CAUCHY INDEPENDENT MEASURES AND ALMOST-ADDITIVITY OF ANALYTIC CAPACITY

By

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Abstract. We show that, given a family of discs centered at a chord-arc curve, the analytic capacity of a union of subsets of these discs (one subset in each disc) is comparable with the sum of their analytic capacities. However, we need the discs in question to be separated, and it is not clear whether the separation condition is essential. We apply this result to find families $\{\mu_j\}$ of measures in $\mathbb{C}$ with the following property. If the Cauchy integral operators $\mathcal{C}_{\mu_j}$ from $L^2(\mu_j)$ to itself are bounded uniformly in j, then $\mathcal{C}_\mu$, $\mu = \sum \mu_j$, is also bounded from $L^2(\mu)$ to itself.

1 Introduction

We consider two properties of families of sets and measures in the complex plane.

1.1 Almost additivity of analytic capacity. The **analytic capacity** $\gamma(F)$ of a compact set $F \subset \mathbb{C}$ is defined by the equality

$$\gamma(F) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus F \to \mathbb{C}$ with $|f| \le 1$ on $\mathbb{C} \setminus F$. Here, $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$. For non-compact F, we set

$$\gamma(F) = \sup\{\gamma(K) : K \text{ compact}, K \subset F\}$$

[G]. For a summary of equivalent definitions, we refer the reader to [To] and [Vo].

In the celebrated paper [To1], Tolsa established the countable semiadditivity of the analytic capacity, i.e.,

$$\gamma\Big(\bigcup F_i\Big) \le C \sum \gamma(F_i),$$

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where C is an absolute constant. But the inverse inequality does not hold in general. To see this, we consider the n-th generation $E_n^{1/4}$ of the corner 1/4-Cantor set constructed in the following way. Start with the unit square (0-th generation). The j-th generation consists of 4^j squares $E_{j,k}$ with side length 4^{-j}; each square $E_{j,k}$ contains four squares of $(j+1)$-th generation, located at the corners of $E_{j,k}$, and so on. It is known [MTV] that

$$\gamma\Big(\bigcup_{k=1}^{4^n} E_{n,k}\Big) = \gamma(E_n^{1/4}) \asymp 1/\sqrt{n}$$

with absolute constants of comparison; here $P \asymp Q$ means that $cP \le Q \le CP$. Positive constants c, C, a, A (possibly with indices) are not necessarily the same at each appearance. On the other hand,

$$\sum_{k=1}^{4^n} \gamma(E_{n,k}) \asymp 4^n \cdot 4^{-n} = 1.$$

Thus, "almost additivity" $\gamma\big(\bigcup F_i\big) \asymp \sum \gamma(F_i)$ of the analytic capacity does not hold in general. During the personal conversation with the first-named author in 2012 (Chebyshev Laboratory, St.-Petersburg), N. A. Shirokov [Sh] posed the question on the validity of this property for the special class of sets described in the following Theorem 1.1.

We say that Γ is a **chord-arc curve** if

$$|t-s| \le A_0|z(t)-z(s)|, \quad A_0 > 1, \tag{1.1}$$

where $z(t)$ is the arc-length parametrization of Γ.

Theorem 1.1. *Let D_j be discs with centers on a chord-arc curve Γ such that $\lambda D_j \cap \lambda D_k = \varnothing$, $j \ne k$, for some $\lambda > 1$. Let $E_j \subset D_j$ be compact sets. Then there exists a constant $c = c(\lambda, A_0)$ such that*

$$\gamma\Big(\bigcup E_j\Big) \ge c\sum_j \gamma(E_j). \tag{1.2}$$

We conclude the present subsection with a useful corollary.

Corollary 1.2. *(a) Let Γ be the union of n chord-arc curves Γ_i with the same constant A_0, and let D_j and E_j be as in Theorem 1.1. Then*

$$\gamma\Big(\bigcup E_j\Big) \ge \frac{c}{n}\sum_j \gamma(E_j), \quad c = c(\lambda, A_0).$$

In particular, Γ might be a circle ($n = 2$).

(b) When Γ is a circle, inequality (1.2) *remains valid if $E_j \subset D_j$, where E_j are compact sets and $\{D_j\}$ is a family of λ-separated discs (that is λD_j are disjoint), $\lambda > 1$, only intersecting Γ, not necessarily with centers on Γ.*

Proof. (a) Let $D_j = D(x_j, r_j)$, and let m be such that

$$\max_{1\le k\le n}\Big\{\sum_{j:x_j\in\Gamma_k}\gamma(E_j)\Big\} = \sum_{j:x_j\in\Gamma_m}\gamma(E_j).$$

Then, by Theorem 1.1,

$$\gamma\Big(\bigcup E_j\Big) \ge \gamma\Big(\bigcup_{j:x_j\in\Gamma_m} E_j\Big) \ge c\sum_{j:x_j\in\Gamma_m}\gamma(E_j) \ge \frac{c}{n}\sum_j \gamma(E_j).$$

(b) It is sufficient to prove that if all discs D_j intersect a semicircle T, then there is another chord-arc curve $\widetilde{\Gamma}$ with a constant $A_0 = A_0(\lambda)$ containing all centers of D_j. We may assume that there are more than one disc D_j. Fix j, and let a_j, b_j be the points of intersection of T and the circle $\partial(\lambda' D_j)$, where $\lambda' = (1+\lambda)/2$ (for two discs D_j, there might be only one such a point). Replace the arc of T with the endpoints a_j and b_j with two line segments $x_j a_j$ and $x_j b_j$ (in the case of one point of intersection, with the only line segment). We claim that the obtained curve is a chord-arc. Indeed, since $|a_j - b_j| \ge c(\lambda) r_j$, the condition (1.1) holds with $A_0 = A_0(\lambda)$ for any points $z(t), z(s) \in \lambda' D_j$ and for any two points in $\widetilde{\Gamma}$ situated on T. If $z(t) \in \lambda' D_i$, $z(s) \in \lambda' D_j$, $i \ne j$, then both parts of (1.1) are comparable with $|x_i - x_j|$. To demonstrate this assertion, we notice that the inequality $|z_1 - z_2| > (\lambda - \lambda')(r_i + r_j)$, $z_1 \in \lambda' D_i$, $z_2 \in \lambda' D_j$, implies the relations $|z_1 - z_2| \asymp |x_i - x_j|$ and $t - s \le C(r_i + |a_i - b_j| + r_j) \le C'|x_i - x_j|$.

Similar arguments yield (1.1) when one point is situated on T. □

Open question. Does the conclusion of Theorem 1.1 hold when $\lambda = 1$?

1.2 Cauchy independence of families of measures. We use the results in the previous subsection to investigate the property of measures described below.

We call a finite Borel measure with compact support in the complex plane a **Cauchy operator measure** if the Cauchy operator $\mathcal{C}_\mu$ is bounded from $L^2(\mu)$ to itself with norm at most 1.

The first natural question is how to interpret the "definition" of $\mathcal{C}_\mu$ as

$$\mathcal{C}_\mu f(z) = \int \frac{f(\xi)d\mu(\xi)}{\xi - z}.$$

One way is to consider the so-called ε-truncations, defined by

$$\mathcal{C}^\varepsilon_\mu f(z) = \int_{\varepsilon<|\xi-z|<\varepsilon^{-1}} \frac{f(\xi)d\mu(\xi)}{\xi - z}.$$

We now say that $\mathcal{C}_\mu$ is **bounded** as an operator from $L^2(\mu)$ to itself if the ε-truncations are bounded from $L^2(\mu)$ to itself uniformly in ε. Moreover, by the norm of $\mathcal{C}_\mu$, we mean $\sup_{\varepsilon>0} \|\mathcal{C}^\varepsilon_\mu\|_\mu =: \|\mathcal{C}_\mu\|_\mu$, where $\|\mathcal{C}^\varepsilon_\mu\|_\mu$ is the norm of $\mathcal{C}^\varepsilon_\mu$ as an operator from $L^2(\mu)$ to itself. We encourage the reader to look at other interpretations in [NTrV1], [To], and [Vo].

The following important fact (which we use repeatedly) demonstrates the connection between analytic capacity and boundedness of the Cauchy operator [To1, To2, To, Vo]: for every compact set $F \subset \mathbb{C}$,

$$\gamma(F) \asymp \sup\{\|\mu\| : \operatorname{supp}\mu \subset F,\ \mu \in \Sigma,\ \|\mathcal{C}_\mu\|_\mu \le 1\}, \tag{1.3}$$

where Σ is the class of non-negative Borel measures μ such that $\mu(D(x,r)) \le r$ for every disc $D(x,r) := \{z \in \mathbb{C} : |z-x| < r\}$.

