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On the Rate of Relaxation for the Landau Kinetic Equation and Related Models

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Abstract We study the rate of relaxation to equilibrium for Landau kinetic equation and some related models by considering the relatively simple case of radial solutions of the linear Landau-type equations. The well-known difficulty is that the evolution operator has no spectral gap, i.e. its spectrum is not separated from zero. Hence we do not expect purely exponential relaxation for large values of time $t > 0$. One of the main goals of our work is to numerically identify the large time asymptotics for the relaxation to equilibrium. We recall the work of Strain and Guo (Arch Rat Mech Anal 187:287–339 2008, Commun Partial Differ Equ 31:17–429 2006), who rigorously show that the expected law of relaxation is $\exp(-ct^{2/3})$ with some $c > 0$. In this manuscript, we find an heuristic way, performed by asymptotic methods, that finds this "law of two thirds", and then study this question numerically. More specifically, the linear Landau equation is approximated by a set of ODEs based on expansions in generalized Laguerre polynomials. We analyze the corresponding quadratic form and the solution of these ODEs in detail. It is shown that the solution has two different asymptotic stages for large values of time t and maximal order of polynomials N: the first one focus on intermediate asymptotics which agrees with the "law of two thirds" for moderately large values of time t and then the second one on absolute, purely exponential asymptotics for very large t, as expected for linear ODEs. We believe that appearance of intermediate asymptotics in finite dimensional approximations must be a generic behavior for different classes of equations in functional spaces (some PDEs, Boltzmann equations for soft potentials, etc.) and that our methods can be applied to related problems.

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1 Introduction

The Landau kinetic equation is one of the most important mathematical objects in Plasma Physics [5,6,9]. It is also a very interesting equation from a mathematical point of view. A brief review of the mathematical properties and main references can be found in recent papers [2,3]. The aim of the present paper is to discuss a particular question of the rate of relaxation to equilibrium for the spatially homogeneous Landau equation and its models. In 2006 and 2008, Strain and Guo [10,11] prove that the rate of relaxation to equilibrium cannot be faster than $\exp(-\lambda t^{2/3})$ for the "true" (Coulomb case) Landau equation. The proof of this fact is rather complicated. We shall try to make this result more "visible" below.

The generalized radial dimensionless Landau equation for the distribution function $f(x,t)$, where $x = |v|^2$, $v \in \mathbb{R}^3$, and t denote squared velocity and time respectively, reads [2]

$$\partial_t f(x,t) = x^{-\theta}\partial_x \int_0^\infty dy \min\left(x^{1+\theta}, y^{1+\theta}\right)(\partial_x - \partial_y) f(x,t) f(y,t) \tag{1}$$

where $\theta = \frac{1}{2}$ for the classical Landau equation with Coulomb interaction. The model equations with $0 \le \theta < \frac{1}{2}$ can be also included into consideration. Properties of these generalized Landau equations are briefly discussed in [2]. In the present paper, we confine ourselves to the simplest case of the linear Landau equation which will be introduced in Sect. 2.

The paper is organized as follows. In Sect. 2, we consider the linear Landau equation for arbitrary $\theta \in [0, \frac{1}{2}]$ and discuss the standard way to obtain the rate of relaxation in the case of purely exponential decay to equilibrium. In Sect. 3, we approximate this equation by a set of linear ordinary differential equations (ODEs) based on an expansion in generalized Laguerre polynomials with maximal order N. Then the approximate solution of the Landau equation is defined by $\exp(-tM)$, where the factor M is a corresponding N by N matrix. Main properties of such matrix M are well-known: it is symmetric and has one zero and $N-1$ positive eigenvalues. The asymptotic behavior for large t (relaxation rate) of the solution of (1) is, then, defined by the smallest positive eigenvalue of M. However, this eigenvalue for our equation tends to zero for increasing values of N and we numerically show this fact in Sect. 4. The only exceptional case $\theta = 0$ is exactly solvable and not very interesting for applications. Sections 5 and 6 are devoted mainly to true Landau equation for Coulomb forces with $\theta = \frac{1}{2}$. First we show in Sect. 5 a heuristic way to guess the fractional exponential rate asymptotics $\exp(-at^{2/3})$ ("law of two thirds") by considering a simplified version of the asymptotic solution from [2]. Finally, in Sect. 6, we study in detail the numerical solution of the corresponding set of N ODEs for large N (up to $N = 100$) and show how to extract the intermediate fractional exponential asymptotics. Note that for any fixed N the solution of such ODEs has purely exponential asymptotics for sufficiently large values of time. However, the intermediate stage is seen very well for large N, as discussed in Sect. 6.

The general question of numerical study of precise asymptotics in the case of absence of the spectral gap is difficult and important not only for the Landau equation. Therefore we hope that our approach based on numerical study of intermediate asymptotics of finite-dimensional approximations of ODEs can be useful for many applied problems.

