Global stability of almost periodic solutions to monotone sweeping processes and their response to non-monotone perturbations

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Abstract

We develop a theory which allows making qualitative conclusions about the dynamics of both monotone and non-monotone Moreau sweeping processes. Specifically, we first prove that any sweeping processes with almost periodic monotone right-hand-sides admits a globally exponentially stable almost periodic solution. And then we describe the extent to which such a globally stable solution persists under non-monotone perturbations.

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1. Introduction

A perturbed Moreau sweeping process reads as

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)), \tag{1}$$

where $N_C(x)$ is a so-called normal cone defined for closed convex $C \in \mathbb{R}^n$ as

$$N_C(x) = \begin{cases} \{\xi \in \mathbb{R}^n : \langle \xi, c - x\rangle \le 0, \text{ for any } c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases} \tag{2}$$

and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ (see [15, 28, 22, 17]). The unboundedness of the right-hand-sides in (1) makes the classical theory of differential inclusions (see e.g. [4, 23]) inapplicable. And despite numerous applications in elastoplasticity (see e.g. [6, 5]) (as well as in problems of power converters [2] and crowd motion [34]), the theory of Moreau's sweeping processes is still in its infancy. Fundamental results on the existence, uniqueness and dependence of solutions on the initial data are proposed in Monteiro Marques [35, Ch. 3], Valadier [42], Castaing and Monteiro Marques [15], Adly-Le [3], Brogliato-Thibault [11], Krejci-Roche [27], Paoli [36]. Dependence of solutions on parameters is covered in Bernicot-Venel [7] and Kamenskiy-Makarenkov [22]. The papers [22, 15] also show the existence of T-periodic solutions for T-periodic in time (1). Optimal control problems for sweeping process (1) and equivalent differential equations with hysteresis operator are addressed in Edmond-Thibault [17], Adam-Outrata [1] (which also discusses applications to game theory), Brokate-Krejci [12]. Numerical schemes to compute the solutions of (1) are discussed through most of the papers mentioned above.

Much less is known about the asymptotic behavior as $t \to \infty$. The known results in this direction are due to Leine and van de Wouw [29, 30], Brogliato [9], and Brogliato-Heemels [10]. Applied to a time-independent sweeping process (1) the statements of [29, Theorem 8.7] (or [30, Theorem 2]), [9, Lemma 2], and [10, Theorem 4.4] imply incremental stability (see Definition 2.1) and global exponential stability of an equilibrium, provided that

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2\rangle \ge \alpha\|x_1 - x_2\|^2, \quad \text{for some fixed } \alpha > 0 \text{ and for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \tag{3}$$

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In particular, the results of [29, 30, 9, 10] do not impose any Lipschitz regularity on $x \mapsto f(t,x)$ and the derivative in (1) is a differential measure, which is capable to deal with solutions x of bounded variation.

This paper is motivated by sweeping processes (1) coming from models of parallel networks of elastoplastic springs (see e.g. Bastein et al [6, 5]), where the right-hand-sides are Lipschitz in all the variables. Here $C(t)$ represents the mechanical loading of the springs and $f(t,x)$ stands for those forces which influence the masses of nodes. Time-periodically changing C and f are most typical in laboratory experiments (see [19, 20, 6]). However, the different nature of $t \mapsto C(t)$ and $t \mapsto f(t,x)$ makes it most reasonable to not rely on the existence of a common period when the two functions receive periodic excitations, but rather to use a theory which is capable to deal with arbitrary different periods of $t \mapsto C(t)$ and $t \mapsto f(t,x)$. The goal of this paper is to develop such a theory.

Specifically, by assuming that both $t \mapsto C(t)$ and $t \mapsto f(t,x)$ are almost periodic, we establish global exponential stability of an almost periodic solution to a monotone sweeping process (14). The corresponding theory for differential equations is available e.g. in Trubnikov-Perov [41] and Zhao [46], that found numerous applications in biology. Moreover, we show that the almost periodic solution found preserves its stability under a wide class of non-monotone perturbations, which is known for differential inclusions with bounded right-hand-sides e.g. from Kloeden-Kozyakin [24] and Plotnikov [25].

The paper is organized as follows. Section 2 establishes (Theorem 2.1) the existence of solutions to (1) defined on the entire $\mathbb{R}$ under the assumption that both $t \mapsto C(t)$ and $(t,x) \mapsto f(t,x)$ are globally Lipschitz functions, but without any use of the monotonicity assumption (3). Note, that for any solution $x(t)$ of (1), one has $x(t) \in C(t)$, so any solution of (1) is uniformly bounded in the domain of its definition, if $C(t)$ is such. When the monotonicity assumption (3) holds, we have (Theorem 2.2) the uniqueness and global exponential stability of a solution defined on the entire $\mathbb{R}$. This result doesn't follow from [9, 10], where the existence of an equilibrium is a consequence of the particular structure of the right-hand-sides. When both $C(t)$ and $f(t,x)$ are constant in t, the existence of an equilibrium to (1) formally follows from [29, 30] which could transform into a solution on $\mathbb{R}$ when $C(t)$ and $f(t,x)$ are time-varying and globally bounded. We provide an independent proof because the proofs of [29, Theorem 8.7] and [30, Lemma 2] rely on Yakubovich [44, Lemma 2]. In turn, [44, Lemma 2] sends the reader to Budak [13, Theorem 2] for the most crucial step of the proof, which is compactness of a sequence $\{x_k\}_{k=1}^{\infty}$ of $C^0(\mathbb{R}, \mathbb{R}^n)$ solutions to (1) corresponding to a converging sequence of initial conditions. Even if one ignores verifying the regularity assumption of Budak [13, Theorem 2], this theorem provides a convergent subsequence on a finite interval and Yakubovich [44, Lemma 2] doesn't explain how the convergence gets extended to the entire $\mathbb{R}$.

Under the assumption that both $t \mapsto C(t)$ and $t \mapsto f(t,x)$ are almost periodic functions and $x \mapsto f(t,x)$ is monotone in the sense of (3), Section 3 shows (Theorem 3.1) that the unique global solution found in Section 2 is almost periodic. Here we follow the standard definitions (see e.g. Levitan-Zhikov [31, p. 1] or Vesely [43]) to introduce the concept of almost periodicity for set-valued functions and for the respective Bochner's theorem. The results of [31] and [43] are developed for functions with values in an arbitrary complete metric space and we take advantage of the completeness of the space of convex closed nonempty sets equipped with the Hausdorff metric (see e.g. Price [38]) to apply the concept of almost-periodicity to sweeping processes. The overall strategy of section 3 originates from the corresponding theory available for differential equations (see e.g. Trubnikov-Perov [41]).

Section 4 considers a sweeping process (1) with a parameter ε under the assumption that the monotonicity condition (3) holds for $\varepsilon = \varepsilon_0$. When $\varepsilon = \varepsilon_0$, the sweeping process has a unique solution x_0 defined on $\mathbb{R}$ by Theorem 2.2. The result of section 4 (Theorems 4.1 and 4.3) proves that the solutions to the perturbed sweeping process with $\varepsilon \neq \varepsilon_0$ and with an initial condition $x_\varepsilon(0) \in C(0)$ approach any given inflation of the solution x_0 (as it is termed in Kloeden-Kozyakin [24]) when the values of time become large and when ε approaches ε_0. Section 4.3 specifies the findings of section 4 for the case where both $t \mapsto C(t)$ and $t \mapsto f(t,x,\varepsilon)$ are almost periodic in time, so that x_0 is almost periodic as well. Instructive examples of Section 4.4 illustrate the domains of applications of Theorems 4.1 and 4.3. Finally, Section 4.5 gives a brief outlook about the potential role of Theorems 4.1 and 4.3 in the analysis of the dynamics of networks of elastoplastic springs that motivated our study.

