BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 3, July 2018, Pages 347–350
http://dx.doi.org/10.1090/bull/1620
Article electronically published on April 19, 2018

COMMENTARY: THREE DECADES AFTER CATHLEEN SYNGE MORAWETZ'S PAPER "THE MATHEMATICAL APPROACH TO THE SONIC BARRIER"

IRENE M. GAMBA

Abstract. Immediately following the commentary below, this previously published article is reprinted in its entirety: Cathleen Synge Morawetz, "The mathematical approach to the sonic barrier", *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 2, 127–145.

Cathleen Synge Morawetz wrote this article in connection with The Josiah Willard Gibbs lecture she presented at the American Mathematical Society meeting in San Francisco, California, January 7, 1981. This is a beautiful piece on a subject at the core of applied mathematical analysis and numerical methods motivated by the pressing engineering technology of the mid-twentieth century and the human urge to travel fast at efficient cost. From the mathematical viewpoint this problem comprises the understanding of models of nonlinear partial differential equations arising in compressible fluid mechanics, as much as understanding how to obtain numerical approximations to a model discretization that result both in finding numerically computed surfaces close to the model's solutions (if such exists) but also in matching these computed model outputs to experiments from engineering or experimental observation viewpoints.

This commentary starts with a description of the state-of-the-art up to 1982, from a very comprehensive explanation for any scientist of what it takes to fly an object with wings and the issues of instabilities that arise as we try to fly too fast, to the description of the adequate model given by the system of Hamilton–Jacobi framework of conservation of mass and momentum for a compressible potential isentropic inviscid fluid, formulated by the coupled nonlinear system of conservation of mass to the Bernoulli law associated to such a fluid model.

More specifically, defining the state variables by density ρ, velocity $\overrightarrow{q}$, and pressure $p = p(\rho)$, consider the relative motion of having an obstacle (such as an airfoil) at rest, so that the velocity at infinity, $\overrightarrow{q}_\infty$, is actually the speed associated to that obstacle. If the flow is irrotational, there exists a potential function ρ that governs the velocity by the relation $\overrightarrow{q} = \nabla\Phi$, and so the flow speed is $\mathbf{q} = |\nabla\Phi|$. The corresponding conservation of momentum relation, renormalized by the density and integrated along noncrossing streamlines (i.e., orthogonal level surfaces to the potential function level surfaces), yields the Bernoulli's law that expresses a

Received by the editors April 9, 2018.
2010 *Mathematics Subject Classification.* Primary 35F20, 35F21, 35M10, 35Q30, 35Q35, 76H05.
The author was supported by NSF grant no. DMS-1715515.

balance law for $\rho, p, |\nabla\Phi|$. Hence, one can obtain a closed system of first-order equations by coupling conservation of mass with Bernoulli's law to obtain a quasilinear Hamilton–Jacobi system for steady, irrotational, inviscid isentropic gas flow

$$\operatorname{div}(\rho\nabla\Phi) = 0 \qquad \text{and} \qquad \frac{|\nabla\Phi|^2}{2} + p(\rho) = K, \tag{1}$$

where K is the isoenergetic constant. Bernoulli's law allows for the relation $\rho = \rho(\nabla\Phi)$, so that the Hamilton–Jacobi system is reduced to a quasilinear scalar equation that, when written in two dimensions, takes the form

$$(c^2 - u^2)\Phi_{xx} - 2uv\Phi_{xy} + (c^2 - v^2)\Phi_{yy} = 0, \tag{2}$$

for $\overrightarrow{q} = \nabla\Phi = (u, v)$ the velocity field, and $c = (dp/d\rho)c(|\nabla\Phi|)$ the *speed of sound*, given through Bernoulli's law. Then the quotient $M = \mathbf{q}/c$, referred to as the *Mach number*, determines the local speed of sound. Then the following occurs (described using a slight change of wording from the Morawetz article [11]):

> ... if $\mathbf{q}$ is small, so both u, v are small, then $\Delta\Phi = 0$ meaning flow is essentially incompressible. Choosing local coordinates with $v = 0$ then, for $\mathbf{q} < c$ the equation is elliptic and for $\mathbf{q} > c$, the equation is hyperbolic. That means the flow is analogous to the incompressible case with locally smooth solutions for $M < 1$, but when $M > 1$, all the difficult features of nonlinear hyperbolic equations occur.

