{\text{cl}} = \tilde{\Delta}\tilde{S}A_{\text{cl}}x_{\text{cl}} + \tilde{\Delta}\tilde{S}B_{\text{cl}}w$ and $z = C_{\text{cl}}x_{\text{cl}} + D_{\text{cl}}w$, where the partitioned graph-diagonal operators A_{cl}, B_{cl}, C_{cl}, and D_{cl} satisfy $A_{\text{cl}} = \bar{A} + \underline{B}J\underline{C}$, $B_{\text{cl}} = \bar{B} + \underline{B}J\underline{D}_{21}$, $C_{\text{cl}} = \bar{C} + \underline{D}_{12}J\underline{C}$, and $D_{cl} = D_{11} + \underline{D}_{12}J\underline{D}_{21}$, with

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0^{(m_T^+,\bar{m}_S^+,\bar{m}_P)\times(m_T,\bar{m}_S,\bar{m}_P)} \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} B_1 \\ 0^{(m_T^+,\bar{m}_S^+,\bar{m}_P)\times n_w} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0 & B_2 \\ I^{(m_T^+,\bar{m}_S^+,\bar{m}_P)} & 0 \end{bmatrix},$$

$$\underline{C} = \begin{bmatrix} 0 & I^{(m_T,m_S,\bar{m}_P)} \\ C_2 & 0 \end{bmatrix},$$

$$J = \begin{bmatrix} \hat{A}^K_{11} & \hat{A}^K_{12} & \hat{B}^K \\ \hat{A}^K_{21} & A^K_{PP} & B^K_P \\ \hat{C}^K & C^K_P & D^K \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 & 0^{n_z\times(m_T,\bar{m}_S,\bar{m}_P)} \end{bmatrix},$$

$$\underline{D}_{12} = \begin{bmatrix} 0^{n_z\times(m_T^+,\bar{m}_S^+,\bar{m}_P)} & D_{12} \end{bmatrix},$$

$$\underline{D}_{21} = \begin{bmatrix} 0^{(m_T,\bar{m}_S,\bar{m}_P)\times n_w} \\ D_{21} \end{bmatrix}.$$

The given parametrization of the closed-loop operators allows us to develop an affine condition in the controller realization J to check whether a given controller $\mathcal{K}_\delta$ ensures the ℓ_2-stability of the closed-loop system and further guarantees an upper bound γ on the ℓ_2-induced norm of the resulting mapping from w to z.

The above equations describe the closed-loop system but are not in the form of (2). Define the partitioned graph-diagonal operator $\Delta^L_P = \operatorname{diag}\left(\Delta^L_1, \ldots, \Delta^L_r\right)$, where, for all $(t,k) \in \mathbb{Z} \times V$ and $j = 1, \ldots, r$, $\Delta^L_j(t,k) = \delta_j(t,k) I_{n^P_j(t,k)+m^P_j(t,k)}$. Let P be an appropriately defined operator such that $P^*P = I$, $PP^* = I$, and $P^*\tilde{\Delta}P = \operatorname{diag}\left(I^{n_T+m_T}, I^{\bar{n}_S+\bar{m}_S}, \Delta^L_P\right) = \Delta^L$; and define A^L, B^L, C^L, and D^L according to $SA^L = P^*\tilde{S}A_{\text{cl}}P$, $SB^L = P^*\tilde{S}B_{\text{cl}}$, $C^L = C_{\text{cl}}P$, and $D^L = D_{\text{cl}}$. The above equations can then be rewritten in the form of (2):

$$\begin{bmatrix} x^L \\ \beta^L \end{bmatrix} = \Delta^L S A^L \begin{bmatrix} x^L \\ \beta^L \end{bmatrix} + \Delta^L S B^L w, \quad z = C^L \begin{bmatrix} x^L \\ \beta^L \end{bmatrix} + D^L w,$$

$$\Delta^L \in \mathbf{\Delta}^L = \{\Delta^L : \|\Delta^L\| \le 1\},$$

where $x^L = \left[(x^L_T)^*, (x^L_1)^*, \ldots, (x^L_d)^*\right]^*$, $\beta^L = \left[(\beta^L_1)^*, \ldots, (\beta^L_r)^*\right]^*$, $x^L_T(t,k) \in \mathbb{R}^{n_T(t,k)+m_T(t,k)}$, $x^L_i(t,k) \in \mathbb{R}^{n^S_i(t,k)+m^S_i(t,k)}$, and $\beta^L_j(t,k) \in \mathbb{R}^{n^P_j(t,k)+m^P_j(t,k)}$, for $i = 1, \ldots, d$ and $j = 1, \ldots, r$.

Definition 2. A controller $\mathcal{K}_\delta$ is said to be a γ-admissible synthesis for plant $\mathcal{G}_\delta$ if the resulting closed-loop system is ℓ_2-stable, and $\|w \to z\| = \|C^L(I - \Delta^L SA^L)^{-1}\Delta^L SB^L + D^L\| < \gamma$, for all $\Delta^L \in \mathbf{\Delta}^L$.

