Coprime Factors Model Reduction of Spatially Distributed LTV Systems Over Arbitrary Graphs

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***Abstract*—This technical note is on the model reduction of distributed systems formed by discrete-time, linear time-varying, heterogeneous subsystems interconnected over arbitrary directed graphs and subjected to communication latency. We give two procedures to construct a strongly stable coprime factorization for a strongly stabilizable and strongly detectable system. One of the procedures ensures the contractiveness of the resulting factorization. Then, we apply the structure-preserving balanced truncation method for distributed systems. We illustrate the proposed methods through an example.**

***Index Terms*—Arbitrary graphs, contractive coprime factorizations, coprime factors reduction, interconnected systems, linear time-varying systems, structure-preserving model reduction.**

I. Introduction

This technical note is on the coprime factors reduction (CFR) of distributed systems formed by discrete-time, linear time-varying (LTV), heterogeneous subsystems interconnected over arbitrary directed graphs and subjected to communication latency. We describe these systems using the framework of [1].

Various works [2]–[9] have appeared that address the problem of structure-preserving balanced truncation (BT) and CFR in the context of linear parameter-varying systems, uncertain systems, and interconnected systems. The methods in these works are based on the existence of block-diagonal solutions to linear matrix inequalities (LMIs), and so, suffer from the ensuing conservatism. The work in [5] identifies a class of interconnected systems with guaranteed structured solutions to the LMIs that appear in [4]. The reader is referred to [10] and the references therein for recent works in a similar direction.

In [2], the authors generalize BT to the considered class of distributed systems. The method is applicable to systems with structured generalized gramians satisfying the generalized Lyapunov inequalities [11], which we refer to as strongly stable systems. BT guarantees the strong stability of the reduced order system and provides an upper bound on the ℓ_2-induced norm of the error system. The CFR method proposed here extends BT to systems that can be represented using a strongly stable pair of coprime factorizations. Strongly stabilizable and strongly detectable systems have strongly stable coprime factorizations, and so can be reduced via the CFR method. That is, CFR extends the range of applicability of BT to strongly stabilizable and strongly detectable systems. However, CFR only guarantees the strong stabilizability and the strong detectability of the reduced order system, and the resulting bound captures the error between the factorizations of the full order and reduced order systems.

The framework of [1] explicitly models the interconnections between the subsystems as spatial states in addition to the states of the subsystems, which we refer to as the temporal states. Like [6], the BT method of [2] allows for the order reduction of both temporal and spatial states. However, unlike [6] where truncation is performed uniformly for all the temporal states and the forward and the backward spatial states, respectively, [2] allows for individually truncating each temporal state and each spatial state. A whole interconnection can even be removed if it is deemed negligible.

We first give a procedure to construct a strongly stable coprime factorization for strongly stabilizable and strongly detectable systems based on [7], [8]. Then, we give an alternative procedure based on [9], which ensures that the obtained factorization is contractive. The associated computational cost is, however, larger. Note that the formal analogy between the system function in [1] and the system functions in [7]–[9] allows for a transparent generalization of the results of [7]–[9] to the systems treated here. In general, both proposed procedures involve solving infinite sequences of LMIs due to the time-varying nature of the subsystems. However, in the case of a finite graph, i.e., a graph with a finite number of vertices and edges, and eventually time-periodic (ETP) subsystems [12], i.e., subsystems with state-space matrices that become time-periodic after some finite time-horizon, both procedures become finite dimensional. Time-invariant, time-periodic, and finite time-horizon subsystems are special cases of ETP subsystems.

A preliminary version of this work appears in [13]. There, we adopt a framework reminiscent of [14], which is equivalent to, yet different from, the framework of [1] adopted here. Sections IV-D, V, and VI are new in this work.

The outline of the technical note is as follows. Section II summarizes the needed notation and the framework of [1]. Section III presents the BT method of [2]. Section IV gives the first procedure for constructing coprime factorizations, while Section V focuses on contractive coprime factorizations. Section VI applies the techniques of the technical note to an illustrative example. The technical note concludes with Section VII.

II. Preliminaries

A. Notation

The sets of nonnegative integers, integers, real numbers, and $n \times n$ symmetric matrices are denoted by $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{S}^n$, respectively. $\mathrm{diag}(M_i)$ refers to the block-diagonal augmentation of the elements of the sequence of operators M_i. Consider a directed graph with set of vertices V and set of directed edges E. We assume a countable number of vertices. The ordered-pair $(i, j) \in E$ represents a directed edge from i to j in V. For each $k \in V$, we define the vertex degree $v(k)$ as the maximum between the indegree and the outdegree. We assume that $v(k)$ is uniformly bounded. A directed graph is said to be d-regular if, for each vertex, both the indegree and the outdegree are equal to d. [1] assumes without loss of generality that the graph

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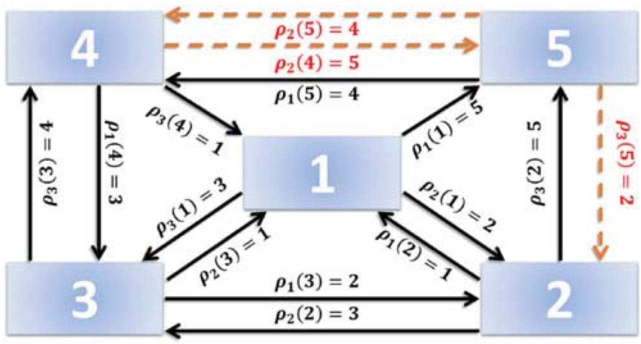

Fig. 1. Arbitrary directed graph made regular by adding virtual edges.

under consideration is d-regular because any arbitrary directed graph can be turned into a "$\max_{k\in V}\, v(k)$"-regular graph by adding, when needed, virtual edges and/or nodes. A d-regular graph permits the definition of d permutations, $\rho_1,\ldots,\rho_d$, of the set of vertices according to the interconnection structure. The permutations $\rho_1,\ldots,\rho_d$ are chosen such that if $(i,j)\in E$, then one $r\in\{1,\ldots,d\}$ satisfies $\rho_r(i)=j$ and $\rho_r^{-1}(j)=i$. Fig. 1 shows a directed graph rendered regular via the addition of virtual (dashed red) interconnections, along with the defined permutations $\rho_1=(15432)$, $\rho_2=(123)(45)$, and $\rho_3=(134)(25)$, where $(a_1\ldots a_m)$ denotes a cycle. For example, $\rho_2(1)=2$, $\rho_2(2)=3$, $\rho_2(3)=1$, $\rho_2(4)=5$, and $\rho_2(5)=4$.

Let H and F be two vector spaces. We denote by $H\oplus F$ the vector space direct sum of H and F. We define the algebra $\mathcal{L}_e(H,F)$ as the space of linear causal operators mapping H to F. We use the simplified notation $\mathcal{L}_e(H)$ when $H=F$. Now, let H and F be Hilbert spaces. We denote the norm and inner product associated with H by $\|.\|_H$ and $\langle .,.\rangle_H$. We drop the subscript when H is clear from context. We use $\mathcal{L}(H,F)$ and $\mathcal{L}_c(H,F)$ to denote the space of bounded linear operators mapping H to F and the space of bounded linear causal operators mapping H to F, respectively. These notations simplify to $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$ when $H=F$. Let X be in $\mathcal{L}(H,F)$. We denote by $\|X\|$ the H to F induced norm of X. The adjoint of X is denoted by X^*. A self-adjoint operator X in $\mathcal{L}(H)$ is negative definite ($X\prec 0$) if, for all nonzero x in H, there exists $\alpha>0$ such that $\langle x, Xx\rangle < -\alpha\|x\|^2$.

