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ON MITTAG-LEFFLER MOMENTS FOR THE BOLTZMANN EQUATION FOR HARD POTENTIALS WITHOUT CUTOFF*

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Abstract. We establish the L^1 weighted propagation properties for solutions of the Boltzmann equation with hard potentials and nonintegrable angular components in the collision kernel. Our method identifies null forms by angular averaging and deploys moment estimates of solutions to the Boltzmann equation whose summability is achieved by introducing the new concept of Mittag-Leffler moments—extensions of L^1 exponentially weighted norms. Such L^1 weighted norms of solutions to the Boltzmann equation are, both, generated and propagated in time, and the characterization of their corresponding Mittag-Leffler weights depends on the angular singularity and potential rates in the collision kernel. These estimates are a fundamental step in order to obtain L^∞ exponentially weighted estimates for solutions of the Boltzmann equation being developed in a follow-up work.

Key words. Boltzmann equation, non-cutoff, moments, exponential moments, Mittag-Leffler moments

AMS subject classifications. 35Q20, 82C40, 76P05

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1. Introduction. We study generation and propagation in time of L^1 exponentially weighted norms, referred to as exponential moments, associated to probability density functions that solve the Boltzmann equation [10, 11] modeling the evolution of monoatomic rarefied gases. Binary interactions of gas particles are described by transition rates from before and after such interactions, usually referred to as collision kernels. Such kernels are modeled as a product of potential functions of local relative speed and functions of the scattering angle between the pre- and postrelative velocities. This angular function may or may not be integrable. When integrable, the collision kernel is said to satisfy an angular cutoff condition. The particular case when the angular part of the kernel is bounded is known as the Grad's cutoff condition [24]. Otherwise, its nonintegrability, referred to as an angular non-cutoff, satisfies specific conditions (for details, see section 2).

The concept of exponential moments is associated to the notion of large energy decay rates for tails. A time-dependent probability distribution function $f(t, v)$ is said to have L^1 exponential moment (tail behavior) of order s and rate $r(t)$ if, for any fixed $t > 0$,

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$$\int_{\mathbb{R}^d} f(t,v) e^{r(t)\,\langle v\rangle^s}\,\mathrm{d}v \quad \text{is positive and finite,} \tag{1.1}$$

where we use the notation

$$\langle v\rangle := \sqrt{1+|v|^2}. \tag{1.2}$$

This concept was introduced by Bobylev in [7, 8] and Gamba, Panferov, and Villani in [20], where they show uniform in time propagation of L^1 Maxwellian tails (i.e., Gaussian in v-space, that is, $s = 2$) for several type of collision kernels ranging from Maxwell-type to hard sphere interactions with angular cutoff conditions and by Bobylev, Gamba, and Panferov in [9] for different values of $s \in (0, 2]$ in the study of inelastic interaction with internal heating sources. These groundbreaking works conceived the idea of controlling exponential moments by proving the summability of power series expansions on a parameter $r(t)$. Such formulation was motivated by formally commuting integration in v-space and the infinite sum derived from the power series of the exponential function in (1.1), upon which one obtains

$$\int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{r^q(t)\langle v\rangle^{sq}}{\Gamma(q+1)}\,\mathrm{d}v \;=\; \sum_{q=0}^{\infty} \frac{r^q(t)\; m_{sq}(t)}{\Gamma(q+1)}. \tag{1.3}$$

The terms $m_{sq}(t)$, called polynomial moments, are $\langle v\rangle^{sq}$-weighted L^1 norms of the distribution function $f(v,t)$ that solves the Boltzmann equation. Representation (1.3) replaces the quest of L^1 exponential integrability with a given order and rate with study of summability of infinite sums (time series forms).

A fundamental technique for accomplishing this task (see [8, 9, 20, 3, 5, 29]) consists of controlling the weak form of the collision operator by the means of angular averaging. These estimates are used to derive a sequence of ordinary differential inequalities for the polynomial moments of the collisional form. These differential inequalities are an algebraic sum of a negative term of moments of highest order and a positive term of bilinear sums of moments of lower orders.

Recently, Alonso et al. [1] introduced a new technique (based on analyzing partial sums corresponding to the infinite sum appearing in (1.3)) to prove the generation of exponential moments with orders up to the potential rate and the propagation of exponential moments with orders up to $s = 2$ under an angular integrability condition. It is interesting to note that these results do not rely on the rate of Povzner estimates for angular averaging, and so the resulting order $r(t)$ may not be optimal.

All results mentioned above were developed for the case of an integrable angular collision kernel. This brings us to the setting of this manuscript, the non-cutoff regime. This manuscript focuses on the study of both generation and propagation in time of exponential moments for solutions to the initial value problem for the d-dimensional Boltzmann equation for elastic collisions, in the space homogeneous case, for hard potentials without the angular cutoff assumption. In this direction, Lu and Mouhot [26] showed generation of exponential moments of order up to the potential rate in the collision kernel. In this work, we considerably extend their result by showing that rates and orders of exponential moments depend on the initial data, as well as potential and angular singularity rates in collision kernels.

In order to treat the non-cutoff regime, we develop angular averaged estimates that account for the cancellation of nonintegrable angular singularities by means of null forms averaging. The other important component is summability of moments, which is achieved by introducing Mittag-Leffler moments.

Indeed, the most significant point of this paper is the introduction of Mittag-Leffler moments as L^1 Mittag-Leffler weighted norms. They enabled us to extend the range of orders of exponential moments that can be propagated uniformly in time for the non-cutoff case. To obtain our result, we encounter the need to study (1.3), where $\Gamma(q+1)$ is replaced with $\Gamma(aq+1)$ for a noninteger $a > 1$ (which is reminiscent of some of the tools used in [9], although no summing of such renormalized moments was performed there):

$$\int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{r^q(t) \langle v \rangle^{sq}}{\Gamma(aq+1)} \, \mathrm{d}v \;=\; \sum_{q=0}^{\infty} \frac{r^q(t)\, m_{sq}(t)}{\Gamma(aq+1)}. \tag{1.4}$$

We observed that the sum appearing on the left-hand side of (1.4) is exactly the well-known Mittag-Leffler function $\mathcal{E}_a(r(t)\langle v \rangle^s)$, where $\mathcal{E}_a$ is defined as

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}. \tag{1.5}$$

In analogy to (1.1), this led us to introduce a concept of Mittag-Leffler moments,

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_a(\alpha^a \, \langle v \rangle^2) \, \mathrm{d}v = \sum_{q=0}^{\infty} \frac{m_{2q}(t)\, \alpha^{aq}}{\Gamma(aq+1)}, \tag{1.6}$$

which are a natural generalization of exponential moments.

Another important aspect of our main result is that the highest order of exponential moment which can be propagated in time depends continuously on the singularity rate of the angular cross section. The less singular the angular kernel is, the higher the order of the exponential moment that can be propagated is. See details in Remark 2.12.

Let us mention one application of L^1 weighted estimates. In [20], Gamba, Panferov, and Villani gave a proof to close the open problem of propagation of L^∞-Maxwellian weighted bounds, uniformly in time, to solutions of the Boltzmann equation with hard potential with a cutoff in the angular kernel. Their result follows from an application of a maximum principle of parabolic type, due to the dissipative nature of the collisional integral, and estimates on the Carleman representation of the gain (positive) part of the collision operator that depend on the L^1-Maxwellian weighted bounds uniformly propagated in time. We mention here that the extension of such result on propagation of $L^\infty(\mathbb{R}^d)$-exponential weights is currently being worked out for the non-cutoff and hard potential case in a forthcoming manuscript [21] using the L^1 weighted estimates obtained in this manuscript.

Organization of the paper. Section 2 presents the Boltzmann equation without the angular cutoff condition, exponential and Mittag-Leffer moments, and the statements of the two main results of the manuscript—the angular averaged Povzner inequalities with angular singularity cancellation in Lemma 2.9 and the generation and propagation of Mittag-Leffler moments in Theorem 2.11. Section 3 contains the proof of the angular averaged Povzner inequalities for nonintegrable angular singularity, i.e., Lemma 2.9. This lemma is the main tool for the formation of ordinary differential inequalities for polynomial moments of all orders, which are covered in section 4. Section 5 provides details of the proof of the propagation of Mittag-Leffler moments, while in section 6 we give a new proof of the generation of exponential moments of order up to the rate of potentials. The final section, the Appendix, gathers known and technical yet fundamental results used throughout this manuscript.

2. Preliminaries and main results.

2.1. The Boltzmann equation. We consider the Cauchy problem for the spatially homogeneous (i.e., x-space independent) Boltzmann equation

$$\begin{cases} \partial_t f(t,v) = Q(f,f)(t,v), \quad t \in \mathbb{R}^+, v \in \mathbb{R}^d, \quad d \geq 2 \\ f(0,v) = f_0(v). \end{cases} \tag{2.1}$$

The function $f(t,v)$ models the particle density at time t and velocity v of a rarefied gas in which particle collisions are elastic and predominantly binary. The collisional operator $Q(f,f)$ is a quadratic integral operator defined via

$$Q(f,f)(t,v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left(f'f'_* - ff_*\right) B(|u|, \hat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*, \tag{2.2}$$

where we use the abbreviated notation $f_* = f(t,v_*)$, $f' = f(t,v')$, and $f'_* = f(t,v'_*)$. Vectors v', v'_* denote precollisional velocities, and v, v_* are their corresponding postcollisional velocities. Relative velocity is denoted by $u = v - v_*$ and its normalization by $\hat{u} = u/|u|$. Being an elastic interaction of reversible character that conserves momentum $v + v_* = v' + v'_*$ and energy $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$, pre- and postcollisional velocities are related by formulas represented in center of mass $V = (v + v_*)/2$ and relative velocity $u = v - v_*$ coordinates as follows:

$$v = V' + \frac{|u'|}{2}\,\sigma, \qquad v_* = V' - \frac{|u'|}{2}\,\sigma, \quad \sigma \in \mathbb{S}^{d-1}. \tag{2.3}$$

The unit vector $\sigma \in \mathbb{S}^{d-1}$, referred to as the scattering direction, has the direction of the precollisional relative velocity $u' = v' - v'_*$. We bring to the reader's attention that the pre- to postcollisional exchange of coordinates satisfy

$$v' - v = \frac{1}{2}(|u|\,\sigma - u),$$
$$v'_* - v_* = -\frac{1}{2}(|u|\,\sigma - u).$$

This representation embodies the relation of the exchange of velocity directions as just functions of the relative velocity u and the scattering direction σ.

The collisional kernel $B(|u|, \hat{u}\cdot\sigma)$ is assumed to take the form

$$B(|u|, \hat{u}\cdot\sigma) = |u|^\gamma\, b(\cos\theta), \tag{2.4}$$

where $\theta \in [0,\pi]$ is the angle between the pre- and postcollisional relative velocities, and thus it satisfies $\cos\theta = \hat{u}\cdot\sigma$. In this manuscript we work in the variable hard potentials case, that is,

$$0 < \gamma \leq 1. \tag{2.5}$$

We assume that the angular kernel is given by a positive measure $b(\hat{u}\cdot\sigma)$ over the sphere $\mathbb{S}^{d-1}$. In many models, this function is nonintegrable over the sphere, while its weighted integral is finite. In this manuscript we assume that for some $\beta \in (0,2]$, the following weighted integral is finite (with $|\mathbb{S}^{d-2}| = \frac{\pi^{(d-2)/2}}{\Gamma((d-1)/2)}$ being the volume of the $d-2$ dimensional unit sphere):

$$A_\beta := \int_{\mathbb{S}^{d-1}} b(\hat{u}\cdot\sigma)\sin^\beta\theta \,\mathrm{d}\sigma$$

$$= |\mathbb{S}^{d-2}| \int_0^\pi b(\cos\theta)\ \sin^\beta\theta\ \sin^{d-2}\theta \,\mathrm{d}\theta < \infty. \tag{2.6}$$

When $\beta = 0$ (a case that we do not consider), this condition is known as the angular cutoff assumption, under which the collisional operator can be split into the gain and loss terms

$$Q(f,f) = Q^+(f,f) - Q^-(f,f), \tag{2.7}$$

where

$$Q^+(f,f)(t,v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f' f'_* \, B(|u|, \hat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*,$$

$$Q^-(f,f)(t,v) = f(v) \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f_* \, B(|u|, \hat{u}\cdot\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*.$$

In 1963, Grad [24] proposed considering a bounded angular kernel $b(\cos\theta)$ and pointed out that different cutoff conditions could be implemented too. Since then, the cutoff theory developed extensively, with the belief that removing the singularity of the angular kernel should not affect properties of the equation. Recently, however, it has been observed (see, for example, [25], [14], [15], [16]) that the singularity of $b(\cos\theta)$ carries regularizing properties. This, in addition to the analytical challenge, motivated further study of the non-cutoff regime.

The typical non-cutoff assumption in the literature is the condition (2.6) with $\beta = 2$. However, we work in the non-cutoff regime where the parameter $\beta \in (0, 2]$ is allowed to vary, and we will see how the strength of the singularity of b influences our main result. In this setting, the splitting (2.7) is not valid, which is one of the technical challenges that the non-cutoff setting brings. In order to address this obstacle, we exploit angular cancellation properties (for details, see section 3).

Remark 2.1. In the physically relevant case corresponding to the dimension $d = 3$, when forces between particles are governed by an inverse-power-law-long range interaction potential $\phi(x) = Cx^{-(p-1)}$, $C > 0$, $p > 2$, the angular kernel $b(\cos\theta)$ has been derived by Grad [24] (see also [12]) and is shown to have the following form:

$$b(\cos\theta)\ \sin\theta \sim C\,\theta^{-1-\nu}, \qquad \theta \to 0^+,$$

$$\nu = \frac{2}{p-1}, \qquad \gamma = \frac{p-5}{p-1}, \qquad p > 2. \tag{2.8}$$

Note that this model satisfies (2.6) with any $r > \nu$.

Weak formulation of the collision operator $Q(f,f)$. Thanks to the symmetries associated to the collisional form $Q(f,f)$, defined in the strong form (2.2), the collisional operator has a weak formulation that is very important for the analytical manipulation of the equation. Indeed, for any test function $\phi(v)$, $v \in \mathbb{R}^d$, one has (see, for example, [12])

$$\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\phi(v)\mathrm{d}v = \frac{1}{2}\iint_{\mathbb{R}^{2d}} f(v)\,f(v_*)\ G_\phi(v,v_*)\,\mathrm{d}v_*\,\mathrm{d}v, \tag{2.9}$$

$$G_\phi(v,v_*) = \int_{\mathbb{S}^{d-1}} \left(\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)\right) B(|u|, \hat{u}\cdot\sigma)\ \mathrm{d}\sigma.$$

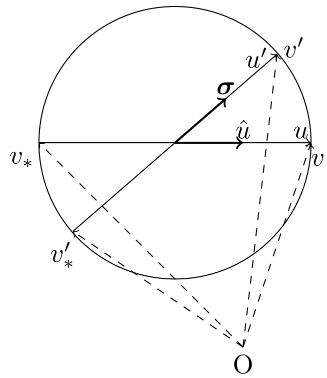


FIG. 1. *Pre and postcollisional velocities.*

The key aspect of the equation in the weak formulation is expressed in the weight G_ϕ, as it carries all the information about collisions through the collisional kernel B, which is averaged over the unit sphere against test functions $\Delta\phi = \phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*)$. Crucial estimates on the function G_ϕ referred to in the Boltzmann equation literature as Povzner estimates are described below.

In the angular cutoff case, positive and negative contributions are treated separately, and such estimates are used to estimate the positive part of G_ϕ. A sharp form of angular averaged Povzner estimates from [8, 9, 20] is obtained for general test functions $\phi(v)$ which are positive and convex. They are crucial for the study of moments summability, the main point of this manuscript.