Without loss of generality, we restrict the discussion to the case $\gamma = 1$ because a γ-admissible synthesis for a plant $\mathcal{G}_\delta$ is 1-admissible for the scaled system $\mathcal{G}_{\text{scaled},\delta}$ in which γ is absorbed into the system operators, namely, $C_{\text{scaled},1} = \frac{1}{\gamma}C_1$, $D_{\text{scaled},11} = \frac{1}{\gamma}D_{11}$, and $D_{\text{scaled},12} = \frac{1}{\gamma}D_{12}$. More details are provided in Section 6.

Theorem 2. *If there exists $X^L \in \mathcal{X}$ that satisfies $H + Q^*J^*R + R^*JQ \prec 0$, then the given controller $\mathcal{K}_\delta$ with realization J is a 1-admissible synthesis for plant $\mathcal{G}_\delta$. In the preceding condition,*

$$R = \begin{bmatrix} \underline{B}^* & 0 & 0 & \underline{D}^*_{12} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & \underline{C} & \underline{D}_{21} & 0 \end{bmatrix},$$

$$X^L_P = PX^LP^*, \text{ and } H = \begin{bmatrix} -\tilde{S}^*\left(X^L_P\right)^{-1}\tilde{S} & \bar{A} & \bar{B} & 0 \\ \bar{A}^* & -X^L_P & 0 & \bar{C}^* \\ \bar{B}^* & 0 & -I & D^*_{11} \\ 0 & \bar{C} & D_{11} & -I \end{bmatrix}.$$

Proof. By Lemma 3, if there exists $X^L \in \mathcal{X}$ that satisfies (4) for $\gamma = 1$ and the closed-loop realization given by A^L, B^L, C^L, and D^L, then $\mathcal{K}_\delta$ is a 1-admissible synthesis for $\mathcal{G}_\delta$. (4) is pre- and post-multiplied by diag (P, I) and its adjoint, respectively, and 'diag (P^*, I) diag $(P, I) = \operatorname{diag}(I, I)$' is inserted as needed, to get

$$\begin{bmatrix} A_{\text{cl}} & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix}^* \begin{bmatrix} \tilde{S}^*X^L_P\tilde{S} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\text{cl}} & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix} - \begin{bmatrix} X^L_P & 0 \\ 0 & I \end{bmatrix} \prec 0.$$

The condition in the theorem is retrieved by applying the Schur complement formula to this inequality and appropriately rearranging the terms in the resulting inequality. □

Recall that $X^L = \operatorname{diag}\left(X_T, X^S_1, \ldots, X^S_d, X^P_1, \ldots, X^P_r\right) \in \mathcal{X}$. For all $(t,k) \in \mathbb{Z} \times V$, we partition $X_T(t,k) = \left[X_{T,ij}(t,k)\right]_{i=1,2;j=1,2}$, where $X_{T,11}(t,k) \in \mathbb{S}^{n_T(t,k)}$, $X_{T,22}(t,k) \in \mathbb{S}^{m_T(t,k)}$, and $X_{T,21}(t,k) = X^*_{T,12}(t,k)$. These partitions define the graph-diagonal operators $X_{T,11}$, $X_{T,12}$, and $X_{T,22}$. We repeat the partitioning process for all $X^S_i(t,k)$ and $X^P_j(t,k)$, and construct $\hat{X}_{11} = \operatorname{diag}\left(X_{T,11}, X^S_{1,11}, \ldots, X^S_{d,11}, X^P_{1,11}, \ldots, X^P_{r,11}\right) \in \mathcal{X}$. $\hat{X}_{22} \in \mathcal{X}$ and $\hat{X}_{12}$ are constructed in a similar way. Then, one can see that $X^L_P = PX^LP^*$ has the structure shown below. Also, since $\left(X^L\right)^{-1} \in \mathcal{X}$, then $\left(X^L_P\right)^{-1} = P\left(X^L\right)^{-1}P^*$ has a similar structure to X^L_P. Namely,

$$X^L_P = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}^*_{12} & \hat{X}_{22} \end{bmatrix}, \qquad \left(X^L_P\right)^{-1} = \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}^*_{12} & \hat{Y}_{22} \end{bmatrix}, \tag{6}$$

where $\hat{Y}_{11}$ is defined similarly to $\hat{X}_{11}$, and so on. Given some $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$, the next result gives necessary and sufficient conditions for the existence of X^L_P and its inverse with the structure defined in (6).

Lemma 4. *Given $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$, there exist $X_P^L \succ 0$ with the structure defined in (6) if and only if, for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \dots, d$, and $j = 1, \dots, r$, the following conditions hold:*

$$\begin{bmatrix} \hat{X}_{11} & I \\ I & \hat{Y}_{11} \end{bmatrix} \succeq 0,$$

$$\operatorname{rank} \begin{bmatrix} X_{T,11}(t,k) & I \\ I & Y_{T,11}(t,k) \end{bmatrix} \leq n_T(t,k) + m_T(t,k),$$

$$\operatorname{rank} \begin{bmatrix} X_{i,11}^S(t,k) & I \\ I & Y_{i,11}^S(t,k) \end{bmatrix} \leq n_i^S(t,k) + m_i^S(t,k),$$

$$\operatorname{rank} \begin{bmatrix} X_{j,11}^P(t,k) & I \\ I & Y_{j,11}^P(t,k) \end{bmatrix} \leq n_j^P(t,k) + m_j^P(t,k).$$

The proof of this result, which includes a procedure for the construction of the required X_P^L and $(X_P^L)^{-1}$, is omitted as it is an immediate generalization of the proof of [12, Lemma 6.2].