We note that condition (3) ensures that the sweeping process (1) is incrementally stable (see [29, Theorem 8.7], [30, Lemma 2], or Theorem 2.2 below), which concept currently attracts an increasing attention in the switched systems literature, see e.g. Lu-di Bernardo [33], Zamani-van de Wouw-Majumdar [45] and references therein. The source for incremental stability in these papers is certain contraction of the right-hand-sides (going back to Demidovich, see [16, Ch. IV, §16] and [37]), which property is also ensured by the monotonicity assumption (3).

2. The existence of an unique globally exponentially stable bounded solution x_0

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipschitz continuous in the sense that

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq L_f \|t_1 - t_2\| + L_f \|x_1 - x_2\|, \qquad \text{for all } t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n, \text{ and for some } L_f > 0. \quad (4)$$

A similar property

$$d_H(C(t_1), C(t_2)) \leq L_C |t_1 - t_2|, \qquad \text{for all } t_1, t_2 \in \mathbb{R}, \text{ and for some } L_C > 0, \quad (5)$$

is assumed for the closed convex-valued function $t \mapsto C(t)$, where the Hausdorff distance $d_H(C_1, C_2)$ between two closed sets $C_1, C_2 \subset \mathbb{R}^n$ is defined as

$$d_H(C_1, C_2) = \max \left\{ \sup_{x \in C_2} \operatorname{dist}(x, C_1), \sup_{x \in C_1} \operatorname{dist}(x, C_2) \right\} \quad \text{with} \quad \operatorname{dist}(x, C) = \inf \{ |x - c| : c \in C \}. \quad (6)$$

Under conditions (4) and (5), for any initial condition $x(t_0) \in C(t_0)$, the sweeping process (1) with nonempty, closed and convex $C(t)$, $t \in \mathbb{R}$, admits (Edmond-Thibault [17, Theorem 1]) a unique absolutely continuous forward solution $x(t)$, in the sense that $x(t)$ satisfies (1) for almost all $t \geq t_0$.

Remark 2.1. If x_0 is a solution to (1) defined on $t \geq t_0$, then $x(t) \in C(t)$, for all $t \geq t_0$, because $N_{C(t)}(x(t))$ is undefined otherwise (the interested reader can see that [17] obtains the solution $x(t)$ as $x(t) = y(t) - \psi(t)$, where $y(t) \in C(t) + \psi(t)$ [17, pp. 352–353]). In particular, if $\|C(t)\| \leq M$ for some $M > 0$ and all $t \in \mathbb{R}$, then

$$\|x(t)\| \leq M, \quad \text{for any solution } x \text{ to (1) with the initial condition } x(t_0) \in C(t_0) \text{ and } t \geq t_0. \quad (7)$$

Theorem 2.1. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the Lipschitz condition (4). Assume that, for any $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex and the map $t \mapsto C(t)$ satisfies the Lipschitz condition (5). If $t \mapsto C(t)$ is globally bounded, then the sweeping process (1) admits at least one absolutely continuous solution x_0 defined on the entire $\mathbb{R}$. The solution x_0 is globally bounded.

Proof. **Step 1:** *Construction of a candidate solution x_0 defined on the entire $\mathbb{R}$.* Let $\{\xi_m\}_{m=1}^{\infty}$ be an arbitrary sequence of elements of $\mathbb{R}^n$ such that $\xi_m \in C(-m)$, $m \in \mathbb{N}$. Let $x_m(t)$ be the solution to (1) with the initial condition $x_m(-m) = \xi_m$. Extend each x_m from $[-m, \infty)$ to $\mathbb{R}$ by defining $x_m(t) = x_m(-m)$ for all $t < -m$. By Edmond-Thibault [17, Theorem 1], the functions of $\{x_m(t)\}_{m=1}^{\infty}$ share same Lipschitz constant $L_k > 0$ on each interval $[-k, k]$, $k \in \mathbb{N}$. Letting $\{x_m^0\}_{m=1}^{\infty} = \{x_m\}_{m=1}^{\infty}$, for each $k \in \mathbb{N}$ we can extract a subsequence $\{x_m^k(t)\}_{m=1}^{\infty}$ of $\{x_m^{k-1}(t)\}_{m=1}^{\infty}$ which converges uniformly on $[-k, k]$. By using this family of subsequences we introduce a sequence $\{x_m^*\}_{m=1}^{\infty}$ by $x_m^*(t) = x_m^m(t)$. The sequence $\{x_m^*\}_{m=1}^{\infty}$ converges uniformly on any fixed interval $[-k, k]$, $k \in \mathbb{N}$. Define $x_0(t)$ by $x_0(t) = \lim_{m \to \infty} x_m^*(t)$.

Step 2: *Proof that x_0 is indeed a solution.* Let $\tau \in \mathbb{R}$ and let v be a solution to (1) with $v(\tau) = x_0(\tau)$. Assume $v(t_0) \neq x_0(t_0)$ for some $t_0 > \tau$, i.e. $\lim_{m \to \infty} x_m^*(t_0) \neq v(t_0)$. Then there exists $\varepsilon_0 > 0$, such that for each $m \in \mathbb{N}$, there exists $m_k > m$ such that $\|x_{m_k}^*(t_0) - v(t_0)\| \geq \varepsilon_0$. On the other hand, by continuous dependence of solutions to (1) on the initial condition (see Edmond-Thibault [17, Proposition 2]), there exists $\delta > 0$ such that if $\|v(\tau) - x_m^*(\tau)\| < \delta$ then $\|v(t) - x_m^*(t)\| < \varepsilon_0$ for all $m \in \mathbb{N}$ with $-m < \tau$ (which ensures that $x_m^*(t)$ is a solution of (1) for $t \geq \tau$) and $t \in [\tau, t_0]$, see Fig. 1. But since $v(\tau) = x_0(\tau) = \lim_{m \to \infty} x_m^*(\tau)$, there exists $N \in \mathbb{N}$ such that $\|v(\tau) - x_m^*(\tau)\| < \delta$ for each $m > N$. Then $\|v(t) - x_m(t)\| < \varepsilon_0$ for all $m > N$ and $t \in [\tau, t_0]$. This contradicts $\lim_{n \to \infty} x_m^*(t_0) \neq v(t_0)$. Therefore $v(t) = x_0(t)$ for each $t \geq \tau$. Hence x_0 is a solution to (1).

The solution x_0 is globally bounded by Remark 2.1. □

Definition 2.1. (see e.g. Leine - van de Wouw [29, Definition 6.22]) The sweeping process (1) is incrementally stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that given an arbitrary $t_0 \in \mathbb{R}$, all solutions x_1 and x_2 of (1) with the initial condition $\|x_1(t_0) - x_2(t_0)\| < \delta$, satisfy $\|x_1(t) - x_2(t)\| < \varepsilon$, for a.e. $t \geq t_0$.

Theorem 2.2. Assume that the conditions of Theorem 2.1 hold. If f satisfies the monotonicity condition (3) then (1) is incrementally stable and (1) admits exactly one absolutely continuous bounded solution x_0 defined on the entire $\mathbb{R}$. Moreover, x_0 is globally exponentially stable.

