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Dwell time for switched systems with multiple equilibria on a finite time-interval

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Abstract. We describe the behavior of solutions of switched systems with multiple globally exponentially stable equilibria. We introduce an ideal attractor and show that the solutions of the switched system stay in any given ε-inflation of the ideal attractor if the frequency of switchings is slower than a suitable dwell time T. In addition, we give conditions to ensure that the ε-inflation is a global attractor. Finally, we investigate the effect of the increase of the number of switchings on the total time that the solutions need to go from one region to another.

Keywords. Switched system, dwell-time, global exponential stability, ideal attractor.

AMS (MOS) subject classification: 93C30; 34D23

1 Introduction

Dwell time is the lower bound on the time between successive switchings of the switched system

$$\dot{x} = f_{u(t)}(x), \qquad u(t) \text{ is a piecewise constant function, } x \in \mathbb{R}^n, \qquad (1)$$

which ensures a required dynamic behavior under the assumption that each of the subsystems

$$\dot{x} = f_u(x), \qquad u \in \mathbb{R},\ x \in \mathbb{R}^n, \qquad (2)$$

possess a globally stable equilibrium x_u. When all the equilibria $\{x_{u(t)}\}_{t \geq t_0}$ coincide, the dwell time $T > 0$ which gives global exponential stability of the common equilibrium x_0 is computed e.g. in Liberzon [5, §3.2.1]. Specifically, the result of [5, §3.2.1] gives a formula for T which makes x_0 globally exponentially stable for any piecewise constant function $u(t)$ whose discontinuities $t_1, t_2, \ldots$ verify

$$|t_i - t_{i-1}| \geq T. \qquad (3)$$

The case where the equilibria are distinct is covered in Alpcan-Basar [1], who offered a dwell time T that ensures global exponential stability of a suitable set $A \supset \{x_{u(t)}\}_{t \geq t_0}$ for any $u(t)$ whose discontinuities verify (3). The problem of stability of switched systems with multiple equilibria appears e.g. in differential games, load balancing, agreement and robotic navigation (see [1, 6] and references therein).

A deeper analysis of the dynamics of switched systems with multiple equilibria was recently carried out in Xu et al [7], who gave a sharp formula for the attractor A in the case of quasi-linear switched systems (1). Assuming that $u(t)$ is periodic and denoting by $t \mapsto X_u(t,x)$ the solution of (2) with the initial condition $X_u(0,x) = x$, the paper [7] investigated the asymptotic attractivity of

$$A = \bigcup_{t \geq t_0,\ \tau \geq t_0} \{X_{u(\tau)}(t, x_{u(\tau)})\}.$$

The motivation for our paper comes from the problem of planning the motion of a 3-D walking robot, where "turn left", "walk straight" and "turn right" correspond to $u(t) = -1$, $u(t) = 0$ and $u(t) = 1$ respectively, see Gregg et al [3]. It is not the asymptotic attractivity of A which is of importance for the robot turning maneuver but rather an appropriate attractivity of A during the time of the maneuver. The goal of this paper is to provide a dwell time which can ensure the required attractivity.

The paper is organized as follows. In the next section of the paper, we prove our main result (Theorem 2.1). Given $\varepsilon > 0$, Theorem 2.1 provides a dwell time $T > 0$ such that the solutions of (1) with the initial conditions in the ε-neighborhood $B_\varepsilon(A)$ of A never leave $B_\varepsilon(A)$ in the future. Theorem 2.1 can be viewed as a version of [7, Theorem 1] for fully nonlinear systems. In section 3, we compute (Theorem 3.1) a dwell time to ensure that the attractor $B_\varepsilon(A)$ is reached asymptotically from any initial condition. The proof of Theorem 3.1 follows the ideas of Alpcan-Basar [1]. However, we offer weaker conditions where the Lyapunov functions of subsystems (2) are not supposed to respect any uniform estimates. A particular case study where the Lyapunov functions of subsystems (2) are shifts of one another is addressed in section 4. In this section, we consider a switched system which switches between two subsystems $u = u_1$ and $u = u_2$ and analyze the solutions of the switched system with the initial conditions in $B_\varepsilon(A)$. Let x_1 and x_2 be the equilibria of subsystems $u = u_1$ and $u = u_2$ respectively. The result of section 4 (Theorem 4.1) clarifies whether or not the solutions from the neighborhood of x_1 reach the neighborhood of x_2 faster if the switching signal is amended in such a way that an additional switching occurs between $u = u_1$ and $u = u_2$. In other words, section 4 investigates whether or not adding more discrete events is alone capable of making the dynamics inside $B_\varepsilon(A)$ faster. Examples 2.1 and 4.1 illustrate the conclusions of Theorems 2.1 and 4.1. Lastly in Appendix A, we derive some explicit formulas that can be used to apply Theorem 3.1.