The first condition can alternatively be expressed in terms of its equivalent sequences of LMIs as follows:

$$\begin{bmatrix} X_{T,11}(t,k) & I \\ I & Y_{T,11}(t,k) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} X_{i,11}^S(t,k) & I \\ I & Y_{i,11}^S(t,k) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} X_{j,11}^P(t,k) & I \\ I & Y_{j,11}^P(t,k) \end{bmatrix} \succeq 0.$$

Theorem 2 allows for checking if a given controller $\mathcal{K}_\delta$ with realization J is a 1-admissible synthesis for plant $\mathcal{G}_\delta$. The next result gives sufficient convex conditions for the existence of a 1-admissible synthesis.

Theorem 3. *There exists a 1-admissible synthesis $\mathcal{K}_\delta$ for plant $\mathcal{G}_\delta$, with dimensions $m_T(t,k) \leq n_T(t,k)$, $m_i^S(t,k) \leq n_i^S(t,k)$, and $m_j^P(t,k) \leq n_j^P(t,k)$, for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \dots, d$, and $j = 1, \dots, r$, if there exist $\hat{X}_{11} \in \mathcal{X}$ and $\hat{Y}_{11} \in \mathcal{X}$ such that*

$$\begin{bmatrix} V_1^* & V_2^* \end{bmatrix} \left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} \hat{Y}_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* - \begin{bmatrix} S^* \hat{Y}_{11} S & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \prec 0, \quad (7)$$

$$\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* \begin{bmatrix} S^* \hat{X}_{11} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \prec 0, \quad (8)$$

$$\begin{bmatrix} \hat{X}_{11} & I \\ I & \hat{Y}_{11} \end{bmatrix} \succeq 0. \quad (9)$$

The partitioned graph-diagonal operators U_1, V_1, and the graph-diagonal operators U_2, V_2 are defined such that $\operatorname{Im}\begin{bmatrix} V_1^* & V_2^* \end{bmatrix}^* = \ker \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}$, $V_1^* V_1 + V_2^* V_2 = I$, $\operatorname{Im}\begin{bmatrix} U_1^* & U_2^* \end{bmatrix}^* = \ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, and $U_1^* U_1 + U_2^* U_2 = I$, where $\operatorname{Im} T$ and $\ker T$ denote the image and the kernel of a linear operator T. Namely,

$$U_1 = \begin{bmatrix} (U_T)^* & (U_1^S)^* & \dots & (U_d^S)^* & (U_1^P)^* & \dots & (U_r^P)^* \end{bmatrix}^*,$$
$$V_1 = \begin{bmatrix} (V_T)^* & (V_1^S)^* & \dots & (V_d^S)^* & (V_1^P)^* & \dots & (V_r^P)^* \end{bmatrix}^*,$$

with $U_T(t,k) \in \mathbb{R}^{n_T(t,k) \times ?}$, $U_i^S(t,k) \in \mathbb{R}^{n_i^S(t,k) \times ?}$, $U_j^P(t,k) \in \mathbb{R}^{n_j^P(t,k) \times ?}$, $V_T(t,k) \in \mathbb{R}^{n_T(t+1,k) \times ?}$, $V_i^S(t,k) \in \mathbb{R}^{n_i^S(t+1,\rho_i(k)) \times ?}$, $V_j^P(t,k) \in \mathbb{R}^{n_j^P(t,k) \times ?}$, $U_2(t,k) \in \mathbb{R}^{n_w(t,k) \times ?}$, and $V_2(t,k) \in \mathbb{R}^{n_z(t,k) \times ?}$.

Proof. By Lemma 4, and for some integers $m_T(t,k)$, $m_i^S(t,k)$, and $m_j^P(t,k)$ such that $m_T(t,k) \leq n_T(t,k)$, $m_i^S(t,k) \leq n_i^S(t,k)$, and $m_j^P(t,k) \leq n_j^P(t,k)$, there exist X_P^L and $(X_P^L)^{-1}$ with the structure defined in (6) since $\hat{X}_{11}$ and $\hat{Y}_{11}$ satisfy (9). Thus, the operator H in Theorem 2 is well-defined; and so, if there exists a solution J to $H + Q^* J^* R + R^* J Q \prec 0$, then the controller $\mathcal{K}_\delta$ with realization J is a 1-admissible synthesis for plant $\mathcal{G}_\delta$. By a generalization of [11, Lemma 3.1] and [13, Lemmas 16, 17] to the class of partitioned graph-diagonal operators, there exists a solution J to the preceding condition if and only if $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$, where W_R and W_Q satisfy $\operatorname{Im} W_R = \ker R$, $W_R^* W_R = I$, $\operatorname{Im} W_Q = \ker Q$, and $W_Q^* W_Q = I$. W_R and W_Q can be chosen as follows:

$$1 = 1.2 W_R = \left[\begin{array}{c|c|c} V_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I^{(n_T, \bar{n}_S, \bar{n}_P, m_T, \bar{m}_S, \bar{m}_P)} & 0 \\ \hline 0 & 0 & I^{n_w} \\ V_2 & 0 & 0 \end{array}\right],$$

$$W_Q = \left[\begin{array}{c|c|c} 0 & I^{(n_T^+, \bar{n}_S^+, \bar{n}_P, m_T^+, \bar{m}_S^+, \bar{m}_P)} & 0 \\ \hline U_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline U_2 & 0 & 0 \\ \hline 0 & 0 & I^{n_z} \end{array}\right].$$