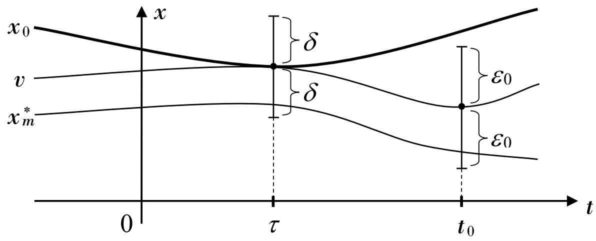

Figure 1: Illustration of the location of curves x_0, v, and x_m^*.

The incremental stability of (1) under condition (3) is proved in Leine - van de Wouw [29, Theorem 8.7] and [30, Lemma 2]. The statements about uniqueness and global stability of the bounded solution x_0 follow from incremental stability. We include a proof of Theorem 2.2 in Appendix for completeness.

Remark 2.2. The global boundedness of $t \mapsto C(t)$ is used in the proof of Theorem 2.1 just to conclude the boundedness of the global solution x_0. Accordingly, the assumption of global boundedness of $t \mapsto C(t)$ and the property of global boundedness of x_0 can be simultaneously dropped in the formulation of Theorem 2.1. But assuming global boundedness of $t \mapsto C(t)$ in Theorem 2.2 cannot be dropped as it is used in the proof in an essential way (to establish the uniqueness of x_0, not to just prove its global boundedness).

3. Almost periodicity of the bounded solution x_0

Let $ck(\mathbb{R}^n)$ be the space of all closed bounded nonempty sets of $\mathbb{R}^n$ equipped with the Hausdorff metric d_H, see (6). The concept of almost periodicity for multi-valued functions with compact values (Definition 3.1) is a combination of the definition of almost periodicity in complete metric spaces (see e.g. Levitan-Zhikov [31, p. 1] or Vesely [43]) and the property of the completeness of the metric space $ck(\mathbb{R}^n)$ (see e.g. Price [38]).

Definition 3.1. A continuous function $\phi : \mathbb{R} \to ck(\mathbb{R}^n)$ is *almost periodic*, if for any $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ with the property that any interval of length $p(\varepsilon) > 0$ of the real line contains at least one point s, such that

$$d_H(\phi(t+s), \phi(t)) < \varepsilon, \quad \text{for all} \quad t \in \mathbb{R}.$$

Theorem 3.1. Let the conditions of Theorem 2.2 hold and let x_0 be the unique absolutely continuous bounded solution given by Theorem 2.2. If both the function $t \mapsto f(t,x)$ and the set-valued function $t \mapsto C(t)$ are almost periodic, then x_0 is almost periodic.

Proof. Let $\{h_m\}_{m=1}^\infty \subseteq \mathbb{R}$. We are going to prove that there exists $\{k_m(x)\}_{m=1}^\infty \subseteq \{h_m\}_{m=1}^\infty$ such that the sequence of

$$x_m(t) = x_0(t + k_m), \quad m \in \mathbb{N},\ t \in \mathbb{R}, \tag{8}$$

converges as $m \to \infty$ uniformly in $t \in \mathbb{R}$, which will imply almost periodicity of x_0 by Bochner's theorem (see e.g. Levitan-Zhikov [31, p. 4]).

Step 1. *The existence of $\{l_m\}_{m=1}^\infty \subseteq \{h_m\}_{m=1}^\infty$ such that $f_m(t,x) = f(t+l_m, x)$ converges as $m \to \infty$ uniformly.* Since $f(t,x)$ is almost periodic, then, for each $x \in \mathbb{R}^n$, Bochner's theorem (see e.g. Levitan-Zhikov [31, p. 4]) implies the existence of $\{l_m(x)\}_{m=1}^\infty \subseteq \{h_m\}_{m=1}^\infty$ such that the sequence of functions $\{f(\cdot + l_m(x), x)\}_{m=1}^\infty$ converges in the sup-norm. The standard diagonal method allows to construct $\{l_m(x)\}_{m=1}^\infty$ independent of x. Indeed, considering $\{x_m\}_{m=1}^\infty = \mathbb{Q}^n$, we first construct sequences $\{l_m(x_1)\}_{m=1}^\infty \supseteq \{l_m(x_2)\}_{m=1}^\infty \supseteq \ldots$, such that each individual sequence $\{f(\cdot + l_m(x_1), x_1)\}_{m=1}^\infty, \{f(\cdot + l_m(x_2), x_2)\}_{m=1}^\infty, \ldots$ converges. And then define $\{l_m\}_{m=1}^\infty \subseteq \{h_m\}_{m=1}^\infty$ as $l_m = l_m(x_m)$, $m \in \mathbb{N}$. Put

$$f_m(t,x) = f(t + l_m, x), \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{Q}^n,\ m \in \mathbb{N}. \tag{9}$$

So constructed, $\{f_m(\cdot, x)\}_{m=1}^\infty$ converges for each fixed $x \in \mathbb{Q}^n$. Let

$$\hat{f}(t,x) = \lim_{m\to\infty} f_m(t,x), \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{Q}^n. \tag{10}$$

By (4) both f_m and $\hat{f}$ are Lipschitz continuous with constant L_f on $\mathbb{R}\times\mathbb{R}^n$ and $\mathbb{R}\times\mathbb{Q}^n$ respectively. Now we extend $\hat{f}$ from $\mathbb{R}\times\mathbb{Q}^n$ to $\mathbb{R}\times\mathbb{R}^n$ by taking an arbitrary sequence $\mathbb{Q}\ni x_k\to x_0\in\mathbb{R}$, as $k\to\infty$, and defining $\hat{f}(t,x_0)=\lim\limits_{k\to\infty}\hat{f}(t,x_k)$. The limit exists because $\{\hat{f}(t,x_k)\}_{k=1}^{\infty}$ is a Cauchy sequence for each fixed $t\in\mathbb{R}$, which follows from Lipschitz continuity of $\hat{f}$ on $\mathbb{R}\times\mathbb{Q}^n$. Lipschitz continuity of $\hat{f}$ extends from $\mathbb{R}\times\mathbb{Q}^n$ to $\mathbb{R}\times\mathbb{R}^n$ by continuity. The latter property also implies that

$$\left\|\hat{f}(t,x_0)-\hat{f}(t,x_k)\right\|\le L_f\|x_0-x_k\|,\quad \text{for all } k\in\mathbb{N}.$$

Finally, to show that

$$f_m(t,x)\to\hat{f}(t,x)\quad \text{as } m\to\infty,\ \text{uniformly in } t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{11}$$

we estimate $f_m(t,x)-\hat{f}(t,x)$ as

$$\left\|f_m(t,x)-\hat{f}(t,x)\right\|\le\|f_m(t,x)-f_m(t,x_*)\|+\left\|f_m(t,x_*)-\hat{f}(t,x_*)\right\|+\left\|\hat{f}(t,x_*)-\hat{f}(t,x)\right\|.$$

Given $x\in\mathbb{R}$ and $\varepsilon>0$, we choose $x_*\in\mathbb{Q}$ so close to x that $\|f_m(t,x)-f_m(t,x_*)\|<\varepsilon/3$ and $\left\|\hat{f}(t,x_*)-\hat{f}(t,x)\right\|<\varepsilon/3$, for all $m\in\mathbb{N}$, $t\in\mathbb{R}$. By (10) we can now select $m_0\in\mathbb{N}$ such that $\left\|f_m(t,x_*)-\hat{f}(t,x_*)\right\|<\varepsilon/3$, for all $m>m_0$ and $t\in\mathbb{R}$. Thus, (11) holds.

