# Dwell time for local stability of switched affine systems with application to non-spiking neuron models

Oleg Makarenkov[a,*], Anthony Phung[a]

[a] *Department of Mathematical Sciences, University of Texas at Dallas, 75080, TX, Richardson, USA*

**Abstract**

For switched systems that switch between distinct globally stable equilibria, we offer closed-form formulas that lock oscillations in the required neighborhood of the equilibria. Motivated by non-spiking neuron models, the main focus of the paper is on the case of planar switched affine systems, where we use properties of nested cylinders coming from quadratic Lyapunov functions. In particular, for the first time ever, we use the dwell-time concept in order to give an explicit condition for non-spiking of linear neuron models with periodically switching current.

*Keywords:* Switched system, dwell-time, trapping region, multiple equlibria, planar switched affine systems, non-spiking, subshreshold oscillations, linear neuron model
*2000 MSC:* 93C30, 34D23, 92C20

## 1. Introduction

Dwell time is the lower bound on the time between successive discontinuities (switchings) of the piecewise constant function $u(t)$, which ensures that the corresponding switched (affine in our case) system

$$\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + B_{u(t)}, \qquad x \in \mathbb{R}^2, \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and $B_u$ is a $u$-dependent vector of $\mathbb{R}^2$, exhibits a required type of stability, under the assumption that each of the subsystems of (1) possess a unique globally asymptotically stable equilibrium $x_u$. Let $V_u$ be some Lyapunov function of subsystem (1) corresponding to $u(t) = x_u$ and let $N_u^k$ be the neighborhood of $x_u$ given by

$$N_u^k = \{x : V_u(x) \le k\}. \tag{2}$$

Extending the pioneering result by Alpcan-Basar [1] (see also Liberzon [7, §3.2.1]), the recent paper [4] by Dorothy and Chung gives an important formula for the dwell time $\tau_d$ which ensures that any solution of (1) with the initial condition $x(t_0) \in N^k_{u(t_0)}$ satisfies

$$x(t_i) \in N^k_{u(t_i)}, \quad i \in \mathbb{N}, \tag{3}$$

as long as the successive discontinuities $t_1, t_2, ...$ of the control signal $u(t)$ verify $t_{i+1} - t_i \ge \tau_d,\ i \in \mathbb{N}$. At the same time, the results of [4] are formulated in general abstract settings and certain work is required to apply those results to particular problems. In the present paper we follow the strategy of [4] when addressing planar switched affine systems, but carry out an independent proof that allows us to get closed-form formulas for the dwell-time $\tau_d$ (i.e. formulas in terms of just coefficients of the affine subsystems).

Relevant significant results have been recently obtained in Xu et al [11] for quasi-linear switched systems (1), but the dwell-time formula [11] is not fully explicit, as it involves the constant of the rate of decay of the matrix exponent of the homogeneous part of subsystems (1).

*Corresponding author
*Email address:* makarenkov@utdallas.edu (Oleg Makarenkov)

Our research is motivated by an application to non-spiking of linear neurons with a periodically switching current. The model of a planar linear neuron reads as (Izhikevich [6, §8.1.1], Hasselmo-Shay [5])

$$\begin{aligned} \dot{v} &= -g_p v + g_h h + I_{in}(t), \\ \dot{h} &= -mv - o_h h, \end{aligned} \tag{4}$$

coupled with the reset law

$$v(t) \mapsto v_R, \ h(t) \mapsto h_R(h(t)), \quad \text{if} \quad v(t) = v_{th}, \tag{5}$$

where $v$ is the neural cell membrane potential, $h$ is the recovery variable, $g_p$ is the rate of passive decay of membrane potential, $g_h$ is the rate of current induced depolarization of the cell, $m > 0$ makes $h$ increasing when $v$ gets negative, $o_h$ is the current decay, $I_{in}$ is a constant current which alternates between switch on and switch off. Though some neurons spike and reset according to vector $v_R$ and function $h_R$ (when reach the firing threshold $v_{th}$) to propagate a message, some others are capable to transmit information without spiking and are not supposed to ever reach the threshold $v_{th}$ (see e.g. Vich-Guillamon [8], Chen et al [2]). The present paper uses the dwell time concept in order to obtain conditions for the model (4)-(5) to never reach the firing threshold $v_R$, i.e. to ensure just subthreshold oscillations. The readers interested in the difference between subthreshold and spiking dynamics are referred to Coombes et al [3].