2 The Linear Landau Equation and Quadratic Form

The linear Landau equation is an equation for a test particle, which collides with equilibriumly distributed "field" particles. It can be obtained by rewriting the homogeneous Landau equation (1) in the form of nonlinear diffusion equations for $f(x,t)$, then replacing the $f(y,t)$ in the integral terms by a constant Maxwellian, say, $M(y) = \exp(-y)$. Finally, we denote $t = (1+\theta)\tilde{t}$.

The generalized linear isotropic Landau equation reads (tildes are omitted)

$$\partial_t f(x,t) = x^{-\theta}\partial_x \left(\mathcal{D}_\theta(x)(\partial_x f(x,t) + f(x,t))\right), \quad x, t \geq 0, \tag{2}$$

where

$$\mathcal{D}_\theta(x) = \int_0^x y^\theta e^{-y} dy, \quad 0 \leq \theta \leq \frac{1}{2}. \tag{3}$$

The "true" 3-D Landau equation corresponds to $\theta = \frac{1}{2}$. The case $\theta = 0$ is exactly solvable through the Laplace transform.

If we assume that

$$f(x,0) = f_0(x), \quad \int_0^\infty x^\theta f_0(x) dx = \Gamma(1+\theta), \tag{4}$$

where the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad z > 0, \tag{5}$$

then, we can expect that, for any $x \geq 0$,

$$\lim_{t\to\infty} f(x,t) = e^{-x}. \tag{6}$$

Our goal is to study the rate of convergence.

Consider a perturbation around the equilibrium, that is,

$$f(x,t) = e^{-x}(1 + \varphi(x,t)). \tag{7}$$

Plugging this back to the generalized linear Landau equation (2) gives an equation for φ

$$x^\theta e^{-x} \partial_t \varphi(x,t) = -\mathcal{L}(\varphi)(x,t), \tag{8}$$

where the linear operator $\mathcal{L}$ reads

$$\mathcal{L}(\varphi)(x,t) = -\partial_x[\mathcal{D}_\theta(x) e^{-x} \partial_x \varphi(x,t)]. \tag{9}$$

We define the weighted L^2 norm as

$$\|\varphi\|^2 = \int_0^\infty x^\theta e^{-x} \varphi^2 dx. \tag{10}$$

Then the equality

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 = -\langle \mathcal{L}(\varphi), \varphi\rangle = -\int_0^\infty \mathcal{D}_\theta(x) e^{-x} [\partial_x \varphi(x,t)]^2 dx, \tag{11}$$

where the *Dirichlet form* $\langle \mathcal{L}(\varphi), \varphi \rangle$ ($\langle \cdot, \cdot \rangle$ denotes the usual unweighted L^2 innner product) is obtained through integration by parts, and so $\langle \mathcal{L}(\varphi), \varphi \rangle$ ($\langle \cdot, \cdot \rangle$ is a non-negative operator.

Obviously, $\mathcal{L}$ is a positive operator. Its smallest eigenvalue is 0 with multiplicity 1 with its eigenspace spanned by constant functions. In particular, we can conclude, at the formal level, that

$$\varphi(x,t) = O\left(e^{-\lambda_\theta t}\right), \quad \text{as } t \to \infty\,, \tag{12}$$

where λ_θ is the *spectral gap*, if exists, of operator $\mathcal{L}$, defined as the minimized Rayleigh quotient of $\mathcal{L}$

$$\lambda_\theta = \min \frac{\langle \mathcal{L}(\varphi), \varphi \rangle}{\|\varphi\|^2} \quad \text{s.t.} \int_0^\infty x^\theta e^{-x} \varphi(x) dx = 0\,, \tag{13}$$

that is, φ is orthogonal to the eigenspace of eigenvalue 0.

That means, we can expect an exponential decay when the state is close to equilibrium, where the decay rate is given by $\lambda_\theta > 0$, if exists. In addition, the existence can be analytically proved for $\theta = 0$. For the case when $\theta > 0$, we study essentially a robust numerical approximation by Laguerre polynomials which can identify the evolution of the spectrum exhibiting not only the lack of a spectral gap, but also the degenerate behavior for the specific choice of $\lambda_\theta = 0.5$ that simulates radial solutions to the true Landau equation.

Remark Note that the simplest way to numerically solve Eqs. (1) or (2) would be to use one of well-known finite difference schemes. There are many choices to find numerical solutions of the Landau-type equations based on such schemes (see e.g. [8] and references there for a review). However, our goal is not to construct one more numerical solution of Equation (2). The goal is to study a precise rate of relaxation of solutions to equilibrium for very large values of time (for both the case when $\lambda_\theta = 0$ and the true radial Landau solutions when $\lambda_\theta = 0.5$). Therefore we use below another way for the discretization of Equation (2) based on Laguerre polynomials.

3 Approximation in Space of Laguerre Polynomials

We have already known that the exponential decay rate is determined by the spectral gap. Thus we need to numerically solve the constraint minimization problem (13). This is done by taking a finite-dimensional approximation space for φ, and examine the behavior for increasing dimensions of the approximate spaces.