2 The local trapping region

Let x_u be the unique equilibrium of (2). We assume that for any u, system (2) admits a global Lyapunov function V_u such that

$$\alpha_u(\|x - x_u\|) \leq V_u(x) \leq \beta_u(\|x - x_u\|), \qquad x \in \mathbb{R}^n, \tag{4}$$

$$(V_u)'(x) f_u(x) \leq -k_u V_u(x), \qquad x \in \mathbb{R}^n, \tag{5}$$

where α, β are strictly monotonically increasing functions with $\alpha_u(0) = \beta_u(0)$, and $k_u > 0$. Introduce the following trapping regions

$$\begin{aligned} N_u^\varepsilon &= \{x : V_u(x) \leq \varepsilon\}, \\ L_{u_1,u_2}^\varepsilon(t) &= \bigcup_{x \in N_{u_1}^\varepsilon} \{X_{u_2}(t,x)\} \end{aligned} \tag{6}$$

and define the dwell time that the solutions need to go from $N_{u_1}^\varepsilon$ to $N_{u_2}^\varepsilon$ as

$$T_{u_1,u_2}^\varepsilon = -\frac{1}{k_{u_2}} \ln \frac{\varepsilon}{\beta_{u_2}\left(\|x_{u_2} - x_{u_1}\| + \alpha_{u_1}^{-1}(\varepsilon)\right)}. \tag{7}$$

Theorem 2.1. Assume that

(A1) $f_u \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for any $u \in \mathbb{R}$,

(A2) for any $u \in \mathbb{R}$, system (2) admits an equilibrium x_u whose Lyapunov function V_u satisfies (4)-(5), where $\alpha, \beta \in C^0(\mathbb{R}, \mathbb{R})$ are strictly increasing functions, $\alpha(0) = \beta(0) = 0$, $k_u > 0$,

(A3) $u : [t_0, \infty) \to \mathbb{R}$ is a piecewise constant function.

Let $\{t_1, t_2, \ldots\} = \{t_i\}_{i \in I}$ be a finite or infinite increasing sequence of points of discontinuity of u and

$$u_i = u(t_i + 0).$$

If

$$t_i - t_{i-1} \geq T_{u_{i-1},u_i}^\varepsilon,$$

then for any solution x of (1) with

$$x(t_{i-1}) \in N_{u_{i-1}}^\varepsilon, \quad i \in I,$$

one has

$$x(t) \in L_{u_{i-1},u_i}^\varepsilon(t), \qquad t_{i-1} \leq t \leq t_i, \quad i \in I, \tag{8}$$

$$x(t_i) \in N_{u_i}^\varepsilon, \qquad i \in I. \tag{9}$$

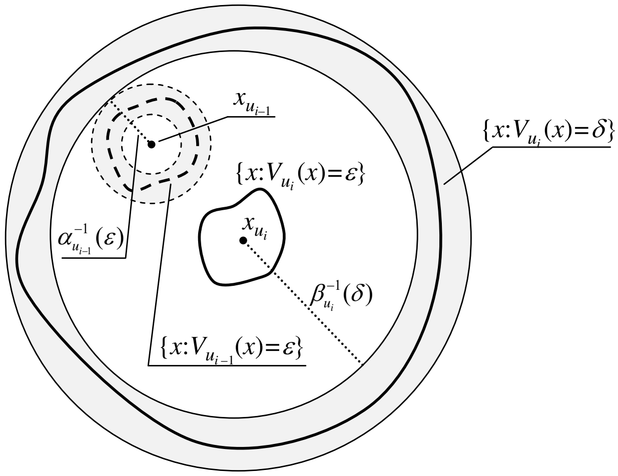

Figure 1: Illustration of the proof of Theorem 2.1. The two gray rings are the estimates for the sets $\{x : V_{u_{i-1}}(x) = \varepsilon\}$ and $\{x : V_{u_i}(x) = \delta\}$ given by condition (4).

Proof. We only have to prove (9) because the validity of (8) follows directly from the definition of $L^{\varepsilon}_{u_{i-1},u_i}$. Let $x(t_{i-1}) \in N^{\varepsilon}_{u_{i-1}}$. Our goal is to show that $x(t_i) \in N^{\varepsilon}_{u_i}$. Given $\varepsilon > 0$, define $\delta > 0$ as

$$\delta = \beta_{u_i}\left(\|x_{u_i} - x_{u_{i-1}}\| + \alpha^{-1}_{u_{i-1}}(\varepsilon)\right).$$

By construction (see Fig. 1), $N^{\delta}_{u_i} \supset N^{\varepsilon}_{u_{i-1}}$, and so $x(t_{i-1}) \in N^{\delta}_{u_i}$. Introduce

$$v(t) = V_{u_i}(x(t)).$$

By (5) we have

$$\begin{aligned} \dot v(t) &\le -k_i v(t), \quad t_{i-1} \le t \le t_i, \\ v(t_{i-1}) &\le \delta. \end{aligned}$$