## 2. The main result

Next theorem offers conditions on the coefficients of the system (1) where the strategy of Dorothy and Chung [4] leads to explicit dwell-time formulas.

**Theorem 2.1.** Assume that $abcd < 0$, $a < 0$, and $d < 0$. Let $k > 0$ be a given constant and let $x$ be any solution of switched system (1) with the control signal $u = u(t)$ and with the initial condition $x(t_0) \in N^k_{u(t_0)}$. If the successive discontinuities $t_1, t_2, ...$ of $u(t)$ verify

$$t_i - t_{i-1} \geq \frac{1}{2\min\{|a|, |d|\}} \ln\left(\frac{k_i}{k}\right), \quad k_i = \left(\sqrt{k} + \sqrt{|c|\left(x_{u(t_i),1} - x_{u(t_{i-1}),1}\right)^2 + |b|\left(x_{u(t_i),2} - x_{u(t_{i-1}),2}\right)_2}\right)^2, \quad i \in \mathbb{N}, \tag{6}$$

with the equilibria $x_u = (x_{u,1}, x_{u,2})^T$ given by $x_u = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} B_u$, then (3) holds, and, moreover,

$$x(t) \in N^{k_i}_{u(t_i)}, \quad t \in [t_{i-1}, t_i], \ i \in \mathbb{N}. \tag{7}$$

The notations and conclusions of the theorem are illustrated in Fig. 1.

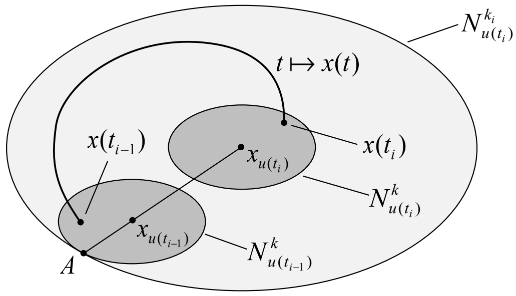


Figure 1: The location of the ellipse $\partial N^{k_i}_{u(t_i)}$ relative to ellipse $\partial N^k_{u(t_{i-1})}$ and the solution $t \mapsto x(t)$ of switched system (1) on the interval $[t_{i-1}, t_i]$.

**Proof.** The condition for $a$, $b$, $c$, $d$ implies that a Lyapunov function for subsystems of (1) can be taken as

$$V_u(x) = |c|(x_1 - x_{u,1})^2 + |b|(x_2 - x_{u,2})^2, \quad x = (x_1, x_2)^T.$$

Specifically,

$$\dot{\omega}_i(t) \le -\varepsilon\omega_i(t), \quad \text{for} \quad \omega_i(t) = V_{u(t_i)}(x(t)), \quad t \in (t_{i-1}, t_i], \tag{8}$$

where $\varepsilon > 0$ is such a constant that

$$\varepsilon \ge -\frac{2a|c|(x_1 - x_{u,1})^2 + 2|b|d(x_2 - x_{u,2})^2}{|c|(x_1 - x_{u,1})^2 + |b|(x_2 - x_{u,2})^2} = 2\frac{|ac|(x_1 - x_{u,1})^2 + |bd|(x_2 - x_{u,2})^2}{|c|(x_1 - x_{u,1})^2 + |b|(x_2 - x_{u,2})^2}. \tag{9}$$

Letting $x_1 - x_{u,1} = r\cos\phi$ and $x_2 - x_{u,2} = r\sin\phi$, the right-hand-side of this inequality takes the form

$$2\frac{|ac|\cos^2\phi + |bd|\sin^2\phi}{|c|\cos^2\phi + |b|\sin^2\phi} =: g(\phi) \quad \text{and} \quad g'(\phi) = \frac{4|bc|(|d| - |a|)\sin\phi\cos\phi}{\left(|c|\cos^2\phi + |b|\sin^2\phi\right)^2}.$$

To find the best (i.e. maximal) possible value of $\varepsilon$ we therefore compute the minimum of $g(\phi)$ on the interval $[0, \pi]$. Analysing $g'(\phi)$, we conclude that $g(\phi)$ has just one critical point $\phi_0 = \pi/2$ on $(0, \pi)$. Therefore,

$$\varepsilon = \min_{\phi\in[0,\pi]} g(\phi) = \min\{g(0), g(\pi/2), g(\pi)\} = 2\min\{|a|, |d|\}. \tag{10}$$