Step 2. *The existence of* $\{k_m\}_{m=1}^{\infty}\subseteq\{l_m\}_{m=1}^{\infty}$*, such that* $C_m(t)=C(t+k_m)$ *converges as* $m\to\infty$ *uniformly.* By Bochner's theorem for almost periodic functions in pseudo-metric spaces (see [43, Theorem 2.4]), there exists $\{k_m\}_{m=1}^{\infty}\subseteq\{l_m\}_{m=1}^{\infty}$, such that $\{C_m(t)\}_{m=1}^{\infty}$ is a Cauchy sequence in $ck(\mathbb{R}^n)$, which is uniform in $t\in\mathbb{R}$. The convergence of $\{C_m(t)\}_{m=1}^{\infty}$ for each individual $t\in\mathbb{R}$ now follows from the completeness of $ck(\mathbb{R}^n)$ (Price [38, the theorem of §3]). The uniformity of the convergence in $t\in\mathbb{R}$ follows along standard lines. Indeed, let

$$\hat{C}(t)=\lim_{m\to\infty}C_m(t).$$

Given $\varepsilon>0$, fix $m_0>0$ such that $d_H(C_m(t),C_{m_*}(t))<\varepsilon/2$ for all $m>m_0$, $m_*>m_0$, and $t\in\mathbb{R}$. For each $t\in\mathbb{R}$ select $m_*(t)>m_0$ such that $d_H\left(C_{m_*(t)}(t),\hat{C}(t)\right)<\varepsilon/2$. Then

$$d_H\left(C_m(t),\hat{C}(t)\right)\le d_H\left(C_m(t),C_{m_*(t)}(t)\right)+d_H\left(C_{m_*(t)}(t),\hat{C}(t)\right)<\varepsilon/2+\varepsilon/2=\varepsilon,\quad \text{for all } m>m_0,\ t\in\mathbb{R}.$$

Note that (5) implies that $\hat{C}$ is globally Lipschitz continuous with constant L_C.

Step 3: *The uniform convergence of* $\{x_m(t)\}_{m=1}^{\infty}$. The function x_m, see (8), is a solution to the sweeping process

$$-\dot{x}(t)\in N_{C_m(t)}(x(t))+f_m(t,x(t)). \tag{12}$$

Along with (12) let us consider

$$-\dot{x}(t)\in N_{\hat{C}(t)}(x(t))+\hat{f}(t,x(t)). \tag{13}$$

Both $\hat{C}$ and $\hat{f}$ are globally bounded and globally Lipschitz continuous. Moreover, by using (9) and (10) one concludes that $\hat{f}$ satisfies the monotonicity property (3). Therefore, by Theorem 2.2, the sweeping process (13) has a unique bounded absolutely continuous solution $\hat{x}$ defined on the entire $\mathbb{R}$. Let $t\in\mathbb{R}$ be such that both $\dot{x}_m(t)$ and $\dot{\hat{x}}(t)$ exist and satisfy the respective relations (12) and (13). Define

$$v_m=\dot{x}_m(t)+f_m(t,x_m(t)),\ \hat{v}=\dot{\hat{x}}(t)+\hat{f}(t,\hat{x}(t)),\ \text{so that } v_m\in-N_{C_m(t)}(x_m(t)),\ \hat{v}\in-N_{\hat{C}(t)}(\hat{x}(t)).$$

Furthermore, introducing $\Delta_m(t)=d_H\left(C_m(t),\hat{C}(t)\right)$ one has

$$x_m(t)\in C_m(t)\subseteq\hat{C}(t)+\bar{B}_{\Delta_m(t)}(0),\ \ \hat{x}(t)\in\hat{C}(t)\subseteq C_m(t)+\bar{B}_{\Delta_m(t)}(0),\quad \text{for all } t\in\mathbb{R}.$$

Therefore, x_m and $\hat{x}$ can be decomposed as

$$x_m(t)=\hat{d}(t)+s_m(t),\ \ \hat{x}(t)=d_m(t)+\hat{s}(t),\quad \text{where } \hat{d}(t)\in\hat{C}(t),\ d_m(t)\in C_m(t),\ \|s_m(t)\|\le\Delta_m(t),\ \|\hat{s}(t)\|\le\Delta_m(t).$$

Let

$$w_m(t)=\|x_m(t)-\hat{x}(t)\|^2.$$

Then,

$$\begin{aligned}
\frac{1}{2}\dot{w}_m(t) &= \langle \dot{x}_m(t) - \dot{\hat{x}}(t), x_m(t) - \hat{x}(t)\rangle \\
&= \langle v_m(t) - f_m(t, x_m(t)) - \hat{v}(t) + \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle \\
&= \langle v_m(t), x_m(t) - d_m(t) - \hat{s}(t)\rangle + \langle \hat{v}(t), \hat{x}(t) - \hat{d}(t) - s_m(t)\rangle - \langle f_m(t, x_m(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle
\end{aligned}$$

By (2) we have $\langle v_m(t), x_m(t) - d_m(t)\rangle \le 0$ and $\langle \hat{v}(t), \hat{x}(t) - \hat{d}(t)\rangle \le 0$. Therefore, for a.e. $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{2}\dot{w}_m(t) &\le -\langle v_m(t), \hat{s}(t)\rangle - \langle \hat{v}(t), s_m(t)\rangle - \langle f_m(t, x_m(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle \\
&\le \|v_m(t)\|\cdot\|\hat{s}(t)\| + \|\hat{v}(t)\|\cdot\|s_m(t)\| - \langle f_m(t, x_m(t)) - f_m(t, \hat{x}(t)) + f_m(t, \hat{x}(t)) - \hat{f}(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle.
\end{aligned}$$

Given $\varepsilon > 0$ we use the conclusions of Steps 1 and 2 to spot an $m_0 > 0$ such that

$$\|\hat{s}(t)\| \le \varepsilon,\ \|s_m(t)\| \le \varepsilon,\ \left\|f_m(t, \hat{x}(t)) - \hat{f}(t, \hat{x}(t))\right\| \le \varepsilon, \quad \text{for all } m \ge m_0,\ t \in \mathbb{R}^n.$$

Almost periodicity in t and the Lipschitz condition (4) imply that the function $f(t, x)$ is uniformly bounded when $t \in \mathbb{R}$ and $\|x\| \le M$, where M is as introduced in Remark 2.1. Therefore, by Edmond-Thibault [17, Theorem 1], there exists $L_0 > 0$ such that

$$\|v_m(t)\| \le L_0,\ \|\hat{v}(t)\| \le L_0.$$

and by using (7) we can estimate $\dot{w}_m(t)$ further as

$$\frac{1}{2}\dot{w}_m(t) \le 2\varepsilon L_0 - \langle f_m(t, x_m(t)) - f_m(t, \hat{x}(t)), x_m(t) - \hat{x}(t)\rangle + 2\varepsilon M, \quad \text{for all } m \ge m_0,\ a.e.\ t \in \mathbb{R}.$$

By referring to the definition (9) of f_m, one observes that f_m satisfies the monotonicity estimate (3), which implies

$$\frac{1}{2}\dot{w}_m(t) \le 2\varepsilon(L_0 + M) - \alpha\|x_m(t) - \hat{x}(t)\|^2 = 2\varepsilon(L_0 + M) - \alpha w_m(t), \quad \text{for all } m \ge m_0 \text{ and a.e. } t \in \mathbb{R}.$$