Given a sequence $n:(t,k)\in\mathbb{Z}\times V\to n(t,k)\in\mathbb{N}_0$, we denote by $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ the vector space of mappings $w:(t,k)\in\mathbb{Z}\times V\to w(t,k)\in\mathbb{R}^{n(t,k)}$. We simply write ℓ when $n(t,k)$ is clear from context. The Hilbert space ℓ_2 is the subspace of ℓ that consists of mappings w with a finite norm $\|w\|=(\sum_{(t,k)} w(t,k)^* w(t,k))^{1/2}$. ℓ_{2e} denotes the subspace of ℓ that satisfies $\sum_k w(t,k)^* w(t,k)<\infty$, for each $t\in\mathbb{Z}$.

We say that an operator $Q:\ell_2\to\ell_2$ is graph-diagonal if $(Qw)(t,k)=Q(t,k)w(t,k)$, for all $(t,k)\in\mathbb{Z}\times V$. Furthermore, we say that an operator $W=[W_{ij}]$ is partitioned graph-diagonal if each partition W_{ij} is a graph-diagonal operator. The mapping $[\![W]\!](t,k)=[W_{ij}(t,k)]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering.

We denote the unitary temporal-shift operator by S_0 and the unitary spatial-shift operators by S_e, for $e=1,\ldots,d$. These shifts operate on ℓ_2 and satisfy

$$(S_0 w)(t,k)=w(t-1,k),\qquad (S_0^{-1}w)(t,k)=w(t+1,k),$$
$$(S_e w)(t,k)=w(t,\rho_e^{-1}(k)),\qquad (S_e^{-1}w)(t,k)=w(t,\rho_e(k)).$$

The definitions of graph-diagonal operators and of the shift operators extend to ℓ and ℓ_{2e}.

B. Operator Theoretic Framework

We now summarize the framework of [1]. A directed graph is used to describe the interconnection structure of a distributed system. Subsystem $G^{(k)}$ is associated with vertex $k\in V$ and the information sent from $G^{(i)}$ to $G^{(j)}$ is associated with edge $(i,j)\in E$. We denote the temporal (standard) state of $G^{(k)}$ by $x_0(t,k)$ with a possibly time-varying dimension $n_0(t,k)$. As for the spatial states, we denote the state corresponding to interconnection (i,j) by $x_e(t,j)$ with dimension $n_e(t,j)$, where $\rho_e(i)=j$. That is, $x_e(t,k)$ represents the information received by $G^{(k)}$ through permutation ρ_e, while $x_e(t,\rho_e(k))$ represents the information sent by $G^{(k)}$ along permutation ρ_e. Note that states of zero dimensions are associated with the virtual edges/nodes. Each subsystem is modeled as a discrete-time LTV system with its own inputs $u(t,k)$ and outputs $y(t,k)$, of dimensions $n_u(t,k)$ and $n_y(t,k)$, respectively. Due to communication latency, the outgoing information reaches the target subsystem at the next time-step. Then, for all $(t,k)\in\mathbb{Z}\times V$, the state-space equations are given by

$$\begin{bmatrix} x_0\,(t+1,k)\\ x_1\,(t+1,\rho_1(k))\\ \vdots\\ \underline{x_d\,(t+1,\rho_d(k))}\\ y\,(t,k)\end{bmatrix}=\left[\begin{array}{c|c}\bar{A}(t,k)&\bar{B}(t,k)\\ \hline \bar{C}(t,k)&\bar{D}(t,k)\end{array}\right]\begin{bmatrix} x_0\,(t,k)\\ x_1\,(t,k)\\ \vdots\\ \underline{x_d\,(t,k)}\\ u\,(t,k)\end{bmatrix}.$$

The state-space matrix-valued functions are known a priori, are assumed to be uniformly bounded, and are partitioned as

$$\bar{A}(t,k)=\left[\,A_{ij}(t,k)\,\right]_{i=0,\ldots,d;\; j=0,\ldots,d},$$
$$\bar{B}(t,k)=\left[\,B_i(t,k)\,\right]_{i=0,\ldots,d},\quad \bar{C}(t,k)=\left[\,C_j(t,k)\,\right]_{j=0,\ldots,d},$$

where $A_{0j}(t,k)$ is an $n_0(t+1,k)\times n_j(t,k)$ matrix, $B_0(t,k)$ is an $n_0(t+1,k)\times n_u(t,k)$ matrix, $C_0(t,k)$ is an $n_y(t,k)\times n_0(t,k)$ matrix, etc. Note that the blocks that correspond to the virtual states have zero dimension(s). The blocks of the state-space matrices, e.g., $A_{ij}(t,k)$, define graph-diagonal operators, e.g., A_{ij}, which, when combined, define the partitioned graph-diagonal operators A, B, and C as follows:

$$A=\left[\,A_{ij}\,\right]_{i=0,\ldots,d;j=0,\ldots,d},\; B=\left[\,B_i\,\right]_{i=0,\ldots,d},\; C=\left[\,C_j\,\right]_{j=0,\ldots,d},$$
$$[\![A]\!](t,k)=\bar{A}(t,k),\;\; [\![B]\!](t,k)=\bar{B}(t,k),\;\; [\![C]\!](t,k)=\bar{C}(t,k).$$

The matrices $\bar{D}(t,k)$ also define a graph-diagonal operator D such that $[\![D]\!](t,k)=D(t,k)=\bar{D}(t,k)$. Introducing the composite-shift operator $S=\mathrm{diag}(S_0, S_0S_1,\ldots,S_0S_d)$, we rewrite the state-space equations in operator form as

$$x=SAx+SBu,\qquad\qquad y=Cx+Du, \tag{1}$$

where $x(t,k)=\left[\,x_0(t,k)^*, x_1(t,k)^*,\ldots,x_d(t,k)^*\,\right]^*$. Then, the input-output map $u\mapsto y$ can be formally expressed as $G=u\mapsto y=C(I-SA)^{-1}SB+D$.

The system in (1) is well-posed if $G\in\mathcal{L}_e(\ell_{2e},\ell_{2e})$, and is stable if $G\in\mathcal{L}_c(\ell_2,\ell_2)$. From [15], the system is well-posed if $\bar{A}(t,k)=0$ for $t<0$. Hereafter, we assume that all state-space matrices are zeros for $t<0$. Next, we give a sufficient condition for stability, and refer to systems that satisfy this condition as strongly stable. Let

$$\mathcal{X}=\{X: X=\mathrm{diag}(X_0,\ldots,X_d),\ X_0,\ldots,X_d \text{ graph-diagonal},$$
$$X=X^*\succ 0, X \text{ and } X^{-1}\in\mathcal{L}\left(\oplus_{i=0}^{d}\ell_2\left(\{\mathbb{R}^{n_i(t,k)}\}\right)\right)\}.$$

Lemma 1 [2, Lemma 3]: The distributed system (1) is strongly stable if and only if there exist X and Y in $\mathcal{X}$ such that

$$AXA^* - S^*XS + BB^* \prec 0, \quad (2)$$

$$A^*S^*YSA - Y + C^*C \prec 0. \quad (3)$$

(2) and (3) are the generalized Lyapunov inequalities, and X and Y in $\mathcal{X}$ are the generalized gramians. Because of the block-diagonal structure of X, S^*XS is also a partitioned graph-diagonal operator with a block-diagonal structure, where $(S_0^*X_0S_0)(t,k) = X_0(t+1,k)$ and $(S_e^*S_0^*X_eS_0S_e)(t,k) = X_e(t+1,\rho_e(k))$, for $e = 1,\ldots,d$. Note that the sequences of LMIs equivalent to (2) and (3) are trivial for $t < 0$ since the state-space matrices are zeros. In the sequel, we write $t \in \mathbb{Z}$ to allow for the use of the operator theoretic framework. However, we focus on $t \in \mathbb{N}_0$. For $t < 0$, the operators become trivial or inconsequential. The next result guarantees an upper bound on the ℓ_2-induced norm of a strongly stable system G.