By the comparison lemma (see e.g. [2, Lemma 16.4]), it holds that

$$v(t) \le p(t),$$

where $p(t)$ is the solution of

$$\begin{aligned} \dot p(t) &= -k_i p(t), \quad t_{i-1} \le t \le t_i, \\ p(t_{i-1}) &= \delta. \end{aligned}$$

At the same time,

$$p(t_i) = e^{-k_{u_i}(t_i - t_{i-1})}\delta \le e^{-k_{u_i}T^{\varepsilon}_{u_{i-1},u_i}}\delta = \frac{\varepsilon}{\beta_{u_i}\left(\|x_{u_i} - x_{u_{i-1}}\| + \alpha^{-1}_{u_{i-1}}(\varepsilon)\right)}\delta = \varepsilon.$$

Therefore, $V_{u_i}(x(t_i)) \le \varepsilon$, which completes the proof. □

Theorem 2.1 suggests the following definition of the ε-inflation A_ε of the ideal attractor A of (1). Given a piecewise constant function $u : [t_0, \infty) \to \mathbb{R}$ and the respective increasing sequence $(t_1, t_2, \ldots) = \{t_i\}_{i \in I}$, let $u_i = u(t_i + 0)$ and

$$A_\varepsilon(t) = \begin{cases} L^\varepsilon_{u_{i-1},u_i}(t), & t_{i-1} \le t < t_i, \quad i \in I, \\ L^\varepsilon_{u_{\max(I)-1},u_{\max(I)}}(t), & t \ge t_{\max(I)}, \qquad \text{if } I \text{ is finite.} \end{cases}$$

Corollary 2.1. Let the assumptions (A1)-(A3) of Theorem 2.1 hold. If

$$t_i - t_{i-1} \ge \sup_{i \in I} T^\varepsilon_{u_{i-1},u_i} =: T^\varepsilon_{loc}, \quad i \in I,$$

then, for any solution x of (1) with the initial condition

$$x(t_0) \in A_\varepsilon(t_0),$$

one has

$$x(t) \in A_\varepsilon(t), \quad t \ge t_0.$$

Note, $\sup_{i \in I} T^\varepsilon_{u_{i-1},u_i}$ is finite when $t \mapsto u(t)$ takes a finite number of values on $[t_0, \infty)$.

Example 2.1. To illustrate Theorem 2.1, we consider the following switched system (slightly modified from Example 2 in [1])

$$\dot{x} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} u \\ 1 \end{pmatrix}, \tag{10}$$

whose unique equilibrium is given by

$$x_u = \frac{1}{2} \begin{pmatrix} u - 1 \\ u + 1 \end{pmatrix}.$$

Introduce the three discrete states u_1, u_2, and u_3 as

$$u_1 = 1, \qquad u_2 = 0, \qquad u_3 = -1,$$

and consider

$$\varepsilon = 0.05.$$

If the Lyapunov function $V_u(x)$ is selected as

$$V_u(x) = \|x - x_u\|^2,$$

then formulas (6) and (7) yield

$$N^\varepsilon_u = \left\{x : \|x - x_u\| \le \sqrt{\varepsilon}\right\}, \quad T^\varepsilon_{loc} \approx 1.426.$$

Therefore, for the control input

$$u(t) = \begin{cases} u_1, & t \in [0, T), \\ u_2, & t \in [T, 2T), \\ u_3, & t \geq 2T, \end{cases} \qquad T = 1.43, \tag{11}$$

and for any solution x of (10), Theorem 2.1 ensures the following:

$$\text{if } x(0) \in N^{\varepsilon}_{u_1}, \text{ then } x(T) \in N^{\varepsilon}_{u_2} \text{ and } x(2T) \in N^{\varepsilon}_{u_3}.$$

Figure 2(left) documents the sharpness of the dwell time T. Indeed, the figure shows that if the initial condition $x(0)$ deviates to the outside of $N^{\varepsilon}_{u_1}$ just a little bit, then the dwell time T is no longer sufficient to get $x(T) \in N^{\varepsilon}_{u_2}$ (though we still have $x(2T) \in N^{\varepsilon}_{u_3}$ for this solution).