Expand $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$, and apply the Schur complement formula to each inequality to get

$$\left(V_1^* A + V_2^* C_1\right) \hat{Y}_{11} \left(A^* V_1 + C_1^* V_2\right) + \left(V_1^* B_1 + V_2^* D_{11}\right)\left(B_1^* V_1 + D_{11}^* V_2\right) - V_1^* S^* \hat{Y}_{11} S V_1 - V_2^* V_2 \prec 0,$$

$$\left(U_1^* A^* + U_2^* B_1^*\right) S^* \hat{X}_{11} S \left(A U_1 + B_1 U_2\right) + \left(U_1^* C_1^* + U_2^* D_{11}^*\right)\left(C_1 U_1 + D_{11} U_2\right) - U_1^* \hat{X}_{11} U_1 - U_2^* U_2 \prec 0,$$

which are (7) and (8), respectively. □

Algorithm 1 shows how to use the synthesis solutions $\hat{X}_{11}$ and $\hat{Y}_{11}$ to construct the desired controller.

Algorithm 1. Given $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ that satisfy (7), (8), and (9), the realization J of a 1-admissible synthesis $\mathcal{K}_\delta$ is constructed as follows. For all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \dots, d$, and $j = 1, \dots, r$:

1. Define the controller dimensions as follows: $m_T(t,k) = \operatorname{rank}\left(Y_{T,11}(t,k) - X_{T,11}(t,k)^{-1}\right)$, $m_i^S(t,k) = \operatorname{rank}\left(Y_{i,11}^S(t,k) - X_{i,11}^S(t,k)^{-1}\right)$, and $m_j^P(t,k) = \operatorname{rank}\left(Y_{j,11}^P(t,k) - X_{j,11}^P(t,k)^{-1}\right)$.
2. Use Lemma 4 to construct X_P^L and its inverse $(X_P^L)^{-1}$ with the structure in (6).
3. Construct H as in Theorem 2, and solve $H + Q^* J^* R + R^* J Q \prec 0$ for J.

This section concludes with a discussion on eventually time-periodic (ETP) subsystems interconnected over finite graphs, for which the analysis and synthesis problems are finite dimensional.

Definition 3. A subsystem $G^{(k)}$ with zero state-space matrices, for $t < 0$, is said to be (h,q)-ETP, for some integers $h \geq 0$ and $q > 0$, if the corresponding state-space matrices, are (h,q)-ETP, that is, they become time-periodic with period q after some finite time-horizon h, namely, for all $t, z \in \mathbb{N}_0$, $[\![W]\!](t+h+zq,k) = [\![W]\!](t+h,k)$ and $W \in \{A, B, C, D\}$.

The class of ETP subsystems includes as special cases the classes of finite time-horizon subsystems and time-periodic subsystems. Standard LPV subsystems are (0, 1)-ETP NSLPV subsystems.

Lemma 5. *If all the subsystems are (h,q)-ETP, then the following hold:*

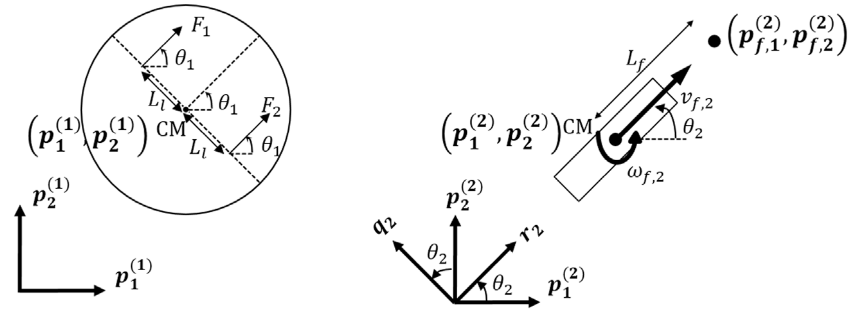

Fig. 4. Left: Depiction of a two-thruster hovercraft. Right: Depiction of the non-holonomic vehicle corresponding to $G^{(2)}$.

1. *There exists a solution $X \in \mathcal{X}$ to (4) (respectively, the inequality in Lemma 2) if and only if there exists an (M, q)-ETP (respectively, an (h, q)-ETP) solution X_{eper}, for some integer $M \geq h$.*
2. *There exist solutions $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ to (7)–(9) if and only if there exist (M, q)-ETP solutions $\hat{X}_{11,\text{eper}}$ and $\hat{Y}_{11,\text{eper}}$, for some integer $M \geq h$.*
3. *In the above propositions, if $h = 0$, then M can also be taken equal to 0.*

The proof is omitted as it parallels the proof of [10, Proposition 21] and the references therein.