Yet, a solution across the two regions with a nonempty contact set needs to be understood as well. This problem is at the core of "passing the sonic barrier".

This regime is called *transonic* when the emergence of strong shocks are expected to be discontinuous solutions to this mixed type system for a stationary flow framework (see Figure 2 and its description from the wind tunnel experiments in the Morawetz paper [11]). While existence of solutions for the transonic flow problem may be rather simple in one space dimension, their nontrivial solutions in two or more dimensions remain one of the most haunting problems in fluid dynamics, with strong implications that range from the modeling of airflow past wing profiles in aerospace applications to wave propagation and singularity formation in relativity theory.

Morawetz's paper is a masterly explanation of why linear methods fail as shown by means of Friedrichs' multiplier method [3] and her own work on [9], but also discussed perturbation theory that yields the Tricomi equation as an approximation to the transonic flow model in equation (2), related to her own contributions [8], [10], [16]. She also presented her vision on how insightful numerical approximations and applied analysis led to significant results that impacted linear and nonlinear wave theory for hyperbolic systems.

Cathleen concludes her 23-page presentation stating,

> We are left with the general weak existence theorem for the full nonlinear problem unsolved. There are lots of approaches to try: Show the difference scheme converges. Extend the variational principles of elliptic theory. Perhaps something quite new...

During the years since 1982, there have been several significant issues that have been addressed and brought progress to this area. There has been progress in

solving the transonic flow model for the full system (1) for small perturbations on potential strength by drawing connections to the obstacle problem by means of solving a suitable free boundary problem. Chen and Feldman constructed full transonic solutions ([1], [2]) for weak shock conditions.

Yet their assumptions do not fully cover the strong discontinuity regime that Morawetz envisioned in her title as "The mathematical approach to the sonic barrier" that would match experimental data, so many aspects of the mathematics for transonic flow models remains unsolved.

Cathleen and I worked for five years in the mid-1990s, and we proposed an approach for solving the problem that would admit large shocks. But certainly we run short of claiming the existence of solution to the steady, irrotational, inviscid isentropic gas flow model in two dimensions for the nontrivial obstacle domain for large shocks. Our techniques are based on a couple of manuscripts that Cathleen had developed in the mid-1980s and the early 1990s on solving a viscous approximation to equations (1) in a nontrivial domain [12] with enough good estimates, uniform in the viscosity parameter, and studying their inviscid limit by means of compensated compactness techniques by Murat, Tartar, and DiPerna, adjusted to system (1) and described in her work [13], [14], [15], and [16]. Later, we were able to construct $C^\infty(\Omega)$-solutions, Ω in $\mathbb{R}^2$, for the vector field $\mathbf{q}$ and density ρ solving an upwind viscous approximation to system (1), somehow inspired by the Jameson numerical approximation, for both the neutral gases cases as well as the case of charged gases, by coupling the fluid equations to an electrostatic mean field potential. It was important to show that such a viscous system was solvable. We accomplished this task by considering non-Newtonian viscosities and nonlinear boundary conditions [5, 6] that allowed sharp control for the speed from above and the density from below [4, 7]. Much is left to do, such as the passage to the inviscid limit, to show that weak entropic solutions do exist in the whole domain.

In our conversations through the last twenty years, Cathleen and I wondered whether the lack of successful progress was due to a lack of more available techniques beyond the ones already used. These available techniques include compactness by comparisons theorem, regularization with degree theory for Leray–Shauder fixed point theorem, optimal uniform bounds rates, and the passage to the inviscid limit by compensated compactness.

