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Shortest circuit covers of signed graphs ☆

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A shortest circuit cover $\mathcal{F}$ of a bridgeless graph G is a family of circuits that covers every edge of G and is of minimum total length. The total length of a shortest circuit cover $\mathcal{F}$ of G is denoted by $SCC(G)$. For ordinary graphs (graphs without sign), the subject of shortest circuit cover is closely related to some mainstream areas, such as, Tutte's integer flow theory, circuit double cover conjecture, Fulkerson conjecture, and others. For signed graphs G, it is proved recently by Máčajová, Raspaud, Rollová and Škoviera that $SCC(G) \leq 11|E|$ if G is s-bridgeless, and $SCC(G) \leq 9|E|$ if G is 2-edge-connected. In this paper this result is improved as follows,

$$SCC(G) \leq |E| + 3|V| + z$$

where $z = \min\{\frac{2}{3}|E| + \frac{4}{3}\epsilon_N - 7,\ |V| + 2\epsilon_N - 8\}$ and ϵ_N is the negativeness of G. The above upper bound can be further reduced if G is 2-edge-connected with even negativeness.

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1. Introduction

For terminology and notations not defined here we follow [4,6,21]. Graphs considered in this paper may have multiple edges or loops. A *circuit cover* of a bridgeless graph G is a family $\mathcal{C}$ of circuits such that each edge of G belongs to at least one member of $\mathcal{C}$. The *length* of $\mathcal{C}$ is the total length of circuits in $\mathcal{C}$. A minimum length of a circuit cover of G is denoted by $SCC(G)$.

For ordinary graphs (graphs without sign), the subject of shortest circuit cover is not only a discrete optimization problem [10], but also closely related to some mainstream areas in graph theory, such as, Tutte's integer flow theory [1,3,7,11,14,18,22], circuit double cover conjecture [15,16], Fulkerson conjecture [8], snarks and graph minors [2,12]. It is proved by Bermond, Jackson and Jaeger [3] that *every graph admitting a nowhere-zero* 4*-flow has* $SCC(G) \leq \frac{4|E|}{3}$. By applying Seymour's 6-flow theorem [20] or Jaeger's 8-flow theorem [13], Alon and Tarsi [1], and Bermond, Jackson and Jaeger [3] proved that *every bridgeless graph G has* $SCC(G) \leq \frac{25|E|}{15}$. One of the most famous open problems in this area was proposed by Alon and Tarsi [1], that *every bridgeless graph G has* $SCC(G) \leq \frac{21|E|}{15}$. It is proved by Jamshy and Tarsi [15] that *the above conjecture implies the circuit double cover conjecture.* The relations between $SCC(G)$ and Fulkerson conjecture, Tutte's 3-flow and 5-flow conjectures were studied by Fan, Jamshy, Raspaud and Tarsi in [8,14,7].

For signed graphs, the following upper bounds for shortest circuit covers were recently estimated in [17].

Theorem 1.1 *(Máčajová, Raspaud, Rollová and Škoviera [17]). Let G be an s-bridgeless signed graph.*

(1) In general, $SCC(G) \leq 11|E|$.

(2) If G is 2*-edge-connected, then* $SCC(G) \leq 9|E|$.

In this paper, Theorem 1.1 is improved as follows.

Theorem 1.2. *Let G be an s-bridgeless signed graph with negativeness* $\epsilon_N > 0$.

(1) In general,

$$SCC(G) \leq |E| + 3|V| + z_1,$$

where $z_1 = \min\{\frac{2}{3}|E| + \frac{4}{3}\epsilon_N - 7,\ |V| + 2\epsilon_N - 8\}$.

(2) If G is 2*-edge-connected and* ϵ_N *is even, then*

$$SCC(G) \leq |E| + 2|V| + z_2,$$

where $z_2 = \min\{\frac{2}{3}|E| + \frac{1}{3}\epsilon_N - 4,\ |V| + \epsilon_N - 5\}$.

Theorem 1.2 is an analog of a result (Theorem 3.4) by Fan [9] that solves a long standing open problem by Itai and Rodeh [10].

Note that, in a connected s-bridgeless signed graph G with $|E_N(G)| = \epsilon_N$, $G - E_N(G)$ is a connected unsigned graph (by Lemma 5.3), and hence $|E| \geq \epsilon_N + |V| - 1$. Therefore Theorem 1.2 implies that if G is an s-bridgeless signed graph with $\epsilon_N > 0$, then

$$SCC(G) \leq \frac{14}{3}|E| - \frac{5}{3}\epsilon_N - 4.$$

This is an analog of a result (Theorem 3.3) by Alon and Tarsi [1] and by Bermond, Jackson and Jaeger [3].

2. Notation and terminology for signed graphs

A *signed graph* is a graph G with a mapping $\sigma : E(G) \to \{1, -1\}$. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* if $\sigma(e) = -1$. The mapping σ, called *signature*, is usually implicit in the notation of a signed graph and will be specified only when needed. For a subgraph H of G, we use $E_N(H)$ to denote the set of all negative edges in H. A circuit C of G is *balanced* if $|E_N(C)| \equiv 0 \pmod 2$, and *unbalanced* otherwise. A *signed circuit* of G is a subgraph of one of the following three types:

(1) a balanced circuit;
(2) a short barbell, the union of two unbalanced circuits that meet at a single vertex;
(3) a long barbell, the union of two disjoint unbalanced circuits with a path that meets the circuits only at its ends.

A *barbell* is either a short barbell or a long barbell. The *length* of a signed circuit C is the number of edges in C.

Definition 2.1. Let $\mathcal{F}$ be a family of signed circuits of a signed graph G and K be a set of some nonnegative integers.

- $\mathcal{F}$ is called a *signed circuit cover* (resp., signed circuit K-cover) of G if each edge e of G belongs to k_e members of $\mathcal{F}$ such that $k_e \geq 1$ (resp., $k_e \in K$). In particular, a signed circuit $\{2\}$-cover is also called a *signed circuit double cover*.
- The *length*, denoted by $\ell(\mathcal{F})$, of $\mathcal{F}$ is the total length of signed circuits in $\mathcal{F}$.
- $\mathcal{F}$ is called a *shortest circuit cover* of G if it is a signed circuit cover of G with minimum length. The length of a shortest circuit cover of G is denoted by $SCC(G)$.