To this goal, we introduce an orthogonal basis $\{\varphi_n(x)\}$, $n = 0, 1, \ldots$, for the weighted L^2 space with norm (10), such that $\varphi_0(x) = \text{const}$, and

$$\langle \varphi_n, \varphi_m \rangle_w = \int_0^\infty x^\theta e^{-x} \varphi_n(x) \varphi_m(x) dx = \delta_{n,m}\,, \tag{14}$$

where $\langle \cdot, \cdot \rangle_w$ denotes the weighted L^2 inner product with weight $w = x^\theta e^{-x}$.

Such requirements are perfectly satisfied by the normalized *generalized Laguerre polynomials* [1],

$$\varphi_n(x) = \frac{L_n^\theta(x)}{\|L_n^\theta\|}, \quad n = 0, 1, \ldots\,. \tag{15}$$

In particular,

$$L_0^\theta = 1, \quad L_1^\theta = 1 + \theta - x, \quad L_n^\theta = \sum_{k=0}^{n} a_k^{(n,\theta)} x^k\,, \tag{16}$$

where the coefficients

$$a_k^{(n,\theta)} = \frac{(-1)^k}{k!}\binom{n+\theta}{n-k} = \frac{(-1)^k}{k!}\,\frac{\Gamma(n+\theta+1)}{\Gamma(n-k+1)\Gamma(\theta+k+1)}\,, \tag{17}$$

and the weighted norm of L_n^θ is given by

$$\|L_n^\theta\|^2 = \int_0^\infty x^\theta e^{-x}[L_n^\theta]^2 dx = \frac{\Gamma(n+\theta+1)}{n!}\,. \tag{18}$$

Thus, for a fixed order of approximation N, we consider the minimization problem (13) for

$$\varphi^{(N)}(x) = \sum_{n=1}^{N} u_n \varphi_n(x)\,, \tag{19}$$

where u_n are the coefficients. Note that, the summation starts from $n = 1$, because $\langle\varphi_0, 1\rangle_w = 0$ for any $n > 1$. This automatically fulfill the constraint in (13). Thus, the Dirichlet form in (11), or equivalently the Rayleigh quotient (13), can be written as a quadratic form varying with the number N of Laguerre polynomials to generate the eigenvalues

$$\lambda_\theta^{(N)} = \min\langle\mathcal{L}(\varphi^{(N)}), \varphi^{(N)}\rangle = \min \mathbf{u}^T G^{(N)}\mathbf{u}\,, \tag{20}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$ is the coefficient vector and entries $G_{nm}^{(N)}$ of the symmetric matrix $G^{(N)}$, for $n, m = 1, \ldots, N$, are obtained by straightforward calculations given by

$$\begin{aligned} G_{nm}^{(N)} &= \int_0^\infty \mathcal{D}_\theta(x) e^{-x} \varphi_n'(x)\varphi_m'(x) dx \\ &= \tfrac{1}{\|L_n^\theta\|\|L_m^\theta\|} \textstyle\sum_{k=1}^n \sum_{l=1}^m k l a_k^{(n,\theta)} a_l^{(m,\theta)} (k+l-2)! S(k+l-2)\,, \end{aligned} \tag{21}$$

with $S(p)$ given by

$$S(p) = \sum_{j=0}^{p} \frac{\Gamma(j+\theta+1)}{2^{j+\theta+1} j!}\,.$$

Since the orthogonality constraint has already been taken into account (19), the smallest eigenvalue of matrix G will be an approximation to the spectral gap.

4 Numerical Results for Quadratic Forms

It is not hard to prove that with increasing order of approximations, the numerical spectral gap is decreasing. Indeed, we can consider Equation (13) and denote by $\lambda_\theta^{(N)}$ the approximate spectral gap (20), which is the minimum taken over the linear space Φ_N =span$\{\varphi_0, \ldots\ldots, \varphi_N\}$, for any fixed $N = 1, 2, \ldots$. Then, since $\Phi_N \subseteq \Phi_{N+1}$ and the Dirichlet form $\langle\mathcal{L}(\varphi), \varphi\rangle$ is positive, therefore $\lambda_\theta^{(N+1)} \le \lambda_\theta^{(N)}$, where each $\lambda_\theta^{(N)}$ is the N^{th} approximating minimization of Dirichlet forms defined on the Laguerre Polynomial basis Φ_N, as defined in (20). Therefore, due to such monotonicity property of the sequence $\{\lambda_\theta^{(N)}\}_N$, for each θ fixed, the corresponding minimization value λ_θ of the Dirichlet forms for the associated linear, radial and generalized Landau equation are well approximated

$$0 \le \lambda_\theta = \lim \lambda_\theta^{(N)}, \quad \text{as } N \to \infty. \tag{22}$$

The existence of this limiting $\lambda_\theta \ge 0$ follows from the monotone convergence theorem for the decreasing sequence of nonnegative terms $\lambda_\theta^{(N)}$, $N = 1, 2, \ldots$, and also from the well-known fact that the generalized Laguerre polynomials form a basis of the corresponding weighted L^2 space defined in (14).