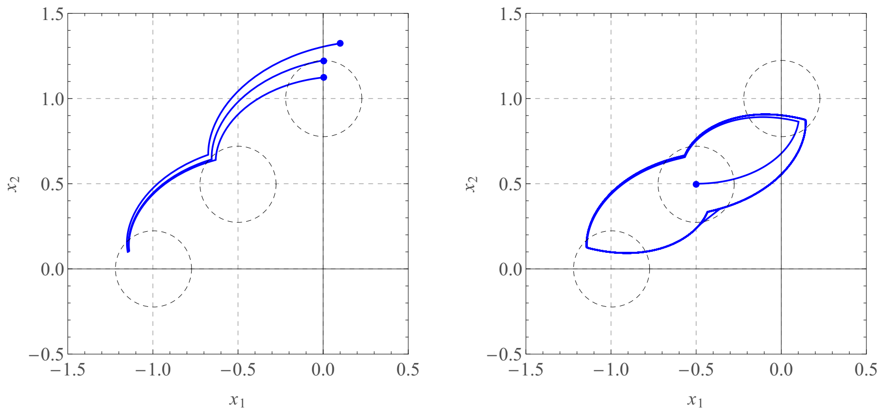

Figure 2: Left: Solutions of switched system (1) with initial conditions (blue dots) inside $N^{0.05}_{u_1}$, on the boundary of $N^{0.05}_{u_1}$, and outside $N^{0.05}_{u_1}$, and for the control input $u(t)$ given by (11). Right: The solution of switched system (1) with the initial condition $(-0.5, 0.5)^T$ for the $4T$-periodic control input (12).

To demonstrate that trapping regions $N^{\varepsilon}_{u_1}$, $N^{\varepsilon}_{u_2}$, $N^{\varepsilon}_{u_3}$ (and thus the ε-inflated attractor A_{ε}, see Corollary 2.1) provide a rather sharp estimate for the location of the attractor of (10), we extend the input $u(t)$ to $[0, 4T]$ as

$$u(t) = \begin{cases} u_1, & t \in [0, T), \\ u_2, & t \in [T, 2T), \\ u_3, & t \in [2T, 3T), \\ u_2, & t \in [3T, 4T), \end{cases} \tag{12}$$

and then continue it to the entire $[0, \infty)$ by $4T$-periodicity. The respective solution x of (10) with the initial condition $x(0) = (-0.5, 0.5)^T$ is plotted in Fig. 2 (right). The drawing shows that the switching points of the solution x are very close to the boundaries of the trapping regions $N^{\varepsilon}_{u_1}$, $N^{\varepsilon}_{u_2}$, $N^{\varepsilon}_{u_3}$, i.e. there is only a little window to reduce the size of those regions.

3 Global attractivity of the local trapping region

Theorem 3.1. Let the assumptions (A1)-(A3) of Theorem 2.1 hold and I be infinite. Fix $\varepsilon > 0$ and suppose that there exists constants $\mu_i(\varepsilon)$ such that

$$\frac{V_{u_{i+1}}(x)}{V_{u_i}(x)} \leq \mu_i(\varepsilon), \quad x \in \mathbb{R}^n \setminus N^{\varepsilon}_{u_i}, \quad i \in \mathbb{N} \cup \{0\}.$$

Finally, assume that

$$\mu_0(\varepsilon) \cdot \ldots \cdot \mu_i(\varepsilon) e^{-\int_{t_0}^{t_{i+1}} k_{u(s)} ds} \to 0, \quad \text{as } i \to \infty.$$

Then, $x(\hat{T}) \in N^{\varepsilon}_{u_i}$ for some $\hat{T} > 0$ and some $i \in \mathbb{N}$.

Proof. Let $W(t) = e^{k_{u(t)} t} V_{u(t)}(x(t))$, where $t \in [t_0, \infty)$. Then, for $t \in [t_i, t_{i+1})$,

$$W'(t) = k_{u_i} W(t) + e^{k_{u_i} t} \frac{d}{dt} V_{u_i}(x(t)) \leq k_{u_i} W(t) - k_{u_i} e^{k_{u_i} t} V_{u_i}(x(t)) = 0,$$

which means that W is decreasing on $[t_i, t_{i+1})$. In particular,

$$W(t_i^+) \geq W(t_{i+1}^-).$$

On the other hand,

$$\frac{W(t_{i+1}^+)}{W(t_{i+1}^-)} = \frac{e^{k_{u_{i+1}} t_{i+1}}}{e^{k_{u_i} t_{i+1}}} \cdot \frac{V_{u_{i+1}}(x(t_{i+1}))}{V_{u_i}(x(t_{i+1}))} \leq \frac{e^{k_{u_{i+1}} t_{i+1}}}{e^{k_{u_i} t_{i+1}}} \cdot \mu_i(\varepsilon) =: \tilde{\mu}_i.$$

Therefore,

$$\tilde{\mu}_i W(t_i^+) \geq \tilde{\mu}_i W(t_{i+1}^-) \geq W(t_{i+1}^+) \tag{13}$$