Thus, for (h, q)-ETP subsystems, we restrict t to some finite time-horizon M and one time-period truncation, i.e., $0 \leq t \leq M + q - 1$, when solving the analysis and the synthesis problems. If, in addition, the interconnection graph is finite, i.e., the sets V and E are finite, then the problems become finite dimensional.

6. Illustrative example

6.1. Problem formulation

Consider a plant $\mathcal{G}_\delta$ formed by four subsystems interconnected as in Fig. 2. The leader $G^{(1)}$ is a two-thruster hovercraft described in [2], and the followers $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$ are non-holonomic vehicles defined in [17]. The leader and subsystem $G^{(2)}$ are depicted in Fig. 4. The leader is to track the eventually time-periodic reference trajectory defined in [2], and the followers are to track the leader. For simplicity, the four vehicles are assumed to be initially on top of each other; any desired formation can be implemented by applying appropriate translations when the controller is executed. Let $(p_1^{(1)}, p_2^{(1)})$ be the position of the center of mass (CM) of the hovercraft, θ_1 be its orientation with respect to the $p_1^{(1)}$-axis, and F_1 and F_2 be the force control inputs applied at a distance $L_l = 0.15$ m from the CM. The forces take values between 0 and 2.5 N. Let $\nu = (p_1^{(1)}, p_2^{(1)}, \theta_1, \dot{p}_1^{(1)}, \dot{p}_2^{(1)}, \dot{\theta}_1)$ and $F = (F_1, F_2)$, then the equations of motion are $\dot{\nu} = f(\nu, F)$, with $f(\nu, F)$ given in [2]. The equations are linearized about the reference trajectory (ν_r, F_r) to obtain

$$\dot{\bar{\nu}} = A_c(t_c)\bar{\nu} + B_{2c}(t_c)\bar{F}, \quad \bar{\nu} = \nu - \nu_r,$$
$$\bar{F} = F - F_r, \quad A_c(t_c) = \left.\frac{\partial f}{\partial \nu}\right|_{(\nu_r, F_r)}, \quad B_{2c}(t_c) = \left.\frac{\partial f}{\partial F}\right|_{(\nu_r, F_r)},$$

where t_c is the continuous-time. Then, the effect of the exogenous disturbances $\tilde{w}(t_c)$, which consist of torques and forces in both the $p_1^{(1)}$ and $p_2^{(1)}$ directions, is added as in $\dot{\bar{\nu}}(t_c) = A_c(t_c)\bar{\nu}(t_c) + B_{1c}\tilde{w}(t_c) + B_{2c}(t_c)\bar{F}(t_c)$, where $B_{1c} = \begin{bmatrix} 0_3 & I_3 \end{bmatrix}^*$. The inputs and disturbances are applied in discrete-time at a sampling frequency of 20Hz. A bilinear transformation [18] is used to obtain a discrete-time trapezoidal approximation for the previous equations. Let $t \in \mathbb{N}_0$ be the discrete-time and $\tau = 0.05$ s be the sampling period, then

$$x_T(t+1, 1) = A_{TT}(t, 1)x_T(t, 1) + B_{T1}(t, 1)w(t, 1) + B_{T2}(t, 1)u(t, 1),$$
$$x_T(t, 1) = \bar{\nu}(t\tau), \quad w(t, 1) = \tilde{w}(t\tau), \quad u(t, 1) = \bar{F}(t\tau).$$

The reference trajectory is (45, 120)-ETP, and so are $A_{TT}(t, 1)$, $B_{T1}(t, 1)$, and $B_{T2}(t, 1)$. The position and the orientation of the hovercraft are measurable at each time-step. For all $t \in \mathbb{N}_0$, we set

$$C_{1T}(t, 1) = \begin{bmatrix} \alpha_1 \begin{bmatrix} I_3 & 0_3 \end{bmatrix} \\ 0_{2\times 6} \end{bmatrix}, \; D_{11}(t, 1) = 0_{5\times 3},$$
$$D_{12}(t, 1) = \begin{bmatrix} 0_{3\times 2} \\ \alpha_2 I_2 \end{bmatrix}, \; C_{2T}(t, 1) = \begin{bmatrix} I_3 & 0_3 \end{bmatrix}, \; D_{21}(t, 1) = 0_3,$$
$$D_{22}(t, 1) = 0_{3\times 2},$$