When $\phi(v) = (1+|v|^2)^{k/2} = \langle v\rangle^k$, these estimates, originally developed by Povzner [31], yield ordinary differential inequalities for moment estimates that lead to an existence theory and generation and propagation of moments as developed in Elmroth [18], Desvillettes [13], Wennberg [34], and Mischler and Wennberg [27]. These estimates were also obtained in the non-cutoff case by Wennberg [33] for hard potentials. Uniqueness theory to solutions of the Boltzmann equation for hard potentials was first developed by Di Blasio in [17].

When the angular part of the collision kernel is not integrable, i.e., the non-cutoff case, one needs to expand $\Delta\phi$ in terms of $v' - v$ and $v'_* - v_*$ since both are multiples of $|u| \sin\theta/2$. For this strategy to succeed, the spherical integration variable $\sigma \in \mathbb{S}^{d-1}$ must be decomposed as $\sigma = \hat{u}\cos\theta + \omega \sin\theta$, corresponding to the polar direction of the relative velocity u and the azimuthal direction $\omega \in \mathbb{S}^{d-1}$ satisfying $u \cdot \omega = 0$. This decomposition also plays a fundamental role in our derivation of the angular averaged Povzner with singularity cancellation in the proof of Lemma 2.9.

Remark 2.2. We note that the identity (2.9) can also be expressed in a double mixing (weighted) convolutional form ([22, 2, 4])

$$\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\phi(v)\mathrm{d}v = \frac{1}{2}\iint_{\mathbb{R}^{2d}} f(v)f(v-u)\;G_\phi(v,u)\,\mathrm{d}u\,\mathrm{d}v$$

$$G_\phi(v,u) = \int_{\mathbb{S}^{d-1}} \left(\phi(v') + \phi(v'-u') - \phi(v) - \phi(v-u)\right) B(|u|, \hat{u}\cdot\sigma)\mathrm{d}\sigma$$

since both v' and v'_* can be written as functions of v, u and σ from (2.3), and so the weight function $G_\phi(v,u)$ is an average over $\sigma \in \mathbb{S}^{d-1}$.

2.2. Moments of solutions to the Boltzmann equation. From the probabilistic viewpoint, moments of a probability distribution density $f(t,v)$ with respect to the variable v are integrals of such density weighted by functions $\phi(v)$. These are important objects to study, as they express average quantities that have significant meaning for the model under consideration. They are the so-called observables. In this sense polynomial moments correspond to such integrals for polynomial weights, and exponential moments are for exponential weights.

We now recall definitions of polynomial and exponential moments, and we here introduce the Mittag-Leffler moments, which are a natural generalization of the exponential moments.

Definition 2.3 (polynomial and exponential moments). *Polynomial moment of order q and exponential moment of order s and rate α are defined, respectively, by*

$$m_q(t) := \int_{\mathbb{R}^d} f(t,v)\ \langle v\rangle^q \,\mathrm{d}v \tag{2.10}$$

$$\mathcal{M}_{\alpha,s}(t) := \int_{\mathbb{R}^d} f(t,v)\ e^{\alpha\,\langle v\rangle^s} \,\mathrm{d}v. \tag{2.11}$$

Remark 2.4. Using the Taylor series expansion, the exponential moment of order s and rate α can also be written as the following sum:

$$\mathcal{M}_{\alpha,s}(t) \ = \ \sum_{q=0}^{\infty} \frac{m_{qs}(t)\ \alpha^q}{q!}. \tag{2.12}$$

Remark 2.5. Polynomial moments can be expressed in terms of the norm of a natural Banach space in the context of the Boltzmann equation. Namely, if we denote

$$L^1_k = \{f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f\langle v\rangle^k \mathrm{d}v = \int_{\mathbb{R}^d} f\left(1+|v|^2\right)^{k/2} \mathrm{d}v < \infty\},$$

then

$$m_q(t) = \|f(t,\cdot)\|_{L^1_q}. \tag{2.13}$$

Also, note that

$$\|f\|_{L^1_q} \leq \|f\|_{L^1_{q'}} \quad \text{for any } q \leq q'. \tag{2.14}$$

Note that this expression is associated to the notion of L^1 exponential tail behavior described in (1.1) and (1.3). Consequently, finiteness of exponential moments can be understood as implying that the function $f(t,v)$ has an exponential tail in v. In this paper, we study whether this property can be generated or propagated in time for the case of variable hard potentials in the non-cutoff case.

Because our summability estimates lead to expressions similar to that of (2.12) yet having $\Gamma(aq+1)$ as a generalization of factorials with noninteger $a>1$, we are motivated to use Mittag-Leffler functions, as they are conceived as a generalization of the Taylor expansion of the exponential function. More precisely, for a parameter $a>0$, the Mittag-Leffler function is defined via

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}\,, \qquad x \geq 0. \tag{2.15}$$

Note that for $a = 1$, the Mittag-Leffler function coincides with the Taylor expansion of the classical exponential function e^x. It is also well known (see, e.g., [19, p. 208] and for more explicit form [23, Theorem 1]) that for any $a > 0$, the Mittag-Leffler function asymptotically behaves like an exponential function of order $1/a$, that is,

$$\mathcal{E}_a(x) \sim e^{x^{1/a}} \qquad \text{as } x \to \infty.$$

In particular, there exists $L > 0$ and positive constants c_L and C_L such that

$$c_L\, e^{x^{1/a}} \le \mathcal{E}_a(x) \le C_L\, e^{x^{1/a}}\,, \qquad x \ge L.$$

Furthermore, one sees from the definition (2.15) that $x \mapsto \mathcal{E}_a(x)$ is increasing and $\mathcal{E}_a(0) = 1$. Therefore,

$$e^{-L^{1/a}} e^{x^{1/a}} \le \mathcal{E}_a(x) \le \mathcal{E}_a(L)\, e^{x^{1/a}}\,, \qquad 0 \le x \le L.$$

As a consequence,

$$c\, e^{x^{1/a}} \le \mathcal{E}_a(x) \le C\, e^{x^{1/a}} \qquad \text{for } x \ge 0 \tag{2.16}$$

with $c := \min\{e^{-L^{1/a}}, c_L\}$ and $C := \max\{\mathcal{E}_a(L), C_L\}$.

Since $\langle v\rangle^2$ is the building block for our calculations, we prefer to have x^2 as the argument of Mittag-Leffler function when generalizing $e^{\alpha\, x^s}$. Hence, using the estimate (2.16),

$$c\, e^{\alpha x^s} \;\le\; \mathcal{E}_{2/s}(\alpha^{2/s} x^2) \;\le\; C\, e^{\alpha x^s}\,, \qquad x \ge 0. \tag{2.17}$$

This motivates our definition of Mittag-Leffler moments.

Definition 2.6 (Mittag-Leffler moment). *The Mittag-Leffler moment of order s and rate $\alpha > 0$ of a function f is introduced via*

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v\rangle^2)\, \mathrm{d}v. \tag{2.18}$$

Remark 2.7. In the rest of the paper we will use the fact that Mittag-Leffler moments can be represented as the following sum (a time series form), which follows from (2.15):

$$\int_{\mathbb{R}^d} f(t,v)\;\; \mathcal{E}_{2/s}(\alpha^{2/s}\, \langle v\rangle^2)\, \mathrm{d}v \;\;=\;\; \sum_{q=0}^{\infty} \frac{m_{2q}(t)\, \alpha^{2q/s}}{\Gamma(\frac{2}{s}\, q + 1)}. \tag{2.19}$$

Remark 2.8. Formally, by taking $k = \frac{2q}{s}$, the above sum becomes

$$\sum_{k\in \frac{2}{s}\mathbb{Z}} \frac{m_{ks}(t)\;\alpha^k}{\Gamma(k+1)},$$

and we show it relates to the time series in (1.3) with the difference being that the summation here goes over the fractions.

2.3. The main results. There are two important results in this manuscript. The first one relates to the angular averaged Povzner estimate with cancellation. It gives an estimate of the weight function G_ϕ in the weak formulation (2.9) when the test function is a monomial $\phi(v) = \langle v\rangle^{rq}$. We denote this weight function by

$$G_{rq} := G_{\langle v\rangle^{rq}} := \int_{\mathbb{S}^{d-1}} \left(\langle v'\rangle^{rq} + \langle v'_*\rangle^{rq} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}\right) B(|u|, \hat{u}\cdot\sigma)\ \mathrm{d}\sigma. \tag{2.20}$$

Lemma 2.9. *Suppose that the angular kernel $b(\cos\theta)$ satisfies the non-cutoff condition* (2.6) *with $\beta = 2$. Let $r, q > 0$. Then the weight function satisfies*

$$\begin{aligned} G_{rq}(v,v_*) \le |v-v_*|^\gamma \Big[&- A_2\left(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\right) + A_2\left(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\right) \\ &+\varepsilon_{qr/2}\, A_2\, \frac{qr}{2}\left(\frac{qr}{2}-1\right)\langle v\rangle^2\langle v_*\rangle^2\left(\langle v\rangle^2+\langle v_*\rangle^2\right)^{\frac{qr}{2}-2}\Big], \end{aligned} \tag{2.21}$$

where $A_2 = |\mathbb{S}^{d-2}|\int_0^\pi b(\cos\theta)\,\sin^d\theta\,\mathrm{d}\theta$ is finite by (2.6)*. The sequence $\varepsilon_{qr/2} =: \varepsilon_{\mathbf{q}}$, defined as*

$$\varepsilon_{\mathbf{q}} := \frac{2}{A_2}|\mathbb{S}^{d-2}| \int_0^\pi \left(\int_0^1 t\left(1-\frac{\sin^2\theta}{2}t\right)^{\mathbf{q}-2}\mathrm{d}t\right) b(\cos\theta)\ \sin^d\theta\,\mathrm{d}\theta, \tag{2.22}$$

has the following decay properties. If $b(\cos\theta)$ satisfies the non-cutoff assumption (2.6) *with $\beta\in(0,2]$, then*

$$0 \ <\ \varepsilon_{\mathbf{q}}\,\mathbf{q}^{1-\frac{\beta}{2}} \ \to 0 \qquad \text{as } \mathbf{q}\to\infty. \tag{2.23}$$

The sequence $\varepsilon_{\mathbf{q}}$ is the same as in [26]. Its decay properties (2.23) are also proved in [26], after invoking angular averaging and the dominated convergence theorem. Condition (2.23) is crucial for finding the highest-order s of the Mittag-Leffler moment that can be propagated in time.

Remark 2.10. This lemma relies on the polynomial inequality presented in Lemma 3.1. The decay rate of $\varepsilon_{\mathbf{q}}$ is fundamental for the success of summability arguments yet is not relevant for the generation and propagation of polynomial moments. In the angular cutoff case when term-by-term techniques were used, the corresponding constant had a rate $\varepsilon_q \approx q^{-r}$, with r depending on the integrability of b; see [8, 9, 20]. When the partial sum technique was employed in [1], the precise rate was not needed any longer. Here, however, in the non-cutoff case, the knowledge of the precise decay rate of $\varepsilon_{\mathbf{q}}$ becomes important again because of extra power of q in the last term of the right-hand side of (2.21).

The second main result, presented as an a priori estimate, consists of two parts. First, under the non-cutoff assumption (2.6) with $\beta = 2$, we provide a new proof of the generation of exponential moments of order $s \in (0,\gamma]$. Second, we show the propagation in time of the Mittag-Leffler moments of order $s\in(\gamma,2)$. When $s\in(\gamma,1]$, $\beta = 2$ in the non-cutoff (2.6) is assumed. When $s\in(1,2)$, the angular kernel is assumed to be less singular. Before we state the theorem, we remind the reader of the following notation:

$$L^1_k = \left\{ f\in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f\langle v\rangle^k \mathrm{d}v < \infty\right\}.$$

This is the natural Banach norm to solve the Boltzmann equation.

THEOREM 2.11 (generation and propagation of exponential-like moments). *Suppose f is a solution to the Boltzmann equation* (2.1) *with the collision kernel of the form* (2.4) *for hard potentials* (2.5) *and with initial data $f_0 \in L^1_2$.*

(a) *(Generation of exponential moments) If the angular kernel satisfies the non-cutoff condition* (2.6) *with $\beta = 2$, then the exponential moment of order γ is generated with a rate $r(t) = \alpha \min\{t,1\}$. More precisely, there are positive constants C, α, depending only on b, γ and initial mass and energy, such that*

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \min\{t,1\}\,|v|^\gamma}\, \mathrm{d}v \le C \quad \textit{for } t \ge 0. \tag{2.24}$$

(b) *(Propagation of Mittag-Leffler moments) Let $s \in (0,2)$, and suppose that the Mittag-Leffler moment of order s of the initial data f_0 is finite with a rate $r = \alpha_0$, that is,*

$$\int_{\mathbb{R}^d} f_0(v)\;\mathcal{E}_{2/s}\Big(\alpha_0^{2/s}\,\langle v\rangle^2\Big)\,\mathrm{d}v < M_0. \tag{2.25}$$

Suppose also that the angular cross section satisfies assumption (2.6)*:*

$$\begin{aligned} &\textit{with } \beta = 2 && \textit{if } s \in (0,1] \\ &\textit{with } \beta = \frac{4}{s} - 2 && \textit{if } s \in (1,2). \end{aligned} \tag{2.26}$$

Then there exist positive constants C, α, depending only on M_0, α_0, b, γ, and initial mass and energy such that the Mittag-Leffler moment of order s and rate $r(t) = \alpha$ remains uniformly bounded in time, that is,

$$\int_{\mathbb{R}^d} f(t,v)\;\mathcal{E}_{2/s}\Big(\alpha^{2/s}\,\langle v\rangle^2\Big)\,\mathrm{d}v < C \quad \textit{for } t \ge 0. \tag{2.27}$$

Remark 2.12. The angular singularity condition $\beta = \frac{4}{s} - 2$ in the case of Mittag-Leffler moments of order $s \in (1,2)$, continuously changes from $\beta = 2$ (for $s = 1$) to $\beta = 0$ (for $s = 2$). Hence, condition $\beta = \frac{4}{s} - 2$ continuously interpolates between the most singular kernel typically considered in the literature, which is (2.6) with $\beta = 2$, and an angular cutoff condition, which corresponds to (2.6) with $\beta = 0$. This also tells us that in the most singular case, one can propagate exponential moments of order $s \le 1$, while in angular cutoff cases, one can propagate exponential moments of order $s \le 2$ (to be completely rigorous, Theorem 2.11 goes up to $\beta > 0$, i.e., $s < 2$, but [1] already established the case $\beta = 0$, i.e., $s = 2$). The less singular the angular kernel is, the higher the order of the exponential moment that can be propagated is.

Remark 2.13. The propagation result of the theorem can be interpreted in two ways. First, for a Mittag-Leffler (or exponential) moment of order s to be propagated, the singularity of b should be such that it satisfies (2.6) with $\beta = \frac{4}{s} - 2$. On the other hand, given an angular kernel b that satisfies condition (2.6) with a parameter $\beta \in (0,2]$, one can propagate Mittag-Leffler (and exponential) moments of order $s \le \frac{4}{\beta+2}$.

Remark 2.14. We note that our result is a priori in the sense that it assumes the existence of solutions. However, two types of solutions can be used in the theorem. One example are weak solutions, whose existence was proven by Arkeryd [6] and later extended by Villani [32], under the assumption that initial data has finite mass, energy, and entropy and a moment of order $2+\delta$ for any $\delta > 0$. Another type of solutions that could be used are measure weak solutions constructed by Lu and Mouhot [26] (see also the result of Morimoto, Wang, and Yang [28]). These solutions exist if initial mass and energy are finite, provided that the angular kernel satisfies the condition $\int_0^\pi b(\cos\theta)\sin^d\theta(1+|\log(\sin\theta)|)\mathrm{d}\theta < \infty$, which automatically holds for kernels that satisfy condition (2.6) with $\beta < 2$.