Let us fix $i \in \mathbb{N}$. Assuming that $x(t_{i-1}) \in N^k_{u(t_{i-1})}$ is established, we now use (8)-(10) in order to prove that $x(t_i) \in N^k_{u(t_i)}$, i.e. to prove that $\omega_i(t_i) \le k$. Specifically, we are going to find $k_i > 0$ satisfying

$$N^k_{u(t_{i-1})} \subset N^{k_i}_{u(t_i)} \qquad \text{(Step 1)} \tag{11}$$

and prove that

$$k_i \mathrm{e}^{-\varepsilon_{u(t_i)}(t_i - t_{i-1})} \le k \qquad \text{(Step 2)} \tag{12}$$

to have $\omega_i(t_i) \le \omega_i(t_{i-1})\mathrm{e}^{-\varepsilon_{u(t_i)}(t_i-t_{i-1})} \le k_i\mathrm{e}^{-\varepsilon_{u(t_i)}(t_i-t_{i-1})} \le k$.

**Step 1.** Note, that the boundary $\partial N^k_u$ of $N^k_u$ is given by $\partial N^k_u = \left\{x \in \mathbb{R}^2 : |c|(x_1 - x_{u,1})^2 + |b|(x_2 - x_{u,2})^2 = k\right\}$. To find $k_i > 0$ satisfying (11) we construct the ellipse $\partial N^{k_i}_{u(t_i)}$ to touch the ellipse $\partial N^k_{u(t_{i-1})}$ (at some point $A \in \mathbb{R}^2$), see Fig. 1. Expressing the point $A$ in the polar coordinates of the ellipses $\partial N^k_{u(t_{i-1})}$ and $\partial N^{k_i}_{u(t_i)}$ we get

$$x_1 - x_{u(t_{i-1}),1} = \sqrt{\frac{k}{|c|}}\cos\phi, \;\; x_2 - x_{u(t_{i-1}),2} = \sqrt{\frac{k}{|b|}}\sin\phi, \;\; x_1 - x_{u(t_i),1} = \sqrt{\frac{k_i}{|c|}}\cos\bar{\phi}, \;\; x_2 - x_{u(t_i),2} = \sqrt{\frac{k_i}{|b|}}\sin\bar{\phi}. \tag{13}$$

The property of the derivative of the curve $\partial N^k_{u(t_{i-1})}$ at $A$ to be parallel to the derivative of the curve $\partial N^{k_i}_{u(t_i)}$ at $A$ leads to $\phi = \bar{\phi}$. Excluding in (13) the unknowns $x_1$ and $x_2$ we get

$$x_{u(t_i),1} - x_{u(t_{i-1}),1} = \left(\sqrt{k} - \sqrt{k_i}\right)\cdot(\cos\phi)\,/\,\sqrt{|c|}, \quad x_{u(t_i),2} - x_{u(t_{i-1}),2} = \left(\sqrt{k} - \sqrt{k_i}\right)\cdot(\sin\phi)\,/\,\sqrt{|b|}, \tag{14}$$

which yields the required formula (6) for $k_i$.

**Step 2.** Combining (6) with (10), the inequality (12) takes form of assumption (6) and so (12) holds true.

The proof of the theorem is complete. □

## 3. Application to non-spiking neuron models

In this section we apply the earlier results to the planar linear system (4) assuming that the current $I_{in}$ is changing according to the law

$$I_{in}(t) = \begin{cases} I, & t \in (0, T_I], \\ 0, & t \in (T_I, T_I + T_0], \end{cases} \qquad \text{where } I > 0 \text{ is given constant}, \tag{15}$$

on $[0, T_I + T_0]$ and is then continued to $[0, \infty)$ by $T_I + T_0$-periodicity. Taking an arbitrary $k > 0$, the equilibrium of (4) with $I_{in}(t) = I$ and the constants $k_i$ in (6) compute as

$$\begin{pmatrix} v_I \\ h_I \end{pmatrix} = \frac{I}{g_p o_h + m g_h}\begin{pmatrix} o_h \\ -m \end{pmatrix}, \qquad k_i = \left(\sqrt{k} + \sqrt{m v_I^2 + g_h h_I^2}\right)^2 =: \bar{k}, \quad i \in \mathbb{N}. \tag{16}$$

One gets the following result as a direct consequence of Theorem 2.1.