We call a collection $\{\mu_j\}$ of finite positive Borel measures with compact supports ***C*-Cauchy independent measures** if a) $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$ (Cauchy operator measures) and b) $\|\mathcal{C}_\mu\|_\mu \le C < \infty$ for $\mu = \Sigma_j \mu_j$. We call such collection **Cauchy independent** if it is C-Cauchy independent for some finite C.

The family $\{\mu_j\}$ can be finite or infinite. Two Cauchy operator measures are always Cauchy independent with an absolute constant C. A short proof of this non-trivial fact is given in [NToV, Proposition 3.1]. So, a finite family is always Cauchy independent for a sufficiently large constant C. But our main interest is in situations in which infinite families are independent (or when C is independent of the number of measures). The main result is the following theorem.

Theorem 1.3. *Suppose that $\lambda > 1$, and measures μ_j are supported on compact sets E_j lying in discs D_j such that λD_j are disjoint. Assume also that measures μ_j are extremal in the sense that $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$ and $\|\mu_j\| \asymp \gamma(E_j)$ with absolute comparison constants. Let $\mu = \sum_j \mu_j$ and $E = \cup E_j$. Then this family is Cauchy independent if and only if for every disc B,*

$$\mu(B) \le C_0 \gamma(B \cap E). \tag{1.4}$$

Remarks. 1. In Section 3, we prove that the condition (1.4) with any disc B is necessary for the bound $\|\mathcal{C}_\mu\|_\mu \le C$ without any additional assumptions on the structure of μ. Example 5.1, given in Section 5, shows that this condition alone is not sufficient, even if μ consists of countably many pieces μ_j and each μ_j gives a bounded Cauchy operator with a uniform bound. Thus, additional conditions on the structure of μ are needed. An example of such assumptions on μ, which seem reasonable, is given in Theorem 1.3, where supports of μ_j are located in separated discs.

2. In general, one cannot discard the requirement that the measures μ_j in Theorem 1.3 are extremal; see Example 5.2 in Section 5.

3. Tolsa [To1, pp. 125–129, 135–146] proved that under the conditions of Theorem 1.3, there exists a piece of measure μ, namely, $\mu' := \chi_{E'} \cdot \mu$, $E' \subset E$, such that $\mu'(E) \geq c\, \mu(E)$, and $\|\mathcal{C}_{\mu'}\|_{\mu'} \leq C < \infty$, where $c > 0$ and C are constants depending only on the parameters in Theorem 1.3. This fact, which is far from trivial, is used in [To1] to approach Painlevé's conjecture. In other words, it is a highly non-trivial problem to prove that under the assumptions of Theorem 1.3, a "good portion" of μ generates a bounded Cauchy operator. It is remarkable that the whole measure μ has, in fact, such a property.

As a corollary of Theorem 1.3 we derive the following independence theorem.

Theorem 1.4. *Let μ_j be measures supported on compacts E_j lying in discs D_j, respectively, such that λD_j are disjoint ($\lambda > 1$), and let $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \leq 1$. If for each disc B,*

$$\sum_j \gamma(B \cap E_j) \leq C_1 \gamma(B \cap E), \quad E = \bigcup E_j, \tag{1.5}$$

then the norm $\|\mathcal{C}_\mu\|_\mu$ is bounded, and the bound depends only on a comparison constant C_1 and on λ. Here, as above, $\mu = \Sigma_j \mu_j$.

Note that unlike Theorem 1.3, this result does not need the additional condition that measures μ_j are extremal. On the other hand, (1.5) is not necessary for the boundedness of $\mathcal{C}_\mu$. For example, if μ_j are the 2-dimensional Lebesgue measure on the squares E_j, $j = 1, 2, \ldots$, defined below in Example 5.1, the operator $\mathcal{C}_\mu$ is obviously bounded, but (1.5) does not hold.

Theorems 1.3 and 1.4, unlike Theorem 1.1, do not have any assumptions on the location of discs D_j. However, condition (1.5) is not completely independent of a geometry of discs. Theorem 1.1 states that if λ-separated discs are situated along a chord-arc curve, then the almost additivity of the analytic capacity holds. We prove a statement which is converse in the sense that almost additivity of analytic capacity in the form of inequality (1.5), together with certain additional assumptions, implies that our discs have to "line-up" along a good (Ahlfors-David regular) curve.

We denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure. A set $G \subset \mathbb{C}$ is called **Ahlfors-David (AD)-regular** if

$$c\,r \leq \mathcal{H}^1(G \cap B(x, r)) \leq Cr, \quad x \in G,\ 0 < r \leq \operatorname{diam} G,$$

for some positive constants c, C.

Corollary 1.5. *Suppose that λ-separated ($\lambda > 1$) discs $D_j = D(x_j, r_j)$ and subsets $E_j \subset D_j$ are such that*

(a) (1.5) *holds,*

(b) $\gamma(E_j) \asymp r_j$,

(c) *the set $\mathcal{T} := \bigcup_j \partial D_j$ is AD-regular.*

Then there exists an AD-regular curve which intersects all discs D_j.

All three assumptions (a)–(c) are required for the conclusion of Corollary 1.5; see Proposition 5.3.

Organization of the paper. To prove Theorems 1.3, 1.4, we need only the special case of Theorem 1.1 in which all discs D_j intersect a real line or a circle, that is, the last assertion of Corollary 1.2. For this case, there is a short proof based only on some classical facts in complex analysis. We give this proof in Section 2. Theorem 1.3 is proved in Section 3, and Theorem 1.4 with Corollary 1.5 in Section 4. In Section 5, we give the examples mentioned above. Section 6 contains the proof of Theorem 1.1 in the full generality, which is completely different from the proof in Section 2. The main tool of this proof is Melnikov–Menger's curvature of a measure. All necessary definitions are given in Section 6. In the last section, Section 7, we formulate an open question.

2 Almost-additivity of analytic capacity: string of beads attached to the real line

For some special sets $\{E_j\}_{j=1}^{\infty}$, a result close to the Theorem 2.1 below was proved (but not stated) in [NV]. Here, we use the approach via the Marcinkiewicz function; the approach in [NV] is a bit more complicated. Unlike in [NV], we do not need any special size properties of these sets.

Theorem 2.1. *Let D_j be discs such that $\lambda D_j \cap \lambda D_k = \varnothing$, $j \neq k$, $\lambda > 1$, and each disc D_j has a non-empty intersection with the real line $\mathbb{R}$. Let $E_j \subset D_j$ be compact sets. Then there exists a constant $c = c(\lambda) > 0$ such that*

$$\gamma\Big(\bigcup E_j\Big) \geq c \sum_j \gamma(E_j).$$

Proof. It suffices to prove the result for finite families of indices j. We first notice that $\gamma_j := \gamma(E_j) \leq \gamma(D_j) = r_j$, where r_j is the radius of D_j. Let y_j be the center of the chord $\mathbb{R} \cap D_j$, and let $\lambda' := (1+\lambda)/2$. For each j, we draw a horizontal line segment $L_j \subset \lambda' D_j$ with center at y_j and with analytic capacity $b(\lambda)\gamma_j$, where

$b(\lambda) = \sqrt{\lambda'^2 - 1}/4$. Thus, the length ℓ_j of L_j satisfies $\ell_j = 4b(\lambda)\gamma_j \le r_j\sqrt{\lambda'^2 - 1}$. In particular, the whole segment L_j lies in $\lambda' D_j$.

Next, let f_j be the function that gives the capacity of E_j. Also let φ_j be the function that gives the capacity of L_j. Then

$$\varphi_j(z) = \int_{L_j} \frac{\varphi_j(x)}{x - z} dx, \qquad \int_{L_j} \varphi_j(x) dx = b(\lambda)\gamma_j.$$

Positive functions $\varphi_j(x)$ have a uniform bound $\|\varphi_j\|_\infty \le A$ with an absolute constant A. In particular, for each subset $\mathcal{F}$ of indices j, we have

$$\left| \operatorname{Im} \sum_{j \in \mathcal{F}} \varphi_j(z) \right| \le A \int_{\bigcup_{j\in\mathcal{F}} L_j} \frac{|\operatorname{Im} z|}{|t - z|^2} \, dt \le \pi A\,, \quad \text{for all } z \in \mathbb{C}. \tag{2.1}$$

Remark. It is important here that the intervals L_j be situated on the real line (or at least not far from $\mathbb{R}$). For any $M > 0$, one can easily construct a chord-arc curve and discs centered on it such that the left-hand side of (2.1) exceeds M. This is the obstacle for extension of these arguments to chord-arc curves.