Lemma 2 [1, Lemma 9]: System (1) is strongly stable and satisfies $\|G\| < \gamma$ if there exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \quad (4)$$

III. Balanced Truncation

In this section, we summarize the BT method of [2].

Definition 1: The realization of a strongly stable distributed system G, denoted by the quadruple (A,B,C,D), is balanced if there exists an operator $\Sigma \in \mathcal{X}$ that simultaneously satisfies (2) and (3), and for all $(t,k) \in \mathbb{Z} \times V$, $[\![\Sigma]\!](t,k)$ is a diagonal matrix. Σ is called the balanced generalized gramian.

Given generalized gramians X and Y, the next algorithm shows how to construct a balanced realization for G.

Algorithm 1:

1) Compute Cholesky Factorizations: $X_i = R_i^*R_i$ and $Y_i = H_i^*H_i$, for $i = 0,\ldots,d$;
2) Perform Singular Value Decompositions: $H_iR_i^* = U_i\Sigma_iV_i^*$;
3) Define blocks of balancing transformation: $T_i = \Sigma_i^{-1/2}U_i^*H_i$ and $T_i^{-1} = R_i^*V_i\Sigma_i^{-1/2}$;
4) Block-diagonally augment transformation blocks: $T = \operatorname{diag}(T_0,\ldots,T_d)$ and $T^{-1} = \operatorname{diag}(T_0^{-1},\ldots,T_d^{-1})$;
5) Define balanced realization: $A_{\text{bal}} = (S^*TS)AT^{-1}$, $B_{\text{bal}} = (S^*TS)B$, $C_{\text{bal}} = CT^{-1}$, and $D_{\text{bal}} = D$.

We now apply the BT method to system G with balanced realization (A,B,C,D). We partition the blocks of Σ into truncated and non-truncated portions. Given integers $r_i(t,k)$, such that $0 \le r_i(t,k) \le n_i(t,k)$, for all $(t,k) \in \mathbb{Z} \times V$ and $i = 0,\ldots,d$, we partition $\Sigma_i(t,k) = \operatorname{diag}(\Gamma_i(t,k),\Omega_i(t,k))$, where $\Gamma_i(t,k) \in \mathbb{S}^{r_i(t,k)}$ are the non-truncated portions and $\Omega_i(t,k)$ are the truncated portions. $\Gamma_i(t,k)$ and $\Omega_i(t,k)$ define graph-diagonal operators Γ_i and Ω_i. We define the augmented operators $\Gamma = \operatorname{diag}(\Gamma_0,\ldots,\Gamma_d)$ and $\Omega = \operatorname{diag}(\Omega_0,\ldots,\Omega_d)$. Next, we partition the blocks of the state-space matrices conformably with the partitioning of the blocks of Σ, e.g., $A_{00}(t,k)$, $B_0(t,k)$, and $C_0(t,k)$ are partitioned conformably with the partitioning of $\Sigma_0(t+1,k)$ and $\Sigma_0(t,k)$ as in $C_0(t,k) = \begin{bmatrix} \hat{C}_0(t,k) & C_{0_2}(t,k) \end{bmatrix}$,

$$A_{00}(t,k) = \begin{bmatrix} \hat{A}_{00}(t,k) & A_{00_{12}}(t,k) \\ A_{00_{21}}(t,k) & A_{00_{22}}(t,k) \end{bmatrix}, \quad B_0(t,k) = \begin{bmatrix} \hat{B}_0(t,k) \\ B_{0_2}(t,k) \end{bmatrix},$$

where $\hat{A}_{00}(t,k)$ is an $r_0(t+1,k) \times r(t,k)$ matrix, $\hat{B}_0(t,k)$ is an $r_0(t+1,k) \times n_u(t,k)$ matrix, and $\hat{C}_0(t,k)$ is an $n_y(t,k) \times r_0(t,k)$ matrix. These partitions define graph-diagonal operators $\hat{C}_0$, C_{0_2}, etc. Then, it is not difficult to see that

$$A = \left[\left[\!\!\left[\begin{matrix} \hat{A}_{ij} & A_{ij_{12}} \\ A_{ij_{21}} & A_{ij_{22}} \end{matrix} \right]\!\!\right] \right]_{i=0,\ldots,d;\ j=0,\ldots,d},$$

$$B = \left[\left[\!\!\left[\begin{matrix} \hat{B}_i \\ B_{i_2} \end{matrix} \right]\!\!\right] \right]_{i=0,\ldots,d}, \quad C = \left[[\![\hat{C}_j \quad C_{j_2}]\!] \right]_{j=0,\ldots,d}.$$

We form the state-space matrices $\bar{A}_r(t,k)$, $\bar{B}_r(t,k)$, $\bar{C}_r(t,k)$, and $\bar{D}_r(t,k)$ of the reduced order system G_r by keeping the partitions marked with a hat. Namely, $\bar{D}_r(t,k) = \bar{D}(t,k)$,

$$A_r = \left[\hat{A}_{ij} \right]_{i=0,\ldots,d; j=0,\ldots,d}, B_r = \left[\hat{B}_i \right]_{i=0,\ldots,d}, C_r = \left[\hat{C}_j \right]_{j=0,\ldots,d},$$

$$\bar{A}_r(t,k) = [\![A_r]\!](t,k),\ \bar{B}_r(t,k) = [\![B_r]\!](t,k),\ \bar{C}_r(t,k) = [\![C_r]\!](t,k).$$

Then, $G_r = C_r(I - SA_r)^{-1}SB_r + D_r$, where $D_r = D$. We do not distinguish between operators S with different associated dimensions. G_r is strongly stable, and the realization (A_r,B_r,C_r,D_r) is balanced with balanced generalized gramian Γ. The bound on the ℓ_2-induced norm of the resulting error system is given by

$$\|(G - G_r)\| < 2\zeta\Big(\operatorname{diag}\Big(\operatorname{diag}\Big([\![\Omega]\!](t,k)\Big)_{t\in\mathbb{Z}}\Big)_{k\in V}\Big). \quad (5)$$

$\zeta(X)$ denotes the sum of distinct diagonal entries of the possibly infinite dimensional matrix X.

The bound in (5) may become infinite if there are infinitely many distinct entries in Ω. However, this bound is guaranteed to be finite in the case of eventually time-periodic (ETP) subsystems interconnected over finite graphs. A partitioned graph-diagonal operator W is said to be (h,q)-ETP, for some integers $h \ge 0$ and $q > 0$, if $[\![W]\!](t+h+zq,k) = [\![W]\!](t+h,k)$ for all $t,z \in \mathbb{N}_0$ and $k \in V$. A distributed system is said to be (h,q)-ETP if all the state-space operators are (h,q)-ETP. In this case, solutions to (2) and (3) exist if and only if (h,q)-ETP solutions exist. Then, in (5), we restrict t to the finite time-horizon and the first time-period truncation, i.e., $0 \le t \le h+q-1$. If, in addition, the graph is finite, i.e., V and E are finite sets, then the bound in (5) is guaranteed to be finite.

IV. Coprime Factors Reduction

A. Strong Stabilizability

A system is said to be strongly stabilizable if it can be rendered strongly stable by some choice of feedback. Next, we give a test to check for strong stabilizablity, and propose an appropriate feedback gain. For brevity, we omit a similar discussion on the dual notion of strong detectability. The results of this section are based on the works of [8] and [16].

Definition 2: A system is strongly stabilizable if there exists a feedback operator F and an operator $P \in \mathcal{X}$ such that

$$(A + BF)P(A + BF)^* - S^*PS \prec 0. \quad (6)$$

$F = \begin{bmatrix} F_0 & \cdots & F_d \end{bmatrix}$ is a partitioned graph-diagonal operator, with F_i in $\mathcal{L}_c(\ell_2(\{\mathbb{R}^{n_i(t,k)}\}), \ell_2(\{\mathbb{R}^{n_u(t,k)}\}))$, for $i = 0,\ldots,d$. We also define the notation $\bar{F}(t,k) = [\![F]\!](t,k)$, for all $(t,k) \in \mathbb{Z} \times V$.