Gronwall-Bellman lemma (see Lemma 6.1 in the Appendix) now allows to conclude that

$$w_m(t) \le w_m(\tau)e^{-2\alpha(t-\tau)} + 4\varepsilon(L_0 + M)\int_\tau^t e^{-2\alpha(t-s)}ds = w_m(\tau)e^{-2\alpha(t-\tau)} + \varepsilon\frac{2(L_0 + M)}{\alpha}\left(1 - e^{-2\alpha(t-\tau)}\right),\ t, \tau \in \mathbb{R},\ m \ge m_0.$$

By passing to the limit as $\tau \to -\infty$ one gets

$$w_m(t) \le \varepsilon \cdot 2(L_0 + M)/\alpha, \quad t \in \mathbb{R},\ m \ge m_0.$$

Therefore, $\|x_m(t) - \hat{x}(t)\| \to 0$ as $m \to \infty$ uniformly in $t \in \mathbb{R}$, and so x_0 is almost periodic by Bochner's theorem. □

Remark 3.1. To fulfill the assumption of global boundedness of $t \mapsto C(t)$ in Theorem 3.1, it is sufficient to assume that $C(t)$ is bounded for each individual $t \in \mathbb{R}$. Indeed, any almost periodic set-valued map $C(t)$ with closed bounded values is globally bounded on $\mathbb{R}$, see e.g. Levitan-Zhikov [31, p. 2] or Vesely [43, Lemma 2.2].

4. Stability of the attractor to non-monotone perturbations

In this section we study the sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t), \varepsilon), \tag{14}$$

which satisfies the monotonicity condition (3) only when $\varepsilon = \varepsilon_0$, i.e.

$$\langle f(t, x_1, \varepsilon_0) - f(t, x_2, \varepsilon_0), x_1 - x_2\rangle \ge \alpha\|x_1 - x_2\|^2, \ \text{ for some fixed } \alpha > 0 \text{ and for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \tag{15}$$

4.1. The case where the dependence of the perturbation on the parameter ε is continuous

Here we assume that

$$\|f(t_1, x_1, \varepsilon) - f(t_2, x_2, \varepsilon)\| \le L_f\|t_1 - t_2\| + L_f\|x_1 - x_2\|, \quad \text{for all } t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R}. \tag{16}$$

Theorem 4.1. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy both the Lipschitz condition (16) and the monotonicity condition (15). Assume that, for any $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex and the globally bounded map $t \mapsto C(t)$ satisfies the Lipschitz condition (5). Finally, assume that $f(t, x, \varepsilon)$ is continuous at $\varepsilon = \varepsilon_0$ uniformly in $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Let $x_0 : \mathbb{R} \to \mathbb{R}^n$ be the unique solution to (14) with $\varepsilon = \varepsilon_0$ provided by Theorem 2.2. Then, given any $\gamma > 0$ there exists $t_1 \in \mathbb{R}$ such that for any solution x_ε to (14) defined on $[0, \infty)$, one has

$$\|x_\varepsilon(t) - x_0(t)\| < \gamma, \qquad t \ge t_1, \tag{17}$$

for all ε sufficiently close to ε_0.

We remind the reader that coresponding results for differential inclusions with bounded right-hand-sides are known e.g. from Kloeden-Kozyakin [24].

The following lemma will be used iteratively throughout the rest of the paper.

Lemma 4.1. Let x_ε be a solution to (14) defined on $[\tau, \infty)$. Let $x_0 = x_{\varepsilon_0}$. If (15) holds, then, for a.e. $t \ge \tau$,

$$\|x_\varepsilon(t) - x_0(t)\|^2 \le e^{-2\alpha(t-\tau)}\|x_\varepsilon(\tau) - x_0(\tau)\|^2 - 2\int_\tau^t e^{-2\alpha(t-s)}\langle f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0), x_\varepsilon(s) - x_0(s)\rangle ds. \tag{18}$$

Proof. For a.e. $t \ge \tau$ and $\varepsilon \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{d}{dt}\|x_\varepsilon(t) - x_0(t)\|^2 &= 2\,\langle \dot{x}_\varepsilon(t) - \dot{x}_0(t), x_\varepsilon(t) - x_0(t)\rangle \\
&\le 2\,\langle -f\,(t, x_\varepsilon(t), \varepsilon)\,, x_\varepsilon(t) - x_0(t)\rangle + 2\,\langle f(t, x_0(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle \\
&= -2\,\langle f(t, x_\varepsilon(t), \varepsilon) - f(t, x_\varepsilon(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle - 2\langle f(t, x_\varepsilon(t), \varepsilon_0) - f(t, x_0(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle \\
&\le -2\alpha\|x_\varepsilon(t) - x_0(t)\|^2 - 2\langle f(t, x_\varepsilon(t), \varepsilon) - f(t, x_\varepsilon(t), \varepsilon_0), x_\varepsilon(t) - x_0(t)\rangle
\end{aligned}$$

and the conclusion follows by applying the Gronwall-Bellman lemma (see Lemma 6.1 in the Appendix). □

Proof of Theorem 4.1. By Lemma 4.1 and (7) one has

$$\|x_\varepsilon(t) - x_0(t)\|^2 \le e^{-2\alpha t}\|x_\varepsilon(0) - x_0(0)\|^2 + \left(\frac{1}{2\alpha} - \frac{e^{-2\alpha t}}{2\alpha}\right)\max_{s\in[0,t]}\|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M, \tag{19}$$

from which the conclusion follows. □

Remark 4.1. The estimate (17) can be extended to the entire $\mathbb{R}$, if x_ε is defined on the entire $\mathbb{R}$ (e.g. if x_ε is that given by Theorem 2.1). Indeed, in this case (19) can be strengthened to

$$\|x_\varepsilon(t) - x_0(t)\|^2 \le e^{-2\alpha(t-\tau)}\|x_\varepsilon(\tau) - x_0(\tau)\|^2 + \left(\frac{1}{2\alpha} - \frac{e^{-2\alpha(t-\tau)}}{2\alpha}\right)\max_{s\in[\tau,t]}\|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M,$$

which gives

$$\|x_\varepsilon(t) - x_0(t)\|^2 \le \frac{1}{2\alpha}\sup_{s\in(-\infty,t]}\|f(s, x_\varepsilon(s), \varepsilon) - f(s, x_\varepsilon(s), \varepsilon_0)\| \cdot M,$$

by passing to the limit as $\tau \to -\infty$.

4.2. The case where the dependence of the perturbation on the parameter ε is ~~only~~ just integrally continuous

In this section we assume that the following version of Lipschitz condition (4) holds:

$$\begin{aligned} \|f(t_1,x,\varepsilon)-f(t_2,x,\varepsilon)\| &\le L_\varepsilon\|t_1-t_2\|, \quad \text{for all } t_1,t_2\in\mathbb{R},\ x\in\mathbb{R}^n,\ \varepsilon\in\mathbb{R}\backslash\{\varepsilon_0\},\\ \|f(t,x_1,\varepsilon)-f(t,x_2,\varepsilon)\| &\le L_f\|x_1-x_2\|, \quad \text{for all } t\in\mathbb{R},\ x_1,x_2\in\mathbb{R}^n,\ \varepsilon\in\mathbb{R}, \end{aligned} \tag{20}$$

where $L_\varepsilon>0$ may depend on $\varepsilon\in\mathbb{R}$ and $L_f>0$ is independent of $\varepsilon\in\mathbb{R}^n$. Following Krasnoselskii-Krein [26] and Demidovich [16, Ch. V, §3], we say that $f(t,x,\varepsilon)$ is *integrally continuous* at $\varepsilon=\varepsilon_0$, if

$$\lim_{\varepsilon\to\varepsilon_0}\int_\tau^t f(s,x,\varepsilon)ds=\int_\tau^t f(s,x,\varepsilon_0)ds, \qquad \text{for all } \tau,t\in\mathbb{R},\ x\in\mathbb{R}^n. \tag{21}$$

The central role in this section is played by a generalization of the theorem on passage to the limit in the integral due to Krasnoselskii-Krein [26] (see also Demidovich [16, Ch. V, §3]). We will formulate this theorem for the case when $f(t,x,\varepsilon)$ satisfies the Lipschitz condition (20).