Our next goal is to find a family $\mathcal{F}$ of indices and absolute positive constants a_1, a_2, such that

$$\sum_{j \in \mathcal{F}} \gamma_j \ge a_1 \sum_j \gamma_j, \tag{2.2}$$

$$\sum_{j \in \mathcal{F}} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le a_2, \quad \text{for all } z \in \mathbb{C} \setminus \bigcup_{j \in \mathcal{F}} (E_j \cup L_j). \tag{2.3}$$

Let us finish the proof of Theorem 2.1, taking these assertions for granted (for a short while). Let $F := \sum_{j\in\mathcal{F}} f_j$. Combining (2.1) and (2.3), we get $|\operatorname{Im} F(z)| \le C_1(\lambda)$, $z \in \mathbb{C} \setminus (\bigcup_{j\in\mathcal{F}} E_j)$. Hence, the function $F_1 := e^{iF} - 1$ is bounded on $\mathbb{C} \setminus (\bigcup_{j\in\mathcal{F}} E_j)$ by a constant $C(\lambda)$. Since $F(\infty) = 0$, we have $|F_1'(\infty)| = |F'(\infty)| = \sum_{j\in\mathcal{F}} \gamma_j$. Thus

$$\gamma\Big(\bigcup_{j \in \mathcal{F}} E_j \Big) \ge \frac{1}{C(\lambda)} \sum_{j \in \mathcal{F}} \gamma_j\,.$$

Combining this with (2.2), we obtain

$$\gamma\Big(\bigcup_j E_j \Big) \ge \gamma\Big(\bigcup_{j \in \mathcal{F}} E_j \Big) \ge a_3 \sum_j \gamma_j,$$

and Theorem 2.1 is proved. So we are left to chose the family $\mathcal{F}$ such that (2.2), (2.3) hold.

By the Schwartz lemma in the form we borrow from [G, p. 12–13],

$$|f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le \frac{Ar_j\gamma_j}{\operatorname{dist}(z, E_j \cup L_j)^2}, \quad z \notin E_j \cup L_j. \tag{2.4}$$

Set

$$Q_i := \lambda' D_i, \quad g_i := \sum_{j:\, j\neq i} \frac{r_j\gamma_j}{D(Q_j, Q_i)^2},$$

where $D(Q_i, Q_j) := \operatorname{dist}(Q_i, Q_j)$.

Remark. Although we do not need this, for the sake of explanation, let us define a function $g = \sum g_j \chi_{Q_j \cap \mathbb{R}}$. This function is often called a **Marcinkiewicz function**. The main trick with Marcinkiewicz functions is to integrate them with respect to a suitable measure. We integrate it with respect to Lebesgue measure on $\mathbb{R}$.

The important point is that we can estimate $\sum_i g_i\gamma_i$. In fact,

$$\begin{aligned}\sum_i g_i\gamma_i &= \sum_i \gamma_i \sum_{j:\, j\neq i} \frac{r_j\gamma_j}{D(Q_j, Q_i)^2} = \sum_j r_j\gamma_j \sum_{i:\, i\neq j} \frac{\gamma_i}{D(Q_i, Q_j)^2} \\ &\le \sum_j r_j\gamma_j \sum_{i:\, i\neq j} \frac{r_i}{D(Q_i, Q_j)^2} \le A_0 \sum_j r_j\gamma_j r_j^{-1} = A_0 \sum_j \gamma_j .\end{aligned}$$

In the last estimate, we used

$$\sum_{i:\, i\neq j} \frac{r_i}{D(Q_i, Q_j)^2} \le A_1 \int_{t: |t-y_j|\ge r_j} \frac{1}{|t-y_j|^2}\, dt \le \frac{2A_1}{r_j}, \quad A_1 = A_1(\lambda),$$

which follows from the facts that the length of $\mathbb{R} \cap Q_j$ exceeds $c(\lambda)r_j$ and $D(Q_i, Q_j) \ge c'(\lambda)|t - y_j|$, $t \in \mathbb{R} \cap Q_i$. Now we apply Tchebychev's inequality. Setting

$$I^* := \{i : g_i > 10A_0\}, \quad I_* := \{i : g_i \le 10A_0\},$$

we see immediately that

$$\sum_{j\in I_*} \gamma_j \ge \frac{9}{10} \sum_j \gamma_j . \tag{2.5}$$

Obviously, by (2.4), for every index i,

$$\sum_{j:\, j\neq i} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le Ag_i, \quad z \in Q_i.$$

This estimate and the choice of I_* imply that

$$\sum_{j:\, j\neq i,\ j\in I_*} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le C(\lambda)g_i \le 10A_0A, \quad \text{for all } i \in I_*, \text{ for all } z \in Q_i.$$

But all functions $|f_i|$, $|\varphi_i|$ are bounded on $\mathbb{C} \setminus (E_i \cup L_i)$ by 1. Therefore, the last inequality implies the estimate

$$\sum_{j:\, j\in I_*} |f_j(z) - b(\lambda)^{-1}\varphi_j(z)| \le 10A_0A + b(\lambda)^{-1} + 1 =: a_2 \tag{2.6}$$

for all $z \in Q_i \setminus (E_i \cup L_i)$, for all $i \in I_*$. The function $\sum_{j\in I_*}(f_j - b(\lambda)^{-1}\varphi_j)$ is analytic on $\mathbb{C} \setminus \bigcup_{i\in I_*}(E_i \cup L_i)$ and vanishes at ∞. Therefore, (2.6) implies (2.3), with $\mathcal{F} := I_*$. Assertion (2.2) is proved in (2.5), and the proof of Theorem 2.1 is completed. □

Corollary 2.2. *The conclusion of Theorem 2.1 remains true if discs D_j intersect a circle instead of the real line.*

Proof. It suffices to consider the unit circle. There are at most K discs D_j of radius $r_j \ge 1/30$ intersecting the unit circle Γ, where K is an absolute constant. Hence, we may assume that $r_j < 1/30$ for each j. Moreover, as in Corollary 1.2, we may restrict ourself to discs intersecting the left semicircle T. Let $h(z) := i\,\frac{1+z}{1-z}$ be the conformal mapping of Γ onto the real line. Let E be a compact subset of $G^0 := \{z : \operatorname{dist}(z, T) < 1/10\}$, and $\mathcal{E} := h(E)$. We prove that

$$\gamma(E) \asymp \gamma(\mathcal{E}) \tag{2.7}$$

with absolute constants of comparison. This relation is a consequence of the general result by Tolsa [To3] about stability of the analytic capacity under bilipschitz maps, but we prefer a direct elementary proof (which possibly is not new) without using Tolsa's very difficult result. Pick a holomorphic function f such that $|f(z)| < 1$ on $\mathbb{C} \setminus E$. Define the sets

$$G := \{z : \operatorname{dist}(z, T) < 1/5\}, \quad \mathcal{G} := h(G), \quad \mathcal{G}^0 := h(G^0),$$

and the function $F(w) := f(g(w))g'(w)$, where $g(w) = \frac{w-i}{w+i}$ is the inverse of h. Clearly, $|F(w)| \le C_1$ as $w \in \mathcal{G} \setminus \mathcal{E}$, and the length of $\partial\mathcal{G}$ does not exceed C_2, where C_1, C_2 are absolute constants. Fix $w \in \mathcal{G}^0 \setminus \mathcal{E}$, and let L^0 be a closed curve in $\mathcal{G}^0 \setminus \mathcal{E}$ that encloses $\mathcal{E}$ but not w and is oriented in such a way that w is "on the left". If $\partial\mathcal{G}$ is oriented in the same way, by Cauchy's formula,

$$F(w) = \frac{1}{2\pi i}\int_{\partial\mathcal{G}} \frac{F(\xi)}{\xi - w}\, d\xi + \frac{1}{2\pi i}\int_{L^0} \frac{F(\xi)}{\xi - w}\, d\xi =: F_1(w) + F_2(w).$$

Since $|\xi - w|$ exceeds an absolute constant for $\xi \in \partial\mathcal{G}$, $|F_1(w)| < C$, as $w \in \mathcal{G}^0 \setminus \mathcal{E}$. But F is bounded on this domain as well. Hence, $F_2(w)$ is bounded on $\mathcal{G}^0 \setminus \mathcal{E}$ and holomorphic on $\mathbb{C} \setminus \mathcal{E}$. By the maximum principle, $F_2(w)$ is holomorphic and bounded on $\mathbb{C} \setminus \mathcal{E}$ by an absolute constant. Let L be a contour in $\mathcal{G} \setminus \mathcal{E}$ inclosing $\mathcal{E}$. Since $F_1(w)$ is holomorphic on $\mathcal{G}$, we have $\int_L F_1(w)\,dw = 0$. Hence,

$$\int_L F_2(w)\,dw = \int_L F(w)\,dw = \int_L f(g(w))\,dg(w) = \int_{g(L)} f(z)\,dz,$$

and $\gamma(E) \le C\gamma(\mathcal{E})$. Similar arguments yield the inverse inequality, if we start with the compact $\mathcal{E}$ (we may assume that $\mathcal{E} \subset \mathcal{G}^0$). Therefore, (2.7) is proved. Corollary 2.2 is a direct consequence of (2.7) and Theorem 2.1. □

3 Proof of Theorem 1.3

3.1 Necessity of the condition (1.4)**.** Suppose that $\|\mathcal{C}_\mu\|_\mu \le C < \infty$ and $\operatorname{supp}\mu \subset E$. One can easily see that $\|\mathcal{C}_{\mu|B}\|_{\mu|B} \le C < \infty$ for every disc B. Moreover, boundedness of $\mathcal{C}_\mu$ implies that $\alpha\mu \in \Sigma$ with α depending only on C; see, for example, [Da]. Thus, the measure $c\mu|B$, $c = c(C) > 0$, participates in the right-hand side of (1.3) with $F = B \cap E$, and we get (1.4).