Replacing i by $i-1$ and combining with (13), one gets $\tilde{\mu}_{i-1}\tilde{\mu}_i W(t_{i-1}^+) \geq W(t_{i+1}^+)$. Continuing this process for $i-2, i-3$, etc. we obtain

$$\tilde{\mu}_0 \cdot \ldots \cdot \tilde{\mu}_i W(t_0^+) \geq W(t_{i+1}^+).$$

Applying $e^{-k_{u_{i+1}} t_{i+1}}$ yields

$$\tilde{\mu}_0 \cdot \ldots \cdot \tilde{\mu}_i e^{-k_{u_{i+1}} t_{i+1}} W(t_0^+) \geq V_{u_{i+1}}(x(t_{i+1})),$$

or, equivalently,

$$V_{u_0}(x(t_0)) e^{-\sum_{j=0}^{i} k_{u_j}(t_{j+1} - t_j)} \mu_0(\varepsilon) \cdot \ldots \cdot \mu_i(\varepsilon) \geq V_{u_{i+1}}(x(t_{i+1})).$$

The left-hand-side approaches 0 as $i \to \infty$ by the assumption of the theorem. Therefore, $V_{u_{i+1}}(x(t_{i+1})) \to 0$ as $i \to \infty$. The proof is complete. □
(For linear switched systems, there is an explicit formula for $\mu(\varepsilon)$. See Theorem A.1 in appendix A.)

Corollary 3.1. (Alpcan-Basar [1]) Let the assumptions (A1)-(A3) of Theorem 2.1 hold and I be infinite. Fix $\varepsilon > 0$ and suppose that there exists a constant $\mu(\varepsilon) > 1$ such that

$$\frac{V_{u(t)}(x)}{V_{u(\tau)}(x)} \le \mu(\varepsilon), \quad x \in \mathbb{R}^n \backslash N^\varepsilon_{u(\tau)}, \quad \tau, t \ge t_0.$$

Finally, assume that $k = \inf_{t \ge t_0} k_{u(t)} > 0$ and consider T^ε_{glob} satisfying

$$T^\varepsilon_{glob} > \frac{\ln(\mu(\varepsilon))}{k}.$$

If

$$t_i - t_{i-1} \ge T^\varepsilon_{glob}, \quad i \in \mathbb{N},$$

then $x(\hat{T}) \in N^\varepsilon_{u_i}$ for some $\hat{T} > 0$ and some $i \in \mathbb{N}$.

Proof. Let $\gamma > 0$ be such that $T^\varepsilon_{glob} = \dfrac{\ln(\mu(\varepsilon) + \gamma)}{k}$. Let $\mu_0(\varepsilon), \ldots \mu_i(\varepsilon)$ be as given by Theorem 3.1. Then

$$e^{-\sum_{j=0}^{i+1} k_{u_j}(t_{j+1} - t_j)} \mu_0(\varepsilon) \cdot \ldots \cdot \mu_{i+1}(\varepsilon) \le e^{-k(i+1)T^\varepsilon_{glob}} \mu(\varepsilon)^{i+1} =$$
$$= (\mu(\varepsilon) + \gamma)^{-(i+1)} \mu(\varepsilon)^{i+1} = \left(\frac{\mu(\varepsilon)}{\mu(\varepsilon) + \gamma} \right)^{i+1} \to 0 \text{ as } i \to \infty. \quad \square$$

One can similarly show the following result.

Corollary 3.2. Let the assumptions (A1)-(A3) of Theorem 2.1 hold and I be infinite. Fix $\varepsilon > 0$ and suppose that there are constants $\mu_i(\varepsilon)$ such that

$$\frac{V_{u_{i+1}}(x)}{V_{u_i}(x)} \le \mu_i(\varepsilon), \quad x \in \mathbb{R}^n \setminus N^\varepsilon_{u_i}, \quad i \in \mathbb{N} \cup \{0\}.$$

If

$$t_i - t_{i-1} > \frac{\ln(\mu_{i-1}(\varepsilon))}{k_{u_{i-1}}}, \quad i \in \mathbb{N},$$

then $x(\hat{T}) \in N^\varepsilon_{u_i}$ for some $\hat{T} > 0$ and some $i \in \mathbb{N}$.

Corollary 3.3. Let the conditions of Corollary 3.1 hold. Let T^ε_{loc} and T^ε_{glob} be those given by Corollaries 2.1 and 3.1. If

$$t_i - t_{i-1} \ge \max\left\{T^\varepsilon_{loc}, T^\varepsilon_{glob}\right\}, \quad i \in \mathbb{N},$$

then, for any solution x of (1), there exists $\hat{T} > t_0$ such that

$$x(t) \in A_\varepsilon(t), \quad t \ge \hat{T}.$$

Example 3.1. To see when Corollary 3.1 cannot be applied, consider the switched system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ u^2 \end{pmatrix}$$

whose unique equilibrium is

$$x_u = \begin{pmatrix} 0 \\ u^2 \end{pmatrix}.$$

Let $\varepsilon = 0.01$. Take any piecewise constant switching signal $u : [t_0, \infty) \to \mathbb{R}$ such that

$$u(t_i) = i, \quad i \in \mathbb{N} \cup \{0\}.$$

Also take the Lyapunov functions

$$V_u(x) = ||x - x_u||^2.$$

Then apply Theorem A.1 to see

$$\mu_i(\varepsilon) = \left(1 + \frac{2i+1}{\sqrt{0.01}}\right)^2 \to \infty \text{ as } i \to \infty.$$

So in this example, we have pairwise $\mu_i(\varepsilon)$ bounds but not a uniform $\mu(\varepsilon)$ bound. Thus we cannot apply Corollary 3.1. Instead, we can use Corollary 3.2 to ensure $x(\hat{T}) \in N^{\varepsilon}_{u_i}$ for some $\hat{T} > 0$ and $i \in \mathbb{N}$.