Remark 2.15. Thanks to the fact (2.17) that Mittag-Leffler functions asymptotically behave like exponential functions, the propagation result (Theorem 2.11 (b)) can be stated in terms of exponential moments. Namely, first note that

$$e^{\alpha|v|^s} \le e^{\alpha\langle v\rangle^s} \le Ce^{\alpha|v|^s},$$

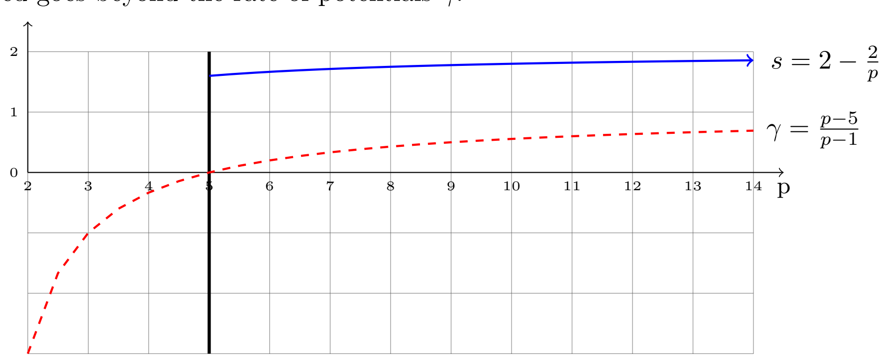
where $C = e^\alpha$ since $|v|^s \le \langle v\rangle^s = (1+|v|^2)^{s/2} \le 1+|v|^2$ for $0 < s/2 < 1$. This, together with (2.17), implies that finiteness of exponential moment of order $s \in (0,2)$ is equivalent to the finiteness of the corresponding Mittag-Leffler moment. This, in turn implies, as a corollary of Theorem 2.11 (b), the propagation of classical exponential moments. More precisely, suppose that $s \in (0,2)$ and that the angular cross section satisfies (2.6) with $\beta = \frac{4}{s} - 2$. Then, if inital data f_0 has finite exponential moment of order s and rate α_0, that is,

$$\int_{\mathbb{R}^d} f_0(v)e^{\alpha_0|v|^s}\mathrm{d}v < M_0,$$

then there exist positive constants C, α (depending only on M_0, α_0, b, γ, initial mass, and energy) such that the exponential moment of order s and rate α of $f(t,v)$ remains uniformly bounded in time, that is,

$$\int_{\mathbb{R}^d} f(t,v)e^{\alpha|v|^s}\mathrm{d}v < C \quad \text{for } t \ge 0.$$

Remark 2.16. In the case of the inverse-power-law model described via (2.8), in which hard potentials correspond to $p > 5$, the non-cutoff condition (2.6) is satisfied for $\beta > \nu$. Hence, Mittag-Leffler moments of orders $s < 2 - \frac{2}{p}$ can be propagated in time. In the graph below, the y-axis represents the order of exponential tails. The dashed red line marks the highest order of exponential moments that can be generated, while the blue line marks the highest order of Mittag-Leffler moments that can be propagated in time. This graph visually confirms that our propagation result indeed goes beyond the rate of potentials γ.



2.4. A strategy for proving Theorem 2.11. Details are provided in sections 5 and 6. The proof is inspired by the recent work [1], where propagation and generation of tail behavior (1.3) is obtained for angular cutoff regimes.

Our goal is to prove that solutions $f(t, v)$ of the Boltzmann equation for hard potentials and angular non-cutoff conditions admit L^1-Mittag-Leffler moments with parameters $a = \frac{2}{s}$ and $\alpha(t) = r(t)$ to be found. Because of the asymptotic behavior (2.16), that would imply that the asymptotic limit for large values of v is, indeed, exponential tail in v-space with order s and rate $r(t) = \alpha(t)$. Thus, our proof is based on studying partial sums of Mittag-Leffler functions $\mathcal{E}_a(\alpha^a x^2)$, with parameter $a = \frac{2}{s}$ and with rate $\alpha(t)$.

To this end, we work with nth partial sums associated to Mittag-Leffler functions, defined as

$$\mathcal{E}_a^n(\alpha, t) = \sum_{q=0}^{n} \frac{m_{2q}(t)\ \alpha^{aq}}{\Gamma(aq+1)}. \tag{2.28}$$

We need to prove that there exits a positive rate $\alpha(t)$ and a positive parameter a, both uniform in n, such that the sequence of finite sums converges as $n \to \infty$. In particular, we need show that $\mathcal{E}_a^n(\alpha, t)$ is bounded by a constant independent of time and independent of n. The values for a, α and the bound of the partial sums are found and shown to depend on data parameters given by the collisional kernel characterization and properties of the initial data.

In order to achieve all of this, we derive a differential inequality for $\mathcal{E}_a^n = \mathcal{E}_a^n(\alpha, t)$. The first step in this direction is to obtain differential inequalities for moments $m_{2q}(t)$ by studying the balance

$$m_{2q}'(t) = \int_{\mathbb{R}^d} Q(f, f)(t, v)\ \langle v\rangle^{2q} \mathrm{d}v, \tag{2.29}$$

which is a consequence of the Boltzmann equation. The right-hand side is estimated by bounding the polynomial moments of the collision operator by nonlinear forms of moments $m_k(t)$ of order up $k = 2q + \gamma$ with $0 < \gamma \le 1$. This requires finding the estimates of the weak formulation (2.9) with test functions $\phi(v) = \langle v\rangle^k$. Consequently, we need to estimate the angular integration within the weight function $G_{\langle v\rangle^{2q}}(v, v')$

$$\int_{\mathcal{S}^{d-1}} \left(\langle v'\rangle^{2q} + \langle v_*'\rangle^{2q} - \langle v\rangle^{2q} - \langle v_*\rangle^{2q}\right) b(\cos\theta)\mathrm{d}\sigma. \tag{2.30}$$

These estimates will lead, thanks to (2.29) and (2.9), to the following differential inequality for polynomial moments:

$$\begin{aligned} m_{2q}' \le{}& -K_1\, m_{2q+\gamma} + K_2\, m_{2q} \\ &+ K_3\, \varepsilon_q\, q\,(q-1) \sum_{k=1}^{k_q} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma}\right), \end{aligned} \tag{2.31}$$

where $K_1 = A_2 C_\gamma$, where A_2 was defined in (2.21) and C_γ just depends on the rate of potentials γ. Similarly K_2 and K_3 depend on these data parameters as well. The key property of this inequality is that the highest-order moment of the right-hand side comes with a negative sign, which is crucial for moments propagation and generation. Another important aspect of this differential inequality is the presence of the factor $q(q-1)$ in the last term, which was absent in angular cutoff cases. Because of this, it will be of great importance to know the decay rate for ε_q.

The second step (section 4) consists in the derivation of a differential inequality for partial sums $\mathcal{E}_a^n = \mathcal{E}_a^n(\alpha, t)$ obtained by adding n inequalities corresponding to (2.31) for renormalized polynomial moments $m_{2q}(t)\alpha^{aq}/\Gamma(aq+1)$. This will yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n \le c_{q_0} + \big(-K_1\,\mathcal{I}_{a,\gamma}^n + K_1\,c_{q_0} + K_2\,\mathcal{E}_a^n + \varepsilon_{q_0}\,q_0^{2-a}\,K_3\,C\,\mathcal{E}_a^n\,\mathcal{I}_{a,\gamma}^n\big)\,. \tag{2.32}$$

In particular we obtain an ordinary differential inequality for the partial sum $\mathcal{E}_a^n$ that depends on a shifted partial sum $\mathcal{I}_{a,\gamma}^n$, defined by

$$\mathcal{I}_{a,\gamma}^n(\alpha,t) = \sum_{q=0}^{n} \frac{m_{2q+\gamma}(t)\;\alpha^{aq}}{\Gamma(aq+1)}. \tag{2.33}$$

The derivation of the last term in the right-hand side of (2.32) requires a decay property of combinatoric sums of Beta functions. These estimates are very delicate and are presented in detail in Lemmas A.6 and A.7 in the Appendix. The constants K_1, K_2, and K_3 only depend on the singularity conditions (2.6), and so they are independent of n and on any moment q. The constant c_{q_0} depends only on a finite number q_0 of moments of the initial data. The choice of q_0 is crucial to control the long time behavior of solutions to inequality (2.32), and it is done such that $\varepsilon_{q_0}\;q_0^{2-a}\,K_3 < K_1/2$ after using condition (2.23) in Lemma 2.9.

Finally, after showing that $\mathcal{I}_{a,\gamma}^n(\alpha,t)$ is bounded below by the sum of two terms depending linearly on $\mathcal{E}_a^n(\alpha,t)$ and on mass m_0 and nonlinearly on the rate α, we obtain the following differential inequality for partial sums in the case of propagation of initial Mittag-Leffler moments:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(t) \le -\frac{K_1}{2\,\alpha^{\frac{\gamma}{2}}}\mathcal{E}_a^n(t) + \frac{K_1\,m_0\,e^{\alpha^{1-a}}}{2\alpha^{\frac{\gamma}{2}}} + \mathcal{K}_0 \quad \textit{(propagation estimate)}.$$

The constant $\mathcal{K}_0$ depends on data parameters characterizing q_0, c_{q_0} and K_i, $i = 1,2,3$. In addition, for the generation case, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\gamma^n \le -\frac{1}{t}\left(\frac{K_1(E_\gamma^n - m_0)}{2\alpha} - C_{q_0}\right) + \mathcal{K}_0 \quad \textit{(generation estimate)}.$$

Thus, the differential inequalities (2.32) are reduced to linear ones. Both inequalities have corresponding solutions for choices on parameters a and α that are independent on n and time t and will depend on q_0, which depends only on data parameters.

3. Angular averaging lemma.

This section is about the proof of the angular averaging with cancellation (i.e., Lemma 2.9), a crucial step for controlling moments and summability of their renormalization by the Gamma function. One of the tools used in the proof is the following estimate on symmetrized convex binomial expansions.

LEMMA 3.1 (symmetrized convex binomial expansions estimate). *Let $a, b \ge 0$, $t \in [0,1]$, and $p \in (0,1] \cup [2,\infty)$. Then*

$$\begin{aligned}&\big(ta + (1-t)b\big)^p + \big((1-t)a + tb\big)^p - a^p - b^p \\ &\quad \le -2t(1-t)\big(a^p + b^p\big) + 2t(1-t)\big(ab^{p-1} + a^{p-1}b\big).\end{aligned} \tag{3.1}$$

Proof. Suppose $p \ge 2$. The case $p \in (0,1]$ can be done analogously. Due to the symmetry of the inequality (3.1), we may without the loss of generality assume that $a \ge b$. Since all the terms have homogeneity p, the inequality (3.1) is equivalent to showing

$$F(z) \ge 0 \qquad \forall z \ge 1,$$

where $F(z)$ is defined by

$$F(z) := \big(1 - 2t(1-t)\big)\big(z^p + 1\big) + 2t(1-t)\big(z + z^{p-1}\big) \\ - \big(tz + (1-t)\big)^p - \big((1-t)z + t\big)^p.$$

It is easy to check that

$$F''(z) = (p-1)\Big[p\big(1 - 2t(1-t)\big)z^{p-2} + 2t(1-t)(p-2)z^{p-3} \\ - pt^2\big(tz + (1-t)\big)^{p-2} - p(1-t)^2\big((1-t)z + t\big)^{p-2}\Big].$$

As $tz + (1-t)$ and $(1-t)z + t$ are two convex combinations of z and 1 and since $z \geq 1$, we have that $tz + (1-t) \leq z$ and $(1-t)z + t \leq z$. Since $p \geq 2$, this implies $(tz + (1-t))^{p-2} \leq z^{p-2}$ and $((1-t)z + t)^{p-2} \leq z^{p-2}$. Therefore,

$$\begin{aligned} \frac{F''(z)}{p-1} &\geq p\big(1 - 2t(1-t)\big)z^{p-2} \ + \ 2t(1-t)(p-2)z^{p-3} - \ pt^2 z^{p-2} \ - \ p(1-t)^2 z^{p-2} \\ &= 2t(1-t)(p-2)z^{p-3} \\ &\geq 0. \end{aligned}$$

Thus, $F''(z) \geq 0$ for $z \geq 1$. So, $F'(z)$ is increasing. Since $F'(1) = 0$, we have that $F'(z) \geq 0$ for $z \geq 1$. Finally using the fact that $F(1) = 0$, we conclude $F(z) \geq 0$ for $z \geq 1$. ∎

We are now ready to prove the new form of the angular averaging with a cancellation-type lemma. For another version, see [26].

Proof of Lemma 2.9. Recall the definition of the weight G_{rq}:

$$G_{rq}(v, v_*) := |v - v_*|^\gamma \int_{\mathcal{S}^{d-1}} b(\cos\theta)\, \sin^{d-2}\theta\, \Delta\langle v\rangle^{rq}\, \mathrm{d}\sigma, \tag{3.2}$$
$$\text{where} \qquad \Delta\langle v\rangle^{rq} = \langle v'\rangle^{rq} + \langle v'_*\rangle^{rq} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}.$$

This integral is rigorous even in cases when $\int_{\mathbb{S}^{d-1}} B(|u|, \cos\theta)\, \mathrm{d}\sigma$ is unbounded by an angular cancellation. A natural way of handling the cancellation is to decompose $\sigma \in \mathbb{S}^{d-1}$ into $\theta \in [0, \pi]$ and its corresponding azimuthal variable $\omega \in \mathbb{S}^{d-2}$, i.e., $\sigma = \cos\theta\, \hat{u} + \sin\theta\, \omega$, where $\mathbb{S}^{d-2}(\hat{u}) = \{\omega \in \mathbb{S}^{d-1} : \omega \cdot \hat{u} = 0\}$. See Figure 2.

This decomposition allows handling the lack of integrability concentrated at the origin of the polar direction $\theta = 0$. However, it requires a specific way of decomposing

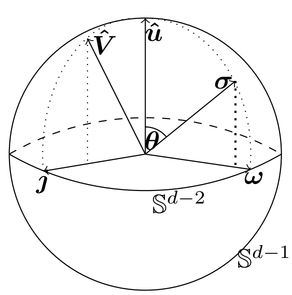


FIG. 2. *Decomposition of* σ.

$\langle v'\rangle^2$ and $\langle v'_*\rangle^2$ that separates the part that depends on ω. More precisely, $\langle v'\rangle^2$ and $\langle v'_*\rangle^2$ are decomposed into a sum of a convex combination of the local energies proportional to a function of the polar angle θ and another term depending on both the polar angle and ω (see the Appendix for details):

$$\begin{aligned}\langle v'\rangle^2 &= E_{v,v_*}(\theta) \;+\; P(\theta,\omega),\\ \langle v'_*\rangle^2 &= E_{v,v_*}(\pi-\theta) \;-\; P(\theta,\omega).\end{aligned} \tag{3.3}$$

Here $P(\theta,\omega) = |v\times v_*|\,\sin\theta\,(j\cdot\omega)$ is a null form in ω by averaging, i.e.,

$$\int\limits_{\mathbb{S}^{d-2}} P(\theta,\omega)\mathrm{d}\omega = 0,$$

and $E_{v,v_*}(\theta)$ is a convex combination of $\langle v\rangle^2$ and $\langle v'_*\rangle^2$ given by

$$E_{v,v_*}(\theta) \;=\; t\,\langle v\rangle^2 \;+\; (1-t)\,\langle v_*\rangle^2, \qquad \text{where} \quad t \;=\; \sin^2\frac{\theta}{2}.$$

These two fundamental properties make the weight function $G_{rq}(v,v_*)$ well defined for every v and v_* for sufficiently smooth test functions ($\phi\in C^2(\mathbb{R}^d)$) even under the non-cutoff assumption (2.6) with $\beta=2$. In fact, Taylor expansions associated to $\langle v'\rangle^{rq}$ are a sum of a power of $E_{v,v_*}(\theta)$, plus a null form in the azimuthal direction, plus a residue proportional to $\sin^2\theta$ that will secure the integrability of the angular cross section with respect to the scattering angle θ. Indeed, Taylor expand $\langle v'\rangle^{rq}$ around $E(\theta)$ up to the second order to obtain

$$\begin{aligned}\langle v'\rangle^{rq} &= \big(E_{v,v_*}(\theta) + h\sin(\theta)\,(j\cdot\omega)\big)^{\frac{rq}{2}}\\ &= \big(E_{v,v_*}(\theta)\big)^{rq/2} + \frac{rq}{2}\big(E_{v,v_*}(\theta)\big)^{\frac{rq}{2}-1}\,h\,\sin\theta\,(j\cdot\omega)\\ &\quad + \frac{rq}{2}\Big(\frac{rq}{2}-1\Big)\,h^2\,\sin^2\theta\,(j\cdot\omega)^2\int_0^1(1-t)\big[E(\theta)+t\,h\,\sin\theta\,(j\cdot\omega)\big]^{\frac{rq}{2}-2}\mathrm{d}t.\end{aligned} \tag{3.4}$$

Similar identity can be obtained for $\langle v'_*\rangle^{rq}$.