3.2 Sufficiency of the condition (1.4)**.** The following result, although not formulated explicitly, was proved in [NToV]; see the last three pages of [NToV, Section 3].

Theorem 3.1. *Suppose that $\{D_j\}$ are discs in the plane and the dilated discs λD_j, $\lambda > 1$, are disjoint. Let ν, σ be positive measures supported in $\bigcup_j D_j$ such that $c_1\nu(D_j) \le \sigma(D_j) \le c_2\nu(D_j)$, $0 < c_1 < c_2 < \infty$. Suppose also that the Cauchy operators $\mathcal{C}_{\sigma_j}$, $\sigma_j = \sigma|D_j$, are uniformly bounded. If ν is a Cauchy operator measure, then $\alpha\sigma$ is also a Cauchy operator measure with a constant α depending only on c_1, c_2, and λ.*

We need some preliminary constructions and notations. Here is an easy lemma, whose proof is omitted.

Lemma 3.2. *For every circle T and disc B,*

$$\gamma(T \cap B) \asymp \mathcal{H}^1(T \cap B)$$

with absolute constants of comparison.

Now we define new sets L_j. We need a number $N = N(\lambda)$, which is defined as follows. Recall that $\lambda > 1$ and $\lambda' = (1 + \lambda)/2$. Let a disc D of radius r be given, and let $A_\lambda := \min(1, \lambda' - 1)/1000$. Fix an integer n, and place n circles of radius

$A_\lambda r$ in $\lambda' D$ tangent to $\partial(\lambda' D)$ and whose centers are vertices of a regular n-gon. Denote by $L^{(n)}$ the union of these n circles and the circle of the same radius $A_\lambda r$ which is concentric to D. Define N to be the minimal integer n such that

(a) every (open) disc that intersects both D and $\mathbb{C} \setminus \lambda D$ contains at least one circle from $L^{(n)}$ (we can easily see that such a finite $N = N(\lambda)$ exists), and

(b) all $N + 1$ circles constituting $L^{(N)}$ are pairwise disjoint.

Set $L = L^{(N)}$. Clearly, the properties (a) with $n = N$ and (b) remain valid (with the same N) if we reduce the radii of circles in L. Since $\gamma(\text{circle}) \asymp \mathcal{H}^1(\text{circle})$, we have the following obvious lemma.

Lemma 3.3. *For the set L defined above with any radius of circles constituting L, $\gamma(L) \asymp \mathcal{H}^1(L)$, where the comparison constants can depend only on N.*

For each D_j, let L_j be the union of $N + 1$ circles lying in $\lambda' D_j$, whose positions are described above and whose radii are defined by the equality

$$\mathcal{H}^1(L_j) = A_\lambda (N + 1)\gamma(E_j). \tag{3.1}$$

Then, in particular, $\mathcal{H}^1(\text{one circle}) = \frac{1}{N+1}\mathcal{H}^1(L_j) = A_\lambda \gamma(E_j) \leq A_\lambda r_j$, since $\gamma(E_j) \leq \gamma(D_j) = r_j$. Hence, the properties (a) with $n = N$, $D = D_j$, and (b) hold for every L_j. Moreover, (3.1) and Lemma 3.3 imply the relation $\gamma(L_j) \asymp \gamma(E_j)$.

We need the following lemma.

Lemma 3.4. *Fix an index j. Let B be a disc such that at least one circle from L_j lies inside B. Then $\gamma(L_j) \asymp \gamma(L_j \cap B)$ with constants depending only on λ. In particular, this is true if $D_j \subset B$, or if B intersects D_j and $\mathbb{C} \setminus \lambda D_j$.*

Proof. Indeed, by the semiadditivity of γ,

$$\gamma(L_j) \leq A(N + 1)\gamma(\text{central circle}) \leq A(N + 1)\gamma(L_j \cap B)$$

□

.

Lemma 3.5. *For every disc B,*

$$\gamma\Big(\bigcup_{j:D_j \subset B} L_j\Big) \asymp \gamma\Big(\bigcup_{j:D_j \subset B} L_j \cap B\Big).$$

with constants depending only on λ.

Proof. By the semiadditivity of γ,

$$\gamma\Big(\bigcup_{D_j \subset B} L_j\Big) \leq A\Big(\gamma\Big(\bigcup_{\lambda' D_j \subset B} L_j\Big) + \gamma\Big(\bigcup_{D_j \subset B,\ \lambda' D_j \not\subset B} L_j\Big)\Big).$$

The first term is the same as $\gamma(\bigcup_{\lambda' D_j \subset B} L_j \cap B)$. For the second term, we notice that $L_j \cap B \subset \lambda' D_j$. Since the discs $\frac{\lambda}{\lambda'}\lambda' D_j$ are pairwise disjoint, we may apply Corollary 2.2 with the new dilation constant $\lambda_{new} := \lambda/\lambda'$. Thus,

$$\begin{aligned}\gamma\Big(\bigcup_{D_j \subset B,\ \lambda' D_j \not\subset B} L_j \cap B\Big) &\geq c \sum_{D_j \subset B,\ \lambda' D_j \not\subset B} \gamma(L_j \cap B) \\ &\geq c_1 \sum_{D_j \subset B,\ \lambda' D_j \not\subset B} \gamma(L_j) \geq c_2 \gamma\Big(\bigcup_{D_j \subset B,\ \lambda' D_j \not\subset B} L_j\Big),\end{aligned}$$

which finishes the proof. In the second inequality, we have used Lemma 3.4. □

Lemma 3.6. *Suppose that a disc B intersects more than one D_j. Then*

$$\gamma\Big(\bigcup_{\mathcal{J}} L_j\Big) \asymp \gamma\Big(\bigcup_{\mathcal{J}} L_j \cap B\Big), \quad \mathcal{J} = \mathcal{J}(B) := \{ j : D_j \cap \partial B \neq \varnothing\},$$

with comparison constants depending only on λ.

Proof. Here again we use Corollary 2.2. Since B intersects more than one D_j, it cannot be contained in λD_j, $j \in \mathcal{J}$. Thus, it contains at least one circle from L_j for each $j \in \mathcal{J}$ (this follows from the choice of N). Call this circle C_j. Then apply Corollary 2.2 to D_j with dilation constant λ to get the estimate

$$\gamma\Big(\bigcup_{\mathcal{J}} L_j \cap B\Big) \geq c \sum_{\mathcal{J}} \gamma(L_j \cap B) \geq \sum_{\mathcal{J}} \gamma(C_j) \geq c_1 \sum_{\mathcal{J}} \gamma(L_j) \geq c_2 \gamma\Big(\bigcup_{\mathcal{J}} L_j\Big),$$

which finishes the proof. □

Finally, we need the following notation. For a disc B, let

$$F_j(B) = \begin{cases} E_j, & D_j \subset B \\ \varnothing, & D_j \not\subset B. \end{cases}, \quad F(B) = \bigcup F_j(B).$$

Remark. In what follows, a disc B is free to change. The constants in further inequalities do not depend on B. In the sequel, we write F_j for $F_j(B)$ and F for $F(B)$.

Our next goal is to prove that under assumptions of Theorem 1.3,

$$\gamma\Big(\bigcup L_j \cap B\Big) \geq c \sum \gamma(L_j \cap B)$$

with a universal constant c (universal means that c does not depend on the disc B). We need the following two lemmas.

We fix a small positive absolute constant ε. How small will be clear from what follows.