4 Dependence of the dwell time on the number of discrete states

Suppose that $u(t)$ switches from u_0 to u_1 at $t = t_0$. According to Theorem 2.1, it takes at most time $T^{\varepsilon}_{u_0,u_1}$ (see formula (7)) for a trajectory x of (1) to go from $N^{\varepsilon}_{u_0}$ to $N^{\varepsilon}_{u_1}$. The next theorem shows that adding more discrete states between u_0 and u_1 makes the travel time from $N^{\varepsilon}_{u_0}$ to $N^{\varepsilon}_{u_1}$ longer.

Theorem 4.1. Let the assumptions (A1)-(A2) of Theorem 2.1 hold and suppose $\alpha_u =: \alpha$, $\beta_u =: \beta$, $k_u =: k$ don't depend on u. Fix $d > 0$ and $r > 0$. Then there exists $\varepsilon_0 > 0$ such that

$$T^{\varepsilon}_{u_0,u_1} < T^{\varepsilon}_{u_0,v} + T^{\varepsilon}_{v,u_1},$$

for any

$$\varepsilon \in (0, \varepsilon_0),\ \|x_{u_0}\| \le d,\ \|x_{u_1}\| \le d,\ \|x_{u_0} - x_v\| \ge r,\ \|x_v - x_{u_1}\| \ge r. \quad (14)$$

Proof. By formula (7) one has

$$\begin{aligned} T^{\varepsilon}_{u_0,u_1} - T^{\varepsilon}_{u_0,v} - T^{\varepsilon}_{v,u_1} &= -\frac{1}{k}\ln\frac{\varepsilon}{\beta\left(\|x_{u_0}-x_{u_1}\|+\alpha^{-1}(\varepsilon)\right)} + \\ &+\frac{1}{k}\ln\frac{\varepsilon}{\beta\left(\|x_{u_0}-x_{v}\|+\alpha^{-1}(\varepsilon)\right)} + \\ &+\frac{1}{k}\ln\frac{\varepsilon}{\beta\left(\|x_{v}-x_{u_1}\|+\alpha^{-1}(\varepsilon)\right)} = \\ &= -\ln\frac{K}{\varepsilon^{1/k}}, \end{aligned} \tag{15}$$

where

$$K = \left(\frac{\beta\left(\|x_v-x_{u_1}\|+\alpha^{-1}(\varepsilon)\right)}{\beta\left(\|x_{u_0}-x_{u_1}\|+\alpha^{-1}(\varepsilon)\right)}\right)^{1/k}\left(\beta\left(\|x_{u_0}-x_v\|+\alpha^{-1}(\varepsilon)\right)\right)^{1/k}.$$

Observe that there exists $K_0 > 0$ such that $K \geq K_0$ for any functions u_0, u_1, v that verify (14) as long as $d > 0$ and $r > 0$ stay fixed. Therefore, it is possible to choose $\varepsilon_0 > 0$ (which depends on just $d > 0$ and $r > 0$) to satisfy $K/\varepsilon^{1/k} > 1$ for all $\varepsilon \in (0, \varepsilon_0)$. The proof is complete. □

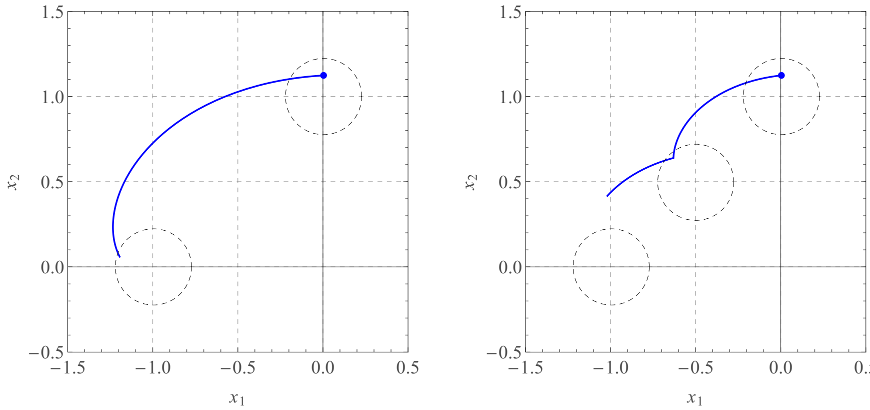

Figure 3: Solutions of switched system (1) with the initial condition in $N^{0.05}_{u_1}$ for the control inputs $\widetilde{u}(t)$ (Left) and $u(t)$ (Right) of Example 4.1.