Theorem 1: A system is strongly stabilizable if and only if there exists $P \in \mathcal{X}$ such that

$$APA^* - S^*PS - BB^* \prec 0. \quad (7)$$

Furthermore, assuming the relevant inverse exists, F can be chosen as follows: $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$.

Proof: For all $(t,k) \in \mathbb{Z} \times V$, we assume that $\operatorname{rank} \bar{B}(t,k) = n_u(t,k) < \left(n_0(t+1,k) + \sum_{e=1}^{d} n_e(t+1,\rho_e(k))\right)$. The case where rank of $\bar{B}(t,k)$ is strictly less than $n_u(t,k)$ can be discarded because it corresponds to the existence of redundant controls, which can be easily removed. If $\bar{B}(t_0,k_0) = 0$ for some $(t_0,k_0) \in \mathbb{Z} \times V$, then $\operatorname{rank} \bar{B}(t_0,k_0) = 0$. In this case, we set $n_u(t_0,k_0) = 0$, i.e., all controls are redundant and hence removed, and $\bar{F}(t_0,k_0)$ is irrelevant. Moreover, the proof for the case of a square, nonsingular $\bar{B}(t,k)$ follows immediately. Given this assumption, we can find an operator $B_\perp$ of the same structure as B that satisfies $B_\perp^* B_\perp = I$ and $B^* B_\perp = 0$, and where the inverse of $\begin{bmatrix} B & B_\perp \end{bmatrix}$ exists and is bounded. Applying the Schur complement formula to (6), we get

$$\underbrace{\begin{bmatrix} -P^{-1} & A^* \\ A & -S^*PS \end{bmatrix}}_{\Psi} + \begin{bmatrix} 0 \\ B \end{bmatrix} F \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{\theta} + \begin{bmatrix} I \\ 0 \end{bmatrix} F^* \underbrace{\begin{bmatrix} 0 & B^* \end{bmatrix}}_{\eta} \prec 0.$$

By a generalization of [8, Lemma 25] to the class of partitioned graph-diagonal operators, we see that a solution F exists to the previous inequality if and only if $W_\theta^* \Psi W_\theta \prec 0$ and $W_\eta^* \Psi W_\eta \prec 0$ for any W_θ and W_η such that $\operatorname{Im} W_\theta = \ker \theta$, $W_\theta^* W_\theta = I$, $\operatorname{Im} W_\eta = \ker \eta$, and $W_\eta^* W_\eta = I$, where $\operatorname{Im} J$ and $\ker J$ denote the image and kernel of a linear operator J, respectively. In particular, we choose $W_\theta = \begin{bmatrix} 0 & I \end{bmatrix}^*$ and $W_\eta = \operatorname{diag}(I, B_\perp)$. The condition $W_\theta^* \Psi W_\theta \prec 0$ is automatically satisfied as $P \succ 0$. As for the condition $W_\eta^* \Psi W_\eta \prec 0$, we expand it and apply the Schur complement formula to obtain $B_\perp^*(-S^*PS + APA^*)B_\perp \prec 0$, which is equivalent by scaling and an application of Finsler's Lemma to (7). Now, we prove that the given choice of F renders the closed-loop strongly stable. Note that F is a well-defined quantity when B^*B has a causal bounded inverse. This can be ensured by removing all redundant controls and properly perturbing B, if necessary, so that $B^*S^*P^{-1}SB$ is invertible. We apply the Schur complement formula twice to (7). By a generalization of the matrix inversion lemma, and since $X^{-1} \succ (I+X)^{-1}$ for any $X \succ 0$, we get

$$\begin{aligned} -A^*S^*P^{-1}SB(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA \\ - P^{-1} + A^*S^*P^{-1}SA \prec 0, \end{aligned}$$

i.e., $-P^{-1} + (A+BF)^*S^*P^{-1}S(A+BF) \prec 0$. We apply the Schur complement formula twice to retrieve (6). □

By comparing the sign of BB^* in (2) and (7), one sees that the notion of strong stabilizability is less stringent than the notion of strong stability. However, the conservatism due to the structure imposed on the solutions of the operator inequalities is still manifest in the notion of strong stabilizability.

B. Coprime Factorizations

Definition 3: Two operators M and N in $\mathcal{L}_c(\ell_2,\ell_2)$ are right coprime if there exist X and Y in $\mathcal{L}_c(\ell_2,\ell_2)$ such that $YM + XN = I$. Similarly, $\tilde{M}$ and $\tilde{N}$ in $\mathcal{L}_c(\ell_2,\ell_2)$ are left coprime if there exist $\tilde{X}$ and $\tilde{Y}$ in $\mathcal{L}_c(\ell_2,\ell_2)$ such that $\tilde{M}\tilde{Y} + \tilde{N}\tilde{X} = I$.

Definition 4: The pair of stable systems (N,M) is a right coprime factorization (RCF) for system G if $M^{-1} \in \mathcal{L}_e(\ell_{2e})$, M and N are right coprime, and $G = NM^{-1}$. The pair of stable systems $(\tilde{N},\tilde{M})$ is a left coprime factorization (LCF) for G if $\tilde{M}^{-1} \in \mathcal{L}_e(\ell_{2e})$, $\tilde{M}$ and $\tilde{N}$ are left coprime, and $G = \tilde{M}^{-1}\tilde{N}$.

Lemma 3: A strongly stabilizable and strongly detectable distributed system G has both a strongly stable RCF and a strongly stable LCF.

Proof: Consider a strongly stabilizable and strongly detectable system G with realization (A,B,C,D). There exist a feedback gain F, e.g., see Theorem 1, and an observer gain K that make the resulting closed-loop systems strongly stable. We define the notations $A_F = A + BF$ and $A_K = A + KC$. Let M and N be distributed systems with realizations (A_F, B, F, I) and $(A_F, B, C+DF, D)$, respectively. Clearly, M and N are in $\mathcal{L}_c(\ell_2,\ell_2)$. We start by showing that M has a causal inverse on ℓ_{2e}. Let R be a distributed system with realization $(A, B, -F, I)$. R is in $\mathcal{L}_e(\ell_e)$ since $\bar{A}(t,k) = 0$ for $t < 0$. By computations similar to the one given next, we show that R satisfies $MR = I$ and $RM = I$, thus proving that $M^{-1} = R$. Then, we prove that $G = NM^{-1} = NR$:

$$\begin{aligned} NR &= \left((C{+}DF)(I{-}SA_F)^{-1}SB{+}D\right)\left(-F(I{-}SA)^{-1}SB{+}I\right) \\ &= -C(I{-}SA_F)^{-1}\left(SBF{-}(I{-}SA)\right)(I{-}SA)^{-1}SB + D \\ &\quad -DF(I{-}SA_F)^{-1}\left(SBF{+}(I{-}SA_F){-}(I{-}SA)\right)(I{-}SA)^{-1}SB \\ &= C(I-SA)^{-1}SB + D = G. \end{aligned}$$

Finally, we show that N and M are right coprime. For that purpose, we consider the strongly stable distributed systems Y and X with realizations $(A_K, B+KD, -F, I)$ and $(A_K, K, F, 0)$, respectively. It is not difficult to show that $YM + XN = I$. Thus, (N,M) is a strongly stable RCF for G. Constructing an LCF for G follows a dual argument. □

C. Coprime Factors Model Reduction Algorithm

Next, we show how to apply the CFR method to a strongly stabilizable and strongly detectable distributed system G. The given algorithm is based on the RCF of G.

Algorithm 2:
1) Find RCF (N,M) for G as in Lemma 3.
2) Define strongly stable system $H = \begin{bmatrix} N^* & M^* \end{bmatrix}^*$.
3) Apply BT to find reduced order model $H_r = \begin{bmatrix} N_r^* & M_r^* \end{bmatrix}^*$.
4) Define reduced order system $G_r = N_r M_r^{-1}$.