Clearly, the signed circuit cover of signed graphs is a generalization of the classic circuit cover of graphs. By the definition of signed circuit cover, a signed graph has a signed circuit cover if and only if every edge of the signed graph is contained in a signed circuit. Such signed graph is called *s-bridgeless*.

In a signed graph, *switching* at a vertex u means reversing the signs of all edges incident with u. It is obvious (see [19]) that the switching operation preserves signed circuits and thus the existence and the length of a signed circuit cover of a signed graph are two invariants under the switching operation.

Definition 2.2. Let G be a signed graph, and $\mathcal{X}$ be the collection of signed graphs obtained from G by a sequence of switching operations. The *negativeness* of G is

$$\epsilon_N(G) = \min\{|E_N(G')| : \forall G' \in \mathcal{X}\}.$$

Definition 2.3. Let b be a bridge of a connected signed graph G and Q_1, Q_2 be the two components of $G - b$. The bridge b is called a *g-bridge* of G if $\epsilon_N(Q_1) \equiv \epsilon_N(Q_2) \equiv 0 \pmod 2$.

Note that a signed graph G is g-bridgeless if and only if every component of G contains no g-bridges, and is s-bridgeless if and only if every component Q of G satisfies $\epsilon_N(Q) \neq 1$ and $\epsilon_N(Q') > 0$ for each bridge b of Q and each component Q' of $Q - b$ (the "only if" part is proved in [5] and the "if" part is easy).

3. Lemmas and outline of the proofs

Since the concept of g-bridge is introduced in Section 2, the part (2) of Theorem 1.2 can be revised as follows in a slightly stronger version.

Theorem 3.1. *Let G be an s-bridgeless signed graph with negativeness $\epsilon_N > 0$.*

(1) In general,

$$SCC(G) \leq |E| + 3|V| + z_1$$

where $z_1 = \min\{\frac{2}{3}|E| + \frac{4}{3}\epsilon_N - 7,\ |V| + 2\epsilon_N - 8\}$.

(2) If G is g-bridgeless and ϵ_N is even, then

$$SCC(G) \leq |E| + 2|V| + z_2$$

where $z_2 = \min\{\frac{2}{3}|E| + \frac{1}{3}\epsilon_N - 4,\ |V| + \epsilon_N - 5\}$.

The following is the major lemma for the proof of Theorem 3.1.

Lemma 3.2. *Let G be an s-bridgeless signed graph with $|E_N(G)| = \epsilon_N(G)$. Then G has a pair of subgraphs $\{G_1, G_2\}$ such that*

(1) $E(G_1) \cup E(G_2) = E(G)$,

(2) G_1 contains no negative edge and is bridgeless, and

(3) $G_2 - E_N(G)$ is acyclic and G_2 has a signed circuit $\{1, 2, \cdots, k\}$-cover, where $k = 2$ if G is g-bridgeless with an even negativeness, and $k = 3$ otherwise.

Lemma 3.2 will be proved in Section 5 after some preparations in Section 4.

The main result, Theorem 3.1, will be proved as a corollary of Lemma 3.2 in Section 6. The following is the outline of the proof. By (1) of Lemma 3.2,

$$SCC(G) \leq SCC(G_1) + SCC(G_2).$$

Lemma 3.2-(3) provides an estimation for $SCC(G_2)$. For the bridgeless unsigned subgraph G_1, we use the following classical results in graph theory.

Theorem 3.3 *(Alon and Tarsi [1], Bermond, Jackson and Jaeger [3]). Let G be a 2-edge-connected graph. Then $SCC(G) \leq \frac{5}{3}|E|$.*

Theorem 3.4 *(Fan [9]). Let G be a 2-edge-connected graph. Then $SCC(G) \leq |E|+|V|-1$.*

4. Signed circuit covers of generalized barbells

In this section, we study signed circuit covers of generalized barbells which play an important role in the proof of Lemma 3.2.

A graph is *eulerian* if it is connected and each vertex is of even degree. For a vertex subset U of a graph G, let $\delta_G(U)$ denote the set of all edges between U and $V(G) - U$. In a graph, a *k-vertex* is a vertex of degree k.

Definition 4.1. A signed graph H is called a generalized barbell if it contains a set of vertex-disjoint eulerian subgraphs $\mathcal{B} = \{B_1, \cdots, B_t\}$ such that

(1) The contracted graph $X = H/(\cup_{i=1}^t B_i)$ is acyclic and
(2) For each vertex x of X (if x is a contracted vertex, then let B_x be the corresponding eulerian subgraph of $\mathcal{B}$; otherwise, simply consider $E(B_x)$ as an empty set),

$$|E_N(B_x)| \equiv |\delta_H(V(B_x))| \pmod 2.$$

We first study signed eulerian graphs with even number of negative edges which is a special case of generalized barbells.

Let T be a closed eulerian trail of a signed eulerian graph. For any two vertices u and v of T, we use uTv to denote the subsequence of T starting with u and ending with v in the cyclic ordering induced by T.

Lemma 4.2. *Every signed eulerian graph with even number of negative edges has a signed circuit double cover.*

Proof. Let B be a counterexample to Lemma 4.2 with $|E(B)|$ minimum. Then the maximum degree of B is at least 4 otherwise B is a balanced circuit. By the minimality of B, B cannot be decomposed into two signed eulerian subgraphs, each contains an even number of negative edges. Thus we have the following observation

Observation. For any eulerian trail $T = u_1e_1u_2e_2 \cdots u_me_mu_1$ of B where $m = |E(G)|$ and for any two integers $i, j \in [1, m]$ with $i < j$ and $u_i = u_j$, u_iTu_j *is a signed eulerian graph with odd number of negative edges.*

Pick an arbitrary eulerian trail $T = u_1e_1u_2e_2 \cdots u_me_mu_1$. We consider the following two cases.

Case 1. For any two integers $i \neq j \in [1, m]$, if $u_i = u_j$, then $|j - i| \equiv 1 \pmod{m}$.

In this case, the resulting graph obtained from B by deleting all loops is either a single vertex or a circuit. Since B has an even number of negative edges, one can check that B has a signed circuit double cover, a contradiction.

Case 2. There are two integers $i, j \in [1, m]$ such that $2 \leq j - i \leq m - 2$ and $u_i = u_j$.