Since the collisional cross section is independent of the azimuthal integration, we will make use of the following property. Any vector j lying in the plane orthogonal to the direction of u is nullified by multiplication and averaging with respect to the azimuthal direction with respect to u, that is, $\int_{\mathbb{S}^{d-2}} j\cdot\omega\,\mathrm{d}\omega = 0$.

Therefore, we can write $G_{rq}(v,v_*)$ as the sum of two integrals on the $\mathbb{S}^{d-1}$ sphere, whose first integrand contains the zero-order order term of the Taylor expansion of both $\langle v'_*\rangle^{rq}$ and $\langle v'\rangle^{rq}$ subtracted by their corresponding unprimed forms, while the second integrand is just the second-order term of the Taylor expansion (3.4):

$$\begin{aligned}G_{rq}(v,v_*) &= I_1 + I_2\\ &= \int_0^\pi\int_{\mathbb{S}^{d-2}}\big(E_{v,v_*}(\theta)^{rq/2} + E_{v,v_*}(\pi-\theta)^{rq/2} - \langle v\rangle^{rq} - \langle v_*\rangle^{rq}\big)\\ &\qquad\times b(\cos\theta)\;\sin^{d-2}\theta\,\mathrm{d}\omega\,\mathrm{d}\theta\\ &\quad + \frac{rq}{2}\Big(\frac{rq}{2}-1\Big)\,h^2\int_0^\pi \sin^d\theta\,b(\cos\theta)\int_{\mathbb{S}^{d-2}}(j\cdot\omega)^2\int_0^1(1-t)\\ &\qquad\times\Big(\big[E_{v,v_*}(\theta)+t\,h\,\sin\theta\,(j\cdot\omega)\big]^{\frac{rq}{2}-2}\\ &\qquad\qquad + \big[E_{v,v_*}(\pi-\theta)-t\,h\,\sin\theta\big]^{\frac{rq}{2}-2}\Big)\,\mathrm{d}t\mathrm{d}\omega\mathrm{d}\theta.\end{aligned} \tag{3.5}$$

At this point we use inequality (3.1) to estimate the first integral I_1. We use it with $a = \langle v\rangle^2$, $b = \langle v_*\rangle^2$, and $t = \cos^2\frac{\theta}{2}$, which yields

$$I_1 \le \left|\mathbb{S}^{d-2}\right| \int_0^\pi -\frac{\sin^2\theta}{2}\left(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\right) b(\cos\theta)\ \sin^{d-2}\theta\,\mathrm{d}\theta \tag{3.6}$$
$$+ \int_0^\pi \frac{\sin^2\theta}{2}\left(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\right) b(\cos\theta)\ \sin^{d-2}\theta\,\mathrm{d}\theta$$
$$= -A_2\left(\langle v\rangle^{rq} + \langle v_*\rangle^{rq}\right) + A_2\left(\langle v\rangle^{rq-2}\langle v_*\rangle^2 + \langle v\rangle^2\langle v_*\rangle^{rq-2}\right).$$

The constant A_2 was defined after (2.21).

For the second-order term I_2, we use that $(j\cdot\omega)^2 \le 1$ and $h = |v \times v_*| \le \langle v\rangle\,\langle v_*\rangle$ and that (see [26])

$$\left|E_{v,v_*}(\theta) + th\sin\theta\,(j\cdot\omega)\right| \le \left(\langle v\rangle^2 + \langle v_*\rangle^2\right)\left(1 - \frac{t}{4}\sin^2\theta\right) \tag{3.7}$$

to conclude

$$I_2 \le \frac{rq}{2}\left(\frac{rq}{2} - 1\right)\langle v\rangle^2\langle v_*\rangle^2\left|\mathbb{S}^{d-2}\right| \int_0^\pi \sin^d\theta\, b(\cos\theta)$$
$$\times \int_0^1 2(1-t)\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{\frac{rq}{2}-2}\left(1 - \frac{1-t}{4}\sin^2\theta\right)^{\frac{rq}{2}-2} \mathrm{d}t\,\mathrm{d}\theta.$$

After a simple change of variables ($t \mapsto 1-t$) and recalling the definition of constant $\varepsilon_{rq/2}$ in (2.22), we see that

$$I_2 \le \varepsilon_{rq/2}\, A_2\, \frac{rq}{2}\left(\frac{rq}{2} - 1\right)\ \langle v\rangle^2\langle v_*\rangle^2\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{\frac{rq}{2}-2}. \tag{3.8}$$

Putting together the estimate for I_1 and for I_2, we obtain the desired estimate on the weight $G_{rq}(v, v_*)$.

4. Ordinary differential inequalities for moments. In this section we present two differential inequalities for polynomial moments (Proposition 4.1) which will be essential for the proof of Theorem 2.11. We also state and prove a result about generation of polynomial moments in the non-cutoff case (Proposition 4.2). Before we state the proposition, we recall the "floor function" of a real number, which in the case of a positive real number $x \in \mathbb{R}^+$ coincides with the integer part of x:

$$\lfloor x\rfloor := \text{integer part of } x. \tag{4.1}$$

PROPOSITION 4.1. *Suppose all the assumptions of Theorem* 2.11 *are satisfied. Let* $q \in \mathbb{N}$, *and define* $k_p = \lfloor\frac{p+1}{2}\rfloor$ *for any* $p \in \mathbb{R}$ *to be the integer part of* $(p+1)/2$. *Then for some constants* $K_1, K_2, K_3 > 0$ *(depending on* γ, $b(\cos\theta)$, *dimension* d*), we have the following two ordinary differential inequalities for polynomial moments of the solution* f *to the Boltzmann equation:*

(a) *The "$m_{\gamma k}$ version" needed for the generation of exponential moments:*

$$m'_{\gamma q}(t) \le -K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + K_3\,\varepsilon_{q\gamma/2}\,\frac{q\gamma}{2}\left(\frac{q\gamma}{2} - 1\right) \tag{4.2}$$
$$\times \sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{\gamma}}} \begin{pmatrix} \frac{q}{2} - \frac{2}{\gamma} \\ k-1 \end{pmatrix} \left(m_{2\gamma k+\gamma}\, m_{\gamma q-2\gamma k} + m_{2\gamma k}\, m_{\gamma q-2\gamma k+\gamma}\right).$$

(b) *The "m_{2k} version" needed for propagation of Mittag-Leffler moments:*

$$m'_{2q} \leq -K_1\, m_{2q+\gamma} + K_2\, m_{2q} + K_3\, \varepsilon_q\, q(q-1) \times \sum_{k=1}^{k_q} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma} \right). \tag{4.3}$$

In both cases, the constant $K_1 = A_2 C_\gamma$, where A_2 was defined in (2.21) and C_γ, to be defined in the proof below, only depends on the γ rate of the hard potentials. Similarly K_2 and K_3 also depend on data through the dependence on A_2 and C_γ.

Proof. We start the proof by analyzing m_{rq} with a general polynomial weight $\langle v \rangle^{rq}$. Then by setting $r = \gamma$ we shall derive (a), and by setting $r = 2$ we shall obtain (b). Recall that after multiplying the Boltzmann equation (2.1) by $\langle v \rangle^{rq}$, the weak formulation (2.9) yields

$$m'_{rq}(t) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} f f_*\ G_{rq}(v, v_*)\, \mathrm{d}v\, \mathrm{d}v_*. \tag{4.4}$$

The weight function G_{rq} can be estimated as in Lemma 2.9, which yields

$$\begin{aligned} m'_{rq}(t) \leq & -\frac{A_2}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\ f_*\ |v - v_*|^\gamma\ \left(\langle v \rangle^{rq} + \langle v_* \rangle^{rq} \right) \mathrm{d}v \mathrm{d}v_* \\ & + \frac{A_2}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\ f_*\ |v - v_*|^\gamma\ \left(\langle v \rangle^{rq-2} \langle v_* \rangle^2 + \langle v \rangle^2 \langle v_* \rangle^{rq-2} \right) \mathrm{d}v \mathrm{d}v_* \\ & + \frac{A_2}{2}\, \varepsilon_{rq/2}\, \frac{rq}{2} \left(\frac{rq}{2} - 1 \right) \\ & \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\ f_*\ |v - v_*|^\gamma\ \langle v \rangle^2 \langle v_* \rangle^2 \left(\langle v \rangle^2 + \langle v_* \rangle^2 \right)^{\frac{rq}{2} - 2} \mathrm{d}v \mathrm{d}v_*. \end{aligned} \tag{4.5}$$

We estimate $|v - v_*|^\gamma$ via elementary inequalities

$$|v - v_*|^\gamma \leq C_\gamma^{-1} \left(\langle v \rangle^\gamma + \langle v_* \rangle^\gamma \right) \qquad \text{and} \qquad |v - v_*|^\gamma \geq C_\gamma \langle v \rangle^\gamma - \langle v_* \rangle^\gamma, \tag{4.6}$$

where $C_\gamma = \min\{1, 2^{1-\gamma}\}$ (see, for example, [1]). As an immediate consequence,

$$\begin{aligned} |v - v_*|^\gamma \left(\langle v \rangle^{rq} + \langle v_* \rangle^{rq} \right) & \geq \left(C_\gamma \langle v \rangle^\gamma - \langle v_* \rangle^\gamma \right) \langle v \rangle^{rq} + \left(C_\gamma \langle v_* \rangle^\gamma - \langle v \rangle^\gamma \right) \langle v_* \rangle^{rq} \\ & = C_\gamma \left(\langle v \rangle^{rq+\gamma} + \langle v_* \rangle^{rq+\gamma} \right) - \left(\langle v \rangle^{rq} \langle v_* \rangle^\gamma + \langle v \rangle^\gamma \langle v_* \rangle^{rq} \right) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & |v - v_*|^\gamma \left(\langle v \rangle^{rq-2} \langle v_* \rangle^2 + \langle v \rangle^2 \langle v_* \rangle^{rq-2} \right) \\ & \quad \leq C_\gamma^{-1} \left(\langle v \rangle^\gamma + \langle v_* \rangle^\gamma \right) \left(\langle v \rangle^{rq-2} \langle v_* \rangle^2 + \langle v \rangle^2 \langle v_* \rangle^{rq-2} \right) \\ & \quad \leq 2 C_\gamma^{-1} \left(\langle v \rangle^{rq} \langle v_* \rangle^\gamma + \langle v \rangle^\gamma \langle v_* \rangle^{rq} \right), \end{aligned} \tag{4.8}$$

where the last inequality uses Lemma A.1. Combining (4.5) with (4.7) and (4.8) we obtain

$$\begin{aligned}
m'_{rq}(t) \le &-\frac{A_2}{2}\, C_\gamma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f f_* \big(\langle v\rangle^{rq+\gamma} + \langle v_*\rangle^{rq+\gamma}\big)\mathrm{d}v\mathrm{d}v_* \\
&+ \frac{A_2}{2}\left(1+2C_\gamma^{-1}\right)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, f_*\big(\langle v\rangle^{rq}\langle v_*\rangle^{\gamma} + \langle v\rangle^{\gamma}\langle v_*\rangle^{rq}\big)\mathrm{d}v\mathrm{d}v_* \\
&+ \frac{A_2\,\varepsilon_{rq/2}}{2\,C_\gamma}\,\frac{rq}{2}\left(\frac{rq}{2}-1\right) \\
&\quad\times \iint_{\mathbb{R}^{2d}} f f_*\big(\langle v\rangle^{\gamma} + \langle v_*\rangle^{\gamma}\big)\langle v\rangle^2\langle v_*\rangle^2\big(\langle v\rangle^2+\langle v_*\rangle^2\big)^{\frac{rq}{2}-2}\mathrm{d}v\mathrm{d}v_* \\
\le &-A_2\, C_\gamma m_0(t) m_{rq+\gamma}(t) + A_2(1+2C_\gamma^{-1}) m_\gamma(t) m_{rq}(t) \\
&+ \frac{A_2\varepsilon_{rq/2}}{2\,C_\gamma}\,\frac{rq}{2}\left(\frac{rq}{2}-1\right) \\
&\quad\times \iint_{\mathbb{R}^{2d}} f f_*(\langle v\rangle^{\gamma} + \langle v_*\rangle^{\gamma})\langle v\rangle^2\langle v_*\rangle^2(\langle v\rangle^2+\langle v_*\rangle^2)^{\frac{rq}{2}-2}\mathrm{d}v\mathrm{d}v_*.
\end{aligned}$$

Therefore, since $0<\gamma\le 1$, by conservation of mass and energy, $m_0(t)=m_0(0)$ and $m_\gamma(t)\le m_2(0)$,

$$\begin{aligned}
m'_{rq}(t) \le &-K_1\, m_{rq+\gamma}(t) + K_2\, m_{rq}(t) + \frac{K_3}{2}\,\varepsilon_{rq/2}\,\frac{rq}{2}\left(\frac{rq}{2}-1\right) \\
&\times \iint_{\mathbb{R}^{2d}} f f_*(\langle v\rangle^{\gamma} + \langle v_*\rangle^{\gamma})\langle v\rangle^2\langle v_*\rangle^2(\langle v\rangle^2+\langle v_*\rangle^2)^{\frac{rq}{2}-2}\mathrm{d}v\mathrm{d}v_*,
\end{aligned} \tag{4.9}$$

where $K_1 = A_2\, C_\gamma\, m_0(0)$, $\quad K_2 = A_2\,(1+2C_\gamma^{-1})\, m_2(0)$, and $\quad K_3 = \frac{A_2}{C_\gamma}$, so these three constants only depend on the initial mass and energy, on the rate of the potential γ, and on the angular singularity condition (2.6) that determines the constant A_2.

From here, we proceed to prove (a) and (b) separately.