Theorem 4.2. (Krasnoselskii-Krein [26]) Assume that $F:\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}\to\mathbb{R}^n$ satisfies (20) and that $t\mapsto F(t,u,\varepsilon_0)$ is continuous for every $u\in\mathbb{R}^k$. Consider a family of continuous functions $\{u_\varepsilon(t)\}_{\varepsilon\in\mathbb{R}}$ defined on an interval $[\tau,T]$ such that $u_\varepsilon(t)\to u_0(t)$ uniformly on $[\tau,T]$. If F verifies the integral continuity property (21), then

$$\lim_{\varepsilon\to\varepsilon_0}\int_\tau^t F(s,u_\varepsilon(s),\varepsilon)ds=\int_\tau^t F(s,u_0(s),\varepsilon_0)ds, \qquad \text{for all } t\in[\tau,T].$$

In this statement, we take $k=n$ when referring to (20) and (21) in the context of the function F.

If a function is integrally continuous at a point, then the function is bounded in the neighborhood of this point, which rigorous formulation is given in the following lemma.

Lemma 4.2. Let $f:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ satisfy the Lipschitz condition (20). Assume, that $f(t,x,\varepsilon)$ is integrally continuous at $\varepsilon=\varepsilon_0$. Then, given any $t_1<t_2$ from $\mathbb{R}$ and any $r>0$, there exist $\Delta>0$ and $K>0$ such that

$$\|f(t,x,\varepsilon)\|\le K, \qquad \text{for all} \quad t\in[t_1,t_2],\ \|x\|\le r,\ \varepsilon\in(\varepsilon_0-\Delta,\varepsilon_0+\Delta).$$

Proof. Assuming the contrary, there exist $t_n\to t_0$, $x_n\to x_0$, $\varepsilon_n\to\varepsilon_0$ as $n\to\infty$, such that $\|f(t_n,x_n,\varepsilon_n)\|\ge\dfrac{1}{n}$. Therefore, for a suitable sequence $\{s_n\}_{n=1}^\infty\subset[t_1,t_2]$, we have

$$\int_{t_1}^{t_2} f(s,x_n,\varepsilon_n)ds=f(s_n,x_n,\varepsilon_n)=\left[f(s_n,x_n,\varepsilon_n)-f(t_n,x_n,\varepsilon_n)\right]+f(t_n,x_n,\varepsilon_n), \qquad n\in\mathbb{N}.$$

The term in the square brackets is bounded by (20), while $\lim\limits_{n\to\infty}f(t_n,x_n,\varepsilon_n)=\infty$, i.e. $\lim\limits_{n\to\infty}\int_{t_1}^{t_2}f(s,x_n,\varepsilon_n)ds=\infty$, which contradicts the integral continuity of $f(t,x,\varepsilon)$. □

We are now in the position to prove the main result of this section.

Theorem 4.3. Let $f:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ satisfy both the Lipschitz condition (20) and the monotonicity condition (15). Assume that, for any $t\in\mathbb{R}$, the set $C(t)\subset\mathbb{R}^n$ is nonempty, closed, and convex, and that the globally bounded map $t\mapsto C(t)$ satisfies the Lipschitz condition (5). Finally, assume that $f(t,x,\varepsilon)$ is integrally continuous at $\varepsilon=\varepsilon_0$. Then, given any $\gamma>0$ there exists $t_1\ge 0$ such that for any solution x_ε to (14) defined on $\mathbb{R}$ (at least one such solution exists by Theorem 2.1) and for any $t_2\ge t_1$, one has

$$\|x_\varepsilon(t)-x_0(t)\|<\gamma, \qquad t\in[t_1,t_2],$$

for all ε sufficiently close to ε_0.

Proof. Let us fix some closed interval $[t_1, t_2]$ and assume that the statement of the theorem is wrong, i.e. assume that there exists $\gamma > 0$ such that

$$\max_{t\in[t_1,t_2]} \|x_{\varepsilon_m}(t) - x_0(t)\| \geq \gamma \tag{22}$$

for some sequence $\varepsilon_m \to \varepsilon_0$ as $m \to \infty$. By (7), we can find $\tau < 0$ such that

$$e^{-2\alpha(t-\tau)}\|x_{\varepsilon_m}(\tau) - x_0(\tau)\|^2 < \frac{\gamma^2}{2}, \quad \text{for all } m \in \mathbb{N},\ t \in [t_1, t_2]. \tag{23}$$

In what follows, we show that the integral term of the estimate (18) can be made smaller than $\gamma^2/2$ on the sequence x_{ε_m} as well. Let $r > 0$ be an arbitrary constant such that

$$\|v\| \leq r, \quad \text{for all} \quad v \in C(t),\ t \in \mathbb{R}.$$

Since, by Lemma 4.2, $f(t, x, \varepsilon)$ is uniformly bounded for $t \in [t_1, t_2]$, $\|x\| \leq r$ and ε close to ε_0 and since C satisfies the global Lipschitz condition (5), using Edmond-Thibault [17, Theorem 1] we have the existence of $L_0 > 0$ such that

$$\|\dot{x}_{\varepsilon_m}(t)\| \leq L_0, \qquad \text{for all } m \in \mathbb{N}, \text{ and a.e. } t \in [\tau, T]$$

where $T > 0$. Since the functions of $\{x_{\varepsilon_m}(t)\}_{m\in\mathbb{N}}$ are uniformly bounded according to (7), the Ascoli-Arzela theorem implies that without loss of generality the sequence $\{x_{\varepsilon_m}(t)\}_{m\in\mathbb{N}}$ can be assumed convergent uniformly on $[\tau, T]$. Introduce

$$F(t, (x_1, x_2)^T, \varepsilon) = \langle f(t, x_1, \varepsilon) - f(t, x_1, \varepsilon_0), x_2\rangle, \quad u_m(t) = \left(x_{\varepsilon_m}(t), e^{2\alpha t}\left(x_{\varepsilon_m}(t) - x_0(t)\right)\right)^T,$$

so that $F : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^n$. Since $f(t, x, \varepsilon)$ is integrally continuous at $\varepsilon = \varepsilon_0$, then

$$\lim_{\varepsilon\to\varepsilon_0} \int_\tau^t F\left(s, (x_1, x_2)^T, \varepsilon\right) ds = 0, \quad \text{for all } (x_1, x_2)^T \in \mathbb{R}^{2n},\ t \in [\tau, T].$$

Furthermore, since $\|x_\varepsilon(t)\| \leq r$, the function F satisfies the same type of Lipschitz condition (20) as f does. The Krasnoselskii-Krein theorem (Theorem 4.2), therefore, implies

$$\lim_{m\to\infty} \int_\tau^t F(s, u_m(s), \varepsilon_m) ds = 0, \quad \text{for all } t \in [\tau, T]. \tag{24}$$

The conclusions (23) and (24) contradict (22) because of (18). The proof follows by Lemma 4.1. □

4.3. A particular case: high-frequency vibrations

In this section we consider a sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + g\left(\frac{t}{\varepsilon}, x(t)\right), \tag{25}$$

where both $t \mapsto C(t)$ and $t \mapsto g(t, x)$ are almost periodic and we use Theorem 4.3 in order to estimate the location of solutions to (25) for large values of time and for small values of ε.