**Corollary 3.1.** Let $g_p, g_h, m, o_h > 0$ and consider $k > 0$. Assume that

$$\min\{T_I, T_0\} \geq \frac{1}{2\min\{g_p, o_h\}} \ln\frac{\bar{k}}{k} =: \tau_d. \tag{17}$$

Then, the dynamics of any solution $t \mapsto (v(t), h(t))$ of (4) with the control function (15) and with the initial condition $(v(0), h(0)) = 0$ satisfies, for $j \in \mathbb{N}$, $t_j = (T_I + T_0)\cdot j$, $\tau_j = (T_I + T_0)\cdot j + T_I$, the relations

$$m(v(t)-v_I)^2 + g_h(h(t)-h_I)^2 \leq \bar{k}, t \in \left[t_j, \tau_j\right], \quad mv(t)^2 + g_h h(t)^2 \leq \bar{k},\ t \in \left[\tau_j, t_{j+1}\right], \quad v(t) \leq v_I + \sqrt{\bar{k}/m},\ \ t \geq 0.$$

In particular, the neuron model (4)-(5) exhibits just sub-threshold oscillations (never develops spiking), if

$$v_{th} > v_I + \sqrt{\bar{k}/m}. \tag{18}$$

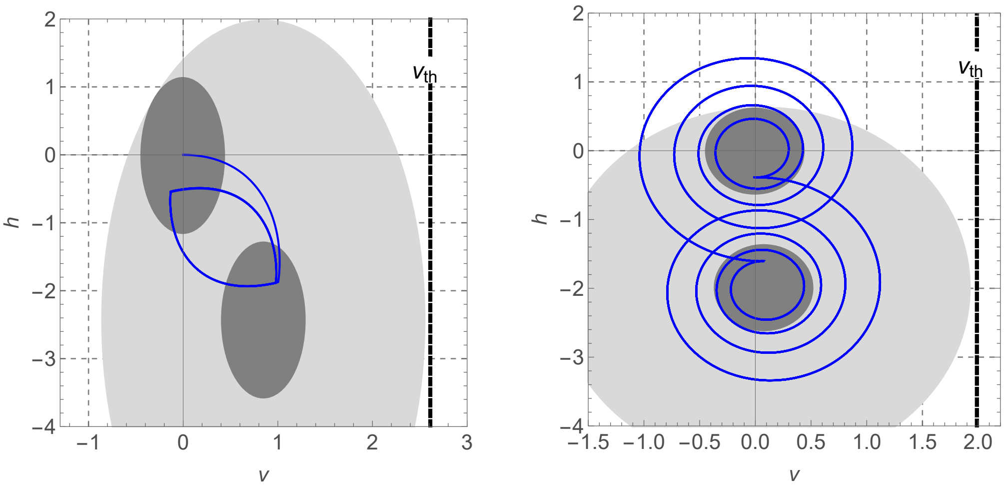


Figure 2: Left: The trajectory $t \mapsto (v(t), h(t))$ of system (4) with the initial condition $(v(0), h(0)) = 0$ for the parameters of $g_p = 0.75$, $g_h = 0.15$, $m = 1$, $o_h = 0.35$ (from Hasselmo-Shay [5]) and with the input $I_{in}(t)$ alternating between 0 and $I = 1$ every $T = 3.84$ units of time (i.e. $T_0 = T_I = 3.84$). Right: The attractor of system (4) for the parameters $g_p = 0.04$, $g_h = 0.5$, $m = 1$, $o_h = 0.04$, whose input $I_{in}(t)$ alternates between 0 and $I = 1$ with period $T_0 = T_I = 35.7$. In both figures the dark gray disks are $N_0^k$ and $N_I^k$, $k = 0.2$, and the light disk is $N_I^{\bar{k}}$, see (16). The line $v = v_{th}$ is an example of firing threshold that doesn't cause spiking (because the line $v = v_{th}$ does intersect $N_I^{\bar{k}}$).