(a) Setting $r=\gamma$ in (4.9), applying the following elementary polynomial inequality which is valid for $\gamma\in(0,1]$

$$\big(\langle v\rangle^2+\langle v_*\rangle^2\big)^{\frac{\gamma q}{2}-2} \le \big(\langle v\rangle^{2\gamma}+\langle v_*\rangle^{2\gamma}\big)^{\frac{q}{2}-\frac{2}{\gamma}}, \tag{4.10}$$

and using the polynomial Lemma A.3 yields

$$\begin{aligned}
m'_{\gamma q}(t) \le &-K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + \frac{K_3}{2}\,\varepsilon_{\gamma q/2}\,\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right) \\
&\times \iint_{\mathbb{R}^{2d}} f\, f_*\big(\langle v\rangle^{\gamma}+\langle v_*\rangle^{\gamma}\big)\langle v\rangle^2\langle v_*\rangle^2\big(\langle v\rangle^{2\gamma}+\langle v_*\rangle^{2\gamma}\big)^{\frac{q}{2}-\frac{2}{\gamma}}\,\mathrm{d}v\mathrm{d}v_* \\
\le &-K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + \frac{K_3}{2}\,\varepsilon_{\gamma q/2}\,\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\iint_{\mathbb{R}^{2d}} f\, f_*\,\big(\langle v\rangle^{\gamma}+\langle v_*\rangle^{\gamma}\big) \\
&\times \sum_{k=0}^{k_{\frac{q}{2}-\frac{2}{\gamma}}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k}\big(\langle v\rangle^{2\gamma k+2}\langle v_*\rangle^{\gamma q-2\gamma k-2} + \langle v\rangle^{\gamma q-2\gamma k-2}\langle v_*\rangle^{2\gamma k+2}\big)\,\mathrm{d}v\mathrm{d}v_*
\end{aligned}$$

$$\leq -K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + K_3\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\cdot$$
$$\times \sum_{k=0}^{k_{\frac{q}{2}-\frac{2}{\gamma}}} \begin{pmatrix} \frac{q}{2}-\frac{2}{\gamma} \\ k \end{pmatrix} \left(m_{2\gamma k+2+\gamma}\, m_{\gamma q-2\gamma k-2} + m_{\gamma q-2\gamma k-2+\gamma}\, m_{2\gamma k+2}\right) \mathrm{d}v\mathrm{d}v_*.$$

Finally, reindexing k to $k-1$ and applying Lemma A.1 yields

$$m'_{\gamma q}(t) \leq -K_1\, m_{\gamma q+\gamma} + K_2\, m_{\gamma q} + K_3\, \varepsilon_{\gamma q/2}\, \frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)$$
$$\times \sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{\gamma}}} \begin{pmatrix} \frac{q}{2}-\frac{2}{\gamma} \\ k-1 \end{pmatrix} \left(m_{2\gamma k+\gamma}\, m_{\gamma q-2\gamma k} + m_{\gamma q-2\gamma k+\gamma}\, m_{2\gamma k}\right) \mathrm{d}v\mathrm{d}v_*,$$

which completes proof of (a).

(b) Now, we set $r=2$ in (4.9) and apply Lemma A.3 to obtain

$$m'_{2q}(t) \leq -K_1\, m_{2q+\gamma} + K_2\, m_{2q} + K_3 \varepsilon_q\, q(q-1) \iint_{\mathbb{R}^{2d}} f\, f_* \left(\langle v\rangle^\gamma + \langle v_*\rangle^\gamma\right) \langle v\rangle^2 \langle v_*\rangle^2$$
$$\times \sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \left(\langle v\rangle^{2k}\, \langle v_*\rangle^{2(q-2)-2k} + \langle v\rangle^{2(q-2)-2k} \langle v_*\rangle^{2k}\right) \mathrm{d}v\mathrm{d}v_*$$
$$= -K_1\, m_{2q+\gamma} + K_2\, m_{2q} + K_3\, \varepsilon_q\, q(q-1) \iint_{\mathbb{R}^{2d}} f\, f_*\, \left(\langle v\rangle^\gamma + \langle v_*\rangle^\gamma\right)$$
$$\times \sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \left(\langle v\rangle^{2k+2} \langle v_*\rangle^{2q-2k-2} + \langle v\rangle^{2q-2k-2} \langle v_*\rangle^{2k+2}\right) \mathrm{d}v\mathrm{d}v_*$$
$$= -K_1\, m_{2q+\gamma} + K_2\, m_{2q} + K_3\, \varepsilon_q\, q(q-1)$$
$$\times \sum_{k=1}^{k_q} \binom{q-2}{k-1} \left(m_{2k+\gamma}\, m_{2q-2k} + m_{2k}\, m_{2q-2k+\gamma}\right).$$

The last equality is obtained by reindexing k to $k-1$ and using that $1+k_{q-2}=k_q$. This completes proof of (b). ∎

Proposition 4.2 (polynomial moment bounds for the non-cutoff case). *Suppose all the assumptions of Theorem* 2.11 *are satisfied. Let f be a solution to the homogeneous Boltzmann equation* (2.1) *associated to the initial data f_0:*

1. *Let the initial mass and energy be finite, i.e., $m_2(0)$ bounded; then for every $p>0$ there exists a constant $\mathbf{B}_{rp} \geq 0$, depending on 2^{rp}, γ, $m_2(0)$, and A_2 from condition* (2.6)*, such that*

$$m_{rp}(t) \leq \mathbf{B}_{rp} \max\{1, t^{-rp/\gamma}\} \quad \textit{for all} \ \ r\in\mathbb{R}^+ \ \ \textit{and}\ t\geq 0\,. \tag{4.11}$$

2. *Furthermore, if $m_{rp}(0)$ is finite, then the control can be improved to*

$$m_{rp}(t) \leq \mathbf{B}_{rp} \quad \textit{for all} \ \ r\in\mathbb{R}^+ \ \ \textit{and}\ t\geq 0. \tag{4.12}$$

Proof. These statements can be shown by studying comparison theorems for initial value problems associated with ordinary differential inequalities of the type

$$y'(t) + Ay^{1+c}(t) \leq By(t)$$

and comparing them to classical Bernoulli differential equations for the same given initial $y(0)$. In our context, these inequalities are a result of estimating moments for variable hard potentials, i.e., $\gamma > 0$ as indicated in (2.5). Comparison with Bernoulli-type differential equations was classically used in angular cutoff cases in [33, 34, 27, 1]. Also, it was used in the proof of propagation of L^1 exponential tails for the derivatives of the solution of the Boltzmann equation by means of geometric series methods in [9, 20, 3].

In fact, the extension to the non-cutoff case follows in a straightforward way from the moment estimates in Proposition 4.1. Indeed, the moment estimates, from either (4.2) or (4.3), show that the only negative contribution is on the highest-order moment, being either $m_{rq+\gamma}$ with $\gamma > 0$ for $r = \gamma$ or 2, respectively. Then, due to the fact that $\gamma > 0$, an application of the classical Jensen inequality with the convex function $\varphi(x) = x^{1+\gamma/(rp)}$ yields

$$m_{rp+\gamma}(t) \ \geq \ m_0^{-\gamma/(rp)}(0)\, m_{rp}^{1+\gamma/(rp)}(t) \ \text{ for all } t > 0.$$

Applying this estimate to the negative term in either (4.2) or (4.3) results in the following estimate:

$$m'_{rp} \ \leq \ B_{rp} m_{rp} - K_1 m_{rp+\gamma} \ \leq \ B_{rp} m_{rp} - K_1 m_{rp}^{1+\gamma/(rp)}, \tag{4.13}$$

with r either γ in (4.2) or 2 in (4.3). The constants are $K_1 = K_1(\gamma, A_2)$ with $0 < \gamma \leq 1$ and A_2 from the angular integrability condition (2.6) and $B_{rp} = B_{rp}(K_2, 2^{rp}K_3)$ after using that $\varepsilon_p \leq 1$, where K_2 and K_3 also depend on the initial data and collision kernel through γ and A_2.

Therefore, as in [33], we set

$$y(t) := m_{rp}(t),\ A := K_1,\ B := B_{rp} \text{ and } c = \gamma/(rp).$$

The bound (4.12) then follows by finding an upper solution that solves the associated Bernoulli ODE

$$y'(t) = By(t) - Ay^{1+c}(t)$$

with finite initial polynomial moment $y(0) = m_{rp}(0)$. This yields that for any $t > 0$,

$$\begin{aligned} m_{rp}(t) &\leq \left[m_{rp}^{-\gamma/(rp)}(0)\, e^{-t\,B\gamma/(rp)} + \frac{A}{B}\left(1 - e^{-t\,B\gamma/(rp)}\right)\right]^{-rp/\gamma} \\ &\leq \left[\frac{A}{B}\left(1 - e^{-t\,B\gamma/(rp)}\right)\right]^{-rp/\gamma} \\ &\leq \left(\frac{A}{B}\right)^{-rp/\gamma} \begin{cases} \left(\dfrac{rp}{B\gamma} e^{B\gamma/rp}\right)^{-rp/\gamma} t^{-rp/\gamma}, & t < 1, \\ \left(1 - e^{-\,B\gamma/(rp)}\right)^{-rp/\gamma}, & t \geq 1. \end{cases} \\ &\leq \mathbf{B}_{rp} \max\{1, t^{-rp/\gamma}\}, \end{aligned} \tag{4.14}$$

where $\mathbf{B}_{rp} := (\frac{K_1}{B_{rp}})^{-rp/\gamma} \max\{(\frac{rp}{\gamma B_{rp}} e^{\gamma B_{rp}/rp})^{-rp/\gamma},\ (1 - e^{-\,\gamma B_{rp}/(rp)})^{-rp/\gamma}\}$.

Now, since $m_{rp}(t)$ is a continuous function of time, if $m_{rp}(0)$ is finite for any $rp \geq 1$, then the bound for strictly positive times we just obtained in (4.14) implies

$$m_{rp}(t) \leq \mathbf{B}_{rp} \tag{4.15}$$

for possibly different constants $\mathbf{B}_{rp}$. We finally stress that constants $\mathbf{B}_{rp}$ depend on $2^{rp}, \gamma, m_2(0)$, and A_2 from condition (2.6). ∎

5. Proof of Mittag-Leffler moments' propagation.

Proof of Theorem 2.11 (b). Let us recall representation (2.19) of the Mittag-Leffler moment of order s and rate α in terms of infinite sums

$$\int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/s}(\alpha^{2/s}\,\langle v\rangle^2)\,\mathrm{d}v \ = \ \sum_{q=0}^{\infty} \frac{m_{2q}(t)\,\alpha^{2q/s}}{\Gamma(\frac{2}{s}\,q+1)}. \tag{5.1}$$

We introduce abbreviated notation $a = \frac{2}{s}$ and note that since $s \in (0,2)$, we have

$$1 < a := \frac{2}{s} < \infty. \tag{5.2}$$

We consider the nth partial sum, denoted by $\mathcal{E}_a^n$, and the corresponding sum, denoted by $\mathcal{I}_{a,\gamma}^n$, in which polynomial moments are shifted by γ. In other words, we consider

$$\mathcal{E}_a^n(\alpha,t) = \sum_{q=0}^{n} \frac{m_{2q}(t)\ \alpha^{aq}}{\Gamma(aq+1)}, \qquad \mathcal{I}_{a,\gamma}^n(\alpha,t) = \sum_{q=0}^{n} \frac{m_{2q+\gamma}(t)\ \alpha^{aq}}{\Gamma(aq+1)}.$$

For each $n \in \mathbb{N}$, define

$$T_n := \sup\left\{t \geq 0 \mid \mathcal{E}_a^n(\alpha,\tau) < 4M_0, \text{ for all } \tau \in [0,t)\right\}, \tag{5.3}$$

where the constant M_0 is the one from the initial condition (2.25).

This parameter T_n is well defined and positive. Indeed, since α will be chosen to be at least smaller than α_0, then at time $t = 0$ we have

$$\mathcal{E}_a^n(0) = \sum_{q=0}^{n} \frac{m_{2q}(0)\ \alpha^{aq}}{\Gamma(aq+1)} \ < \ \sum_{q=0}^{\infty} \frac{m_{2q}(0)\ \alpha_0^{aq}}{\Gamma(aq+1)} = \int f_0(v)\mathcal{E}_{2/s}(\alpha_0^{2/s}\,\langle v\rangle^2)\,\mathrm{d}v < 4\,M_0$$

uniformly in n. Therefore, since partial sums are continuous functions of time (they are finite sums and each $m_{2q}(t)$ is also continuous function in time t), we conclude that $\mathcal{E}_a^n(\alpha,t) < 4M_0$ holds for t on some positive time interval denoted $[0,t_n)$ with $t_n > 0$ (and hence $T_n > 0$).

Next, we look for an ordinary differential inequality that the partial sum $\mathcal{E}_a^n(\alpha,t)$ satisfies, following the steps presented in subsection 2.4. We start by splitting $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t)$ into the following two sums, where index q_0 will be fixed later, and then apply the moment differential inequality (4.3):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(\alpha,t) &= \sum_{q=0}^{q_0-1} \frac{m'_{2q}(t)\,\alpha^{aq}}{\Gamma(aq+1)} + \sum_{q=q_0}^{n} \frac{m'_{2q}(t)\ \alpha^{aq}}{\Gamma(aq+1)} \\
&\leq \sum_{q=0}^{q_0-1} \frac{m'_{2q}(t)\,\alpha^{aq}}{\Gamma(aq+1)} - K_1 \sum_{q=q_0}^{n} \frac{m_{2q+\gamma}(t)\,\alpha^{aq}}{\Gamma(aq+1)} + K_2 \sum_{q=q_0}^{n} \frac{m_{2q}(t)\,\alpha^{aq}}{\Gamma(aq+1)} \\
&\quad + K_3 \sum_{q=q_0}^{n} \frac{\varepsilon_q\,q(q-1)\,\alpha^{aq}}{\Gamma(aq+1)} \sum_{k=1}^{k_q} \binom{q-2}{k-1} \big(m_{2k+\gamma}\,m_{2(q-k)} + m_{2k} m_{2(q-k)+\gamma}\big) \\
&=: S_0 - K_1\,S_1 + K_2\,S_2 + K_3\,S_3.
\end{aligned} \tag{5.4}$$

We estimate each of the four sums S_0, S_1, S_2, and S_3 separately with the goal of comparing each of them to the functions $\mathcal{E}_a^n(\alpha, t)$ and $\mathcal{I}_{a,\gamma}^n(\alpha, t)$. We remark that the most involving term is S_3. It resembles the corresponding sum in the angular cutoff case [1], with a crucial difference that our sum S_3 has two extra powers of q, namely, $q(q-1)$. Therefore, a very sharp calculation is required to control the growth of S_3 as a function of the number q of moments. This is achieved by an appropriate renormalization of polynomial moments within S_3 and also by invoking the decay rate of associated combinatoric sums of Beta functions developed in the Appendix A.

The term S_0 can be bounded by terms that depend on the initial data and the parameters of the collision cross section. Indeed, from Proposition 4.2, the propagated polynomial moments can be estimated as follows:

$$m_p \leq \mathbf{B}_p \quad \text{and} \quad m'_p \leq B_p\, \mathbf{B}_p \qquad \text{for any } p > 0, \tag{5.5}$$

where the constant $\mathbf{B}_p$ defined in (4.14) depends on γ, the initial p-polynomial moment $m_p(0)$, and A_2 from condition (2.6).