G_r is always well-posed since we assume zero state-space matrices for $t < 0$, in addition to the communication latency between the subsystems. Also, G_r is both strongly stabilizable and strongly detectable. (N_r, M_r) is a strongly stable RCF for G_r. Note that (5) gives an upper bound on the ℓ_2-induced norm of $(H - H_r)$ (not $(G - G_r)$).

We now detail the steps in Algorithm 2. In Step 1, we find $P \in \mathcal{X}$ such that $APA^* - S^*PS - BB^* \prec 0$. Then, we choose the feedback gain F as in Theorem 1, while ensuring that the choice is well-defined. In Step 2, we construct system H with realization

$$(A_H, B_H, C_H, D_H) = \left(A + BF, B, \begin{bmatrix} C + DF \\ F \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix}\right).$$

In Step 3, we apply BT to H. We find X in $\mathcal{X}$ such that $A_H X A_H^* - S^*XS + B_H B_H^* \prec 0$ and Y in $\mathcal{X}$ such that $A_H^* S^* Y S A_H - Y + C_H^* C_H \prec 0$. To obtain useful error bounds, we solve for the generalized gramians with minimum trace. From Algorithm 1, we find a balanced realization for H. Based on the absolute and the relative orders of the entries of Σ, the resultant error bound (5), and the upper bound on $\|H\|$ from Lemma 2, we choose how much to reduce each temporal and each spatial state. We denote the resulting realization of H_r by $(A_{H,r}, B_{H,r}, C_{H,r}, D_{H,r})$. In Step 4, we define the realization (A_r, B_r, C_r, D_r) of G_r from $A_{H,r} = A_r + B_r F_r$, $B_{H,r} = B_r$, $C_{H,r} = \begin{bmatrix} (C_r + D_r F_r)^* & F_r^* \end{bmatrix}^*$, and $D_{H,r} = \begin{bmatrix} D_r^* & I \end{bmatrix}^*$. Note that $D_{H,r} = D_H$, $D_r = D$, and F_r is a strongly stabilizing feedback gain for G_r.

D. Computational Complexity

In summary, we need to solve 4 semi-definite programming (SDP) problems, which are of comparable sizes, namely, checking for strong stabilizability and strong detectability of G and finding the minimum trace generalized gramians for H. Assume that the graph is finite and has N_s subsystems, i.e., $k = 1, \dots, N_s$, and N_g interconnections, and assume that we want to find (h, q)-ETP solutions to the following problem:

$$\min \sum_{t=0}^{h+q-1} \sum_{k=1}^{N_s} \sum_{i=0}^{d} \text{trace } X_i(t,k) \quad \text{subject to:}$$

$$\bar{A}(t,k)\,\text{diag}(X_0(t,k),\dots,X_d(t,k))\,\bar{A}(t,k)^*$$
$$-\text{diag}(X_0(t\oplus 1,k), X_1(t\oplus 1,\rho_1(k)),\dots,X_d(t\oplus 1,\rho_d(k)))$$
$$+\bar{B}(t,k)\bar{B}(t,k)^* + \mu(t,k)I \preceq 0,$$
$$-X_i(t,k) + \nu_i(t,k)I \preceq 0, \quad i = 0,\dots,d, t = 0,\dots,h+q-1,$$

where $\oplus$ denotes the (h, q)-eventually periodic addition, i.e., $j \oplus 1 = j+1$ for $j = 0,\dots,h+q-2$, and $j \oplus 1 = h$ for $j = h+q-1$. The small positive quantities $\mu(t,k)$ and $\nu_i(t,k)$ are added because the sequences of constraints are non-strict inequalities. These sequences specify the block structure of the problem, which can then be exploited by the solver to solve the problem at a faster rate. This kind of SDP problems is typically solved by commercial software using customized primal-dual interior point methods [17]. Suppose that we are to solve the given problem using the modeling language Yalmip [18] and the solver SDPT3 [19]. Yalmip models the problem in dual format and then passes it to SDPT3. The dual problem corresponding to the preceding primal problem is given next.

$$\max \sum_{t=0}^{h+q-1} \sum_{k=1}^{N_s} \Big(\text{trace}\left(\bar{B}(t,k)^*\phi(t,k)\bar{B}(t,k)\right)$$
$$+\,\mu(t,k)\ \text{trace}\ \phi(t,k) + \sum_{i=0}^{d} \nu_i(t,k)\ \text{trace}\ \psi_i(t,k)\Big)$$

subject to: $\phi(t,k) \succeq 0, \quad \psi_i(t,k) \succeq 0,$

$$I + \bar{A}(0,k)^*\phi(0,k)\bar{A}(0,k) - \text{diag}\big(\psi_0(0,k),\dots,\psi_d(0,k)\big) = 0,$$
$$I + \bar{A}(h,k)^*\phi(h,k)\bar{A}(h,k) - \text{diag}\big(\psi_0(h,k),\dots,\psi_d(h,k)\big)$$
$$-\,\text{diag}\big(\phi_{00}(h-1,k), \phi_{11}(h-1,\rho_1^{-1}(k)),\dots,\phi_{dd}(h-1,\rho_d^{-1}(k))\big)$$
$$-\,\text{diag}\big(\phi_{00}(h+q-1,k),\dots,\phi_{dd}(h+q-1,\rho_d^{-1}(k))\big) = 0,$$
$$I + \bar{A}(t,k)^*\phi(t,k)\bar{A}(t,k) - \text{diag}\big(\psi_0(t,k),\dots,\psi_d(t,k)\big)$$
$$-\,\text{diag}\big(\phi_{00}(t-1,k),\dots,\phi_{dd}(t-1,\rho_d^{-1}(k))\big) = 0, \text{for } t \neq 0, t \neq h.$$

$\psi_i(t,k)$ is in $\mathbb{S}^{n_i(t,k)}$, and $\phi_{ii}(t,k)$ corresponds to the ii-th block on the diagonal of $\phi(t,k)$, where $\phi_{00}(t,k)$ is in $\mathbb{S}^{n_0(t\oplus 1,k)}$ and $\phi_{ee}(t,k)$ are in $\mathbb{S}^{n_e(t\oplus 1,\rho_e(k))}$, for $e = 1,\dots,d$. Thus, the size of the SDP variable is $\sum_{(t,k)}\left(\sum_{i=0}^{d} n_i(t\oplus 1,k) + \sum_{i=0}^{d} n_i(t,k)\right)$, the number of SDP blocks is $(h+q)(2N_s+N_g)$, and the number of constraints is $\sum_{(t,k)}\sum_{i=0}^{d}\frac{1}{2}n_i(t,k)(n_i(t,k)+1)$. We carried out sample computations for random problems of various sizes in Matlab 8.3.0.532 (The MathWorks Inc., Natick, Massachusetts, USA) on a Dell computer with 4 Intel Cores, 3.07 GHz processors, and 6 GB of RAM running Windows 7. In the time-invariant case, i.e., $h = 0$ and $q = 1$, the largest problem we are able to solve before running into memory problems has $N_s = 49, N_g = 1164, n_0 = 6, n_e = 3, n_u = 4, n_y = 3$. The solution time is 1010 seconds (CPU time 970 seconds).

V. Contractive Coprime Factorizations

In this section, we seek to construct a contractive coprime factorization (CCF) for a given strongly stabilizable and strongly detectable system G with realization (A, B, C, D). We focus on the right CCF of system G. Our results generalize their counterparts for uncertain systems in [9].

Definition 5: The pair of stable systems (N^c, M^c) is a CCF (resp., an α-expansive coprime factorization) for system G if (N^c, M^c) is an RCF for G and satisfies $(N^c)^*N^c + (M^c)^*M^c \preceq I$ (resp., $(N^c)^*N^c + (M^c)^*M^c \succeq \alpha^2 I, \alpha > 0$).