Example 4.1. In order to illustrate Theorem 4.1, we refer to Example 2.1 again. Figure 3 shows the graphs of the solutions x of (10) for two control inputs

$$\widetilde{u}(t) = \begin{cases} u_1, & t \in [0, T^{\varepsilon}_{u_1,u_3}), \\ u_3, & t \geq T^{\varepsilon}_{u_1,u_3}, \end{cases} \qquad u(t) = \begin{cases} u_1, & t \in [0, T^{\varepsilon}_{u_1,u_2}), \\ u_2, & t \in [T^{\varepsilon}_{u_1,u_2}, T^{\varepsilon}_{u_1,u_2} + T^{\varepsilon}_{u_2,u_3}), \\ u_3, & t \geq T^{\varepsilon}_{u_1,u_2} + T^{\varepsilon}_{u_2,u_3}, \end{cases}$$

over the time interval $[0, T^{\varepsilon}_{u_1,u_3}]$. The plotting documents that $T^{\varepsilon}_{u_1,u_2} + T^{\varepsilon}_{u_2,u_3}$ turns out to be a longer time compared to $T^{\varepsilon}_{u_1,u_3}$.

5 Conclusion

In this paper we considered a switched system of differential equations under the assumption that the time between two successive switchings is greater than a certain number T called dwell time. We proved (Theorem 2.1) that a suitable choice of the dwell time makes the solution stay within a required neighborhood A_ε of a so-called ideal attractor. We further proved that the solutions reach A_ε asymptotically if the initial conditions don't belong to A_ε. By doing that we obtained a new integral condition (Theorem 3.1) for global stability which didn't seem to appear in the literature before. Finally, we addressed a case study where the Lyapunov functions of different subsystems are just shifts of one another. Here we used the dwell time formulas from Theorem 2.1 to estimate the time that the trajectories need to go from the neighborhood of an equilibrium of one subsystem to the neighborhood of an equilibrium of another subsystem (i.e. we considered a switched system with two discrete states). We proved (Theorem 4.1) that adding more discrete states makes this travel time longer. Examples 2.1 and 4.1 show that our theoretical conclusions agree with numeric simulations.

For future work, it would be interesting to explore the relationship between local results like Theorem 2.1 and global results like Theorem 3.1. For example, under what conditions are the local and global dwell times the same? Under what conditions would it be possible to have local stability and not global convergence and vice versa?

A Computing μ for linear switched systems

Consider a linear switched system

$$\dot{x} = A_u(x - x_u) \tag{16}$$

where x_u is asymptotically stable for each u subsystem. Then (16) admits quadratic Lyapunov functions of the form

$$V_u(x) = \langle x - x_u, P_u(x - x_u)\rangle, \quad u \in \mathbb{R}$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$ and $P_u \in \mathbb{R}^{n\times n}$ is a symmetric matrix with positive eigenvalues

$$0 < \lambda^u_{min} = \lambda^u_1 \le \lambda^u_2 \le \ldots \le \lambda^u_n = \lambda^u_{max}.$$

(See [4] Theorem 3.6, p.127.)

Theorem A.1. Let $\varepsilon > 0$. Then for all $a, b \in \mathbb{R}$

$$\frac{V_b(x)}{V_a(x)} < \left(\sqrt{\frac{\lambda^b_{max}}{\lambda^a_{min}}} + \frac{\sqrt{\langle x_b - x_a, P_b(x_b - x_a)\rangle}}{\sqrt{\varepsilon}}\right)^2 := \mu^b_a(\varepsilon), \quad x \in \mathbb{R}^n \setminus N^\varepsilon_a.$$

Proof. Set $||x||_u = \sqrt{\langle x - x_u, P_u(x - x_u)\rangle}$. Consider the spectral decomposition

$$P_u = U\Lambda U^T$$

of P_u. Recall Λ is the diagonal matrix whose nonzero entries are the eigenvalues of P_u and $UU^T = I$. Then

$$\langle x, P_u x\rangle = \langle x, U\Lambda U^T x\rangle = \langle U^T x, \Lambda U^T x\rangle = \sum_{i=1}^n \lambda_i \langle U^T x, U^T x\rangle = \sum_{i=1}^n \lambda_i \langle x, UU^T x\rangle$$
$$= \sum_{i=1}^n \lambda_i \langle x, x\rangle, \quad x \in \mathbb{R}^n.$$

Thus

$$\lambda^u_{min}\langle x, x\rangle \le \langle x, P_u x\rangle \le \lambda^u_{max}\langle x, x\rangle, \quad x \in \mathbb{R}.$$