In particular, for $0 < \gamma < 1$, we can fix q_0, to be chosen later, such that the constant

$$c_{q_0} := \max_{p \in I_{q_0}} \{\mathbf{B}_p,\, B_p\, \mathbf{B}_p\} \quad \text{with} \quad I_{q_0} = \{0, \ldots, 2q_0 + 1\} \tag{5.6}$$

depends only on q_0, γ, A_2 from condition (2.6) and the initial polynomial moments $m_q(0)$ for $q \in I_{q_0}$. Thus, due to the monotoncity of L_k^1 norms with respect to k as presented in (2.14), both the $2q$-moments and its derivatives, as well as the shifted moments of order $2q + \gamma$, are controlled by c_{q_0} as follows:

$$m_{2q}(t),\, m_{2q+\gamma}(t),\, m'_{2q}(t) \leq c_{q_0} \quad \text{for all } q \in \{0, 1, 2, \ldots q_0\}. \tag{5.7}$$

Therefore, for q_0 fixed, to be chosen later, S_0 is estimated by

$$\begin{aligned} S_0 := \sum_{q=0}^{q_0-1} \frac{m'_{2q}\, \alpha^{aq}}{\Gamma(aq+1)} &\leq c_{q_0} \sum_{q=0}^{q_0-1} \frac{\alpha^{aq}}{\Gamma(aq+1)} \\ &\leq c_{q_0} \sum_{q=0}^{q_0-1} \frac{(\alpha^a)^q}{\Gamma(q+1)} \leq c_{q_0}\, e^{\alpha^a} \leq 2c_{q_0} \end{aligned} \tag{5.8}$$

for the parameter α small enough to satisfy

$$\alpha < (\ln 2)^{1/a} \qquad \text{or equivalently} \quad e^{\alpha^a} \leq 2. \tag{5.9}$$

The second term S_1 is crucial, as it brings the negative contribution that will yield uniform-in-n and global-in-time control to an ordinary differential inequality for $\mathcal{E}_a^n(\alpha, t)$. In fact, S_1 is controlled from below by $\mathcal{I}_{a\gamma}^n(\alpha, t)$ as follows:

$$S_1 := \sum_{q=q_0}^{n} \frac{m_{2q+\gamma}\, \alpha^{aq}}{\Gamma(aq+1)} = \mathcal{I}_{a,\gamma}^n - \sum_{q=0}^{q_0-1} \frac{m_{2q+\gamma}\, \alpha^{aq}}{\Gamma(aq+1)}.$$

So, Using (5.7) and the estimate just obtained for S_0 in (5.8) yields the bound from below:

$$S_1 \geq \mathcal{I}_{a,\gamma}^n - c_{q_0} \sum_{q=0}^{q_0-1} \frac{\alpha^{aq}}{\Gamma(aq+1)} \geq \mathcal{I}_{a,\gamma}^n - 2c_{q_0}. \tag{5.10}$$

The sum S_2 is a part of the partial sum $\mathcal{E}_a^n$, so

$$S_2 \leq \mathcal{E}_a^n. \tag{5.11}$$

While this term is positive, it will need to be lower order than the one in the negative part of the right-hand side.

Finally, we estimate S_3 and show that it can be bounded by the product of $\mathcal{E}_a^n(\alpha, t)$ and $\mathcal{I}_{a,\gamma}^n(\alpha, t)$. We work out the details of the first term in the sum $S_3 := S_{3,1} + S_{3,2}$, that is, the one with $m_{2k+\gamma}\, m_{2(q-k)}$. The other sum with $m_{2k}\, m_{2(q-k)+\gamma}$ can be bounded by following a similar strategy. In order to generate both the partial sum $\mathcal{E}_a^n(\alpha, t)$ and the shifted one $\mathcal{I}_{a,\gamma}^n(\alpha, t)$, we make use of the following well-known relations between Gamma and Beta functions (see also Appendix A):

$$\begin{aligned} B(ak+1, a(q-k)+1) &= \frac{\Gamma(ak+1)\,\Gamma(a(q-k)+1)}{\Gamma(\,(ak+1)+(a(q-k)+1))} \\ &= \frac{\Gamma(ak+1)\,\Gamma(a(q-k)+1)}{\Gamma(aq+2)}. \end{aligned} \tag{5.12}$$

Therefore, multiplying and dividing products of moments $m_{2k+\gamma} m_{2(q-k)}$ in $S_{3,1}$ by $\Gamma(ak+1)\Gamma(a(q-k)+1)$ yields

$$\begin{aligned} S_{3,1} &:= \sum_{q=q_0}^{n} \frac{\varepsilon_q\, q\,(q-1)\,\alpha^{aq}}{\Gamma(aq+1)} \sum_{k=1}^{k_q} \binom{q-2}{k-1} m_{2k+\gamma}\, m_{2(q-k)} \\ &= \sum_{q=q_0}^{n} \varepsilon_q\, q\,(q-1) \sum_{k=1}^{k_q} \binom{q-2}{k-1} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\ &\quad \times B(ak+1, a(q-k)+1) \frac{\Gamma(aq+2)}{\Gamma(aq+1)}. \end{aligned}$$

Note that the factors $\frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)}$ and $\frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)}$ are the building blocks of $\mathcal{I}_{a,\gamma}^n(\alpha, t)$ and $\mathcal{E}_a^n(\alpha, t)$, respectively.

Next, since $\Gamma(aq+2)/\Gamma(aq+1) = aq+1$, using the inequality $\sum_k a_k\, b_k \leq \sum_k a_k \sum_k b_k$, it follows that

$$\begin{aligned} S_{3,1} &\leq \sum_{q=q_0}^{n} \varepsilon_q\,(aq+1)\,q\,(q-1) \left(\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \right) \\ &\quad \times \left(\sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \right). \end{aligned} \tag{5.13}$$

Next we show that the factor

$$(aq+1)\,q\,(q-1) \left(\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \right)$$

on the right-hand side of (5.13) grows at most as q^{2-a}. Indeed, using Lemma A.6, the sum of the Beta functions is bounded by $C_a (aq)^{-(1+a)}$. Therefore, $S_{3,1}$ is estimated by

$$S_{3,1} \leq C_a \sum_{q=q_0}^{n} \varepsilon_q\, q^{2-a} \left(\sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \right), \tag{5.14}$$

where C_a is a (possibly different) constant that depends on a. Now, by Lemma 2.9, the factor $\varepsilon_q \, q^{2-a}$ decreases monotonically to zero as $q \to \infty$ if the angular kernel $b(\cos\theta)$ satisfies (2.6) with $\beta = 2a-2$. Hence,

$$\varepsilon_q \, q^{2-a} \leq \varepsilon_{q_0} \, q_0^{2-a} \qquad \text{for any } q \geq q_0 \,, \tag{5.15}$$

and thus the term $S_{3,1}$ is further estimated by

$$S_{3,1} \leq C_a \, \varepsilon_{q_0} \; q_0^{2-a} \sum_{q=q_0}^{n} \sum_{k=1}^{k_q} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \, \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)}.$$

Finally, inspired by [1], we bound this double sum by the product of partial sums $\mathcal{E}_a^n \, \mathcal{I}_{a,\gamma}^n$. To achieve that, change the order of summation to obtain

$$\begin{aligned} S_{3,1} &\leq C_a \, \varepsilon_{q_0} \; q_0^{2-a} \sum_{k=0}^{k_n} \sum_{\max\{q_0, 2k-1\}}^{n} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\ &\leq C_a \varepsilon_{q_0} \; q_0^{2-a} \sum_{k=0}^{k_n} \frac{m_{2k+\gamma}\alpha^{ak}}{\Gamma(ak+1)} \sum_{\max\{q_0, 2k-1\}}^{n} \frac{m_{2(q-k)}\alpha^{a(q-k)}}{\Gamma(a(q-k)+1)} \\ &\leq C_a \, \varepsilon_{q_0} \; q_0^{2-a} \, \mathcal{I}_{a,\gamma}^n \, \mathcal{E}_a^n, \end{aligned} \tag{5.16}$$

obtaining the expected control of $S_{3,1}$. As mentioned above, the estimate of the companion sum $S_{3,2}$ follows in a similar way, so we can assert

$$S_3 \leq C_a \, \varepsilon_{q_0} \; q_0^{2-a} \, \mathcal{E}_a^n(t) \, \mathcal{I}_{a,\gamma}^n(t). \tag{5.17}$$

Next we obtain an ordinary differential inequality for $\mathcal{E}_a^n(t)$ depending only on data parameters and $\mathcal{I}_{a,\gamma}^n(t)$. Indeed, combining (5.8), (5.10), (5.11), and (5.16) with (5.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n \leq -K_1 \, \mathcal{I}_{a,\gamma}^n + 2 \, c_{q_0}(1+K_1) + K_2 \, \mathcal{E}_a^n + \varepsilon_{q_0} \; q_0^{2-a} \, C_a \, K_3 \, \mathcal{I}_{a,\gamma}^n \, \mathcal{E}_a^n \,. \tag{5.18}$$

Since, by the definition of time T_n, the partial sum $\mathcal{E}_a^n$ is bounded by the constant $4M_0$ on the time interval $[0, T_n]$, we can estimate, uniformly in n, the following two terms in (5.18):

$$2 \, c_{q_0}(1+K_1) + K_2 \, \mathcal{E}_a^n \leq 2 \, c_{q_0}(1+K_1) + 4K_2 \, M_0 =: \mathcal{K}_0, \tag{5.19}$$

where $\mathcal{K}_0$ depends only on the initial data and q_0 (still to be determined).

Thus, factoring out $\mathcal{I}_{a,\gamma}^n$ from the remaining two terms in (5.18) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n &\leq -\mathcal{I}_{a,\gamma}^n \left(K_1 - \varepsilon_{q_0} \; q_0^{2-a} \, C_a \, K_3 \, \mathcal{E}_a^n\right) + \mathcal{K}_0 \\ &\leq -\mathcal{I}_{a,\gamma}^n \big(K_1 - 4\varepsilon_{q_0} \; q_0^{2-a} \, C_a \, K_3 \, M_0\big) + \mathcal{K}_0, \end{aligned} \tag{5.20}$$

where in the last inequality we again used that, by the definition of T_n, we have $\mathcal{E}_a^n \leq 4M_0$ on the closed interval $[0, T_n]$. Now, since $\varepsilon_{q_0} \; q_0^{2-a}$ converges to zero as q_0 tends to infinity (by Lemma 2.9, as $b(\cos\theta)$ satisfies (2.6) with $\beta = 2a-2$), we can choose large enough q_0 (depending on γ, angular kernel (through the constant A_2), inital mass $m_0(0)$, initial value M_0 of the Mittag-Leffler moment of order s, and s) so that

$$K_1 - 4\varepsilon_{q_0}\, q_0^{2-a}\, C_a\, K_3\, M_0 > \frac{K_1}{2}. \tag{5.21}$$

For such choice of q_0 we then have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n \le -\frac{K_1}{2}\,\mathcal{I}_{a,\gamma}^n + \mathcal{K}_0. \tag{5.22}$$

The final step consists in finding a lower bound for $\mathcal{I}_{a,\gamma}^n$ in terms of $\mathcal{E}_a^n$. The following calculation follows from a revised form of the lower bound given in [1]:

$$\begin{aligned}
\mathcal{I}_{a,\gamma}^n(t) &:= \sum_{q=0}^{n} \frac{m_{2q+\gamma}\ \alpha^{aq}}{\Gamma(aq+1)} \ge \sum_{q=0}^{n} \int_{\langle v\rangle \ge \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q+\gamma}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\,\mathrm{d}v \\
&\ge \frac{1}{\alpha^{\gamma/2}} \sum_{q=0}^{n} \int_{\langle v\rangle \ge \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\,\mathrm{d}v \\
&= \frac{1}{\alpha^{\gamma/2}} \left(\sum_{q=0}^{n} \int_{\mathbb{R}^d} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\,\mathrm{d}v - \sum_{q=0}^{n} \int_{\langle v\rangle < \frac{1}{\sqrt{\alpha}}} \frac{\langle v\rangle^{2q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\,\mathrm{d}v \right) \\
&\ge \frac{1}{\alpha^{\gamma/2}} \left(\mathcal{E}_a^n(t - \sum_{q=0}^{n} \int_{\mathbb{R}^d} \frac{\alpha^{-q}\ \alpha^{aq}}{\Gamma(aq+1)}\ f(t,v)\,\mathrm{d}v \right) \\
&\ge \frac{1}{\alpha^{\gamma/2}} \left(\mathcal{E}_a^n(t) - m_0 \sum_{q=0}^{\infty} \frac{\alpha^{q(a-1)}}{\Gamma(aq+1)} \right) \\
&> \frac{1}{\alpha^{\frac{\gamma}{2}}}\,\mathcal{E}_a^n(t) - \frac{1}{\alpha^{\frac{\gamma}{2}}}\, m_0\, e^{\alpha^{a-1}}.
\end{aligned} \tag{5.23}$$

Therefore, applying inequality (5.23) to (5.22) yields the following linear differential inequality for the partial sum $\mathcal{E}_a^n$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_a^n(t) \le -\frac{K_1}{2\,\alpha^{\frac{\gamma}{2}}}\mathcal{E}_a^n(t) + \frac{K_1\, m_0\, e^{\alpha^{1-a}}}{2\alpha^{\frac{\gamma}{2}}} + \mathcal{K}_0.$$

Then, by the maximum principle for ordinary differential inequalities,

$$\begin{aligned}
\mathcal{E}_{2/s}^n(t) = \mathcal{E}_a^n(t) &\le M_0 + \frac{2\,\alpha^{\gamma/2}}{K_1} \left(\frac{K_1\, m_0\, e^{\alpha^{1-a}}}{2\alpha^{\dfrac{\gamma}{2}}} + \mathcal{K}_0. \right) \\
&= M_0 + m_0\, e^{\alpha^{1-a}} + \frac{2\,\alpha^{\gamma/2}}{K_1}\,\mathcal{K}_0 \\
&\le 4M_0
\end{aligned}$$

provided that $\alpha = \alpha_1$ is chosen sufficiently small so that

$$m_0\, e^{\alpha_1^{1-a}} + \frac{2\,\alpha_1^{\gamma/2}}{K_1}\,\mathcal{K}_0 < 3M_0, \tag{5.24}$$

which is possible since $a > 1$.

In conclusion, if q_0 is chosen according to (5.21) and hence depending only on the initial data, the initial Mittag-Leffler moment, γ, and A_2 from (2.6) and if $\alpha = \min\{\alpha_0, (\ln 2)^{1/\alpha}, \alpha_1\}$, from (5.24), we have that the *strict* inequality $\mathcal{E}_a^n(t) < 4M_0$ holds on the *closed* interval $[0, T_n]$ uniformly in n. Therefore, invoking the global continuity of $\mathcal{E}_a^n(t)$ once more, the set of time t for $\mathcal{E}_a^n(t) < 4M_0$ holds on a slightly larger half-open time interval $[0, T_n + \mu)$ with $\mu > 0$. This would contradict maximality of the definition of T_n unless $T_n = +\infty$. Hence, we conclude that $T_n = +\infty$ for all n. Therefore, we in fact have that

$$\mathcal{E}_a^n(\alpha, t) < 4M_0 \quad \text{for all } t \geq 0 \ \text{ for all } n \in \mathbb{N}.$$

Thus, by letting $n \to +\infty$, we conclude that $\mathcal{E}_a^\infty(\alpha, t) \leq 4M_0$ for all $t \geq 0$. That is,

$$\int_{\mathbb{R}^d} f(t, v) \ \ \mathcal{E}_{2/s}(\alpha^{2/s} \langle v \rangle^2) \, \mathrm{d}v \ \leq 4M_0 \quad \text{for all } t \geq 0. \tag{5.25}$$

Estimate (5.25) shows that the solution of the Boltzmann equation with the finite initial Mittag-Leffler moment of order s and rate α_0 will propagate Mittag-Leffler moments with the same order s and rate α satisfying $\alpha = \min\{\alpha_0, (\ln 2)^{1/\alpha}, \alpha_1\}$. This concludes the proof part(b) of Theorem 2.11. ∎

Part(a) of Theorem 2.11 concerns the generation of Mittag-Leffler or exponential moments. This is proven in the next section.

6. Proof of exponential moments' generation.

Proof of Theorem 2.11 (a). Notation and strategy are similar to those in the proof of Theorem 2.11 (b) contained in section 5. The goal is to find a positive and bounded real valued number α such that the solution $f(v, t)$ of the Boltzmann equation will have an exponential moment, of order γ and rate $\alpha \min\{t, 1\}$, generated for every positive time t, from the fact that the initial data $f_0(v)$ has finite energy given by $M_0^* := m_2(0)$.