Lemma 3.7 (The first case)**.** *Suppose that $\gamma(F) \le \varepsilon\gamma(E \cap B)$. Then there exists a constant c, that can depend only on N, ε, and other universal constants, such that*

$$\gamma\Big(\bigcup L_j \cap B\Big) \ge c \sum \gamma(L_j \cap B).$$

Proof. Suppose that B intersects only one λD_j. Then the $\bigcup$ and the $\sum$ have only one term, and there is nothing to prove. So, we can assume that B intersects at least two of λD_j's. Notice also that by this assumption, by the fact that λD_i are pairwise disjoint and by the choice of N, if B intersects D_j, then at least one circle from L_j lies inside B. Let $\mathcal{J}$ be as in Lemma 3.6. Using Lemma 3.4 and Corollary 2.2, we get

$$\sum_{\mathcal{J}} \gamma(L_j) \le A_1 \sum_{\mathcal{J}} \gamma(L_j \cap B) \le A_2 \gamma\Big(\bigcup_{\mathcal{J}} L_j \cap B\Big). \tag{3.2}$$

On the other hand, by the assumption of Theorem 1.3,

$$\sum_{D_j \subset B} \gamma(L_j) \le C \sum_{D_j \subset B} \gamma(E_j) \le C' \sum_{D_j \subset B} \mu_j(D_j) \le C'\mu(B) \le C'C_0\gamma(E \cap B). \tag{3.3}$$

Also, for some absolute constant A,

$$\gamma(E \cap B) \le A\bigg(\gamma(F) + \gamma\Big(\bigcup_{\mathcal{J}} E_j \cap B\Big)\bigg) \le \varepsilon A\gamma(E \cap B) + A\gamma\Big(\bigcup_{\mathcal{J}} E_j \cap B\Big).$$

Thus, if ε is small enough (notice that the smallness depends only on A), we have

$$\gamma(E \cap B) \le C\gamma\Big(\bigcup_{\mathcal{J}} E_j \cap B\Big). \tag{3.4}$$

Therefore, combining (3.3), (3.4), and (3.2), we obtain

$$\begin{aligned} \sum_{D_j \subset B} \gamma(L_j) &\le C\gamma\Big(\bigcup_{\mathcal{J}} E_j \cap B\Big) \le C_1 \sum_{\mathcal{J}} \gamma(E_j \cap B) \\ &\le C_2 \sum_{\mathcal{J}} \gamma(L_j) \le C_3 \gamma\Big(\bigcup_{\mathcal{J}} L_j \cap B\Big). \end{aligned} \tag{3.5}$$

We now combine (3.2) and (3.5) to get

$$\gamma\Big(\bigcup L_j \cap B\Big) \ge \gamma\Big(\bigcup_{\mathcal{J}} L_j \cap B\Big) \ge c \sum_{D_j \subset B} \gamma(L_j) + c \sum_{\mathcal{J}} \gamma(L_j) = c \sum_{D_j \cap B \ne \varnothing} \gamma(L_j). \tag{3.6}$$

Obviously,

$$\gamma\Big(\bigcup L_j \cap B\Big) \ge \gamma\Big(\bigcup_{\mathcal{J}_1} L_j \cap B\Big), \quad \mathcal{J}_1 := \{ j : D_j \cap B = \varnothing,\ L_j \cap B \ne \varnothing\}. \tag{3.7}$$

For $j \in \mathcal{J}_1$, we again consider the new dilation constants $\lambda_{new} := \lambda/\lambda'$ and discs $D'_j := \lambda' D_j$. The discs $\lambda_{new} D'_j$, $j \in \mathcal{J}_1$, are disjoint, and D'_j intersects B for $j \in \mathcal{J}_1$. By Corollary 2.2 applied to $L_j \cap B$ playing the role of E_j, $j \in \mathcal{J}_1$, we get

$$\gamma\Big(\bigcup_{\mathcal{J}_1} L_j \cap B\Big) \geq c \sum_{\mathcal{J}_1} \gamma(L_j \cap B). \tag{3.8}$$

The combination of (3.6)–(3.8) finishes the proof. □

Lemma 3.8 (The second case)**.** *Suppose that $\gamma(F) \geq \varepsilon\gamma(E \cap B)$ with ε from the previous lemma. Then there exists a universal constant c such that*

$$\gamma\Big(\bigcup L_j \cap B\Big) \geq c \sum \gamma(L_j \cap B).$$

Proof. By Theorem 2.1, or rather Corollary 2.2, we need only prove the inequality

$$\gamma\Big(\bigcup_{D_j \subset B} L_j \cap B\Big) \geq c \sum_{D_j \subset B} \gamma(L_j \cap B). \tag{3.9}$$

Using the assumption of our lemma as well as the conditions (1.4) and $\|\mu_j\| \asymp \gamma(E_j)$ of Theorem 1.3, we get

$$\gamma(F) \geq \varepsilon\gamma(E \cap B) \geq \varepsilon c \mu(B) \geq \varepsilon c \sum_{D_j \subset B} \mu_j(B) \geq \varepsilon c' \sum_j \gamma(F_j). \tag{3.10}$$

We denote by ν the measure on F participating in (1.3) for which $\|\nu\| \asymp \gamma(F)$. Let $d\nu_j = \chi_{F_j} d\nu$. Then $\mathcal{C}_{\nu_j}$ is bounded on $L^2(\nu_j)$ (with norm at most 1), and (1.3) yields the estimate

$$\|\nu_j\| \leq C\gamma(F_j) \leq C_1\gamma(L_j) \leq C_2\mathcal{H}^1(L_j) =: C_2\ell_j.$$

We call j **good** if $D_j \subset B$ and $\|\nu_j\| \geq \tau\ell_j$. The choice of τ will be clear from the next steps. However, we emphasize now that this choice is universal. By (3.10), we have

$$\begin{aligned}\varepsilon c' A^{-1} \sum \gamma(F_j) \leq A^{-1}\gamma(F) \leq \|\nu\| = \sum \|\nu_j\| &\leq C_2 \sum_{j \text{ is good}} \ell_j + \tau \sum_{D_j \subset B} \ell_j \\ &\leq C_2 \sum_{j \text{ is good}} \ell_j + C_3\tau \sum \gamma(F_j)\end{aligned}$$

(in the last inequality, we have used (**??**)). Therefore,

$$\sum_{j \text{ is good}, D_j \subset B} \ell_j \geq c \sum \gamma(F_j) \geq c_1 \sum_{D_j \subset B} \ell_j. \tag{3.11}$$

Actually, here is where τ is chosen. We see that it indeed depends only on universal constants such as A and ε. Recall that C_j denotes the central circle of each L_j. We set

$$d\sigma_g := \sum_{j \text{ is good}, D_j \subset B} \chi_{C_j} d\mathcal{H}^1, \quad d\nu_g := \sum_{j \text{ is good}, D_j \subset B} d\nu_j.$$

Then for good j,

$$\sigma_g(D_j) = \mathcal{H}^1(C_j) \asymp \mathcal{H}^1(L_j) = \ell_j \asymp \nu_g(D_j).$$

In the last relation, the comparison constants can depend on previous universal constants and τ. The operators $\mathcal{C}_{\sigma_g|D_j}$ are uniformly bounded. By the choice of ν, the operator $\mathcal{C}_{\nu_g}$ is bounded as well with norm at most 1. Thus, we may apply Theorem 3.1 and conclude that $\mathcal{C}_{\sigma_g}$ is also bounded with a certain absolute bound of the norm. Therefore, using (3.11), we obtain

$$\gamma\Big(\bigcup_{D_j \subset B} L_j\Big) \geq \gamma\Big(\bigcup_{j \text{ is good}, D_j \subset B} L_j\Big) \geq c\|\sigma_g\| \geq c_1 \sum_{j \text{ is good}, D_j \subset B} \ell_j \geq c_2 \sum_{D_j \subset B} \ell_j.$$

In Lemma 3.5, we have proved that $\gamma\big(\bigcup_{D_j \subset B} L_j\big) \asymp \gamma\big(\bigcup_{D_j \subset B} L_j \cap B\big)$. Moreover, for every j such that $D_j \subset B$, we have $\ell_j = \mathcal{H}^1(L_j) \asymp \mathcal{H}^1(L_j \cap B) \asymp \gamma(L_j \cap B)$. Thus, we obtain (3.9), and Lemma 3.8 is proved. □

The Main Theorem of [NV] says the following.