CCFs are motivated in [3], [7], [9] and the references therein. We summarize the main points next. The bound from CFR is on $\|(H - H_r)\|$. As we discuss in Section VII, this bound can be interpreted in terms of the robust stability of the closed-loop system. In the case of single linear time-invariant (LTI) systems, normalized coprime factorizations (NCFs) become of particular interest as they lead to the least conservative robustness conditions when the error bound is interpreted in the gap metric. However, for systems with an uncertainty structure or a spatial structure, the construction of NCFs becomes more difficult, and so CCFs are pursued instead. CCFs are desirable because they can be constructed from solutions to LMIs. Also, even in the case of single discrete-time LTI systems, applying BT to an NCF of the full order system results in a CCF for the reduced order system, and so one may start with a CCF for the full order system. Alternatively, one can look at $\alpha = 1$-expansive factorizations as they allow for the generalization of the robustness results of NCFs. Such an approach, however, results in a non-convex optimization problem when imposing the strong stability condition for the employed factorizations. Next, we restrict our discussion to CCFs. We show that the obtained coprime factorization of the reduced order system is also contractive. In order to establish the connection between the resulting error bound and the gap metric, we need to additionally impose α-expansiveness on the employed factorizations, for some $\alpha < 1$, which is a difficult task. Instead, we use the heuristic (P_1) to make the factorizations approach normalized ones.

The following theorem gives an equivalent characterization for strong stabilizability.

Theorem 2: System G is strongly stabilizable if and only if there exist $P \in \mathcal{X}$ and an operator Q of the same form as F, as given in Definition 2, satisfying

$$\begin{bmatrix} -P & (AP+BQ)^* & Q^* & (CP+DQ)^* \\ (AP+BQ) & -S^*PS & 0 & 0 \\ Q & 0 & -I & 0 \\ (CP+DQ) & 0 & 0 & -I \end{bmatrix} \prec 0. \quad (8)$$

Proof: By definition, a system is strongly stabilizable if and only if there exist $Y \in \mathcal{X}$ and F^c satisfying (6). By scalability and homogeneity with respect to Y, there exist solutions Y and F^c if and only if there exist solutions $X \in \mathcal{X}$ and F^c to

$$(A+BF^c)^*S^*XS(A+BF^c) - X + \begin{bmatrix} C+DF^c \\ F^c \end{bmatrix}^* \begin{bmatrix} C+DF^c \\ F^c \end{bmatrix} \prec 0.$$

We pre- and post- multiply the previous inequality by X^{-1}, and then apply the Schur complement formula to obtain (8), where $P = X^{-1}$ and $Q = F^cX^{-1}$. Conversely, let $P \in \mathcal{X}$ and Q be solutions to (8). Using the Schur complement formula, and pre- and post- multiplying the resulting inequality by P^{-1}, one can verify that $Y = P^{-1}$ and $F^c = QP^{-1}$ satisfy (6). □

Thus, for any P and Q satisfying (8), $F^c = QP^{-1}$ strongly stabilizes G. F^c relaxes the full rank assumption on $\bar{B}(t,k)$ in Theorem 1. For all $(t,k) \in \mathbb{Z} \times V$, let $\bar{F}^c(t,k) = [\![F^c]\!](t,k)$.

Theorem 3: Let $P \in \mathcal{X}$ and Q satisfy (8), and define

$$Z = I + D^*D + B^*S^*P^{-1}SB, F^c = -Z^{-1}(B^*S^*P^{-1}SA + D^*C).$$

Then, the pair of strongly stable distributed systems N^c and M^c with realizations $(A + BF^c, BZ^{-\frac{1}{2}}, C + DF^c, DZ^{-\frac{1}{2}})$ and $(A + BF^c, BZ^{-\frac{1}{2}}, F^c, Z^{-\frac{1}{2}})$, respectively, forms a CCF of system G.

To prove this theorem, we need the following result.

Lemma 4: If $P \in \mathcal{X}$ and Q satisfy (8), and Z is defined as in Theorem 3, then P and Q^c satisfy (8), where $Q^c = -Z^{-1}(B^*S^*P^{-1}SAP + D^*CP)$.

Proof: We apply the Schur complement formula to (8), then add and subtract the following quantity:

$$(PC^*D + PA^*S^*P^{-1}SB)Z^{-1}(D^*CP + B^*S^*P^{-1}SAP).$$

Grouping terms together, we rewrite (8) equivalently as

$$\begin{aligned}&\left[Q^* + (PC^*D + PA^*S^*P^{-1}SB)Z^{-1}\right] Z\\ &\quad\times \left[Q + Z^{-1}(D^*CP + B^*S^*P^{-1}SAP)\right]\\ &- (PC^*D + PA^*S^*P^{-1}SB)Z^{-1}(D^*CP + B^*S^*P^{-1}SAP)\\ &\quad - P + PC^*CP + PA^*S^*P^{-1}SAP \prec 0. \qquad (9)\end{aligned}$$

That is, P and Q satisfy (8) if and only if they satisfy (9). The given Q^c cancels out the first term in (9), and P satisfies the resulting inequality since it satisfies (8). Then, P and Q^c satisfy (9), and by equivalence, P and Q^c satisfy (8). □

Proof of Theorem 3: By Lemma 4, P and Q^c satisfy (8). Then, N^c and M^c are strongly stable, since $F^c = Q^cP^{-1}$. Noting the scaling by $Z^{-\frac{1}{2}}$, we verify that (N^c, M^c) is an RCF of G as in Lemma 3. Finally, we show that this RCF is contractive, i.e., the system $H^c = \begin{bmatrix}(N^c)^* & (M^c)^*\end{bmatrix}^*$ satisfies $\|H^c\| \leq 1$. This is true since $P^{-1} \in \mathcal{X}$ satisfies (4) for the realization of H^c and all $\gamma > 1$. Namely, we expand (4) and denote the result by $\left[\mathcal{N}_{ij}\right]_{i=1,2;j=1,2}$. From Theorem 2, $\mathcal{N}_{11} =$

$$(A{+}BF^c)^*S^*P^{-1}S(A{+}BF^c) - P^{-1} + \begin{bmatrix}C{+}DF^c\\ F^c\end{bmatrix}^*\begin{bmatrix}C{+}DF^c\\ F^c\end{bmatrix} \prec 0.$$

Also, $\mathcal{N}_{22} = Z^{-\frac{1}{2}}(B^*S^*P^{-1}SB + D^*D + I)Z^{-\frac{1}{2}} - \gamma^2 I = (1-\gamma^2)I \prec 0$, for all $\gamma > 1$. Finally, $\mathcal{N}_{12} = \mathcal{N}_{21} = ((A + BF^c)^*S^*P^{-1}SB + (C + DF^c)^*D + (F^c)^*)\, Z^{-\frac{1}{2}} = 0$. □

Now, we apply Algorithm 2 to the strongly stable CCF (N^c, M^c) of system G. In Step 3, we find X and $Y \in \mathcal{X}$ such that

$$(A + BF^c)X(A + BF^c)^* - S^*XS + BZ^{-1}B^* \prec 0, \qquad (10)$$

$$(A + BF^c)^*S^*YS(A + BF^c) - Y + \begin{bmatrix}C{+}DF^c\\ F^c\end{bmatrix}^*\begin{bmatrix}C{+}DF^c\\ F^c\end{bmatrix} \prec 0.$$