Let $B_1 = u_iTu_j$ and $B_2 = u_jTu_i$. Then, by Observation, both B_1 and B_2 are signed eulerian subgraphs of B with $B = B_1 \cup B_2$ such that $|E(B_k)| \geq 2$ and $|E_N(B_k)| \equiv 1 \pmod{2}$ for each $k = 1, 2$.

If $V(B_1) \cap V(B_2) = \{u_i\}$, then for each $k = 1, 2$, let B'_k be the resulting graph obtained from B_k by adding a negative loop e'_k at u_i. Clearly, B'_k remains eulerian, $|E(B'_k)| < |E(B)|$, and $|E_N(B'_k)|$ is even. By the minimality of B, B'_k has a signed circuit double cover $\mathcal{F}_k$. Since e'_k is a negative loop of B'_k, it is covered by two barbells, say C^1_k and C^2_k, in $\mathcal{F}_k$. Let $C^\ell = \cup_{k=1}^2 (C^\ell_k - e'_k)$ for each $\ell = 1, 2$. Since $V(B_1) \cap V(B_2) = \{u_i\}$, both C^1 and C^2 are two barbells of B, and so B has a signed circuit double cover $\cup_{k=1}^2 (\mathcal{F}_k - \{C^1_k, C^2_k\}) \cup \{C^1, C^2\}$, a contradiction.

If $V(B_1) \cap V(B_2) \neq \{u_i\}$, then there are two integers s and t such that $s \in [i, j]$, $t \notin [i, j]$, and $u_s = u_t$. By Observation, $|E_N(u_sTu_t)| \equiv 1 \pmod{2}$. Let T^* be a new closed eulerian trail of B obtained from T by reversing the subsequence u_iTu_j in T. Then $E(u_sT^*u_t)$ is the disjoint union of $E(u_iTu_s)$ and $E(u_jTu_t)$ and thus $E_N(u_sT^*u_t)$ is the disjoint union of $E_N(u_iTu_s)$ and $E_N(u_jTu_t)$. Since $|E_N(u_iTu_j)| \equiv 1 \pmod{2}$ and $|E_N(u_sTu_t))| \equiv 1 \pmod{2}$, $|E_N(u_iTu_s)| \equiv |E_N(u_jTu_t)| \pmod{2}$. Therefore $|E_N(u_sT^*u_t)| \equiv 0 \pmod{2}$, a contradiction to Observation. This completes the proof of the lemma. □

The following lemma is a generalization of Lemma 4.2.

Lemma 4.3. *Every generalized barbell has a signed circuit double cover.*

Proof. Let H be a generalized barbell. Let $\{B_1, \cdots, B_t\}$ be a set of disjoint eulerian subgraphs of H and $X = H/(\cup_{i=1}^t B_i)$ as described in Definition 4.1. We will prove by induction on $|E(H) - \cup_{i=1}^t E(B_i)|$.

If $E(H) - \cup_{i=1}^t E(B_i) = \emptyset$, then by the definition of generalized barbell, each component of H is a signed eulerian graph with an even number of negative edges. Thus H has a signed circuit double cover by Lemma 4.2.

Now assume that $E(H) - \cup_{i=1}^{t} E(B_i) \neq \emptyset$. Let $uv \in E(H) - \cup_{i=1}^{t} E(B_i)$ and H' be the new signed graph obtained from H by deleting uv and adding negative loops e_u and e_v at u and v, respectively. By the definition, H' remains as a generalized barbell. Since X is acyclic, H' has more components than H, and thus by induction to each component of H', H' has a signed circuit double cover $\mathcal{F}'$. Let $\{C_u^1, C_u^2\}$ and $\{C_v^1, C_v^2\}$ be the sets of barbells in $\mathcal{F}'$ containing e_u and e_v, respectively. Since e_u and e_v belong to two distinct components of H', $C^i = (C_u^i - e_u) \cup (C_v^i - e_v) + uv$ $(i = 1, 2)$ is a barbell in H. Hence

$$(\mathcal{F}' - \{C_u^1, C_u^2, C_v^1, C_v^2\}) \cup \{C^1, C^2\}$$

is a signed circuit double cover of H. □

Lemma 4.4. *Let H be a generalized barbell with a set of vertex-disjoint eulerian subgraphs $\mathcal{B} = \{B_1, \cdots, B_t\}$, and assume that $\{B_1, \cdots, B_s\}$ $(2 \leq s \leq t)$ is the set of eulerian subgraphs corresponding to the 1-vertices of the contracted graph $X = H/(\cup_{i=1}^{t} B_i)$. If each B_i $(1 \leq i \leq t)$ is a circuit, then there is a family of signed circuits $\mathcal{F}$ in H such that each edge e of H belongs to*

(a) *exactly one member of $\mathcal{F}$ if $e \in \cup_{i=1}^{s} E(B_i)$,*
(b) *one or two members of $\mathcal{F}$ if $e \in \cup_{i=s+1}^{t} E(B_t)$, and*
(c) *at most one member of $\mathcal{F}$ if $e \in E(H) - \cup_{i=1}^{t} E(B_i)$.*

Proof. Assume that H is embedded in the plane and let $\overline{X^*}$ be a graph obtained from X by first clockwise splitting each vertex x with even degree into $\frac{1}{2} d_X(x)$ 2-vertices, and replacing each maximal subdivided edge with a single edge. Then each vertex of $\overline{X^*}$ is of odd degree. By the definition of generalized barbell, $\overline{X^*}$ is a forest and $V(\overline{X^*})$ corresponds to the set of unbalanced circuits of $\mathcal{B}$. Thus $\overline{X^*}$ has a spanning subgraph satisfying that each component is a star graph with at least two vertices. Let K_{1,r_i} $(i = 1, \cdots, \ell)$ be all such star subgraphs.