Theorem 3.9. *Let $L \subset \mathbb{R}^2$ be a compact set of positive and finite Hausdorff measure $\mathcal{H}^1$, and let $\sigma = \mathcal{H}^1|L$. Then $\mathcal{C}_\sigma$ is bounded if and only if there exists a finite constant C_0 such that $\sigma(B \cap L) \leq C_0\gamma(B \cap L)$ for every disc B.*

Starting with the main assumption of Theorem 1.4 (the inequality $\mu(B) \leq C_0\gamma(B \cap E)$ for every disc B) we have proved in Lemmas 3.7 and 3.8 that the uniform in B almost-additivity of γ holds for the union of all sets $\{L_j \cap B\}$. Namely, we have proved that for all B,

$$\gamma(B \cap L) \geq c_1 \sum_j \gamma(B \cap L_j) \geq c_2 \sum_j \sigma(B \cap L_j) = c_2\sigma(B \cap L), \tag{3.12}$$

with uniform positive c_2, where $\sigma := \mathcal{H}^1|L$. Hence the measure σ satisfies Theorem 3.9. So the boundedness of Cauchy operator on the union of circles is obtained. The measures $\sigma|L_j$ and μ_j are supported on $\lambda' D_j$, the discs $\frac{\lambda}{\lambda'} \cdot (\lambda' D_j)$ are disjoint, and $\sigma(L_j) \asymp \mu_j$ (see (3.1)). We may apply Theorem 3.1 again to establish the boundedness of $\mathcal{C}_\mu$ in $L^2(\mu)$.

4 Proof of Theorem 1.4 and Corollary 1.5

Proof of Theorem 1.4. We would like to reduce the assumptions of Theorem 1.4 to the conditions of Theorem 1.3. For every E_j, we choose an "extremal" measure μ'_j, supported on E_j and such that $\|\mathcal{C}_{\mu'_j}\|_{\mu'_j} \leq 1$ and

$$c'_1\mu'_j(E_j) \leq \gamma(E_j) \leq c'_2\mu'_j(E_j), \quad 0 < c'_1 < c'_2 < \infty$$

(the existence of such measures μ'_j follows from (1.3)).

Now let us consider new measures $\widetilde{\mu}_j := \mu_j + \mu'_j$, $\widetilde{\mu} = \sum_j \widetilde{\mu}_j$. We prove that these measures satisfy all assumptions of Theorem 1.3 (up to an absolute constant). Indeed, we have seen in Section 3.1 that the assumption $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \leq 1$ implies the inequality $\mu_j(E_j) \leq C\gamma(E_j)$. Hence, the new measures $\widetilde{\mu}_j$ also satisfy the "extremality" condition (μ_j does not satisfy it in general)

$$c_1\widetilde{\mu}_j(E_j) \leq \gamma(E_j) \leq c_2\widetilde{\mu}_j(E_j), \quad 0 < c_1 < c_2 < \infty.$$

Moreover, since two Cauchy operator measures are always Cauchy independent (see Section 1.2), $\|\mathcal{C}_{\widetilde{\mu}_j}\|_{\widetilde{\mu}_j} \leq C$. Now the relation (1.3) implies that

$$\widetilde{\mu}(B \cap E_j) = \widetilde{\mu}_j(B \cap E_j) \leq C\gamma(B \cap E_j)$$

for every disc B. Therefore, by (1.5),

$$\widetilde{\mu}(B) = \sum \widetilde{\mu}(B \cap E_j) \leq C \sum \gamma(B \cap E_j) \leq C_0\gamma(B \cap E), \quad C_0 = C_0(C_1),$$

where the latter inequality is the condition (1.4) of Theorem 1.3 for $\widetilde{\mu}$.

We apply Theorem 1.3 to $c\widetilde{\mu}_j$ and $c\widetilde{\mu} = c\sum_j \widetilde{\mu}_j$ with a sufficiently small absolute constant $c > 0$ and obtain $\|\mathcal{C}_{\widetilde{\mu}}\|_{\widetilde{\mu}} \leq C(C_1, \lambda)$. But $\mu = \sum_j \mu_j$ is a part of $\widetilde{\mu}$, and hence $\|\mathcal{C}_\mu\|_\mu \leq C(C_1, \lambda)$. □

Proof of Corollary 1.5. Recall that above we built L_j for each D_j, and each L_j contains a "central circle". Let $\mathcal{T}'$ be the union over j of central circles in L_j. By (3.1), the radii of these circles are comparable with r_j. Since $\mathcal{T}$ is AD-regular and the discs D_j are λ-separated, $\mathcal{T}'$ is AD-regular as well (with other constants c, C). We have proved in the previous section that the Cauchy operator $\mathcal{C}_\sigma$, $\sigma := \mathcal{H}^1|L$, is bounded from $L^2(\sigma)$ to itself. Hence, the operator $\mathcal{C}_{\sigma'}$, $\sigma' := \mathcal{H}^1|\mathcal{T}'$, is bounded too. By a theorem of Mattila, Melnikov and Verdera [MMV], the set $\mathcal{T}'$ is contained in an AD-regular curve. □

5 Examples

We have seen in Section 3 that the condition

$$\mu(B) \le C_0 \gamma(B \cap E) \text{ for every disc } B \tag{5.1}$$

is necessary for the boundedness of the Cauchy operator $\mathcal{C}_\mu$ with any Borel measure μ. It is not difficult to see that this condition alone is not sufficient for the boundedness of $\mathcal{C}_\mu$. Indeed, let $\mu_n^{1/4}$ be the probability measure uniformly distributed on the set $E_n^{1/4}$ defined in the Introduction. Let $\mu^{1/4}$ be the weak limit of some weakly convergent subsequence $\{\mu_{n_k}^{1/4}\}$, $E^{1/4} = \bigcap E_n^{1/4}$, E be the initial unit square, and $\mu := \mu^{1/4} + \mathcal{H}^2|E$. Then μ satisfies (5.1), but $\mathcal{C}_\mu$ is unbounded; see, for example, [MT, MTV]. We demonstrate more: in general, the condition (5.1) is not sufficient for the boundedness even if μ consists of countably many pieces and each piece gives a bounded Cauchy operator.

Example 5.1. There exists a family of measures $\{\mu_j\}_{j=0}^\infty$, supported on squares E_j, with the following properties:

(a) $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \le 1$;
(b) $\|\mu_j\| \asymp \gamma(E_j)$;
(c) (c) $2E_j \cap 2E_k = \varnothing$, $j \ne k$, $j, k \ge 1$;
(d) the measure $\mu = \sum_{j=0}^\infty \mu_j$ satisfies (5.1);
(e) $\|\mathcal{C}_\mu\|_\mu = \infty$.

We use an idea of David-Semmes; see [VE, Example 8.7] for a more detailed exposition. Let $N_0 = 0$, and let $\{N_k\}_{k=0}^\infty$ be a sequence of natural numbers such that $N_{k+1} - N_k \to \infty$ as $k \to \infty$. Start the construction with the unit square E_0 and make $N_1 - N_0$ steps of the construction of the corner 1/4-Cantor set $E^{1/4}$. We get $4^{N_1 - N_0}$ squares with side length 4^{-N_1}. Choose one (any) of them, denote it by Q_1, and continue the construction with only this square. The other $4^{N_1-N_0} - 1$ squares are the sets E_j that have already been defined. For the chosen square Q_1, we make next $N_2 - N_1$ steps of the construction of $E^{1/4}$, obtaining $4^{N_2-N_1}$ squares with side length 4^{-N_2}. Again, continue the construction for only one of them, say, for a square Q_2, and so on.

Let μ_j, $j = 0, 1, \ldots$, be the 2-dimensional measure uniformly distributed on E_j such that $\|\mu_j\| = c\ell_j$, where ℓ_j is the side length of E_j, and the absolute constant c is chosen in such a way that $\|\mathcal{C}_{\mu_j}\|_{\mu_j} = 1$. Then properties (a), (b), (c) are obvious. To demonstrate (d), we notice that $E := \bigcup_{j \ge 0} E_j$ is equal to E_0, and thus $\gamma(B \cap E) \asymp \operatorname{diam}(B \cap E) =: d_0$. On the other hand, for any $j \ge 0$ and $d_j := \operatorname{diam}(B \cap E_j)$, we have $\mu(B \cap E_j) \le c\ell_j^{-1} d_j^2 < C d_j$ (the density of μ_j is equal to c/ℓ_j). Hence, $\mu(B \cap E) < C \sum_{j=0}^\infty d_j \asymp d_0$, and (d) is established.

Finally, to prove (e), we apply the operator $\mathcal{C}_\mu$ to the characteristic functions χ_{Q_k}, $k = 0, 1, \dots$, obtaining

$$\|\mathcal{C}_\mu(\chi_{Q_k})\|_{L^2(\mu)} = \|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|_{L^2(\mu)} \geq \|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|_{L^2(\mu|Q_k)}.$$

But

$$\|\mathcal{C}_{\mu|Q_k}(\mathbf{1})\|^2_{L^2(\mu|Q_k)} \geq c(N_{k+1} - N_k)4^{-N_k}$$

with an absolute constant c; see [MT]. Hence, $\|\mathcal{C}_\mu\|_\mu \geq c(N_{k+1} - N_k) \to \infty$, and (e) is proved.