Simulations of Figs. 2-3 illustrate the accuracy of the predictions of Corollary 3.1. At Fig. 2(left) we drew the solution of the linear neuron model (4) with the parameters of Hasselmo-Shay [5] ($g_p = 0.75$, $g_h = 0.15$, $m = 1$, $o_h = 0.35$), $I = 1$, $k = 0.2$ and the periods $T_0 = T_I = 3.84$, that was computed using the dwell-time formula (17) (which gives $\tau_d = 3.836$). Formula (18) provides the estimate $v_{th} > 2.56$ for the firing threshold to ensure non-spiking. A possible firing threshold $v_{th}$ is drawn in Fig. 2(left). The figure also illustrates the construction beyond the estimate (18) whose role is to locate the cylinder $N_I^{\bar{k}}$ to the left from the line $v = v_{th}$. Fig. 2(left) is an example where Corollary 3.1 leads to a rather conservative estimate for $v_{th}$. The figure shows that the value $v_{th}$ can actually be much smaller than $v_{th} = 2.56$ (roughly $v_{th} = 1.3$) for sub-threshold oscillations to not spike. The sharpness of the estimates of Corollary 3.1 is seen e.g. with the parameters $g_p = 0.04$, $g_h = 0.5$, $m = 1$, $o_h = 0.04$, $k = 0.2$, $I = 1$. The dwell time $\tau_d$ given by Corollary 3.1 is now $\tau_d = 35.621$, which was used in simulations of Fig. 2(right) (we took $T_0 = T_I = 35.7$) where the respective attractor of model (4) is shown. First of all, one can see that the switchings (corners of the trajectory) occur very close to the boundary of the cylinders $N_0^k$ and $N_I^k$. Moreover, Fig. 3(left) shows that the switching points are no longer in $N_0^k$ and $N_I^k$, if $T_0$ and $T_I$ reduce to $T_0 = T_I = 32$, which confirms that $\tau_d = 35.621$ is a relatively sharp dwell time bound. Finally, Fig. 3(d) illustrates that the estimate (18) for the maximal current is also accurate, i.e. a trajectory with the initial condition in $N_0^k$ can pass quite close to the rightmost point of the ellipse $N_I^{\bar{k}}$.

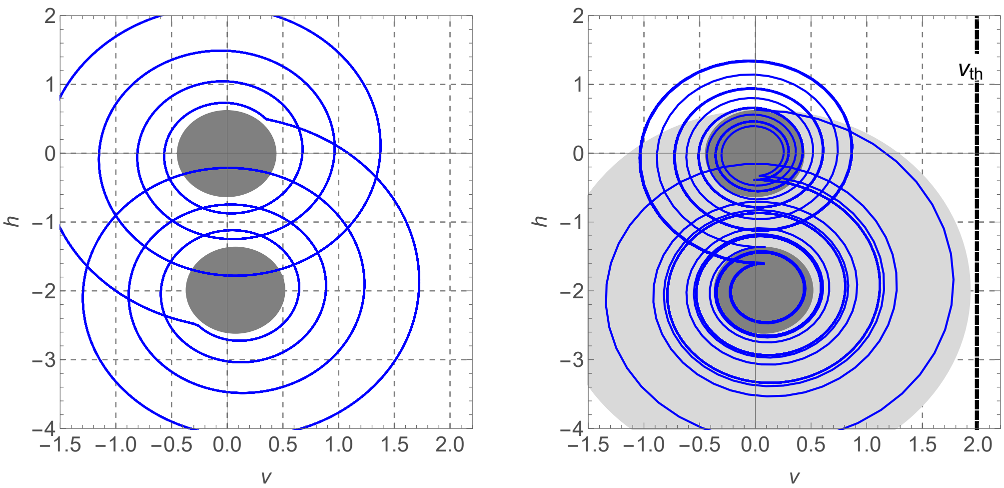


Figure 3: Both the figures are plotted with the parameters of Fig. 2(right) except for $T_0$ and $T_I$. Left: Attractor of (4) for $T_0 = T_I = 32$. Right: The solution of (4) with the initial condition at the top of $N_0^k$ and $T_0 = T_I = 35.7$. The meaning of gray disks is the same as in Fig. 2.

## 4. Conclusion

We used the dwell-time concept to give explicit conditions for the attractor of a linear planar switched system to never cross a given threshold. As a consequence, we gave a closed-form formula for a linear neuron model to never reach the firing threshold, i.e. to operate in just a sub-threshold mode. To the best of our knowledge, the dwell-time approach has never been used in the context of neuroscience before. Extending the ideas of the paper to wider classes of neuron models and switched systems (such as those in [7, 9, 10, 12, 13]), and utilizing more relaxed dwell-time concepts is a subject of future research.

## 5. Acknowledgments.

OM is supported by NSF Grant CMMI-1436856 and by Burroughs Wellcome Fund Collaborative Grant #1017453. The authors are grateful to anonymous referees, whose comments helped to improve the paper.

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