For example, when $N = 1$, the matrix $G^{(1)}$ in (21) reduces to one single entry

$$\lambda_\theta = G_{11}^{(1)} = \frac{1}{1+\theta} 2^{-(1+\theta)} .$$

In particular, the above first order approximation yields a rough approximation $\lambda_0 \approx \frac{1}{2}$ which is rather inaccurate. More basis functions are clearly needed to get a good approximation of the minimum of the Rayleigh quotient sequences $\lambda_\theta^{(N)}$ in (22), for each θ fixed. Therefore, one can compute numerically $\lambda_\theta^{(N)}$ and study its asymptotic behavior with increasing N, for any $0 \leq \theta \leq 1/2$.

We restrict our numerical calculations in this section to the two cases, namely $\theta = 0$ and $\theta = 1/2$. The corresponding results are presented on Figs. 1 and 2 (some details of the calculations are discussed at the end of this section). The case $\theta = 0$ is special, since there exists a spectral gap $\lambda_0 = 1/4$ in this case. This means that Eqs. (7), (12) are valid with $\lambda_0 = 1/4$, i.e. we obtain a purely exponential convergence to equilibrium. In fact, the value $\lambda_0 = 1/4$ can be found analytically since the solution of the corresponding eigenvalue problem can be expressed in terms of hypergeometric functions [1].

The second case of $\theta = 1/2$ corresponds to the most important physical case of the linear Landau equation for Coulomb forces. It is clearly seen from Fig. 2 that numerically we obtain in the limit the value $\lambda_0 = 0$. In fact the same result can be obtained also for all $0 < \theta < 1/2$.

Hence, the spectral gap gap does not exist in these cases and therefore one cannot expect a purely exponential relaxation to equilibrium. An alternative law of relaxation will be discussed in the next sections.

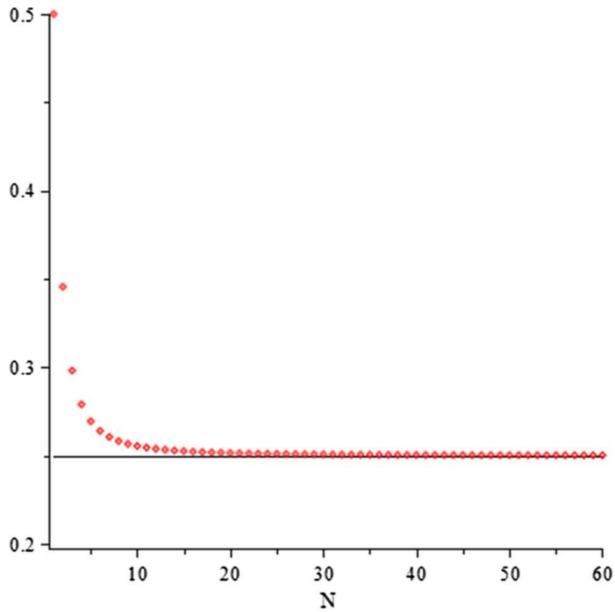


Fig. 1 The numerical spectral gap of generalized linear Landau operator for $\theta = 0$ with increasing N converges to 0.25

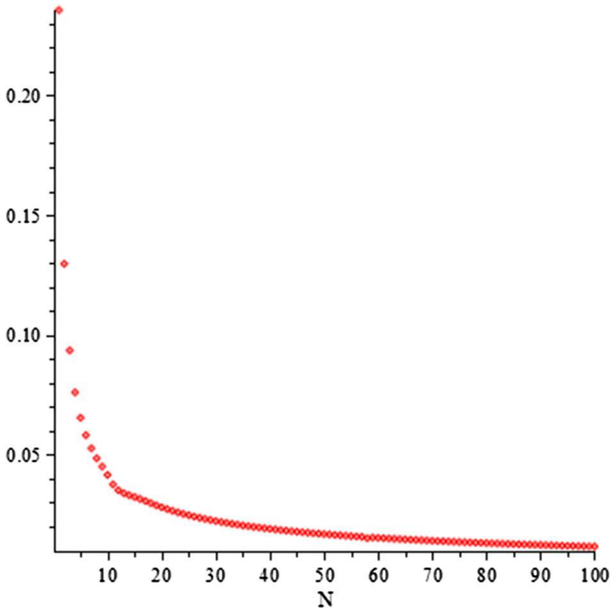


Fig. 2 The numerical spectral gap of the generalized linear Landau operator for 3-dimensional radial solutions corresponds to $\theta = 1/2$ with increasing N, exhibiting the lack of a spectral gap

For $\theta = 0$, the convergence to the analytical value $1/4$ can be observed. While for $\theta = 0.5$, the "gap" goes all the way down to zero with increased orders of basis Laguerre polynomials. Only results with respect to N up to 100 are plotted, but the "gap" will continue to decrease towards zero when order $N > 100$, which implies there is no such a spectral gap.