The proof works with the exponential forms of order γ. From this viewpoint, the difference with respect to the propagation of the Mittag-Leffler moments result obtained in the previous section is that the propagation result had to be established for every order $s \in (0, 2)$, while now the generation of Mittag-Leffler moments of order s and rate α implies generation of such moments for all smaller orders $0 < s$. Hence, it suffices to consider just the order $\boldsymbol{s = \gamma}$.

First, for an arbitrary positive and bounded number α, we denote the nth partial sum of the exponential moment of order γ by $E_\gamma^n(\alpha t, t)$ and the corresponding one in which polynomial moments are shifted by γ by $I_{\gamma,\gamma}^n(\alpha t, t)$, that is,

$$E_\gamma^n(\alpha t, t) = \sum_{q=0}^{n} \frac{m_{\gamma q}(t) \ (\alpha t)^q}{\Gamma(q+1)} = \sum_{q=0}^{n} \frac{m_{\gamma q}(t) \ (\alpha t)^q}{q!} \tag{6.1}$$

$$\mathcal{I}_{\gamma,\gamma}^n(\alpha t, t) = \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t) \ (\alpha t)^q}{\Gamma(q+1)} = \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t) \ (\alpha t)^q}{q!} \,. \tag{6.2}$$

The form $E_\gamma^n(\alpha t, t)$ is the exponential moment of order γ with rate α of the probability density f in the Mittag-Leffler representation.

Define the time T_n^* as follows:

$$T_n^* := \min\left\{1, \ \sup\left\{t \geq 0 \mid E_\gamma^n(\alpha\tau, \tau) < 4M_0^*, \quad \text{for all } \tau \in [0, t) \ \right\}\right\}. \tag{6.3}$$

T_n^* is well defined where now the constant M_0^* is the sum of the initial conserved mass and energy, i.e., $M_0^* := M_0^*(t) = \int f(v,t)\langle v\rangle^2 \mathrm{d}v = \int f_0(v)\langle v\rangle^2 \mathrm{d}v$ as in the initial condition for the generation of the Mittag-Leffler moments estimate (2.24). Since moments are uniformly in time generated for the hard potential case, even for angular non-cutoff regimes (see [33]), then every finite sum $\mathcal{E}_a^n(\alpha t, t)$ is well defined and continuous in time. Note that for $t = 0$, we have that $E_\gamma^n(\alpha 0, 0) = m_0 < 4M_0^*$. Then, as in the previous case, continuity in time of partial sums $\mathcal{E}_a^n(\alpha t, t)$ implies that $\mathcal{E}_a^n(\alpha t, t) < 4M_0^*$ holds for t on some positive time interval $[0, t_n^*)$, which implies that $T_n^* > 0$. In addition, the definition (6.3) implies that $T_n^* \le 1$ for all $n \in \mathbb{N}$.

As we did in the previous section for the proof of propagation of Mittag-Leffler moments, we search for an ordinary differential inequality for $E_\gamma^n(\alpha t, t)$, depending only on data parameters and on $\mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)$, for a positive and bounded real valued α to be found and characterized.

To this end, we start by computing

$$\begin{aligned}\text{(6.4)}\quad \frac{\mathrm{d}}{\mathrm{d}t} E_\gamma^n(\alpha t, t) &= \alpha \sum_{q=1}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^{q-1}}{(q-1)!} + \sum_{q=0}^{n} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} \\ &= \alpha \sum_{q=1}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^{q-1}}{(q-1)!} + \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} + \sum_{q=q_0}^{n} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!},\end{aligned}$$

where index q_0 will be fixed later. The first sum in this identity is reindexed from $q-1$ to q and estimated by $I_{\gamma,\gamma}^n(\alpha t, t)$ (defined in (6.2)) as follows:

$$\sum_{q=0}^{n-1} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} \le \sum_{q=0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} = I_{\gamma,\gamma}^n(\alpha t, t).$$

Next, we replace the term $m'_{\gamma q}(t)$ by the upper bound in the ordinary differential inequality (4.2) just on the sums starting from q_0, for $\alpha > 0$, and for

$$\text{(6.5)}\qquad k_{q^*} := \left\lfloor \frac{q}{4} - \frac{1}{\gamma} + \frac{3}{2} \right\rfloor := \text{integer part of } \frac{q}{4} - \frac{1}{\gamma} + \frac{3}{2},$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E_\gamma^n(\alpha t, t) &\le \alpha I_{\gamma,\gamma}^n(\alpha t, t) + \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!} \\ \text{(6.6)}\qquad &\quad - K_1 \sum_{q=q_0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} + K_2 \sum_{q=q_0}^{n} \frac{m_{\gamma q}(t)\,(\alpha t)^q}{q!} \\ &\quad + K_3 \sum_{q=q_0}^{n} \frac{\varepsilon_{\gamma q/2} \frac{\gamma q}{2}(\frac{\gamma q}{2}-1)\,(\alpha t)^q}{q!} \sum_{k=1}^{k_{q_*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} \\ &\qquad \times ((m_{2\gamma k+\gamma}(t)\, m_{\gamma q-2\gamma k}(t) + m_{2\gamma k}(t)\, m_{\gamma q-2\gamma k+\gamma}(t)\,) \\ &=: \alpha \mathcal{I}_{\gamma,\gamma}^n(\alpha t, t) + S_0 - K_1\, S_1 + K_2\, S_2 + K_3\, S_3.\end{aligned}$$

We stress that the positive constant $K_1 = A_2\, C_\gamma$ depends only on the collision cross section with A_2 defined in (2.21) and C_γ only depending on $0 < \gamma \le 1$. In the following, we will estimate the terms in (6.6) to show that the negative one is of higher order uniformly in time t for a choice of α and q_0 that depend only on the initial and collision kernel data.

The term S_0 can be bounded by terms that depend on the initial data and the parameters of the collision cross section. Indeed, as was the case for the propagation estimates, from Proposition 4.2, setting $r = \gamma$ in (4.14), the generated polynomial moments can be estimated by

$$m_{\gamma q}(t) \le \mathbf{B}_{\gamma q} \max_{t>0}\{1, t^{-q}\} \qquad \text{and} \tag{6.7}$$
$$m'_{\gamma q}(t) \le B_{\gamma q} m_{\gamma q}(t) \le B_{\gamma q}\,\mathbf{B}_{\gamma q} \max_{t>0}\{1, t^{-q}\},$$

where the constant $\mathbf{B}_{\gamma q}$, now from (4.14), also depends on $m_2(0)$, γ, q, and A_2 from condition (2.6). Next, for q_0 fixed, to be chosen later, set

$$c^*_{q_0} := \max_{q\in\{0,\dots,q_0-1\}} \{\mathbf{B}_{\gamma q},\, B_{\gamma q}\,\mathbf{B}_{\gamma q}\}\,, \tag{6.8}$$

and then both the $2q$-moments and its derivatives are controlled by $c^*_{q_0}$ as follows:

$$m_{\gamma q}(t), m'_{\gamma q}(t) \le c^*_{q_0} \max_{t>0}\{1, t^{-q}\} \quad \text{for all } \ q \in \{0, \dots, q_0-1\}\,. \tag{6.9}$$

Thus, we can estimate S_0, for a fixed q_0, to be defined later, by

$$S_0 := \sum_{q=0}^{q_0-1} \frac{m'_{\gamma q}(t)\,(\alpha t)^q}{q!}$$
$$\le c^*_{q_0} \max_{t>0}\{1, t^{-q}\} \sum_{q=0}^{q_0-1} \frac{(\alpha t)^q}{q!}$$
$$\le c^*_{q_0} \max_{t>0}\{t^q, 1\} \sum_{q=0}^{q_0-1} \frac{\alpha^q}{q!} \tag{6.10}$$
$$\le c^*_{q_0} e^{\alpha} \le 2\, c^*_{q_0} \tag{6.11}$$

uniformly in $t \in [0, T_n^*] \subset [0,1]$ for any $\alpha \le \ln 2$. To obtain inequality (6.10) we used that $t \le T_n^* \le 1$.

The sum S_2 is a part of the partial sum E^n_γ; hence,

$$S_2 := \sum_{q=q_0}^{n} \frac{m_{\gamma q}(\alpha t)^q}{q!} \le E^n_\gamma(\alpha t, t). \tag{6.12}$$

The sum S_1 needs to be bounded from below because of the negativity of the term $K_1\, S_1$. To this end, using again the time-dependent estimates for moments from Proposition 4.2, the estimate from below follows for $t \in (0, T_n^*] \subset (0,1]$ as

$$S_1 := \sum_{q=q_0}^{n} \frac{m_{\gamma q+\gamma}(t)\,(\alpha t)^q}{q!} = I^n_{\gamma,\gamma}(\alpha t, t) - \sum_{q=0}^{q_0-1} \frac{m_{\gamma q+\gamma}(\alpha t)^q}{q!} \tag{6.13}$$
$$\ge I^n_{\gamma,\gamma}(\alpha t, t) - c^*_{q_0} \sum_{q=0}^{q_0-1} \frac{\max_{0<t\le 1}\{1, t^{-(\gamma q+\gamma)/\gamma}\}(\alpha t)^q}{q!}$$
$$\ge I^n_{\gamma,\gamma}(\alpha t, t) - c^*_{q_0} \sum_{q=0}^{q_0-1} \frac{t^{-q-1}(\alpha t)^q}{q!}$$

$$
\begin{aligned}
&= I^n_{\gamma,\gamma}(\alpha t, t) - \frac{c^*_{q_0}}{t}\sum_{q=0}^{q_0-1}\frac{\alpha^q}{q!}\\
&\geq I^n_{\gamma,\gamma}(\alpha t, t) - \frac{c^*_{q_0}}{t}e^{\alpha}\\
&\geq I^n_{\gamma,\gamma}(\alpha t, t) - \frac{2c^*_{q_0}}{t}.
\end{aligned}
$$

The estimate for the double sum term in S_3 uses an analogous treatment to the one in the previous section to obtain the Mittag-Leffler moment's propagation. More precisely, set $S_3 := S_{3,1} + S_{3,2}$, and we make use of the identity (A.4) written in the format

$$
\Gamma(2k+1)\Gamma(q-2k+1) = B(2k+1, q-2k+1)\,\Gamma(q+2) \tag{6.14}
$$

to obtain

$$
\begin{aligned}
S_{3,1} :=& \sum_{q=q_0}^{n} \varepsilon_{\gamma q/2}\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\\
&\times \sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}\\
&\times B(2k+1, q-2k+1)\frac{\Gamma(q+2)}{\Gamma(q+1)}\\
\leq& \, \varepsilon_{\gamma q_0/2}\sum_{q=q_0}^{n}(q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\\
&\times\left(\sum_{k=1}^{k_{q*}} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}\right)\\
&\times\left(\sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} B(2k+1, q-2k+1)\right).
\end{aligned} \tag{6.15}
$$

The last inequality was obtained via the inequality $\sum_k a_k b_k \leq \sum_k a_k \ \sum_k b_k$ and the fact that ε_q decreases in q. Again, using the estimate of Lemma A.7, the sum of the Beta functions is bounded by Cq^{-3}, with C a uniform constant independent of q. Therefore,

$$
\begin{aligned}
&(q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)\left(\sum_{k=1}^{k_{q*}} \binom{\frac{q}{2}-\frac{2}{\gamma}}{k-1} B(2k+1, q-2k+1)\right)\\
&\leq (q+1)\frac{\gamma q}{2}\left(\frac{\gamma q}{2}-1\right)q^{-3} \ \leq \ C_\gamma
\end{aligned} \tag{6.16}
$$

uniformly in q. Then, estimating the right-hand side of (6.15) by the estimate (6.16) just above yields

$$
S_{3,1} \leq K_3\, C_\gamma\, \varepsilon_{\gamma q_0/2}\sum_{q=q_0}^{n}\left(\sum_{k=1}^{k_{q*}} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(t)\,(\alpha t)^{q-2k}}{\Gamma(q-2k+1)}\right). \tag{6.17}
$$

Finally, as was the case for the propagation estimates in the previous section, changing the order of summation in the right-hand side of (6.17) yields a control by a factor $E_\gamma^n(\alpha t, t)\,\mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)$ as follows. Recalling the definition of k_{q_*} from (6.5) and evaluating it for n instead of q yields

$$\begin{aligned}
S_{3,1} &\leq C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\lfloor \frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\rfloor} \sum_{q=\max\{q_0,4k-2\}}^{n} \frac{m_{2\gamma k+\gamma}\,(\alpha t)^{2k}}{\Gamma(2k+1)} \frac{m_{\gamma q-2\gamma k}(\alpha t)^{q-2k}}{\Gamma(q-2k+1)} \\
&= C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\lfloor \frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\rfloor} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)} \left(\sum_{q=\max\{q_0,4k-2\}}^{n} \frac{m_{\gamma q-2\gamma k}(t)(\alpha t)^{q-2k}}{\Gamma(q-2k+1)} \right) \\
&\leq C_\gamma\, \varepsilon_{\gamma q_0/2} \sum_{k=0}^{\lfloor \frac{n}{4}+\frac{3}{2}-\frac{1}{\gamma}\rfloor} \frac{m_{2\gamma k+\gamma}(t)\,(\alpha t)^{2k}}{\Gamma(2k+1)}\, E_\gamma^n(\alpha t, t) \\
&\leq C_\gamma\, \varepsilon_{\gamma q_0/2}\, \mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)\, E_\gamma^n(\alpha t, t).
\end{aligned}$$

An analogous estimate can be obtained for $S_{3,2}$, so overall we have

$$S_3 \;\leq\; 2C_\gamma \varepsilon_{\gamma q_0/2}\, \mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)\; E_\gamma^n(\alpha t, t). \tag{6.18}$$

Therefore, combining estimates (6.11), (6.13), (6.12), and (6.18) with (6.6) yields the following differential inequality for $E_\gamma^n = E_\gamma^n(\alpha t, t)$ depending on $\mathcal{I}_{\gamma,\gamma}^n = \mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\gamma^n \leq 2c_{q_0}^* + \left(-K_1\, \mathcal{I}_{\gamma,\gamma}^n + K_1\, \frac{2\, c_{q_0}^*}{t} + K_2\, E_\gamma^n + 2\varepsilon_{\gamma q_0/2} C_\gamma\, K_3 E_\gamma^n\, \mathcal{I}_{\gamma,\gamma}^n \right) + \alpha \mathcal{I}_{\gamma,\gamma}^n.$$

This inequality is the analog to the one in (5.18) for the propagation argument. Since the partial sum $E_\gamma^n(\alpha t, t)$ is bounded by $4M_0^*$ on the interval $[0, T_n^*]$ uniformly in n and $T_n^* \leq 1$, then the right-hand side of the above inequality is controlled by

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\gamma^n(\alpha t, t) \leq -\mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)\big(K_1 - 8M_0^*\, \varepsilon_{\gamma q_0/2} C_\gamma K_3 - \alpha\big) + 4M_0^*\, K_2 + \frac{2\, K_1\, c_{q_0}^*}{t} + 2c_{q_0}^*.$$

Next, since $t \leq T_n^* \leq 1$, $t^{-1} \geq 1$, so the above estimate is further bounded by

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\gamma^n(\alpha t, t) \leq -\mathcal{I}_{\gamma,\gamma}^n(\alpha t, t)\big(K_1 - 8M_0^*\, \varepsilon_{\gamma q_0/2} C_\gamma K_3 - \alpha\big) + \frac{\mathcal{K}_{q_0}}{t}$$

with $0 < \mathcal{K}_{q_0} = 2c_{q_0}^* + 4M_0^* K_2 + 2K_1 c_{q_0}^*$ only depending on data parameters, including q_0, independent of n.