Note that $V(\overline{X^*}) = \cup_{i=1}^{\ell} V(K_{1,r_i})$ corresponds to the set of unbalanced circuits of $\mathcal{B}$. For $1 \leq i \leq \ell$, one can check that the subgraph of H corresponding to K_{1,r_i} has a signed circuit cover $\mathcal{F}_i$ such that each edge of the unbalanced circuits corresponding to 1-vertices of K_{1,r_i} is covered by $\mathcal{F}_i$ exactly once and each edge of the unbalanced circuit corresponding to the unique vertex of K_{1,r_i} with degree $r_i \geq 2$ is covered by $\mathcal{F}_i$ once or twice. Therefore the union of $\cup_{i=1}^{\ell} \mathcal{F}_i$ together with the set of balanced circuits of $\mathcal{B}$ is a desired family $\mathcal{F}$ of signed circuits of H. □

Given a family of sets $\{A_1, \cdots, A_t\}$, their *symmetric difference*, denoted by $\Delta_{i=1}^{t} A_i$, is defined as the set consisting of elements contained in an odd number of A_i's.

The following result states that a generalized barbell has a signed circuit $\{1, 2\}$-cover with some edges covered only once.

Lemma 4.5. *For each generalized barbell, it either*

(i) *can be decomposed into balanced circuits, or*
(ii) *has a signed circuit $\{1,2\}$-cover $\mathcal{F}$ such that there are two edge-disjoint unbalanced circuits C_1 and C_2 whose edges are covered by $\mathcal{F}$ exactly once.*

Proof. Let H be a counterexample to Lemma 4.5 with $|E(H)|$ minimum. Thus H is connected. Otherwise each component of H satisfies either (i) or (ii). This implies that H satisfies either (i) or (ii), a contradiction to the choice of H.

Claim 4.1. *H is eulerian and therefore contains an even number of negative edges.*

Proof of Claim 4.1. By the definition of generalized barbell, it is sufficient to show that H is bridgeless. Suppose to the contrary that H has a bridge. By Lemma 4.3, H has a signed circuit double cover $\mathcal{F}'$. Since H has bridges, $\mathcal{F}'$ contains a barbell C with two unbalanced circuits C_1 and C_2. Then $\mathcal{F} = \mathcal{F}' - \{C\}$ is a signed circuit $\{1,2\}$-cover of H and covers C_1 and C_2 exactly once, a contradiction. This proves the claim. □

Since H is eulerian by Claim 4.1, H has a decomposition

$$\mathcal{C} = \{C_1, \cdots, C_h, C_{h+1}, \cdots, C_{h+m}, C_{h+m+1}, \cdots, C_{h+m+n}\},$$

where h, m and n are three nonnegative integers, and each C_i is an unbalanced circuit if $1 \le i \le h$, a short barbell if $h+1 \le i \le h+m$, and a balanced circuit otherwise. We choose such a decomposition that
(a) $h + 2m + n$ is as large as possible,
(b) subject to (a), n is as large as possible, and
(c) subject to (a) and (b), m is as large as possible.

Claim 4.2. *$h \ge 2$ is even and $|V(C_i) \cap V(C_j)| = 0$ for $1 \le i < j \le h$.*

Proof of Claim 4.2. If $h = 0$, then $\mathcal{C}$ satisfies (i) if $m = 0$ and the multiset $\mathcal{C} \cup (\mathcal{C} \setminus \{C_1\})$ satisfies (ii) otherwise. Thus $h > 0$. Since $|E_N(H)| = \sum_{i=1}^{h+m+n} |E_N(C_i)|$ is even and $|E_N(C_i)|$ is even for $h+1 \le i \le h+m+n$, we have $\sum_{i=1}^{h} |E_N(C_i)|$ is even. But each $|E_N(C_i)|$ is odd for $1 \le i \le h$, and so h is even and $h \ge 2$.

Let C_i and C_j be two circuits in $\mathcal{C}$ with $1 \le i < j \le h$. If $|V(C_i) \cap V(C_j)| \ge 3$, then $C_i \cup C_j$ can be decomposed into three or more circuits (balanced or unbalanced), a contradiction to (a). So $|V(C_i) \cap V(C_j)| \le 2$. If $|V(C_i) \cap V(C_j)| = 2$, then $C_i \cup C_j$ has a decomposition into two balanced circuits since both C_i and C_j are unbalanced circuits, which contradicts (b). If $|V(C_i) \cap V(C_j)| = 1$, then $C_i \cup C_j$ is a short barbell, which contradicts (c). So the claim is true. □

Let $H' = H/(\cup_{i=1}^{h} C_i)$ and for $1 \leq i \leq h$, let c_i be the vertex of H' corresponding to C_i. Let T' be a spanning tree of H' since H is connected. By Claim 4.2, $h \geq 2$ is even. Let P_j $(1 \leq j \leq \frac{h}{2})$ be a path in T' from c_{2j-1} to c_{2j} and let

$$F' = T'[\Delta_{j=1}^{\frac{h}{2}} E(P_j)]$$

Then F' is a forest and $\{c_1, \cdots, c_h\}$ is the set of vertices of F' with odd degree. By the definition, the subgraph of H corresponding to F' is a generalized barbell satisfying the conditions in Lemma 4.4, and thus, by Lemma 4.4, it has a family $\mathcal{F}^*$ of signed circuits such that $\mathcal{F} = \mathcal{F}^* \cup \{C_{h+1}, \cdots, C_{h+m+n}\}$ is a signed circuit $\{1, 2\}$-cover of H and at least two unbalanced circuits in $\{C_1, \cdots, C_h\}$ are covered by $\mathcal{F}$ exactly once, a contradiction. This completes the proof of Lemma 4.5. □

5. Proof of Lemma 3.2

In this section, we complete the proof of Lemma 3.2. For a signed graph G, we use $B(G)$ to denote the set of bridges of G and for each $e \in E_N(G)$, define

$$S_G(e) = \{e\} \cup \{f : \{e, f\} \text{ is a 2-edge-cut of } G\}.$$

Let $B_g(G)$ be the subset of $B(G)$ such that, for each $b \in B_g(G)$, at least one component of $G - b$ contains an odd number of negative edges, and let $B_s(G)$ be the subset of $B(G)$ such that, for each $b \in B_s(G)$, each component of $G - b$ contains negative edges. We need the following lemmas.

Lemma 5.1. *Let H be a signed graph satisfying that $|E_N(H)| \geq 2$ and $H - E_N(H)$ is a spanning tree of H. If $|E_N(H)|$ is even, then H has a generalized barbell containing all edges of $B_g(H) \cup (\cup_{e \in E_N(H)} S_H(e))$.*

Proof. Let $T = H - E_N(H)$. Then $E(H)$ is the disjoint union of $E(T)$ and $E_N(H)$. For each $e \in E_N(H)$, let C_e be the unique circuit of $T + e$.