Since g is almost periodic in the first variable, the following property holds uniformly in $a \in \mathbb{R}$ (see Bohr [8, p. 44])

$$g_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T g(\tau, x) d\tau = \lim_{T\to\infty} \frac{1}{T} \int_a^{T+a} g(\tau, x) d\tau, \tag{26}$$

where both limits exist. Therefore,

$$\lim_{\varepsilon\to 0} \int_\tau^t g\left(\frac{s}{\varepsilon}, x\right) ds = \lim_{T\to\infty} (t - \tau)\frac{1}{T} \int_{\tau T/(t-\tau)}^{T+\tau T/(t-\tau)} g(s, x) ds = \int_\tau^t g_0(x) ds.$$

In other words, the function

$$f(t,x,\varepsilon) = \begin{cases} g\left(\dfrac{t}{\varepsilon}, x\right), & \text{if } \varepsilon \neq 0, \\ g_0(x), & \text{if } \varepsilon = 0, \end{cases}$$

is integrally continuous at $\varepsilon = 0$ in the sense of (21).

We arrive to the following corollary of Theorem 3.1, Remark 3.1, and Theorem 4.3.

Corollary 4.1. Assume that, for each $t \in \mathbb{R}$, the set $C(t) \subset \mathbb{R}^n$ is nonempty, closed, convex, and bounded. Let $t \mapsto C(t)$ be an almost periodic set-valued function that satisfies the global Lipschitz condition (5). Assume that, for each $x \in \mathbb{R}^n$, the function $t \mapsto g(t,x)$ is almost periodic and satisfies the global Lipschitz condition

$$\|g(t_1,x_1) - g(t_2,x_2)\| \leq L_g|t_1 - t_2| + L_g\|x_1 - x_2\|, \qquad \text{for all } t_1,t_2 \in \mathbb{R},\ x_1,x_2 \in \mathbb{R}^n.$$

Finally, assume that for some $\alpha > 0$ the function g_0 given by (26) satisfies the monotonicity condition

$$\langle g_0(x_1) - g_0(x_2), x_1 - x_2\rangle \geq \alpha\|x_1 - x_2\|^2, \qquad \text{for all } x_1,x_2 \in \mathbb{R}^n.$$

If x_ε is an arbitrary solution to (25) defined on $\mathbb{R}$, then uniformly on any time-interval $[t_1,t_2]$ with sufficiently large t_1, the family $\{x_\varepsilon(t)\}_{\varepsilon\in\mathbb{R}}$ converges, as $\varepsilon \to 0$, to the unique globally exponentially stable almost periodic solution $x_0(t)$ to the averaged sweeping process

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + g_0(x(t)).$$

4.4. Instructive examples

The examples of this section illustrate how the results of the paper are supposed to be used in applications.

Example 4.1. Consider a one-dimensional sweeping process

$$-\dot{x}(t) \in N_{[\sin(t),\sin(t)+1]}(x(t)) + \varepsilon x^2(t) + \left(\sin\left(\sqrt{2}\cdot t\right) + 2\right)x(t). \tag{27}$$

The sweeping process (27) satisfies the monotonicity property (3) when $\varepsilon = 0$. Theorems 3.1 and 4.1 imply that for any $\gamma > 0$ there exists $t_1 > 0$ such that any solution x_ε of (27), with $x_\varepsilon(0) \in [0,1]$, satisfies $\|x_\varepsilon(t) - x_0(t)\| \leq \gamma$ for all $t \geq t_1$ and for all $|\varepsilon|$ sufficiently small, where x_0 is the unique globally exponentially stable almost periodic solution to

$$-\dot{x}(t) \in N_{[\sin(t),\sin(t)+1]}(x(t)) + \left(\sin\left(\sqrt{2}\cdot t\right) + 2\right)x(t).$$

Example 4.2. Let now the monotonicity of a sweeping process get broken by a high-frequency ingredient as follows

$$-\dot{x}(t) \in N_{[\sin(t),\sin(t)+1]}(x(t)) + \sin\left(\frac{t}{\varepsilon}\right)x^2(t) + \left(\sin\left(\sqrt{2}\cdot t\right) + 2\right)x(t). \tag{28}$$

The non-monotonic term $\sin\left(\dfrac{t}{\varepsilon}\right)$ no longer approaches 0 as it took place in Example 4.1, and Theorem 4.1 is inapplicable. However, $\sin\left(\dfrac{t}{\varepsilon}\right)$ approaches 0 as $\varepsilon \to 0$ integrally (i.e. in the sense to (21)) on any bounded time interval $[t_1,t_2]$. Therefore, Corollary 4.1 ensures that given any $\gamma > 0$ there exists $t_1 > 0$ such that for any $t_2 > t_1$ and for any solution x_ε of (28) with $x_\varepsilon(0) \in [0,1]$ and defined on $\mathbb{R}$ one has $\|x_\varepsilon(t) - x_0(t)\| \leq \gamma$ on $[t_1,t_2]$ for all $|\varepsilon|$ sufficiently small, where x_0 is the unique globally exponentially stable almost periodic solution to the averaged sweeping process

$$-\dot{x}(t) \in N_{[\sin(t),\sin(t)+1]}(x(t)) + \left(\sin\left(\sqrt{2}\cdot t\right) + 2\right)x(t).$$

To summarize, Examples 4.1 and 4.2 establish useful qualitative properties of non-monotone sweeping processes without any need of actual computing of solutions. Numerical computation of solutions to (27) and (28) (e.g. using the catching-up algorithm of Edmond-Thibault [17]) is thus outside the scope of this paper.

4.5. *Applications in elastoplasticity*

The perturbation term of the sweeping processes that models networks of elastoplastic springs (like those in Bastein et al [5]) does not generally satisfy the monotonicity property (3) because it always contains oscillatory terms coming from springs. In applications of Theorems 4.1 and 4.3 to elastoplasticity, one can expect monotonicity (caused by viscous friction) only when the eigenfrequencies of all springs vanish. The magnitudes of these eigenfrequencies is, therefore, a natural choice for the small parameter ε. The eigenfrequencies of the springs can be viewed small compared to other parameters, if the masses of nodes of the network (i.e. inertial forces) are large. However, setting the so-selected small parameter ε to 0 will ensure monotonicity and global asymptotic stability for velocity-like variables only, not for the position-like variables. This can be intuitively seen from a simple oscillator $\ddot{x} + c\dot{x} + \varepsilon h(t, x) = 0$, whose solutions approach those of $\ddot{x} + c\dot{x} = 0$ as $\varepsilon \to 0$ (assuming that $h(t, x)$ stays bounded). The solutions of the reduced oscillator asymptotically approach the line $\mathbb{R} \times \{0\}$ because of the monotonicity provided by the friction term. As a consequence, the solutions of the original oscillator stay close to $\mathbb{R} \times \{0\}$ for small values of $\varepsilon > 0$. Coming back to the sweeping processes of elastoplasticity, we expect that for large inertial forces the methods of Theorems 4.1 and 4.3 will predict convergence to the manifold of equilibria that correspond to infinite inertial forces. Pursuing this plan is subject of a different paper, that we are working on.