Remark The definition of the entries of the matrix G involves arithmetics among numbers of widely different magnitudes such as factorials and Gamma functions. Thus, in order to avoid large numerical errors caused by the fixed-precision floating point arithmetic, quite standard in C/C++, we include the package GNU MPFR (for GNU Multiple Precision Floating-Point Reliably [4]), which is a portable C library for arbitrary-precision binary floating-point computation with correct rounding, based on GNU Multi-Precision Library.

5 Asymptotic form of the Equation and the "Law of Two Thirds"

Thus, the absence of spectral gap for the main case $\theta = \frac{1}{2}$ shows that it is impossible to get any estimate like $O(\exp(-\lambda t))$, with $\lambda > 0$, for the rate of relaxation. In fact, one expects to obtain the order $O(\exp(-\lambda t^{2/3}))$ based on rigorous results of Strain and Guo [10,11]. We shall see below how the "law of two thirds" can be guessed on the basis of some heuristic arguments.

Start from considering Eq. (2) for $x \to \infty$. Then the asymptotic behavior of the diffusion coefficient

$$\mathcal{D}_\theta(x) \xrightarrow[x\to\infty]{} \int_0^\infty dx x^\theta e^{-x} = \Gamma(\theta + 1), \quad 0 \le \theta \le \frac{1}{2}. \tag{23}$$

Then, in the domain values for $x \gg 1$, and times $\tilde{t} = t\Gamma(\theta+1)$, it yields

$$\begin{aligned} \partial_{\tilde{t}} \tilde{f}(x,\tilde{t}) &= x^{-\theta}(\partial_x^2 + \partial_x)\tilde{f}(x,\tilde{t})\,, \\ \tilde{f}(x,\tilde{t}) &= f(x,t). \end{aligned} \tag{24}$$

The same asymptotic equation form (24) can also be obtained for the nonlinear case (see [2]).

Next, for sake of simplicity omit tildes, and set

$$f(x,t) = e^{-x}F(x,t)\,, \quad F(x,t) \xrightarrow[t\to\infty]{} 1. \tag{25}$$

Then, it follows that

$$F_t + x^{-\theta}F_x = x^{-\theta}F_{xx}\,, \quad x \gg 1. \tag{26}$$

While our results are mostly motivated by the Coulomb case $\theta = \frac{1}{2}$, all considerations remain valid for any $0 < \theta < \frac{1}{2}$.

Since the frequency of Coulomb collisions decays as $x^{-3/2}$ for large x, then one can expect that the relaxation process has two stages:

(a) fast relaxation for thermal velocities $x \simeq O(1)$ provided that the initial data has a compact support concentrated in thermal domain;
(b) slow propagation to the domain of large velocities, i.e $x \gg 1$.

Therefore, assuming that for a time $t_0 \gg 1$, the relaxation process for small and moderately large velocities (energies) $x \in [0, x_0]$, $x_0 \gg 1$, is practically achieved, and that

$$F(x,t_0) \approx \eta(x_0 - x), \tag{27}$$

where $\eta(x)$ stands for the unit step function; then we need to solve the above equation just for $t > t_0$.

At the first approximation, we neglect the diffusion term and obtain

$$F_t + x^{-\theta}F_x = 0\,, \quad F|_{t=t_0} = \eta(x_0 - x)\,, \quad t > t_0. \tag{28}$$

Then the solution reads (traveling wave)

$$F(x,t) = \eta[x_f(t) - x]\,, \tag{29}$$

where

$$x_f(t) = [x_0 + (1+\theta)t]^{\frac{1}{1+\theta}} \tag{30}$$

corresponds to the velocity of the wave front. The corresponding distribution function reads

$$f(x,t) = e^{-x}F(x,t). \tag{31}$$

Its distance to the equilibrium distribution e^{-x} is given by

$$d(f, e^{-x}) = \|f(x,t) - e^{-x}\|_{L^\infty} = \sup_{x > x_f(t)} e^{-x} = \exp(-x_f(t)). \tag{32}$$

Hence, for large t we obtain,

$$d(f, e^{-x}) = \exp\left(-c_\theta t^{\frac{1}{1+\theta}}\left(1 + O(t^{-1})\right)\right), \tag{33}$$

where $c_\theta = (1+\theta)^{\frac{1}{1+\theta}}$. If $\theta = \frac{1}{2}$, we obtain the correct power $q = \frac{2}{3}$. This is probably the simplest heuristic way to "derive" this result from the Landau equation. The more detailed study of asymptotic Landau equation in the nonlinear case was performed in [2]. The result shows that the diffusion term influences only the shape of the wave. Therefore, the conclusion is the same: the rate of relaxation for the Landau equation (linear or nonlinear) is of order of $\exp(-ct^{2/3})$ with some $c > 0$. An interesting fact is that this conclusion can be obtained heuristically from asymptotics for large velocities.