Or perhaps the quasilinear system (1) is either too simple or incomplete to describe the phenomena observed by the wind tunnel experiments from Figure 2 of [11]. In such a case, model corrections and new experiments may be needed.

Toward the end of the article, Cathleen wrote

> Let me close now... to say that I have left a lot unsaid and a lot unquoted. But I would like to thank for their help my transonic colleagues, Kurt O. Friedrichs, Lipman Bers, Paul R. Garabedian and Antony Jameson.

I hope many of my colleagues would have been able to interact as I have done with Morawetz and these champions whose minds were filled day after day with the mathematics and numerics of transonic flow models. And I hope for more individuals whose curiosity will arise to complete Morawetz's envisioned mathematical path.

References

[1] G.-Q. Chen and M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type*, J. Amer. Math. Soc. **16** (2003), no. 3, 461–494. MR1969202

[2] G.-Q. Chen and M. Feldman, *Steady transonic shocks and free boundary problems for the Euler equations in infinite cylinders*, Comm. Pure Appl. Math. **57** (2004), no. 3, 310–356. MR2020107

[3] K. O. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math. **11** (1958), 333–418. MR0100718

[4] I. M. Gamba, *An existence and uniqueness result of a nonlinear two-dimensional elliptic boundary value problem*, Comm. Pure Appl. Math. **48** (1995), no. 7, 669–689. MR1342379

[5] I. M. Gamba, *Sharp uniform bounds for steady potential fluid-Poisson systems*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 3, 479–516. MR1453279

[6] I. M. Gamba and C. S. Morawetz, *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow*, Comm. Pure Appl. Math. **49** (1996), no. 10, 999–1049. MR1404324

[7] I. M. Gamba and C. S. Morawetz, *Viscous approximation to transonic gas dynamics: flow past profiles and charged-particle systems*, Modelling and computation for applications in mathematics, science, and engineering (Evanston, IL, 1996), Numer. Math. Sci. Comput., Oxford Univ. Press, New York, 1998, pp. 81–102. MR1677377

[8] C. S. Morawetz, *On the non-existence of continuous transonic flows past profiles. I*, Comm. Pure Appl. Math. **9** (1956), 45–68; *II* Comm. Pure Appl. Math. **10** (1957), 107–132; *III* Comm. Pure Appl. Math. **11** (1958), 129–133; see also Comm. Pure Appl. Math. **17** (1964), 357–367. MR0078130; MR0088253; MR0096478; MR0184522

[9] C. S. Morawetz, *A weak solution for a system of equations of elliptic-hyperbolid type*, Comm. Pure Appl. Math. **11** (1958), 315–332. MR0096893

[10] C. S. Morawetz, *The Dirichlet problem for the Tricomi equation*, Comm. Pure Appl. Math. **23** (1970), 587–601. MR0280062

[11] C. S. Morawetz, *The mathematical approach to the sonic barrier*, Bull. Amer. Math. Soc. (N.S.), **6** (1982), no. 2, 127–145. MR0640941

[12] C. S. Morawetz, *On a weak solution for a transonic flow problem*, Comm. Pure Appl. Math. **38** (1985), no. 6, 797–817. MR812348

[13] C. S. Morawetz, *Transonic flow and compensated compactness*, Wave motion: theory, modelling, and computation (Berkeley, Calif., 1986), Math. Sci. Res. Inst. Publ., vol. 7, Springer, New York, 1987, pp. 248–258. MR920838

[14] C. S. Morawetz, *An alternative proof of DiPerna's theorem*, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 1081–1090. MR1127051

[15] C. S. Morawetz, *On steady transonic flow by compensated compactness*, Methods Appl. Anal. **2** (1995), no. 3, 257–268. MR1362016

[16] C. S. Morawetz, *Mixed equations and transonic flow*, J. Hyperbolic Differ. Equ. **1** (2004), no. 1, 126. MR2052469

Department of Mathematics and The Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, Texas

Email address: gamba@math.utexas.edu