5. Conclusion

In this paper we established the existence and global exponential stability of bounded and almost periodic solutions of Moreau's sweeping process (1). We proved that non-monotone sweeping processes with ~~bounded~~ Lipschitz right-hand-sides admit at least one solution defined on the entire $\mathbb{R}$. When the moving constraint $C(t)$ is globally bounded and the sweeping process satisfies the monotonicity property (3), we proved the existence of exactly one bounded solution defined on $\mathbb{R}$ which is almost periodic when the right-hand-sides of (1) are almost periodic.

When the right-hand-sides of (1) are non-monotone, but close to monotone, we discovered that all the solutions to (1) are close to the unique bounded (or almost periodic) solution of the respective monotone process for large values of time. In particular, we initiated the development of the averaging theory for Moreau sweeping process (1) with high-frequency almost periodic excitation $g\left(\frac{t}{\varepsilon}, x\right)$, where only monotonicity of the average $g_0(x) = \lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T g(s, x)ds$ is required. This result can be used for the design of vibrational control strategies for Moreau sweeping processes (see e.g. Bullo [14] for the respective theory in the case of differential equation).

The approach of this paper finds applications in the problem of global stabilization of parallel networks of elastoplastic springs where the period of the mechanical loading (e.g. stretching/compressing) of springs doesn't coincide with the period of the force that excites the masses at nodes, as we discussed in the Introduction and in Section 4.5.

Further potential applications of the results of this paper are in studying the dynamics of a circuit involving devices like diodes, thyristors and diacs (see Addi et al [2]) when ampere-volt characteristics (for the set function) and voltage supply (for the perturbation) receive time-periodic excitations of different periods. Such a study will require extending our theory to sweeping processes with state-dependent convex constraints.

6. Appendix

The following version of Gronwall-Bellman lemma and its proof are taken from Trubnikov-Perov [41, Lemma 1.1.1.5].

Lemma 6.1. (Gronwall-Bellman) Let an absolutely continuous function $a : [0, T] \to \mathbb{R}$ satisfies

$$\dot{a} \le \lambda a(t) + b(t), \qquad \text{for a.e. } t \in [0, T], \tag{29}$$

where $b : [0, T] \to \mathbb{R}$ is an integrable function and $\lambda \in \mathbb{R}$ is a constant. Then

$$a(t) \le e^{\lambda t} a(0) + \int\limits_0^t e^{\lambda(t-s)} b(s)ds, \qquad \text{for all } t \in [0, T].$$

Proof. By introducing

$$\psi(t) = e^{\lambda t}a(0) + \int_0^t e^{\lambda(t-s)}b(s)ds,$$

one has

$$\psi(t)e^{-\lambda t} - \int_0^t e^{-\lambda s}b(s)ds = a(0)$$

and so

$$\frac{d}{dt}\left[\psi(t)e^{-\lambda t} - \int_0^t e^{-\lambda s}b(s)ds\right] = 0, \qquad \text{for a.e. } t \in [0, T],$$

which implies

$$\dot\psi(t) - \lambda\psi(t) = b(t) \geq \dot a(t) - \lambda a(t).$$

If now

$$u(t) = a(t) - \psi(t),$$

then $\dot u(t) \leq \lambda u(t)$ and so $\frac{d}{dt}\left[u(t)e^{-\lambda t}\right] = e^{-\lambda t}(\dot u - \lambda u) \leq 0$, i.e. $u(t)e^{-\lambda t} \leq u(0)$. Therefore, $u(t) \leq 0$ and

$$a(t) \leq \psi(t) = e^{\lambda t}a(0) + \int_0^t e^{\lambda(t-s)}b(s)ds.$$

□

The following proof is known (see e.g. [29, Theorem 8.7] and [30, Lemma 2]), but we add a proof in terms of sweeping process (1) for completeness.

Proof of Theorem 2.2. **Step 1:** *Incremental stability.* Let x_1 and x_2 be solutions to (1) with the initial conditions $x_1(t_0), x_2(t_0) \in C(t_0)$. Assuming that $t \geq t_0$ is such that both $\dot x_1(t)$ and $\dot x_2(t)$ exist and verify (1), one has

$$\langle -\dot x_1(t) - f(t, x_1(t)), x_1(t) - x_2(t)\rangle \geq 0.$$

Therefore $\langle -f(t, x_1(t)), x_1(t) - x_2(t)\rangle \geq \langle \dot x_1(t), x_1(t) - x_2(t)\rangle$. By analogy, $-\dot x_2(t) - f(t, x_2(t)) \in N_{C(t)}(x_2(t))$ implies $\langle -\dot x_2(t), x_1(t) - x_2(t)\rangle \leq \langle f(t, x_2(t)), x_1(t) - x_2(t)\rangle$ Therefore,

$$\begin{aligned}
\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 &= 2\langle \dot x_1(t) - \dot x_2(t), x_1(t) - x_2(t)\rangle \\
&= 2\langle \dot x_1(t), x_1(t) - x_2(t)\rangle - 2\langle \dot x_2(t), x_1(t) - x_2(t)\rangle \\
&\leq -2\langle f(t, x_1(t)), x_1(t) - x_2(t)\rangle + 2\langle f(t, x_2(t)), x_1(t) - x_2(t)\rangle \\
&= -2\langle f(t, x_1(t)) - f(t, x_2(t)), x_1(t) - x_2(t)\rangle \\
&\leq -2\alpha\|x_1(t) - x_2(t)\|^2
\end{aligned}$$

and by Gronwall-Bellman lemma (see Lemma 6.1 in the Appendix), $\|x_1(t) - x_2(t)\|^2 \leq e^{-2\alpha(t-t_0)}\|x_1(t_0) - x_2(t_0)\|^2$, for a.e. $t \geq t_0$. Since both x_1 and x_2 are continuous functions,

$$\|x_1(t) - x_2(t)\|^2 \leq e^{-2\alpha(t-t_0)}\|x_1(t_0) - x_2(t_0)\|^2, \quad \text{for all } t \geq t_0. \tag{30}$$

Step 2. *Uniqueness of the bounded solution* x_0. Let v be another bounded solution to (1) defined on the entire $\mathbb{R}$. Then, given any $\tau \in \mathbb{R}$, the inequality (30) yields

$$\|x_0(t) - v(t)\|^2 \leq e^{-2\alpha(t-\tau)}\|x_0(\tau) - v(\tau)\|^2, \qquad \text{for all } t \geq \tau.$$

Thus $\|x_0(t) - v(t)\| \leq 2Me^{-\alpha(t-\tau)}$, for all $t \geq \tau$, where M is as defined in (7). Now we fix $t \in \mathbb{R}$ and pass to the limit as $\tau \to -\infty$, obtaining $\|u(t) - v(t)\|^2 \leq 0$. Thus $u(t) = v(t)$ for all $t \in \mathbb{R}$.

Step 3. *Global exponential stability of* x_0 follows from (30). Indeed, (30) implies that $\|x_0(t) - v(t)\| \leq e^{-\alpha(t-\tau)}\|x_0(\tau) - v(\tau)\|$, for any solution v to (1) and for any $t \geq \tau$. □

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