Hence

$$||x||_b^2 = \langle x, P_b x\rangle \le \lambda^b_{max}\langle x, x\rangle \le \lambda^b_{max}\frac{\lambda^a_{min}}{\lambda^a_{min}}\langle x, x\rangle \le \frac{\lambda^b_{max}}{\lambda^a_{min}}\langle x, P_a x\rangle$$
$$\le \frac{\lambda^b_{max}}{\lambda^a_{min}}||x||_a^2, \quad x \in \mathbb{R}.$$

So in particular,

$$||x||_b \le \sqrt{\frac{\lambda^b_{max}}{\lambda^a_{min}}}||x||_a.$$

Therefore let $x \in \mathbb{R}^n \setminus N_a^\varepsilon$. Then apply the above inequality to get

$$\frac{V_b(x)}{V_a(x)} = \frac{(||x - x_b||_b)^2}{(||x - x_a||_a)^2} = \left(\frac{||x - x_b||_b}{||x - x_a||_a}\right)^2$$
$$\le \left(\frac{||x - x_a||_b + ||x_b - x_a||_b}{||x - x_a||_a}\right)^2$$
$$\le \left(\sqrt{\frac{\lambda^b_{max}}{\lambda^a_{min}}} + \frac{||x_b - x_a||_b}{\sqrt{V_a(x)}}\right)^2$$
$$< \left(\sqrt{\frac{\lambda^b_{max}}{\lambda^a_{min}}} + \frac{||x_b - x_a||_b}{\sqrt{\varepsilon}}\right)^2.$$

□

As a consequence of the above result, we get sufficient conditions for when the local dwell time of Theorem 2.1 also ensures global convergence to the attractor A_ε for linear switched systems.

Corollary A.1. Let $\varepsilon > 0$. Suppose the linear switched system (16) satisfies assumptions (A1)-(A3) in Theorem 2.1 with quadratic Lyapunov functions of the form

$$V_u(x) = \langle x, P_u x \rangle$$

and a single constant $k_u \equiv k$. Let $u : [t_0, \infty) \to \mathbb{R}$ be a piecewise constant function with discontinuities $\{t_1, t_2, \dots\}$. Assume

$$t_i - t_{i-1} > -\frac{1}{k} \ln \frac{\varepsilon}{(||x_{u_i} - x_{u_{i-1}}|| + \sqrt{\varepsilon})^2} = T^{\varepsilon}_{u_{i-1},u_i}, \quad i \in \mathbb{N}.$$

Suppose in addition

(i) $||x|| \geq ||x||_u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}$, and

(ii) $\lambda^a_{min} = \lambda^b_{max} \quad a, b \in \mathbb{R}$

where $||x||_u = \sqrt{\langle x, P_u x \rangle}$ and λ^u_{min} and λ^u_{max} are the smallest and largest eigenvalues of P_u. Then $x(\hat{T}) \in N^{\varepsilon}_{u_i}$ for some $\hat{T} > 0$ and some $i \in \mathbb{N}$.

Proof. The goal of the proof is to show that the local dwell time from Theorem 2.1 is larger than the global dwell time from Theorem 3.1. More concretely, we need to verify

$$t_i - t_{i-1} > \frac{\ln(\mu(\varepsilon))}{k}, \quad i \in \mathbb{N}$$

from Corollary 3.1. So let $a, b \in \mathbb{R}$. By Theorem A.1, we need to check

$$\frac{1}{k} \ln \frac{(||x_b - x_a|| + \sqrt{\varepsilon})^2}{\varepsilon} \geq \frac{1}{k} \ln \left(\sqrt{\frac{\lambda^b_{max}}{\lambda^a_{min}}} + \frac{||x_b - x_a||_b}{\sqrt{\varepsilon}} \right)^2.$$

Apply (ii) and simplify this expression to get

$$\frac{(||x_b - x_a|| + \sqrt{\varepsilon})^2}{\varepsilon} \geq \left(1 + \frac{||x_b - x_a||_b}{\sqrt{\varepsilon}}\right)^2$$
$$\frac{||x_b - x_a|| + \sqrt{\varepsilon}}{\sqrt{\varepsilon}} \geq 1 + \frac{||x_b - x_a||_b}{\sqrt{\varepsilon}}$$
$$\frac{||x_b - x_a||}{\sqrt{\varepsilon}} \geq \frac{||x_b - x_a||_b}{\sqrt{\varepsilon}}.$$

Apply (i) to finish the proof. □

Example A.1. To illustrate this result, let us revisit Example 2.1. In this example,

$$V_u(x) = ||x - x_u||^2 = \langle x - x_u, x - x_u \rangle.$$

So $||x||_u = ||x||$ and $P_u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all u. So moreover P_u has the eigenvalue $\lambda = 1$ for all u. Thus (i) and (ii) from Corollary A.1 hold. Therefore taking the local dwell time $T = 1.43$ with switching signal given by (12) ensures global convergence to A_ε in this case.

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