Remark. The measures $\{\mu_j\}_{j=1}^\infty$ satisfy all the assumptions of Theorem 1.3 except for (5.1). Therefore, we have to add μ_0 and change the structure of μ.

Now we demonstrate that the condition $\|\mu_j\| \asymp \gamma(E_j)$ in Theorem 1.3 is essential.

Example 5.2. There exists a family of measures $\{\mu_j\}_{j=1}^\infty$ with the following properties:

(a) $\|\mathcal{C}_{\mu_j}\|_{\mu_j} \leq 1$;
(b) $\|\mu_j\| \leq C\gamma(E_j)$, where $E_j = \operatorname{supp} \mu_j$, and E_j is either a square or a disc;
(c) $2E_j \cap 2E_k = \varnothing$, $j \neq k$;
(d) the measure $\mu = \sum_{j=1}^\infty \mu_j$ satisfies (5.1);
(e) $\|\mathcal{C}_\mu\|_\mu = \infty$.

We use the same construction as in Example 5.1, but with the following modifications. 1. The initial square E_0 now is not included in the collection $\{E_j\}$ of squares; thus, the squares are separated. 2. Besides the same squares E_j and measures μ_j, as in Example 5.1, we add additional discs $\widetilde{E}_j$ and measures $\widetilde{\mu}_j$ to satisfy (5.1) (otherwise (5.1) does not hold after exclusion E_0).

As before, we set $N_0 = 0$, choose a sequence $\{N_k\}_{k=0}^\infty$ of natural numbers such that $N_{k+1} - N_k \to \infty$ as $k \to \infty$, and make $N_1 - N_0$ steps of the construction of the corner 1/4-Cantor set $E^{1/4}$, starting with the unit square E_0. Let $E_{n,k}$, $k = 0, \dots, 4^n$, be the squares forming the nth generation in this construction (not all of them are included in $\{E_j\}$). In each square $E_{n,k}$, $n = 0, \dots, N_1 - 1$, $k = 1, \dots, 4^n$, place the disc $\widetilde{D}_{n,k}$ concentric with $E_{n,k}$ and with radius $\ell_n/10 = 4^{-n}/10$. On $\widetilde{D}_{n,k}$, uniformly distribute a measure $\mu_{n,k}$ with $\|\mu_{n,k}\| = 2^{-n} \cdot 4^{-n}$. Then for n larger than a certain absolute n_0, we automatically have $\|\mathcal{C}_{\mu_{n,k}}\|_{\mu_{n,k}} \leq 1$. For $n \leq n_0$, we might need a small absolute positive c' such that $\|\mathcal{C}_{c'\mu_{n,k}}\|_{c'\mu_{n,k}} \leq 1$. We put then $\widetilde{\mu}_{n,k} := c'\mu_{n,k}$ and achieve that $\|\mathcal{C}_{\widetilde{\mu}_{n,k}}\|_{\widetilde{\mu}_{n,k}} \leq 1$. After $N_1 - N_0$ steps, choose one (any) of the squares $E_{N_1,k}$, and denote it by Q_1. The other $4^{N_1 - N_0} - 1$ squares $E_{N_1,k}$ and discs $\widetilde{D}_{n,k}$, $n = 0, \dots, N_1 - 1$, $k = 1, \dots, 4^n$, are the sets E_j that have already been defined. As before, $\mu_{N_1,k}$ is the 2-dimensional measure uniformly distributed on $E_{N_1,k}$ such that $\|\mu_{N_1,k}\| = c\ell_{N_1}$ and $\|\mathcal{C}_{\mu_{N_1,k}}\|_{\mu_{N_1,k}} = 1$.

For the chosen square Q_1, we continue the construction and make the next N_2-N_1 steps of the construction of $E^{1/4}$, obtaining squares $E_{N_2,k}$, $k = 1, \dots, 4^{N_2-N_1}$ with side length $\ell_{N_2} := 4^{-N_2}$ and with measures $\mu_{N_2,k}$ such that $\|\mu_{N_2,k}\| = c\ell_{N_2}$. Besides these squares, we get discs $\widetilde{D}_{n,k}$, $n = N_1, \dots, N_2 - 1$, $k = 1, \dots, 4^{n-N_1}$ of radii $4^{-n}/10$, concentric with $E_{n,k}$ and supporting the measures $\widetilde{\mu}_{n,k}$, $\|\widetilde{\mu}_{n,k}\| = c'2^{-n} \cdot 4^{-n}$. Again, we continue the construction for only one of these squares, and so on.

We have to prove only (d). Fix a disc B. Suppose that $\widetilde{D}_{0,1} \subset B$. Since $\mu(B) \le \|\mu\| \le C$ and $\gamma(B \cap E) \ge \gamma(\widetilde{D}_{0,1}) = 1/10$, (5.1) holds. Suppose now that $\widetilde{D}_{0,1} \not\subset B$, and that B contains at least one disc $\widetilde{D}_{n,k}$. Choose a maximal disc in B, say, $\widetilde{D}_{n_1,k_1}$. Then “the parent” $E_{n_1-1,k_1'}$ of the square E_{n_1,k_1} (that is, the square of the previous generation containing E_{n_1,k_1}) does not lie in B; otherwise, $\widetilde{D}_{n_1,k_1}$ would not be a maximal disc in B. Set $G_1 := E_{n_1-1,k_1'} \cap B$. Now choose a maximal disc $\widetilde{D}_{n_2,k_2} \subset (B \setminus G_1)$ (if such a disc exists). Its “parent” $E_{n_2-1,k_2'}$ and the set G_1 are disjoint. By the same reason as above, $E_{n_2-1,k_2'} \not\subset B$. Set $G_2 := E_{n_2-1,k_2'} \cap B$, and choose a maximal disc $\widetilde{D}_{n_3,k_3} \subset (B \setminus (G_1 \cup G_2))$ (if any). Continuing in this way, we obtain a sequence $\{G_j\}$ of sets with the following properties:

(i) $2G_i \cap 2G_j = \varnothing$, $i \ne j$, and one may place them into λ-separated discs, $\lambda > 1$;

(ii) for each G_j,

$$\mu(G_j) \le C\ell_{n_j} = C4^{-n_j} \asymp \gamma(\widetilde{D}_{n_j,k_j}) \le \gamma(G_j);$$

(iii) all the squares $E_{N_i,k}$ and discs $\widetilde{D}_{n,k}$ contained in B are contained in $\bigcup_j G_j$.

‘Also, it might be that some discs $\widetilde{D}_{n,k}$ and squares $E_{N_i,k}$ intersect B and are not contained in the sets $E_{n_j-1,k_j'}$ considered above. Each of these discs and squares forms a separate set $G_j := \widetilde{D}_{n,k} \cap B$ or $E_{N_i,k} \cap B$. These sets are λ-separated as well, and $\mu(G_j) \le C \operatorname{diam}(G_j) \le C'\gamma(G_j)$; see Example 5.1. Moreover, all the sets G_j satisfy the conditions of Corollary 2.2, which yields the estimate

$$\mu(B) = \sum_j \mu(G_j) \le C \sum_j \gamma(G_j) \le C'\gamma(B \cap E).$$

and the example has been established.

Now we demonstrate the sharpness of Corollary 1.5.

Proposition 5.3. *The conclusion of Corollary 1.5 is incorrect, in general, if any of the assumptions (a)–(c) is missing.*

Proof. *(a): the assumption* (a) *is missing*. We may use the same sets $\{E_j\}_{j=1}^{\infty}$ as in Example 5.1, only without the initial square E_0. For D_j, we take discs containing E_j and slightly larger than E_j. A proof that (b) and (c) hold is not difficult, and we leave it to the reader. At the same time,

$$\text{(5.2)}\quad \text{(length of any curve connecting all discs in } Q_k)/4^{-N_k} \to \infty \text{ as } k \to \infty$$

(the squares Q_k are defined in the proof of Example 5.1).