From the proof of Theorem 2, $Y = P^{-1}$; thus, we only solve for $X \in \mathcal{X}$ that satisfies (10). We use Algorithm 1 to construct the transformation T and the operators $A_{\rm bal}$, $B_{\rm bal}$, $C_{\rm bal}$, and $D_{\rm bal} = D$ as in Step 5. We also let $F^c_{\rm bal} = F^cT^{-1}$. Then, the realization

$$\left(A_{\rm bal} + B_{\rm bal}F^c_{\rm bal}, B_{\rm bal}Z^{-\frac{1}{2}}, \begin{bmatrix}C_{\rm bal} + DF^c_{\rm bal}\\ F^c_{\rm bal}\end{bmatrix}, \begin{bmatrix}DZ^{-\frac{1}{2}}\\ Z^{-\frac{1}{2}}\end{bmatrix}\right)$$

is a balanced realization for $H^c = \begin{bmatrix}(N^c)^* & (M^c)^*\end{bmatrix}^*$. We apply BT to system H^c and obtain the reduced order system H^c_r. There exists an operator L such that $L^*\Sigma L = \mathrm{diag}(\Gamma, \Omega)$, where Γ and Ω are defined in Section III; see [8, Lemma 10]. The following also hold: $F^c_{\rm bal}L = \begin{bmatrix}F^c_r & \bar{F}_2\end{bmatrix}$, $C_{\rm bal}L = \begin{bmatrix}C^c_r & \bar{C}_2\end{bmatrix}$,

$$L^*SA_{\rm bal}L = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix}\begin{bmatrix}A^c_r & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}, \quad L^*SB_{\rm bal} = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix}\begin{bmatrix}B^c_r\\ \bar{B}_2\end{bmatrix}.$$

Then, the realization

$$\left(A^c_r + B^c_rF^c_r, B^c_rZ^{-\frac{1}{2}}, \begin{bmatrix}C^c_r + DF^c_r\\ F^c_r\end{bmatrix}, \begin{bmatrix}DZ^{-\frac{1}{2}}\\ Z^{-\frac{1}{2}}\end{bmatrix}\right)$$

is a balanced realization for $H^c_r = \begin{bmatrix}(N^c_r)^* & (M^c_r)^*\end{bmatrix}^*$ with balanced generalized gramian Γ. Moreover, (A^c_r, B^c_r, C^c_r, D) is a realization for the reduced order system $G^c_r = N^c_r(M^c_r)^{-1}$. (N^c_r, M^c_r) constitutes a strongly stable RCF for G^c_r.

Theorem 4: The RCF (N^c_r, M^c_r) of G^c_r is contractive.

Proof: From the proof of Theorem 3, P^{-1} satisfies (4) for the initial/unbalanced realization of system H^c and all $\gamma > 1$. Noting that Σ can be expressed as TXT^* and as $(T^*)^{-1}P^{-1}T^{-1}$, we verify that Σ satisfies (4) for the balanced realization of system H^c and all $\gamma > 1$. We pre- and post- multiply the resulting inequality by $\mathrm{diag}(L^*, I)$ and its adjoint, and insert $\mathrm{diag}(L, I)\mathrm{diag}(L^*, I) = \mathrm{diag}(I, I)$ where needed, to retrieve the terms A^c_r, B^c_r, C^c_r, and F^c_r. We see that Γ satisfies (4) for the realization of H^c_r and all $\gamma > 1$. That is, $\|H^c_r\| \leq 1$. □

We conclude this section by giving one heuristic to make the CCF (N^c, M^c) approach normalization [9, Remark 18].

$$(P_1):\ \min \sum_{(t,k)}\sum_{i=0}^{d} \mathrm{trace}\, U_i(t,k),\ P \in \mathcal{X} \text{ and Q satisfy (8)}$$

$$\text{and } U_i \text{ satisfy} \begin{bmatrix}U_i & I\\ I & P_i\end{bmatrix} \succ 0, \text{ for } i = 0, \ldots, d;$$

$$(P_2):\ \min \sum_{(t,k)}\sum_{i=0}^{d} \mathrm{trace}\, X_i(t,k),\quad X \in \mathcal{X} \text{ satisfies (10)}.$$

Solving (P_1) induces a larger computational burden than solving the problems of Section IV-D. Let the graph be finite, and assume that we seek (h, q)-ETP solutions for (P_1). Then, the size of the SDP variable, the number of SDP blocks, and the number of constraints are, respectively,

$$\sum_{(t,k)}\left(3\sum_{i=0}^{d} n_i(t,k) + \sum_{i=0}^{d} n_i(t \oplus 1, k) + n_u(t,k) + n_y(t,k)\right),$$

$$(h+q)(2N_s + N_g), \text{ and} \sum_{(t,k)}\sum_{i=0}^{d} n_i(t,k)(n_i(t,k) + 1 + n_u(t,k)).$$

For the same system parameters as in Section IV-D, the largest (P_1) which can be solved before running into memory problems has $N_s = 31$ and $N_g = 451$.

VI. Illustrative Example

Consider a distributed system G with 5 agents interconnected as shown in Fig. 1, and having the dynamics

$$r(t+1,k) = r(t,k) + w(t,k)\tau,\ \ w(t+1,k) = w(t,k) + v(t,k)\tau,$$

where k is the subsystem index, $t \geq 0$ the discrete time-step, $\tau = 0.05\,$s the sampling time, and scalar-valued r, w, and v the position, velocity, and applied input. If $(i, j) \in E$, then agent i sends its position and velocity to agent j. Because of the communication delay, the information is received at the next time-step. The input $v(t, k)$ includes both the consensus control protocol [20] and the disturbance d acting on the agent:

$$v(t,k) = d(t,k) + p_{1,k} \sum_{e=1}^{3} a_{e,k} \left(r(t-1, \rho_e^{-1}(k)) - r(t,k) \right)$$

$$- p_{0,k} w(t,k) + p_{2,k} \sum_{e=1}^{3} a_{e,k} \left(w(t-1, \rho_e^{-1}(k)) - w(t,k) \right).$$

The gains $p_{0,k}$, $p_{1,k}$, $p_{2,k}$ are set as follows: $p_{0,1} = 15$, $p_{0,2} = p_{0,3} = p_{0,4} = p_{0,5} = 2$, $p_{1,3} = 1.5$, $p_{1,5} = p_{2,3} = p_{2,5} = 0.1$, with the rest equal to unity. $a_{e,k} > 0$ is the weight associated with edge $(\rho_e^{-1}(k), k)$. Weights $a_{1,2}, a_{1,3}, a_{1,5}, a_{2,3}$ are equal to 0.0001, and the rest are equal to 1. We reformulate the problem in our framework by defining the temporal state $x_0(t,k)$ and the outgoing spatial states $x_e(t, \rho_e(k))$, $e = 1,2,3$, as $(r(t,k), w(t,k))$. The output $y(t,k) = x_0(t,k)$. The input $u(t,k)$ reduces to $d(t,k)$ because the other terms in $v(t,k)$ are associated with the spatial states. Since the subsystems are LTI ($(0,1)$-ETP), we drop the time parameter from the state-space matrices. For all $k \in V$, we have $\bar{D}(k) = 0$. The resultant system is strongly stabilizable and strongly detectable.

We start by applying the CFR method of Section IV. We find a $(0,1)$-ETP solution P to (7), and compute F according to Theorem 1. Then, we apply BT to the strongly stable system H whose state-space matrices are defined as $\bar{A}_H(k) = \bar{A}(k) + \bar{B}(k)\bar{F}(k)$, $\bar{B}_H(k) = \bar{B}(k)$, $\bar{C}_H(k) = \begin{bmatrix} \bar{C}(k)^* & \bar{F}(k)^* \end{bmatrix}^*$, and $\bar{D}_H(k) = \begin{bmatrix} 0 & I \end{bmatrix}^*$. We solve for $(0,1)$-ETP generalized gramians X and Y that minimize $\mathrm{trace}(\sum_{k=1}^{5}\sum_{e=1}^{3} X_e(k))$ and $\mathrm{trace}(\sum_{k=1}^{5}\sum_{e=1}^{3} Y_e(k))$, respectively, because we are only interested in simplifying the interconnection structure. Then, we find the ℓ_2-induced norm $\|H\|$ of system H. We find an upper bound γ using Lemma 2. Then, we treat system H as a global LTI system with 5 inputs and 15 outputs and find its H_∞-norm, which is a lower bound on $\|H\|$ because the spatial structure is relaxed. The upper and lower bounds are almost equal, and so, $\|H\| = 1.9985$.