$\alpha_1 = 0.3$, $\alpha_2 = 0.2$. There are no incoming edges to vertex 1 and there are no parameters affecting $G^{(1)}$, and so, for all $t \in \mathbb{N}_0$, $A_{TS}(t, 1)$, $A_{SS}(t, 1)$, $A_{PS}(t, 1)$, $C_{1S}(t, 1)$, $C_{2S}(t, 1)$, $A_{TP}(t, 1)$, $A_{SP}(t, 1)$, $A_{PP}(t, 1)$, $C_{1P}(t, 1)$, $C_{2P}(t, 1)$, $A_{PT}(t, 1)$, $B_{P1}(t, 1)$, and $B_{P2}(t, 1)$ are non-existent. The leader sends $(p_1^{(1)}, p_2^{(1)}) = (\bar{p}_1^{(1)} + p_{1,r}^{(1)}, \bar{p}_2^{(1)} + p_{2,r}^{(1)})$ to the followers, i.e., $x_1(t+1, 2) = x_2(t+1, 3) = (p_1^{(1)}((t+1)\tau), p_2^{(1)}((t+1)\tau))$. $w(t, 1)$ is augmented by $(10\,p_{1,r}^{(1)}((t+1)\tau), 10\,p_{2,r}^{(1)}((t+1)\tau))$. Let $\mathcal{W} = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}$, and define

$$A_{ST}(t, 1) = \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} A_{TT}(t, 1),$$
$$B_{S1}(t, 1) = \begin{bmatrix} \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} B_{T1}(t, 1) & \begin{bmatrix} 0.1I_2 \\ 0.1I_2 \end{bmatrix} \end{bmatrix},$$
$$B_{S2}(t, 1) = \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} B_{T2}(t, 1).$$

Moreover, $B_{T1}(t, 1)$, $D_{11}(t, 1)$, and $D_{21}(t, 1)$ are augmented by two zero columns.

For the followers, we focus on modeling $G^{(2)}$ since $G^{(3)}$ and $G^{(4)}$ are treated similarly. As in [17,19], a point $(p_{f,1}^{(2)}, p_{f,2}^{(2)})$ at a fixed distance $L_f = 0.02$ m ahead of the center of mass $(p_1^{(2)}, p_2^{(2)})$ of the vehicle is considered. Let θ_2 be the orientation of $G^{(2)}$ with respect to the $p_1^{(2)}$-axis, $v_{f,2} = v_{c,2} + v_{d,2}$ be the forward velocity, and $\omega_{f,2} = \omega_{c,2} + \omega_{d,2}$ be the angular velocity, where $v_{c,2}$ and $\omega_{c,2}$ are the control inputs and $v_{d,2}$ and $\omega_{d,2}$ are the disturbances. In the (r_2, q_2)-reference frame, the equations of motion are $\dot{r}_2 = v_{f,2} + \omega_{f,2}q_2$, $\dot{q}_2 = \omega_{f,2}L_f - \omega_{f,2}r_2$, and $\dot{\theta}_2 = \omega_{f,2}$. As in [17], the third equation is disregarded during control design. Assume $\omega_{f,2} \in [-5, 5]$ rad/sec, and define $\eta_2 = \frac{1}{5}\omega_{f,2}$. Then, the previous equations can be expressed as

$$\begin{bmatrix}\dot{r}_2\\ \dot{q}_2\end{bmatrix}=\begin{bmatrix}0 & 5\eta_2\\ -5\eta_2 & 0\end{bmatrix}\begin{bmatrix}r_2\\ q_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & L_f\end{bmatrix}\begin{bmatrix}v_{d,2}\\ \omega_{d,2}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & L_f\end{bmatrix}\begin{bmatrix}v_{c,2}\\ \omega_{c,2}\end{bmatrix}.$$

These equations are written in an LFT form by taking η_2 as the parameter. Since τ is sufficiently small, the bilinear transformation in [18] is used to obtain a discrete-time trapezoidal approximation of the equations, assuming the parameters and the parameter states to be constants over each time-interval $[t\tau,(t+1)\tau)$, for all $t\in\mathbb{N}_0$, with the parameter states treated as exogenous inputs. We define $x_T(t,2)=(r_2(t\tau),q_2(t\tau))$, $w(t,2)=(v_{d,2}(t\tau),\omega_{d,2}(t\tau))$, $u(t,2)=(v_{c,2}(t\tau),\omega_{c,2}(t\tau))$, and $\delta_1(t,2)=\eta_2(t\tau)$. $G^{(2)}$ receives the spatial state $x_1(t,2)=(p_1^{(1)}(t\tau),p_2^{(1)}(t\tau))$. The output measurements of subsystem $G^{(2)}$ are then defined as

$$y(t,2)=\big(r_2(t\tau)-L_f-(p_1^{(1)}(t\tau)\cos\theta_2(t\tau)+p_2^{(1)}(t\tau)\sin\theta_2(t\tau)),\; q_2(t\tau)-(-p_1^{(1)}(t\tau)\sin\theta_2(t\tau)+p_2^{(1)}(t\tau)\cos\theta_2(t\tau))\big).$$

The performance outputs are taken as $z(t,2)=(\alpha_3 y(t,2),\alpha_4 u(t,2))$, where $\alpha_3=0.3$ and $\alpha_4=0.5$. The equations for $y(t,2)$ and $z(t,2)$ are written in an LFT framework by defining $\delta_2(t,2)=\cos(\theta_2(t\tau))$ and $\delta_3(t,2)=\sin(\theta_2(t\tau))$. Choosing cosine and sine functions as scheduling parameters may be conservative, but it reduces the computational complexity significantly. Then, $\underline{\Delta}(t,2)=\mathrm{diag}(\delta_1(t,2)I_2,\delta_2(t,2)I_2,\delta_3(t,2)I_2)$. We augment $w(t,2)$ by $10L_f$. $G^{(2)}$ sends (r_2,q_2) to $G^{(4)}$, and so $x_2(t+1,4)=(r_2((t+1)\tau),q_2((t+1)\tau))$. Thus, the time-invariant (0, 1)-ETP state-space matrices of $G^{(2)}$ are formed as follows. For all $t\in\mathbb{N}_0$, $C_{2P}(t,2)=\begin{bmatrix}0_2 & I_2 & I_2\end{bmatrix}$, $C_{2T}(t,2)=I_2$, $A_{PP}(t,2)=0_6$, $A_{TS}(t,2)=0_2$, $D_{22}(t,2)=0_2$,