Let $H' = \triangle_{e \in E_N(H)} C_e$ and $O_{H'}$ be the set of all components of H' containing an odd number of negative edges. Since $|E_N(H)|$ is even, so is $|O_{H'}|$. Let $\{v_1, v_2, \cdots, v_{2t}\}$ be the set of vertices of the contracted graph H/H' corresponding to $O_{H'}$. For $i = 1, \cdots, t$, there is a shortest path P_i in H/H' from v_{2i-1} to v_{2i}. Note that $E_N(H) \subseteq E(H')$ and hence $E(P_i) \subseteq E(H/H') \subseteq E(T)$. Since T is a tree of H, $H'' = H' \cup (\triangle_{i=1}^{t} P_i)$ is a generalized barbell.

For every bridge $b \in B_g(H)$, each component of $H - b$ contains an odd number of negative edges since $|E_N(H)|$ is even, and thus contains an odd number of members of $O_{H'}$. This fact implies that b must belong to an odd number of members of $\{P_1, \cdots, P_t\}$ and thus $b \in E(H'')$. Hence $B_g(H) \subseteq E(H'')$. For every $e \in E_N(H)$, it is obvious that $S_H(e) \subseteq E(C_e)$ and $S_H(e) \cap E(C_f) = \emptyset$ for any $f \in E_N(H) - \{e\}$, which implies that $S_H(e) \subseteq E(H')$. Therefore, $\cup_{e \in E_N(H)} S_H(e) \subseteq E(H') \subseteq E(H'')$. □

Lemma 5.2. *Let H be a signed graph satisfying that $|E_N(H)| \geq 2$ and $H - E_N(H)$ is a spanning tree of H. Then H has a signed circuit $\{0,1,2,3\}$-cover such that each edge of $B_s(H) \cup (\cup_{e \in E_N(H)} S_H(e))$ is covered at least once and each negative loop (if any) is covered precisely twice.*

Proof. Let H be a counterexample with $|E(H)|$ minimum.

Claim 5.1. $B(H) = \emptyset$.

Proof of Claim 5.1. Suppose to the contrary that $B(H) \neq \emptyset$. Let $b = u_1u_2 \in B(H)$ and Q_1 and Q_2 be the two components of $H - b$ such that $u_i \in Q_i$ for $i = 1, 2$. Since $H - E_N(H)$ is connected, we have that $b \notin E_N(H)$.

If $b \in B(H) - B_s(H)$, then there is one member in $\{Q_1, Q_2\}$, without loss of generality, say Q_1, satisfying that $B_s(Q_1) = B_s(H)$ and $E_N(Q_1) = E_N(H)$. By the minimality of H, Q_1 (and thus H) has a desired signed circuit $\{0,1,2,3\}$-cover, a contradiction.

If $b \in B_s(H)$, then $|E_N(Q_1)| \geq 1$ and $|E_N(Q_2)| \geq 1$. For each $i = 1, 2$, let Q_i^* be the graph obtained from Q_i by adding a negative loop e_i at u_i. It is easy to see that $B_s(Q_1^*) \cup B_s(Q_2^*) = B_s(H) - \{b\}$ and $\cup_{i=1}^2 (E_N(Q_i^*) - \{e_i\}) = E_N(H)$. By the minimality of H, each Q_i^* has a signed circuit $\{0,1,2,3\}$-cover $\mathcal{F}_i^*$ which covers each edge of $B_s(Q_i^*) \cup E_N(Q_i^*)$ at least once and covers each negative loop of Q_i^* exactly twice. Let C_i^1 and C_i^2 be the two signed circuits in $\mathcal{F}_i^*$ containing e_i. Since e_i is a negative loop, C_i^j $(j = 1, 2)$ is a barbell of Q_i^*, and so $C^j = (C_1^j - e_1) \cup (C_2^j - e_2) + b$ is also a barbell of H. Therefore, $\mathcal{F} = (\mathcal{F}_1^* - \{C_1^1, C_1^2\}) \cup (\mathcal{F}_2^* - \{C_2^1, C_2^2\}) \cup \{C^1, C^2\}$ is a desired signed circuit $\{0,1,2,3\}$-cover of H, a contradiction. □

Claim 5.1 implies that H is 2-edge-connected. So Lemma 5.2 follows from Lemmas 5.1 and 4.3 if $|E_N(H)|$ is even. Since $|E_N(H)| \geq 2$, in the following, we assume that $|E_N(H)| \geq 3$ is odd.

Let $T = H - E_N(H)$. Note that T is a spanning tree of H and $E(H)$ is the disjoint union of $E(T)$ and $E_N(H)$. For each $e \in E_N(H)$, let C_e be the unique circuit of $T + e$.

Claim 5.2. *For every $e \in E_N(H)$, H has a signed circuit containing all edges of $S_H(e)$.*

Proof of Claim 5.2. Let $e \in E_N(H)$ and $f \in E_N(H) - \{e\}$. Note that $S_H(e) \subseteq E(C_e)$, $S_H(f) \subseteq E(C_f)$ and $S_H(e) \cap S_H(f) = \emptyset$ (it can be checked easily since $T = H - E_N(H)$ is a spanning tree of H). If $|V(C_e) \cap V(C_f)| \leq 1$, then there is a shortest path P in T joining C_e to C_f (note that P is a single vertex if $|V(C_e) \cap V(C_f)| = 1$), and so $C_e \cup C_f \cup P$ is a desired signed circuit. If $|V(C_e) \cap V(C_f)| \geq 2$, since T is a spanning tree of H, then $C_e \cap C_f$ is a path containing no edges of $S_H(e)$. Thus $C_e \Delta C_f$ is a balanced circuit as desired. □

Claim 5.3. *Each edge $e \in E_N(H)$ is contained in a 2-edge-cut of H.*

Proof of Claim 5.3. Suppose to the contrary then there is a negative edge $e \in E_N(H)$ such that $H_0 = H - e$ remains 2-edge-connected. If H contains negative loops, we choose e which is a negative loop.