By looking at the diagonal entries of Σ and comparing their relative orders, and from the predicted error upper bound in (5) as compared to $\|H\|$, we decide to eliminate edges $(1,5), (4,3), (2,3), (3,2)$ and to reduce the dimension of the remaining balanced spatial states from 2 to 1 each. For example, $\Sigma_1(5) = 10^{-3}\mathrm{diag}(0.15, 0.02)$, $\Sigma_3(5) = \mathrm{diag}(0.01, 0.0001)$, and $\Sigma_1(1) = \mathrm{diag}(0.19, 0.0001)$. From (5), $\|(H - H_r)\| < 0.0094 = 0.4712\,\%\|H\|$. We form the reduced order system G_r from the state-space matrices of H_r by partitioning $\bar{C}_{H,r}(k)$ as $\begin{bmatrix} \bar{C}_r(k)^* & \bar{F}_r(k)^* \end{bmatrix}^*$, setting $\bar{B}_r(k) = \bar{B}_{H,r}(k)$, and defining $\bar{A}_r(k) = \bar{A}_{H,r}(k) - \bar{B}_r(k)\bar{F}_r(k)$.

By looking at the first entry of $\Sigma_3(5)$, one may choose to additionally remove the first variable of $x_3(t,5)$, i.e., remove interconnection $(2,5)$ altogether, especially because the bound in (5) only adds up to $1.5435\,\%\|H\|$. However, there will no more be interconnections feeding into agent 5, and so, the resulting reduced order system $G_{r,1}$ will have a significantly different behavior from G. If, however, one groups the entries of Σ according to their orders of magnitude, one sees that 0.01 does not belong to the group of the previously truncated entries. In fact, it is one order of magnitude larger. Also, the new error bound, while still small, is 3 times larger than the previous bound, i.e., truncating one additional variable beyond what was truncated previously results in a jump in the error upper bound. Weighing in these new observations, one concludes that the first variable of $x_3(t,5)$ should not be truncated.

We now apply the method of Section V. We start by finding $(0,1)$-ETP solutions to (P_1) that minimize the sum of traces of the spatial terms $U_e(k)$ only. We define F^c and Z as in Theorem 3, and construct the state-space matrices of H^c as follows:

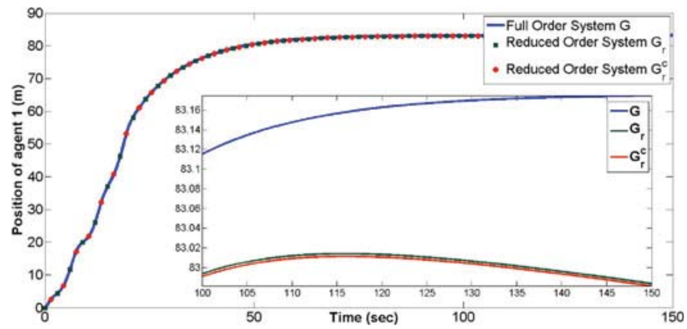

Fig. 2. Comparison of the reduced order systems with the full order system.

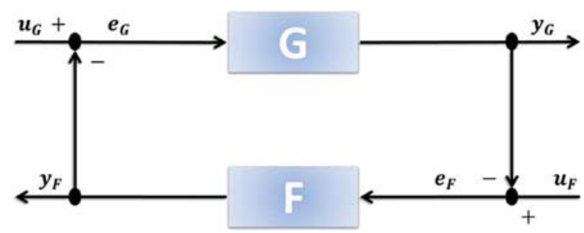

Fig. 3. Feedback Interconnection.

$\bar{A}_H^c(k) = \bar{A}(k) + \bar{B}(k)\bar{F}^c(k)$, $\bar{B}_H^c(k) = \bar{B}(k)Z(k)^{-\frac{1}{2}}$, $\bar{C}_H^c(k) = \begin{bmatrix} \bar{C}(k)^* & \bar{F}^c(k)^* \end{bmatrix}^*$, and $\bar{D}_H^c(k) = \begin{bmatrix} 0 & Z(k)^{-\frac{1}{2}} \end{bmatrix}^*$. $\|H^c\|$ turns out to be approximately equal to 1. We find $X \in \mathcal{X}$ that satisfies (10) and minimizes $\mathrm{trace}(\sum_{k=1}^{5}\sum_{e=1}^{3} X_e(k))$. We construct a balanced realization for H^c, and form the reduced order system H_r^c by truncating the same variables as before. (5) gives $\|(H^c - H_r^c)\| < 12.5351\,\%\|H^c\|$. The same observations as before regarding the relative order of the truncated singular values and the bound hike still hold, but the CCF error bound is much larger in this particular example ($12.5351\%\|H^c\|$ vs $0.4712\,\%\|H\|$). Finally, we form the reduced order system G_r^c from the state-space matrices of H_r^c as follows: $\bar{C}_{H,r}^c(k) = \begin{bmatrix} \bar{C}_r^c(k)^* & \bar{F}_r^c(k)^* \end{bmatrix}^*$, $\bar{B}_r^c(k) = \bar{B}_{H,r}^c(k)Z(k)^{\frac{1}{2}}$, and $\bar{A}_r^c(k) = \bar{A}_{H,r}^c(k) - \bar{B}_r^c(k)\bar{F}_r^c(k)$.

We subject G, G_r, and G_r^c to the same mix of sinusoidal and exponential disturbances for $20\,\mathrm{sec}$, and then let the systems evolve on their own. Fig. 2 shows that the three systems exhibit a similar behavior. However, by zooming in, one sees that the error in the position between the agents of G_r and G slowly increases as time elapses because the agents of G_r do not reach consensus in position. They only reach a nonzero consensus in velocity, with a consensus value of the order 10^{-3}. So, while G is stable (but not strongly stable) in the sense that the zero-input response converges to a common state, G_r is not. This remark does not contradict CFR theory, which only guarantees the strong stabilizability and detectability of the reduced order model. A similar observation holds for system G_r^c.

VII. Conclusion

We gave two methods for structure-preserving/simplifying CFR of strongly stabilizable and strongly detectable distributed systems. The second method guarantees the contractiveness of the factorizations, but is computationally more expensive. We now conclude the technical note by giving an interpretation of the bound on $\|(H - H_r)\|$ in terms of robust feedback stability. We say that the closed-loop system in Fig. 3 is stable (resp., well-posed) if the map from (u_G, u_F) to (e_G, e_F) is in $\mathcal{L}_c(\ell_2 \oplus \ell_2)$ (resp., $\mathcal{L}_e(\ell_{2e} \oplus \ell_{2e})$). We assume that the plant G has a strongly stable RCF (N, M). We also assume that the controller F has an LCF $(\tilde{N}_F, \tilde{M}_F)$. In fact, many synthesis techniques [1], [15] construct a distributed controller, with the same structure as the plant, which guarantees the strong stability of the closed-loop system. This means that F strongly stabilizes G, but also that G strongly stabilizes F, and so, F has an LCF.

Theorem 5: Assume that F stabilizes G and that $\|\begin{bmatrix} \tilde{N}_F & \tilde{M}_F \end{bmatrix}\| < \frac{1}{\epsilon}$. If G_r is obtained from applying CFR to G, and satisfies $\|(H - H_r)\| \leq \epsilon$, then F stabilizes G_r.

This theorem is proved by extending results from [21], [22] to the class of distributed systems. Given the assumption that F has an LCF, such an extension is conceivably transparent because of the similarity between the system function in the adopted framework and in standard state-space systems.

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