Finally, since $\varepsilon_{\gamma q_0/2}$ converges to zero as q_0 goes to infinity, we can choose large enough q_0 and small enough α so that $b(\cos\theta)$ satisfies (2.6) with $\beta = 2a - 2$),

$$K_1 - 8\varepsilon_{q_0}\, q_0^{2-a}\, K_3 - \alpha > \frac{K_1}{2}, \tag{6.19}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_a^n(\alpha_1 t, t) \leq -\frac{K_1}{2} \mathcal{I}_{a,\gamma}^n(\alpha t, t) + \frac{\mathcal{K}_{q_0}}{t}. \tag{6.20}$$

Therefore, the final step consists in finding a lower bound for $\mathcal{I}^n_{a,\gamma}(\alpha t, t)$ in terms of $\mathcal{E}^n_a(\alpha t, t)$ as follows:

$$\begin{aligned} I^n_{\gamma,\gamma}(\alpha t, t) &= \sum_{q=0}^{n} \frac{m_{\gamma(q+1)}(t)\ (\alpha t)^q}{q!} = \sum_{q=1}^{n+1} \frac{m_{\gamma q}(t)\ (\alpha t)^q}{q!} \frac{q}{\alpha t} \\ &\geq \frac{1}{\alpha t} \sum_{q=3}^{n} \frac{m_{\gamma q}(t)\ (\alpha t)^q}{q!} = \frac{E^n_\gamma(t, \alpha t) - M_0^*}{\alpha t}. \end{aligned} \tag{6.21}$$

Combining (6.20) and (6.21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E^n_\gamma(\alpha t, t) \leq -\frac{1}{t}\left(\frac{K_1(E^n_\gamma - M_0^*)}{2\alpha} - \mathcal{K}_{q_0}\right) = -\frac{K_1}{2\alpha t}\left(E^n_\gamma - M_0^* - \frac{2\alpha}{K_1}\mathcal{K}_{q_0}\right).$$

Then choosing a small enough α such that

$$M_0^* + \frac{2\alpha}{K_1}\mathcal{K}_{q_0} < 2M_0^* \qquad \text{or equivalently} \qquad \alpha < \frac{K_1 M_0^*}{2\mathcal{K}_{q_0}} \tag{6.22}$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E^n_\gamma(\alpha t, t) \leq -\frac{K_1}{2\alpha t}\left(E^n_\gamma(\alpha t, t) - 2M_0^*\right) \quad \forall t \in [0, T_n^*]. \tag{6.23}$$

This differential inequality can be integrated using the integrating factor $t^{\frac{K_1}{2\alpha}}$ for $t > 0$. Indeed, inequality (6.23) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\frac{K_1}{2\alpha}} E^n_\gamma(\alpha t, t)\right) \leq 2M_0^* \frac{K_1}{2\alpha} t^{\frac{K_1}{2\alpha}-1} \qquad \forall t \in (0, T_n^*]. \tag{6.24}$$

Integrating in $0 < \epsilon \leq t \leq T_n^*$, one readily gets

$$\begin{aligned} E^n_\gamma(\alpha t, t) &\leq \left(\frac{\epsilon}{t}\right)^{\frac{K_1}{2\alpha}} E^n_\gamma(\alpha\epsilon, \epsilon) + 2M_0^*\left(1 - \left(\frac{\epsilon}{t}\right)^{\frac{K_1}{2\alpha}}\right) \\ &\leq \max\left\{E^n_\gamma(\alpha\epsilon, \epsilon), 2M_0^*\right\} \qquad \forall t \in [\epsilon, T_n^*]. \end{aligned} \tag{6.25}$$

Sending $\epsilon \to 0$ in (6.25), it follows that $E^n_\gamma(\alpha t, t) \leq \max\{E^n_\gamma(0,0), 2M_0^*\} \leq 2M_0^*$ for any $t \in [0, T_n^*]$. For the last inequality we recall that $E^n_\gamma(0,0) = m_0 < 2M_0^*$. Therefore, the *strict* inequality $E^n_\gamma(\alpha t, t) \leq 2M_0^* < 4M_0^*$ holds uniformly on the closed interval $[0, T_n^*]$. By continuity of the partial sum, this strict inequality $E^n_\gamma(\alpha t, t) < 4M_0^*$ then holds on a slightly larger interval, which would contradict maximality of T_n^* from the definition (6.3), unless $T_n^* = 1$. Hence, we conclude that $T_n^* = 1$ for all n.

Therefore, we in fact have that

$$E^n_\gamma(\alpha t, t) < 4M_0^* \quad \text{for all } t \in [0,1] \ \text{ for all } n \in \mathbb{N}.$$

Thus, by letting $n \to +\infty$, we conclude that $E^\infty_\gamma(\alpha t, t) \leq 4M_0^*$ for all $t \in [0,1]$. That is,

$$\int_{\mathbb{R}^d} f(t,v)\ \ \mathcal{E}_{2/\gamma}((\alpha t)^{2/\gamma} \langle v \rangle^2)\ \mathrm{d}v \leq 4M_0 \quad \text{for all } t \in [0,1]. \tag{6.26}$$

To finalize the proof, first set $\alpha = \min\{\ln 2, \alpha_1\}$ from (6.11) and with α_1 satisfying condition (6.22) that depends on the initial data, γ, the collisional kernel, and A_2 from the integrability condition (2.6). This α is a positive and bounded real number.

Then note that the above inequality implies that at the time $t = 1$, the Mittag-Leffler moment of order γ and rate $\alpha t = \alpha$ is finite. Now, starting the argument from $t = 1$ on, we bring ourselves into the setting of the propagation and conclude that for $t \geq 1$, the Mittag-Leffler moment of the same order γ and potentially smaller α than the one found on time interval $[0, 1]$ remain uniformly bounded for all $t \geq 1$.

In conclusion,

$$\int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/\gamma}((\alpha t)^{2/\gamma} \langle v \rangle^2)\, \mathrm{d}v < C \quad \text{for all } t \in [0,1] \tag{6.27}$$

and

$$\int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/\gamma}(\alpha^{2/\gamma} \langle v \rangle^2)\, \mathrm{d}v < C \quad \text{for all } t \geq 1. \tag{6.28}$$

Therefore, we conclude that for all $t \geq 0$, we have

$$\int_{\mathbb{R}^d} f(t,v)\ \mathcal{E}_{2/\gamma}((\alpha \min\{1,t\})^{2/\gamma} \langle v \rangle^2)\, \mathrm{d}v\ < C. \tag{6.29}$$

In particular, this asserts that the solution of the Boltzmann equation with an initial mass and energy will develop Mittag-Leffler moments or equivalently exponential high energy tails of order γ with rate $r(t) = \alpha \min\{t, 1\}$. Therefore, the proof of Theorem 2.11 is now complete. ∎

Appendix A. We gather technical results used throughout this manuscript. The first two lemmas focus on elementary polynomial inequalities that will be used to derive ordinary differential inequalities for polynomial moments in section 4.

LEMMA A.1 (polynomial inequality I). *Let $b \leq a \leq \frac{s}{2}$. Then for any $x, y \geq 0$,*

$$x^a y^{s-a} + x^{s-a} y^a \ \leq\ x^b y^{s-b} + x^{s-b} y^b. \tag{A.1}$$

Remark A.2. This lemma is useful for comparing products of moments. Namely, as its consequence, we have that for a fixed s, the sequence $\{m_k\ m_{s-k}\}_k$ is decreasing in k for $k = 1, 2, \ldots, \lfloor s/2 \rfloor :=$ Integer Part of $s/2$. For example, if $s \geq 4$, then $m_2 m_{s-2} \leq m_1 m_{s-1}$.

Proof. Note that a, b, and s satisfy $a - b \geq 0$ and $s - a - b \geq 0$. Therefore,

$$\left(y^{a-b} - x^{a-b}\right) x^b y^b \left(y^{s-a-b} - x^{s-a-b}\right) \geq 0,$$

which is easily checked to be equivalent to the inequality (A.1). ∎

LEMMA A.3 (polynomial inequality II, Lemma 2 in [9]). *Assume $p \geq 3$, and let $k_p = \lfloor (p+1)/2 \rfloor$. Then for all $x, y > 0$, the following inequalities hold:*

$$\sum_{k=1}^{k_p - 1} \binom{p}{k} \left(x^k y^{p-k} + x^{p-k} y^k\right) \leq (x+y)^p - x^p - y^p \leq \sum_{k=1}^{k_p} \binom{p}{k} \left(x^k y^{p-k} + x^{p-k} y^k\right).$$

Remark A.4. Using this lemma, it is easy to see a rough but useful estimate:

$$\sum_{k=0}^{k_p} \binom{p}{k} \left(x^k y^{p-k} + x^{p-k} y^k\right) \leq 2(x+y)^p. \tag{A.2}$$

Next, we recall the basic definitions and properties of Gamma $\Gamma(x)$ and Beta $B(x,y)$ functions that are useful for the next estimates. They are defined via

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, \mathrm{d}t \qquad \text{and} \qquad B(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\, \mathrm{d}t, \tag{A.3}$$

respectively. Two fundamental properties of these well-known functions are

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \text{and} \qquad B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{A.4}$$

The following classic result for estimates of generalized Laplace transforms will be needed to estimate the combinatoric sums of Beta functions to be shown in the subsequent Lemma A.6.

LEMMA A.5. *Let $0 < \alpha, R < \infty$, $g \in C([0,R])$, and $S \in C^1([0,R])$ be such that $S(0) = 0$ and $S'(x) < 0$ for all $x \in [0,R]$. Then for any $\lambda \geq 1$, we have*

$$\int_0^R x^{\alpha-1}\, g(x)\, e^{\lambda S(x)}\, dx = \Gamma(\alpha)\, \left(\frac{1}{-\lambda\, S'(0)}\right)^{\alpha}\, (g(0) + o(1)).$$

The proof of this estimate is a direct application of the Laplace method for asymptotic expansion of integrals that can be found in [30, p. 81, Theorem 7.1].

The next two lemmas estimate a combinatoric sum of Beta functions. These estimates are inspired by the work in Lemma 4 in [9] and Lemma 3.3 in [26]. However, in our context, the arguments of Beta functions are shifted, so we compute exact decay rates for our situation. These estimates are crucial to control the growth in q of the ordinary differential inequality of partial sums of renormalized moments.

The first lemma will be used for the proof of propagation of moments with $a = 2/s$, while the second will be used for the generation of moments with $s = \gamma$.

LEMMA A.6 (first estimate on combinatoric sums of Beta functions). *Let $q \geq 3$ and $k_q = [(q+1)/2]$. Then for any $a > 1$, we have*

$$\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \leq C_a\, \frac{1}{(aq)^{1+a}}, \tag{A.5}$$

where the constant C_a depends only on a.

Proof. Reindexing the summation from $k=1$ to $k=0$ by changing $k-1$ into k and rearranging the integral forms defining Beta functions yields

$$\begin{aligned}
&\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \\
&\quad = \sum_{k=0}^{k_q-1} \binom{q-2}{k}\, B(a(k+1)+1, a(q-k-1)+1) \\
&\quad = \frac{1}{2} \int_0^1 \sum_{k=0}^{k_q-1} \binom{q-2}{k} \left(x^{a(k+1)}\,(1-x)^{a(q-k-1)} + x^{a(q-k-1)}\,(1-x)^{a(k+1)} \right)\, dx
\end{aligned}$$

$$= \frac{1}{2}\int_0^1 x^a(1-x)^a \sum_{k=0}^{k_{q-2}} \binom{q-2}{k} \left(x^{ak}\,(1-x)^{a(q-2-k)} + x^{a(q-2-k)}\,(1-x)^{ak}\right)\,dx$$
$$= \frac{1}{2}\int_0^1 x^a(1-x)^a \sum_{k=0}^{k_p} \binom{p}{k} \left(x^{ak}\,(1-x)^{a(p-k)} + x^{a(p-k)}\,(1-x)^{ak}\right)\,dx$$

after setting $q-2=p$ in the last integral. The estimate (A.2) then yields

$$\begin{aligned}
&\sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \\
&\quad \le \frac{1}{2}\int_0^1 x^a(1-x)^a\; 2\;\left(x^a+(1-x)^a\right)^p\,dx \\
&\quad = \int_0^1 x^a(1-x)^a \left(x^a+(1-x)^a\right)^{q-2}\,dx \\
&\quad = 2\int_0^{1/2} x^a\; g(x)\; e^{qS(x)}\,dx,
\end{aligned}$$

where $g(x) = (1-x)^a\left(x^a+(1-x)^a\right)^{-2}$ and $S(x) = \log(x^a+(1-x)^a)$ for $x \in [0,1/2]$. Finally, applying Lemma A.5 for these $g(x)$ and $S(x)$ as indicated and noting that $g(0)=1$ and $S'(0)=-a$ yields the desired estimate:

$$\text{(A.6)}\qquad \sum_{k=1}^{k_q} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \;\le\; C_a\,\Gamma(a+1)\,\left(\frac{1}{aq}\right)^{a+1}. \qquad \square$$

Lemma A.7 (second estimate on combinatoric sums of Beta functions). *Let $0 < s \le 1$ and $q \ge 3$. Then there exits a constant C, independent on q, such that*

$$\text{(A.7)}\qquad \sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{s}}} \binom{\frac{q}{2}-\frac{2}{s}}{k-1}\; B(2k+1,\; q-2k+1) \le C\frac{1}{q^3}.$$

Proof. First we note a simple property of binomial coefficients. For any integer $k \in \mathbb{N}_0$ and any real numbers $\tilde{a}, a \in \mathbb{R}$ that satisfy $\tilde{a} \ge a \ge k$,

$$\text{(A.8)}\qquad \binom{a}{k} \le \binom{\tilde{a}}{k}.$$

This is easily proved by noting that the binomial coefficient $\binom{a}{k}$ (and similarly $\binom{\tilde{a}}{k}$) can be computed as

$$\binom{a}{k} = \frac{a\;(a-1)\;(a-2)\cdots(a-k+1)}{k!}.$$

Next, since $s \le 1$,

$$\text{(A.9)}\qquad \frac{q}{2}-\frac{2}{s} \le \frac{q}{2}-2.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{1+k_{\frac{q}{2}-\frac{2}{s}}} \binom{\frac{q}{2}-\frac{2}{s}}{k-1} B(2k+1,\ q-2k+1) \\
&\le \sum_{k=1}^{1+k_{\frac{q}{2}-2}} \binom{\frac{q}{2}-2}{k-1} B(2k+1,\ q-2k+1) \\
&= \sum_{k=1}^{k_{\frac{q}{2}}} \binom{\frac{q}{2}-2}{k-1} B\left(2k+1,\ 2\left(\frac{q}{2}-k\right)+1\right).
\end{aligned} \tag{A.10}$$

Now applying (A.5) yields (A.7). □

Appendix B. Finally, for completeness we include detailed calculation of deriving the representation of energies from (3.3). Recall that

$$v' = \frac{v+v_*}{2} + \frac{1}{2}|u|\sigma.$$

Hence,

$$\begin{aligned}
\langle v'\rangle^2 &= 1 + \frac{|v+v_*|^2}{4} + \frac{|v-v_*|^2}{4} + \frac{1}{2}|u|\sigma \\
&= 1 + \frac{|v|^2+|v_*|^2}{2} + \frac{1}{2}|u|(v+v_*)\cdot(\hat{u}\cos\theta + \omega\sin\theta) \\
&= 1 + \frac{|v|^2+|v_*|^2}{2} + \frac{1}{2}(v+v_*)\cdot(v-v_*)\cos\theta + \frac{1}{2}|u||V|\sin\theta(\hat{V}\cdot\omega) \\
&= 1 + |v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2} + \frac{1}{2}|u||V|\sin\theta(j\cdot\omega)\sin\alpha \\
&= \langle v\rangle^2\cos^2\frac{\theta}{2} + \langle v_*\rangle^2\sin^2\frac{\theta}{2} + |v\times v_*|\sin\theta(j\cdot\omega),
\end{aligned}$$

which coincides with the representation in (3.3).

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