Note that $H_0 - E_N(H_0) = H - E_N(H)$ is a spanning tree of H (and thus H_0). Since H_0 is 2-edge-connected and $|E_N(H_0)| = |E_N(H) - \{e\}| \geq 2$ is even, Lemma 5.1 implies that H_0 has a generalized barbell H_1 containing all edges of $\cup_{f \in E_N(H_0)} S_{H_0}(f)$. Let $\mathcal{F}_1$ be a signed circuit double cover of H_1 by Lemma 4.3. Note that $S_H(e) = \{e\}$ and $S_H(f) \subseteq S_{H_0}(f)$ for any $f \in E_N(H_0) = E_N(H) - \{e\}$. Thus $\cup_{f \in E_N(G)} S_H(f) \subseteq \{e\} \cup (\cup_{f \in E_N(H_0)} S_{H_0}(f))$.

If e is not a negative loop of H, then H has no loop, but has a signed circuit C containing e by Claim 5.2. Thus $\mathcal{F} = \mathcal{F}_1 \cup \{C\}$ is a signed circuit $\{0, 1, 2, 3\}$-cover of H covering all edges of $\cup_{f \in E_N(H)} S_H(f)$, a contradiction.

Assume that e is a negative loop of H and let u denote the unique endvertex of e.

If $\mathcal{F}_1$ contains a barbell C, then let C_1 and C_2 be the two unbalanced circuits of C. Since H is 2-edge-connected, there are two edge-disjoint paths in H from u to C_1 and C_2, denoted by P_1 and P_2, respectively. Then $C_i' = C_i \cup P_i + e_0$ for $i = 1, 2$ is a barbell of H. Since $\mathcal{F}_1$ is a signed circuit double cover of H_1, $\mathcal{F} = (\mathcal{F}_1 - C) \cup \{C_1', C_2'\}$ is a desired signed circuit $\{0, 1, 2, 3\}$-cover of H, a contradiction.

If $\mathcal{F}_1$ contains no barbells, then e is the unique loop of H. Note that H_1 is a generalized barbell. By Lemma 4.5, H_1 has either a decomposition $\mathcal{F}_1'$ into balanced circuits or a signed circuit $\{1, 2\}$-cover $\mathcal{F}_1''$ and two edge-disjoint unbalanced circuit C_1 and C_2 such that each edge in $E(C_1) \cup E(C_2)$ is covered by $\mathcal{F}_1''$ exactly once. In the former case, let C' be a signed circuit containing e by Claim 5.2. Then the family $\mathcal{F} = \mathcal{F}_1' \cup \{C', C'\}$ is a desired signed circuit $\{0, 1, 2, 3\}$-cover of H. In the latter case, since H is 2-edge-connected, there are two edge-disjoint paths of H from u to C_1 and C_2, denoted by P_1 and P_2, respectively. Similar to the case when $\mathcal{F}_1$ contains a barbell, we can construct a desired signed circuit $\{0, 1, 2, 3\}$-cover of H, and thus obtain a contradiction. □

By Claim 5.3, H contains no negative loops and $|S_H(e)| \geq 2$ for each $e \in E_N(H)$. For every $e \in E_N(G)$, let $\mathcal{M}_e$ denote the set of all components of the subgraph $H - S_H(e)$.

Claim 5.4. *For two distinct $e, e' \in E_N(H)$, $S_H(e')$ is contained in exactly one member of $\mathcal{M}_e$.*

Proof of Claim 5.4. Note that each member of $\mathcal{M}_e$ is 2-edge-connected, and $S_H(e) \cap S_H(e') = \emptyset$ since $H - E_N(H)$ is a spanning tree of H. Then $S_H(e') \subseteq \cup_{M \in \mathcal{M}_e} E(M)$. Let e^* be an arbitrary edge in $S_H(e') - \{e'\}$. If there are two distinct members M_i and M_j of $\mathcal{M}_e$ such that $e' \in E(M_i)$ and $e^* \in E(M_j)$, then both $M_i - e'$ and $M_j - e^*$ are connected, and so $H - \{e', e^*\}$ is also connected. This contradicts that $\{e', e^*\}$ is a 2-edge-cut of H. So e' and e^* are contained in a common member of $\mathcal{M}_e$. The arbitrariness of e^* implies that the claim holds. □

For every $e \in E_N(H)$, let $m_e = \max\{|E_N(H) \cap E(M)| : M \in \mathcal{M}_e\}$. It is obvious that $m_e \le |E_N(H)| - 1$ since $e \notin \cup_{M \in \mathcal{M}_e} E(M)$.

Claim 5.5. $\max\{m_e : e \in E_N(H)\} = |E_N(H)| - 1$.

Proof of Claim 5.5. Let $e_0 \in E_N(H)$ and $M_{01} \in \mathcal{M}_{e_0}$ such that $m_{e_0} = |E_N(H) \cap E(M_{01})| = \max\{m_e : e \in E_N(H)\}$. Suppose that $m_{e_0} < |E_N(H)| - 1$. Then there is a member $M_{02} \in \mathcal{M}_{e_0} - \{M_{01}\}$ such that M_{02} contains a negative edge e_1 of H.

By Claim 5.4, $S_H(e_1) \subseteq E(M_{02})$ and there is a member $M_{11} \in \mathcal{M}_{e_1}$ such that $S_H(e_0) \subseteq E(M_{11})$. So

$$\{e_0\} \cup E(M_{01}) \subseteq S_H(e_0) \cup (\cup_{M \in \mathcal{M}_{e_0} - \{M_{02}\}} E(M)) \subseteq E(M_{11}),$$

which implies that

$$m_{e_1} \ge |E_N(H) \cap E(M_{11})| \ge 1 + |E_N(H) \cap E(M_{01})| = 1 + m_{e_0}.$$

This contradicts the choice of e_0, and so the claim holds. □

By Claim 5.5, there is an edge $e \in E_N(H)$ such that $E_N(H) - \{e\}$ is contained in exactly one member of $\mathcal{M}_e$. Let $\mathcal{M}_e = \{M_1', \cdots, M_s'\}$. Without loss of generality, assume that $E_N(H) - \{e\} \subseteq E(M_1')$ and all edges of M_i' $(i = 2, \cdots, s)$ are positive. Since H is 2-edge-connected, it follows from the definition of $S_H(e)$ that $H / \cup_{i=1}^{s} M_i'$ is a circuit, and each M_i' is also 2-edge-connected. Since $|E_N(M_1')| = |E_N(H)| - 1 \ge 2$ is even, M_1' has a generalized barbell H_1' containing all edges of $\cup_{f \in E_N(M_1')} S_{M_1'}(f)$ by Lemma 5.1, and H_1' has a signed circuit double cover $\mathcal{F}_1$ by Lemma 4.3.