$$A_{TT}(t,2)=A_{ST}(t,2)=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$$A_{TP}(t,2)=A_{SP}(t,2)=\begin{bmatrix}\begin{bmatrix}5\tau&0\\0&5\tau\end{bmatrix} & 0_{2\times4}\end{bmatrix},$$

$$D_{21}(t,2)=\begin{bmatrix}0_2 & \begin{bmatrix}-0.1\\0\end{bmatrix}\end{bmatrix},$$

$$B_{T1}(t,2)=B_{S1}(t,2)=\begin{bmatrix}\tau&0&0\\0&\tau L_f&0\end{bmatrix},$$

$$B_{T2}(t,2)=B_{S2}(t,2)=\begin{bmatrix}\tau&0\\0&\tau L_f\end{bmatrix},$$

$$B_{P1}(t,2)=0_{6\times3},\quad B_{P2}(t,2)=0_{6\times2},$$

$$C_{1T}(t,2)=\begin{bmatrix}\alpha_3 I_2\\ 0_2\end{bmatrix},\quad C_{1S}(t,2)=0_{4\times2},$$

$$C_{2S}(t,2)=0_2,$$

$$C_{1P}(t,2)=\alpha_3\begin{bmatrix}\begin{bmatrix}0_2 & I_2 & I_2\end{bmatrix}\\ 0_{2\times6}\end{bmatrix},\quad D_{12}(t,2)=\begin{bmatrix}0_2\\ \alpha_4 I_2\end{bmatrix},$$

$$A_{PT}(t,2)=\begin{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}\\ 0_{4\times2}\end{bmatrix},\; A_{SS}(t,2)=0_2,$$

$$A_{PS}(t,2)=\begin{bmatrix}0_2\\ -\begin{bmatrix}I_2\\ \begin{bmatrix}0&1\\-1&0\end{bmatrix}\end{bmatrix}\end{bmatrix},$$

$$D_{11}(t,2)=\begin{bmatrix}0_{4\times2} & \begin{bmatrix}-0.1\alpha_3\\ 0_{3\times1}\end{bmatrix}\end{bmatrix}.$$

For $G^{(4)}$, $\underline{\Delta}(t,4)=\mathrm{diag}(\delta_1(t,4)I_2,\delta_2(t,4)I_2,\delta_3(t,4)I_2,\delta_4(t,4)I_2,\delta_5(t,4)I_2)$, $z(t,4)=(0.1y(t,4),\alpha_4 u(t,4))$,

$$y(t,4)=\begin{bmatrix}r_4(t\tau)-(r_3(t\tau)\delta_2(t,4)+q_3(t\tau)\delta_3(t,4))\\ q_4(t\tau)-(-r_3(t\tau)\delta_3(t,4)+q_3(t\tau)\delta_2(t,4))\\ r_4(t\tau)-(r_2(t\tau)\delta_4(t,4)+q_2(t\tau)\delta_5(t,4))\\ q_4(t\tau)-(-r_2(t\tau)\delta_5(t,4)+q_2(t\tau)\delta_4(t,4))\end{bmatrix},$$

and

$$\begin{bmatrix}\delta_1(t,4)\\ \delta_2(t,4)\\ \delta_3(t,4)\\ \delta_4(t,4)\\ \delta_5(t,4)\end{bmatrix}=\begin{bmatrix}\eta_4(t\tau)\\ \cos(\theta_4(t\tau)-\theta_3(t\tau))\\ \sin(\theta_4(t\tau)-\theta_3(t\tau))\\ \cos(\theta_4(t\tau)-\theta_2(t\tau))\\ \sin(\theta_4(t\tau)-\theta_2(t\tau))\end{bmatrix}.$$