We consider below only the value $\theta = \frac{1}{2}$ in Eq. (8). The numerical evidence for the "step function"- like traveling wave $F(x, t)$ can be found in [2]. Here, under the setting of Laguerre polynomial expansions and with the diffusion term being kept, we shall try, both, to numerically confirm this observation and to verify the "two thirds" fractional relaxation rate as well.

Remark The main focus of this manuscript is to described the relaxation to the equilibrium Maxwellian in the "physical" case $\theta = 1/2$. However, our arguments can be applied to any $0 < \theta < 1/2$, in which case the expected rate to equilibrium is then described by a "Law of $1/(1+\theta)$", and can use the same approach, as in Sect. 6, to the numerical verification of this law.

6 Numerical Results for the Landau Equation

The target equation to be computed is the one for perturbation $\varphi(x, t)$, i.e Eqs. (8–9), with exponent $\theta = 1/2$ corresponding to the radial solutions to the Landau equation in three dimensions. And thus $F(x, t) = 1 + \varphi(x, t)$ in Eq. (31). The $\varphi(x, t)$ is then approximated by expansions of Laguerre polynomials as in (15). Then, multiplying both sides of Eq. (8) by test function $\varphi_n(x)$ yields the linear ODE system

$$\frac{d}{dt}U(t) = -G^{(N)}U \tag{34}$$

where $U = (u_1, \ldots, u_N)^T$ is the vector of coefficients; matrix $G^{(N)}$ has already been analytically given in (21).

Solving the above linear system (34) is relatively simple. However, this system is extremely stiff due to the asymptotically vanishing spectral gap for matrix G, as observed in numerical experiment (see Fig. 2). Therefore, an implicit numerical method should be applied, and so we adopt a second order implicit trapezoidal scheme

$$U(t + dt) = U(t) - \frac{dt}{2}(G^{(N)}U(t + dt) + G^{(N)}U(t)), \tag{35}$$

with dt being the time step.

Another issue worthy of careful attention is the fact that with increasing order of Laguerre polynomials and expanding tails in x-space, the extremely contrasting magnitudes of float-pointing numbers from factorials, Gamma functions and large order polynomials, will easily invalidate the calculations. Based on these considerations, we choose to numerically solve (35) with Maple [7]. Maple applies a hybrid symbolic-numeric approach. It supports, both, hardware precision and infinite precision calculations. In addition, by increasing the precision, round-off error is reduced and problems of catastrophic cancellation are avoided.

The initial function $\varphi(x, 0)$ can be arbitrarily chosen. Here, if assume the ultimate equilibrium state $\exp(-x)$, then initial $\varphi(x, 0)$ should satisfy condition (4) due to the conservation of mass. Suppose we take an initial distribution $f_0(x) = \exp(-x)(1 + \varphi(x, 0)) = a\exp(-bx)$. With $b > 1$ fixed, condition (4) gives $a = b^{3/2}$. We project $\varphi(x, 0)$ onto space of Laguerre polynomials up to order N and get

$$f_0(x) = \exp(-x)\left(1 + \sum_{n=1}^{N} u_n^0 \varphi_n(x)\right). \tag{36}$$

Note, the summation starts from $n = 1$, rather than $n = 0$, due to condition (4).

By considering the orthogonality of Laguerre polynomials (14), we can easily calculate the coefficients

$$u_n^0 = a \int_0^\infty x^{\frac{1}{2}} \exp(-bx)\varphi_n(x), \qquad n = 1 \ldots N \tag{37}$$

Thus, an initial vector for system (35) would be given as $U(t = 0) = (u_1^0, \ldots, u_N^0)$. In order to better capture the intermediate asymptotics, in the following numerical experiments, we choose $b = 6$ for a more "concentrated" initial distribution function.

Despite the above heuristic discussions in the sense of L^∞ distance in (32), the result of Strain and Guo [10,11] is expected to be valid for any reasonable norm, in particular the standard L^2 distance. We denote

$$d(x,t) = e^{-x}\varphi(x,t) = e^{-x} \sum_{n=1}^{N} u_n \varphi_n(x) \tag{38}$$

If we adopt weighted L^2 norm with weight $e^x x^{1/2}$, then

$$\|d(x,t)\|_w^2 = \sum_{n,m=1}^{N} u_n u_m \int_0^\infty e^{-x} x^{1/2} \varphi_n(x)\varphi_m(x)dx = \sum_{n=1}^{N} u_n^2 \tag{39}$$

This will be adopted in the following as the measurement of distance to equilibrium.