Since $E_N(M_1') = E_N(H) - \{e\}$ and $S_{M_1'}(f) \supseteq S_H(f)$ for any $f \in E_N(M_1')$,

$$\cup_{f \in E_N(H)} S_H(f) \subseteq S_H(e) \cup (\cup_{f \in E_N(M_1')} S_{M_1'}(f)).$$

By Claim 5.2, H has a signed circuit C containing all edges of $S_H(e)$, and so $\mathcal{F} = \mathcal{F}_1 \cup \{C\}$ is a desired signed circuit $\{0, 1, 2, 3\}$-cover of H, a contradiction. This completes the proof of the lemma. □

By the definition of the switching operations, we have the following observation.

Observation 5.3. *Let G be a signed graph. Then $|E_N(G)| = \epsilon_N(G)$ if and only if for every edge cut T of G,*

$$|E_N(G) \cap T| \le \frac{|T|}{2}.$$

We now prove Lemma 3.2.

Proof of Lemma 3.2. Let G be an s-bridgeless signed graph with $|E_N(G)| = \epsilon_N(G)$. Without loss of generality, we further assume that G is connected. Since G is s-bridgeless, $|E_N(G)| \neq 1$. If $|E_N(G)| = 0$, then G is a 2-edge-connected unsigned graph. The lemma is trivial, and thus assume that $|E_N(G)| \geq 2$.

Let

$$G_1 = G - B(G) - (\cup_{e\in E_N(G)} S_G(e)).$$

Then G_1 contains no negative edges of G since $E_N(G) \subseteq \cup_{e\in E_N(G)} S_G(e)$ by the definition of $S_G(e)$.

Moreover, we claim that G_1 is bridgeless. Let $G_1' = G - B(G)$ and $G_1'' = G_1' - E_N(G)$. Then G_1' is bridgeless. Note that by Observation 5.3, $B(G) \cap E_N(G) = \emptyset$. Thus $E_N(G_1') = E_N(G)$. Therefore by the definition of G_1'', f is a bridge of G_1'' if and only if there is an edge-cut S of G_1' (and of G too) containing f such that $S \setminus \{f\} \subseteq E_N(G)$. On the other hand, if $f \in S_G(e)$ for some $e \in E_N(G)$, then f is a bridge of G_1''. Thus by Observation 5.3, f is a bridge of G_1'' if and only if $f \in S_G(e)$ for some $e \in E_N(G)$. This implies $G_1 = G_1'' - B(G_1'')$ and thus is bridgeless.

To construct G_2, let $H = T + E_N(G)$, where T is a spanning tree of $G - E_N(G)$ (the existence of T is guaranteed by Observation 5.3). Note that we have the following simple facts:

(1) $E_N(G) = E_N(H)$;
(2) $B_g(G) \subseteq B_g(H)$;
(3) $B_s(G) \subseteq B_s(H)$;
(4) $S_G(e) \subseteq S_H(e)$ for each $e \in E_N(G)$.

By Lemma 5.2, H has a signed circuit $\{0, 1, 2, 3\}$-cover $\mathcal{F}_2$ such that each edge of $B_s(H) \cup (\cup_{e\in E_N(H)} S_H(e))$ $(\supseteq B_s(G) \cup (\cup_{e\in E_N(G)} S_G(e)))$ is covered by $\mathcal{F}_2$ at least once. Let $G_2 = G[\cup_{C\in\mathcal{F}_2} E(C)]$. Since G is s-bridgeless, $B_s(G) = B(G)$, and so $E(G) = E(G_1) \cup E(G_2)$. It is obvious that $G_2 - E_N(G)$ is acyclic and $\mathcal{F}_2$ is a desired signed circuit $\{1, 2, 3\}$-cover of G_2.

In particular, if G is g-bridgeless with even negativeness, then $B_g(G) = B(G)$ and by Lemma 5.1, H has a generalized barbell, denoted by G_2, containing all edges of $B_g(H) \cup (\cup_{e\in E_N(H)} S_H(e))$ $(\supseteq B_g(G) \cup (\cup_{e\in E_N(G)} S_G(e)))$. Thus $E(G) = E(G_1) \cup E(G_2)$, $G_2 - E_N(G)$ is acyclic and by Lemma 4.3, G_2 has a signed circuit double cover. This proves Lemma 3.2. □

6. Proof of Theorem 3.1

In this section, we complete the proof of Theorems 3.1 by applying Lemma 3.2. Let G be an s-bridgeless signed graph with $\epsilon_N(G) > 0$. We only need to consider the case $|E_N(G)| = \epsilon_N(G)$ since the existence and the length of a signed circuit cover are two invariants under the switching operations.

Since G is s-bridgeless and $\epsilon_N(G) > 0$, we have that $|E_N(G)| = \epsilon_N(G) \geq 2$. If G contains positive loops, then we may consider the subgraph obtained from G by deleting all positive loops. Thus we further assume that G contains no positive loops.

By Lemma 3.2, G has a bridgeless unsigned subgraph G_1 and a signed subgraph G_2 such that $E(G_1) \cup E(G_2) = E(G)$, $G_2 - E_N(G)$ is acyclic and G_2 has a signed circuit $\{1, 2, \cdots, k\}$-cover $\mathcal{F}_2$, where $k = 2$ if G is g-bridgeless with even negativeness and $k = 3$ otherwise.

Note that $E(G_1) \subseteq G - E_N(G)$ and thus $E(G_1) \cap E(G_2) \subseteq E(G_2) - E_N(G)$ is acyclic. Hence we have the following two inequalities.

$$|E(G_1)| + |E(G_2)| = |E(G_1) \cup E(G_2)| + |E(G_1) \cap E(G_2)| \leq |E(G)| + |V(G)| - 1 \tag{1}$$

$$|E(G_2)| \leq (|V(G)| - 1) + |E_N(G)| = |V(G)| - 1 + \epsilon_N(G). \tag{2}$$

Let $\mathcal{F}_2'$ be a subset of $\mathcal{F}_2$ such that $\mathcal{F}_2'$ is still a signed circuit cover of G_2 and the number of signed circuits of $\mathcal{F}_2'$ is as small as possible. We have the following claim.