6.2. Synthesis conditions

We now construct a γ-admissible synthesis $\mathcal{K}_\delta$ for plant $\mathcal{G}_\delta$. In Section 5, we assume $\gamma=1$ without loss of generality because a γ-admissible synthesis for $\mathcal{G}_\delta$ is 1-admissible for the scaled plant $\mathcal{G}_{\mathrm{scaled},\delta}$ in which γ is absorbed into the system operators, i.e., $C_{\mathrm{scaled},1}=\frac{1}{\gamma}C_1$, $D_{\mathrm{scaled},11}=\frac{1}{\gamma}D_{11}$, and $D_{\mathrm{scaled},12}=\frac{1}{\gamma}D_{12}$. Using these operators, we incorporate γ into the conditions of Theorem 3; and using the Schur complement formula, we reformulate the resulting conditions so that they are linear in γ^2; see [20, Remark 10]. Denote by $\gamma_{\min}$ the minimum value of γ for which there exist (45, 120)-ETP solutions to the semi-definite programming (SDP) problem formed by the sequences of LMIs equivalent to the synthesis conditions; see for instance [10, Section 8.2]. In the present SDP, there are additional parameter blocks $X_j^P(t,k)$. The SDP problem is modeled using Yalmip [21] and solved using SDPT-3 [22]. The total number of constraints is 24751, the dimension of the SDP variable is 33000, the dimension of the linear variable is 1, and the number of SDP blocks is 4455. The computations are carried out in Matlab R2016a on a Dell computer with 4 Intel Cores, 3.07 GHz processors, and 6 GB of RAM running Windows 10. The wall-clock time is 378 s (CPU time 67 s). We obtain $\gamma_{\min}\approxeq 64.86$. This value is relaxed to 66, and the problem is re-solved as a feasibility problem.

6.3. Simulation

From the synthesis solutions, the controller realization J is constructed as in Algorithm 1. The Matlab function ‘basiclmi’ is used in Step 3. The controller is applied to the nonlinear plant and the resulting closed-loop system is simulated. In the simulation, the parameters η_i, for $i=2,3,4$, are set equal to $\omega_{c,i}$ because the disturbances $\omega_{d,i}$ are not measurable. The subsystems start with their CM at (0, 0.825). The leader is subjected to random force and torque disturbances in $[-0.5,+0.5]$N and $[-0.05,0.05]$ N.m, respectively. The followers are subjected to random disturbances which lie within $\pm15\%$ of the nominal inputs. The position of the CM of the four agents is shown in Fig. 5. Observe that even in the presence of disturbances, the leader and the followers track the desired trajectory fairly well. Fig. 6 shows plots of various performance outputs.

7. Conclusion

This paper provides an operator theoretic framework for working with distributed NSLPV systems, along with convex analysis and synthesis conditions. The conditions are in general infinite dimensional because of the explicit time dependence in the system equations, but become finite dimensional for eventually time-periodic subsystems interconnected over finite graphs. The sizes of the analysis and synthesis problems scale with the numbers and dimensions of the temporal, spatial, and parameter states, which calls for structure-preserving model reduction techniques, such as balanced truncation and coprime factors reduction.

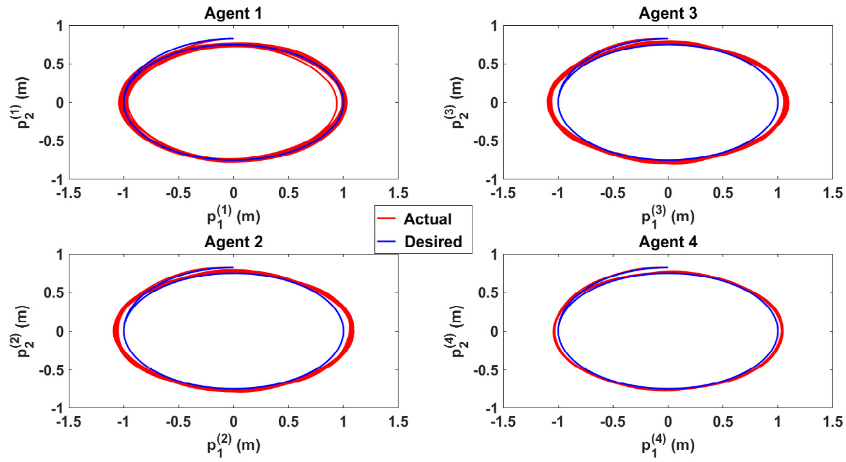

Fig. 5. Closed-loop system response.

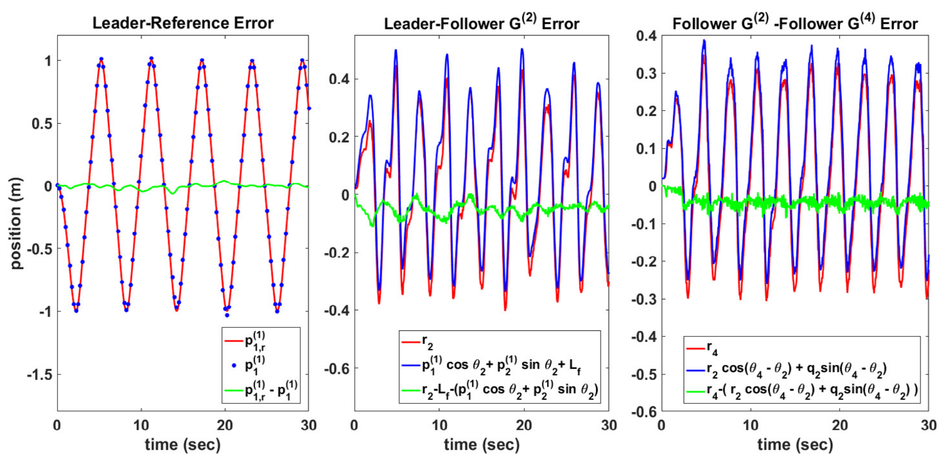

Fig. 6. Plots of various performance outputs.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant Number CMMI-1333785.

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