Figure 3 shows the traveling wave, which is the normalized solution $f(x,t)$ by the equilibrium $\exp(-x)$, i.e the function $F(x,t)$ given in Eq. (31). It agrees with what has been

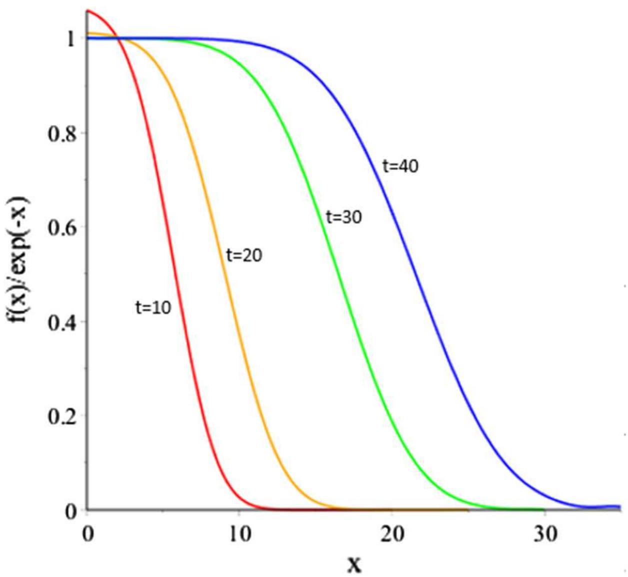


Fig. 3 Case $\theta = 1/2$. The travelling waves, $N = 50$

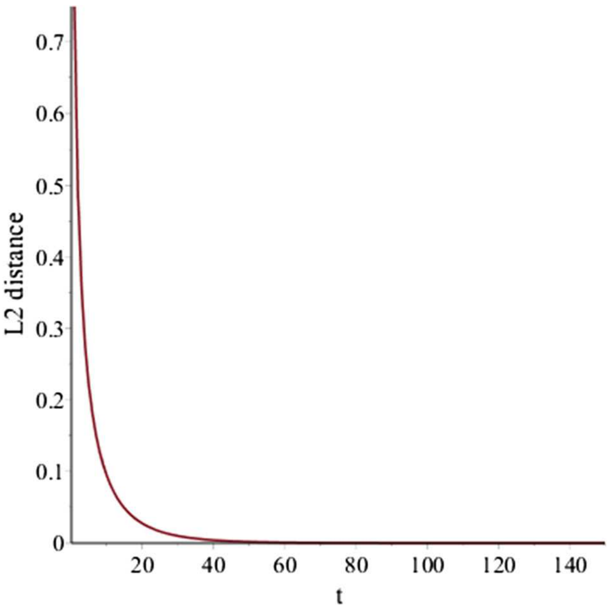


Fig. 4 Case $\theta = 1/2$. The distance to equilibrium, $N = 50$. Only plot up to t = 150

predicted in [2]. It asymptotically behaves like a step function and keeps constant width for much large time and velocities out of the thermal domain.

Figure 4 is the L^2 distance to equilibrium (39). The numerical results for the above function $d(x, t)$ show a rapid decay in the thermal domain and then slow relaxation for large velocities. More detailed intermediate and large time asymptotics can be observed by exploring its double-logarithm measurements, which will be done later.

The next step is to numerically verify the fractional power in the exponential decay. As discussed above, after relaxing for moderately long time, the $t^{2/3}$ term is well believed to dominate in the exponential relaxation. Then, for the moderate "far" tails, we take the double-logarithm of the distance $d(f, e^{-x})$, that is,

$$\log\left(-\log\left(d(f, e^{-x})\right)\right) \sim c + \frac{2}{3}\log t, \quad t \gg 1 \tag{40}$$

which is expected to be a linear relationship, with a slope being close to $\frac{2}{3}$. However, in actual numerical simulations, this fractional exponential decay is just an intermediate asymptotics, and will only last for a moderately long time. If we let the relaxation proceeds for sufficiently long time, we should expect a pure exponential decay with a decay rate determined by the spectral gap of G^N in (34). This decay rate, or spectral gap, will be smaller and smaller with increasing order N, as shown in Fig. 2. In other words, if measured in double-log (40), there should be another linear segment after sufficiently long time, with a slope close to 1.0. And with increasing order N, it should be slower to enter the domain of pure exponential decay.

Remark Since there is no definite boundary between the fractional and pure exponential decaying realm, we explain here how we numerically capture in the double-log plot the two linear segments. The double-log curve will be linearly fitted starting from zero, piece by piece, with each piece spanning, e.g 100 units. During the process of sliding the fitting line