Claim 6.1. $\ell(\mathcal{F}_2') \leq k|E(G_2)| - 2(k-1)$.

Proof of Claim 6.1. Let t be the number of signed circuits in $\mathcal{F}_2'$. Since $|E_N(G_2)| = |E_N(G)| \geq 2$, $t \geq 1$. By the choice of $\mathcal{F}_2'$, every signed circuit in $\mathcal{F}_2'$ has an edge which is covered by $\mathcal{F}_2'$ exactly once, and so G_2 has at least t edges which are covered by $\mathcal{F}_2'$ exactly once. Note that $k = 2$ or 3, and each signed circuit in $\mathcal{F}_2$ is of length at least 2 since G has no positive loops. If $t = 1$, then G_2 is the unique signed circuit in $\mathcal{F}_2'$, and so $\ell(\mathcal{F}_2') = |E(G_2)| \leq k|E(G_2)| - 2(k-1)$. If $t \geq 2$, then $\ell(\mathcal{F}_2') \leq k(|E(G_2)| - t) + t = k|E(G_2)| - (k-1)t \leq k|E(G_2)| - 2(k-1)$. □

Since G_1 is bridgeless and unsigned, by Theorems 3.3 and 3.4, G_1 has a circuit cover $\mathcal{F}_1$ with total length

$$\ell(\mathcal{F}_1) \leq \min\{\frac{5}{3}|E(G_1)|, |E(G_1)| + |V(G_1)| - 1\}. \tag{3}$$

Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2'$ is a signed circuit cover of G and by Claim 6.1 and Equation (3) together with Equations (1) and (2), the total length of $\mathcal{F}$ satisfies that

$$\begin{aligned}
\ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2') \\
&\leq \min\{\frac{5}{3}|E(G_1)|, |E(G_1)| + |V(G_1)| - 1\} + k|E(G_2)| - 2(k-1) \\
&\leq \min\{\frac{5}{3}(|E(G)| + |V(G)| - 1) + (k - \frac{5}{3})(|V(G)| - 1 + \epsilon_N(G)) - 2(k-1), \\
&\quad (|E(G)| + |V(G)| - 1) + (|V(G)| - 1)
\end{aligned}$$

$$
\begin{aligned}
&+ (k-1)(|V(G)| - 1 + \epsilon_N(G)) - 2(k-1)\} \\
= \min\{&\frac{5}{3}|E(G)| + k|V(G)| + (k-\frac{5}{3})\epsilon_N(G) - (3k-2), \\
&|E(G)| + (k+1)|V(G)| + (k-1)\epsilon_N(G) - (3k-1)\}.
\end{aligned}
$$

This completes the proof of Theorem 3.1.

References

[1] N. Alon, M. Tarsi, Covering multigraphs by simple circuits, SIAM J. Algebr. Discrete Methods 6 (1985) 345–350.
[2] B. Alspach, L.A. Goddyn, C.-Q. Zhang, Graphs with the circuit cover property, Trans. Amer. Math. Soc. 344 (1994) 131–154.
[3] J.C. Bermond, B. Jackson, F. Jaeger, Shortest covering of graphs with cycles, J. Combin. Theory Ser. B 35 (1983) 297–308.
[4] J.A. Bondy, U.S.R. Murty, in: Graph Theory, in: GTM, vol. 244, Springer, 2008.
[5] A. Bouchet, Nowhere-zero integral flows on bidirected graph, J. Combin. Theory Ser. B 34 (1983) 279–292.
[6] R. Diestel, Graph Theory, fourth edn., Springer-Verlag, 2010.
[7] G.-H. Fan, Short cycle covers of cubic graphs, J. Graph Theory 18 (1994) 131–141.
[8] G.-H. Fan, A. Raspaud, Fulkerson's conjecture and circuits covers, J. Combin. Theory Ser. B 61 (1994) 133–138.
[9] G.-H. Fan, Proofs of two minimum circuit cover conjectures, J. Combin. Theory Ser. B 74 (1998) 353–367.
[10] A. Itai, M. Rodeh, Covering a graph by circuits, in: Automata, Languages and Programming, in: Lecture Notes in Computer Science, vol. 62, Springer-Verlag, Berlin, 1978, pp. 289–299.
[11] B. Jackson, Shortest circuit covers and postman tours of graphs with a nowhere-zero 4-flow, SIAM J. Comput. 19 (1990) 659–665.
[12] B. Jackson, Shortest circuit covers of cubic graphs, J. Combin. Theory Ser. B 60 (1994) 299–307.
[13] F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory Ser. B 26 (1979) 205–216.
[14] U. Jamshy, A. Raspaud, M. Tarsi, Short circuit covers for regular matroids with nowhere-zero 5-flow, J. Combin. Theory Ser. B 43 (1987) 354–357.
[15] U. Jamshy, M. Tarsi, Shortest cycle covers and the cycle double cover conjecture, J. Combin. Theory Ser. B 56 (1992) 197–204.
[16] A.V. Kostochka, The $\frac{7}{5}$-conjecture strengthens itself, J. Graph Theory 19 (1995) 65–67.
[17] E. Máčajová, A. Raspaud, E. Rollová, M. Škoviera, Circuit covers of signed graphs, J. Graph Theory 81 (2) (2016) 120–133.
[18] E. Máčajová, A. Raspaud, M. Tarsi, X.-D. Zhu, Short cycle covers of graphs and nowhere-zero flows, J. Graph Theory 68 (2011) 340–348.
[19] A. Raspaud, X.-D. Zhu, Circular flow on signed graphs, J. Combin. Theory Ser. B 101 (2011) 464–479.
[20] P.D. Seymour, Nowhere-zero 6-flows, J. Combin. Theory Ser. B 30 (1981) 130–135.
[21] D.B. West, Introduction to Graph Theory, Upper Saddle, Prentice Hall, River, NJ, 1996.
[22] C.-Q. Zhang, Minimum cycle coverings and integer flows, J. Graph Theory 14 (1990) 537–546.