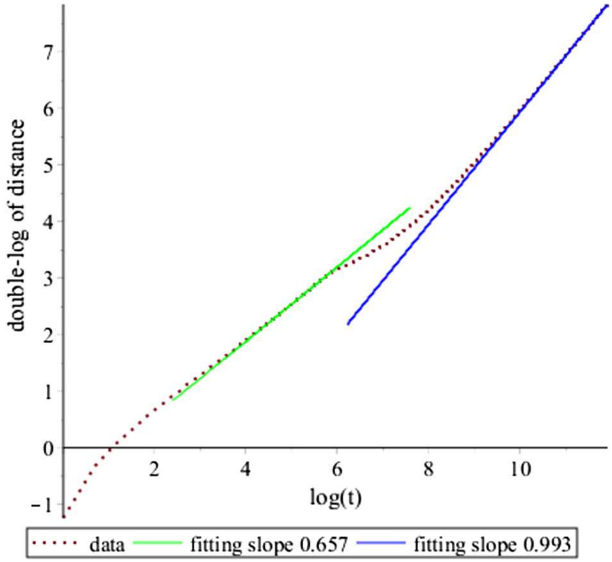
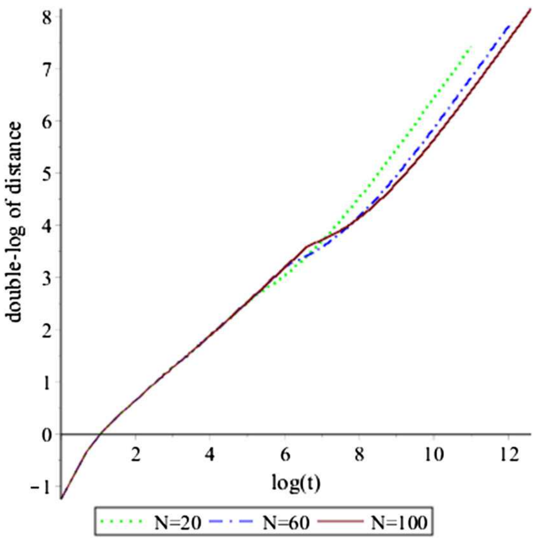


Fig. 5 Case $\theta = 1/2$. Two linear segments numerically captured in the double-log plot for $N = 50$, representing the fractional and pure expoential decaying realms respectively

Fig. 6 Case $\theta = 1/2$. Overlay of double-log distance plots for $N = 20$, 60 and 100



toward the end of curve, we will reach several pieces with a steady (almost constant) slope. Such a segment will be treated as a line. There should be two such segments for each curve (each fixed N), representing the fractional and pure exponential decaying realm respectively. Figure 5 shows the two line segments numerically captured in the double-log plot for $N = 50$.

Figure 6 is the overlay of the double-log plots for different N. For each N, the two linear segments are visually clear, as already been discussed. The trailing lines for each N always have a slope close to 1.0, as long as the simulation time is long enough. However, as readers

Table 1 The steady fitting slopes captured for the intermediate asymptotics

Order N	20	30	40	50	60	70	80	90	100
Intermediate fitting slope	0.638	0.648	0.654	0.657	0.659	0.660	0.662	0.663	0.663

can see from Fig. 6, as N increases, it takes longer time to enter the domain of pure exponential decay. This is also expected due to smaller spectral gaps.

The slope of the intermediate line is of most interest to us. Table 1 lists the sliding fitting results introduced in the aforementioned remark. It is evident that the steady slopes of the intermediate line segments approach $\frac{2}{3}$, which numerically verifies the intermediate asymptotics of the fractional exponential decay, i.e the "law of two thirds" .

7 Summary

In this manuscript we study the rate of relaxation to equilibrium for Landau kinetic equation and related models. Though there exists a rather complicated rigorous proof by Strain and Guo [10,11] of the "law of two thirds" for fractional exponential relaxation applicable to the case of radial solutions to the Landau equation in three dimension. Our analysis, based on approximations by series expansions of Laguerre polynomials extend to the case of equations the generalized the radial Landau solution in three dimensions, when the parameter θ in Eqs. (8–9) lays in $(0, 1/2)$. This result is clear through a series of heuristic arguments as well as numerical experiments on the linear radial dimensionless generalized Landau equation. We approximate its solution by a linear combination of N generalized Laguerre polynomials, having in mind the corresponding weighted L^2 space when $N \to \infty$. The rate of (purely exponential) relaxation for corresponding system of N linear ODEs can be easily obtained through studying numerically its Rayleigh quotients for any fixed N. The difficulty, however, appears when the corresponding smallest positive eigenvalue tends to zero as $N \to \infty$. Then the limiting spectral gap does not exist. It is exactly the case for the classical Landau equation. In this case we have considered the time-dependent solution of the corresponding set of N ODEs and show numerically the existence of intermediate fractional exponential asymptotics, which is clearly separated for large N and t from the purely exponential absolute asymptotics. The intermediate asymptotics yields precisely the "law of two thirds", as clearly seen from Table 1 at the end of Sect. 6. This can be considered as the main result of the paper. Note that to prove it numerically we need to deal with exponentially small (for large values of time t) numbers. Such computations involve delicate handling of extremely contrasting magnitudes of float-pointing numbers. To validate such subtle computations, we make use of hybrid symbolic-numeric approach, which was impossible in previous conventional numerical treatments. Finally we note that, to the best of our knowledge, the very existence of intermediate asymptotics for a system of linear ODEs is not studied in literature. On the other hand, such asymptotics must be typical, for example, for finite dimensional approximations of kinetic equations which do not have a spectral gap. The well-known examples are linear and linearized Boltzmann equations for soft potentials with cut-off. We believe that our methods can be also applied to these and similar equations.

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