(b): the assumption (b) *is missing*. Let D_j be the discs $\widetilde{D}_{n,k}$ in Example 5.2, enumerated in the non-increasing order by the radius r_j. In each D_j, we place the disc E_j concentric with D_j and of radius $4^{-j}/20$. To prove that (a) holds, fix a disc B. For discs E_j intersecting ∂B, (1.5) holds by Corollary 2.2 (with $B \cap E_j$ as E_j). Let $j_0 := \{\min j : E_j \subset B\}$. Then

$$\sum_{j:B_j \subset B} \gamma(B \cap E_j) \le \sum_{j=j_0}^{\infty} \gamma(E_j) = \sum_{j=j_0}^{\infty} \frac{1}{20} 4^{-j}$$
$$= \frac{1}{15} 4^{-j_0} = \frac{8}{3} \gamma(E_{j_0}) \le \frac{8}{3} \gamma(B \cap E),$$

and (a) holds. The proof that (c) holds is easy, and we omit it. At the same time, (5.2) holds for discs D_j in Q_k.

(c): assumption (c) *is missing*. The counterexample in this case is not based on Cantor-type constructions. Given $\ell > 0$, $N \in \mathbb{N}$, $N \ge 4$, consider the square $Q_\ell = [0, \ell] \times [0, \ell]$ and points

$$x_i = \frac{\ell}{N-1} i, \quad y_j = \frac{\ell}{N-1} j, \quad i, j = 0, 1, \dots, N-1.$$

Let E_{ij} be the disc centered at the point (x_i, y_j) of radius ℓ/N^2, and let $\mathcal{E} := \bigcup_{i,j} E_{ij}$, $D_{ij} := 2E_{ij}$. Fix a disc B. Set $\mathcal{E}_B = \{\bigcup E_{ij} : E_{ij} \subset B\}$ ($\mathcal{E}_B$ might be empty). If $\mathcal{E}_B \ne \varnothing$, we have

$$\left| \int \frac{d(\mathcal{H}^1 | \partial \mathcal{E}_B)(\xi)}{\xi - z} \right| < C, \quad z \in \mathbb{C}.$$

Hence,

$$\gamma(\mathcal{E} \cap B) \ge \gamma(\mathcal{E}_B) \ge c\mathcal{H}^1(\partial \mathcal{E}_B) = c' \sum_{E_{ij} \subset B} \gamma(E_{ij}) \ge c'' \sum \gamma(E_{ij} \cap B).$$

If $\mathcal{E}_B = \varnothing$, (1.5) holds as well (for example, by Corollary 2.2). At the same time, the length of any curve intersecting all discs D_{ij} is at least $C\ell N$.

Now we construct a series of squares Q_{ℓ_k} with $\ell_k = 2^{-k}/10$, $N_k = k2^k$, centered at points $1/k^2$, and the corresponding sets $\mathcal{E}^{(k)}$ and discs $D_{ij}^{(k)}$, $E_{ij}^{(k)}$. One may place $\mathcal{E}^{(k)}$ in λ-separated discs centered at the point $1/k^2$. Set $E := \bigcup_k \mathcal{E}^{(k)}$. By Theorem 2.1,

$$\gamma(B \cap E) \ge c \sum_k \gamma(B \cap \mathcal{E}^{(k)}).$$

Thus, to prove (1.5), we have to establish almost additivity of γ for each $\mathcal{E}^{(k)}$ separately, and that was done above. That (b) holds is obvious. But all the discs $D_{ij}^{(k)}$ in E cannot be connected by an AD-regular curve. □

6 Proof of Theorem 1.1

It is known that a compact chord-arc curve is a bi-lipschitz image of a straight segment; see [Po, Chapter 7]. On the other hand analytic capacity can be only finitely distorted by bi-lipschitz maps. This is a non-trivial result by X. Tolsa, [To3]. So if we allow the separation constant $\lambda > 1$ to depend on the Lipschitz constant of our chord-arc curve (so the separation of the discs is large if the constant of the curve is large), we can obtain Theorem 1.1 directly from Theorem 2.1. However, we want to avoid the dependence of the separation constant on the chord-arc constant. Thus, we need another proof, which follows.

The **Melnikov–Menger curvature** of a positive Borel measure μ in $\mathbb{C}$ is defined as

$$c^2(\mu) = \iiint \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z),$$

where $R(x,y,z)$ is the radius of the circle passing through the points $x, y, z \in \mathbb{C}$ and $R(x,y,z) = \infty$ if x, y, z are collinear (in particular, if two of them coincide). This notion was introduced by Melnikov [M]. The following relation characterizes the analytic capacity in terms of the curvature of a measure [To1], [To2, p. 104], [Vo], [To]. For a compact set $F \subset \mathbb{C}$,

$$\gamma(F) \asymp \sup\{\mu(F) : \ \operatorname{supp}\mu \subset F,\ \mu \in \Sigma,\ c^2(\mu) \le \mu(F)\}, \tag{6.1}$$

where Σ is the class of measures of linear growth defined in (1.3).

Lemma 6.1 (Main Lemma). *Let $D_j = D(x_j, r_j)$ be discs with centers on a chord-arc curve Γ such that $\lambda D_j \cap \lambda D_k = \varnothing$, $j \ne k$, for some $\lambda > 1$. Let μ_j be positive measures such that*

(1) $\operatorname{supp}\mu_j \subset D_j$,

(2) $\mu_j(D_j) =: \|\mu_j\| \le r_j$.

Then, for $\mu = \sum \mu_j$,

$$c^2(\mu) \le \sum_j c^2(\mu_j) + C\|\mu\|, \quad C = C(\lambda, A_0), \tag{6.2}$$

where A_0 is the constant of Γ.

To begin, let us show that Theorem 1.1 is a direct consequence of Main Lemma and (6.1).

Proof of Theorem 1.1. Consider measures μ_j participating in (6.1) for $F = E_j$, $j = 1, \dots$. Then $\mu(D(x,r)) \le Cr$ for any disc D, where $C = C(A_0)$ and $\mu = \sum \mu_j$. To prove this assertion, we fix a disc $D = D(x,r)$ and divide all discs D_j onto two groups:

$$\mathcal{D}_1 := \{D_j : D_j \cap D \neq \varnothing,\ r_j \le r\} \quad \text{and} \quad \mathcal{D}_2 := \{D_j : D_j \cap D \neq \varnothing,\ r_j > r\}.$$

Since Γ is chord-arc, $\sum_{D_j \in \mathcal{D}_1} r_i \le Cr$, $C = C(A_0)$. It is easy to see that $\mathcal{D}_2$ consists of at most six discs D_j. Hence,

$$\mu(D) \le \sum_{D_j \in \mathcal{D}_1} \mu(D_j) + \sum_{D_j \in \mathcal{D}_2} \mu(D_j \cap D) \le \sum_{D_j \in \mathcal{D}_1} r_j + 6r < Cr.$$

Furthermore, Main Lemma implies the inequality $c^2(\mu) \le C\|\mu\|$, $C = C(\lambda, A_0)$. Thus, the measure $c\mu$ with an appropriate constant c depending on λ, A_0, participates in (6.1) for $F = E = \bigcup E_j$. So, $\gamma(E) \ge c\|\mu\|$, which implies Theorem 1.1. $\square$

Proof of Lemma 6.1. It suffices to consider a finite set of discs D_j, $j = 1, \dots, N$. We assume that these discs are enumerated in increasing order by the natural parameters of their centers.

Let Γ_j be arcs of Γ such that $\Gamma_j \subset D_j$ and $\mathcal{H}^1(\Gamma_j) = \mu(D_j)$. Let $\sigma_j := \mathcal{H}^1|\Gamma_j$ and $\sigma := \sum \sigma_j$, so that $\sigma(D_j) = \mu(D_j)$. Obviously,

$$c^2(\mu) = \left(\sum_j \iiint_{D_j^3} + \iiint_{\mathbb{C}^3 \setminus \bigcup_j D_j^3} \right) \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) =: I_1 + I_2.$$

Since $I_1 = \sum_j c^2(\mu_j)$, we have only to estimate I_2. Our proof is based on the comparison of I_2 and the corresponding integral with respect to σ:

$$\bar{I_2} := \iiint_{\mathbb{C}^3 \setminus \bigcup_j D_j^3} \frac{1}{R^2(x,y,z)}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z).$$

Notice that

$$\bar{I_2} < c^2(\sigma) \le C\|\sigma\|, \quad C = C(A_0). \tag{6.3}$$

The last inequality is a consequence of two well-known facts. The first is the boundedness of the Cauchy operator $\mathcal{C}_{\mathcal{H}^1|\Gamma}$ on chord-arc curves; see [MV, p. 330]. In particular,

$$\|\mathcal{C}_\sigma^\varepsilon \mathbf{1}\|_{L^2(\sigma)}^2 \le \|\mathcal{C}_{\mathcal{H}^1|\Gamma}(\chi_{\cup\Gamma_j})\|_{L^2(\mathcal{H}^1|\Gamma)}^2 \le C\|\chi_{\cup\Gamma_j}\|_{L^2(\mathcal{H}^1|\Gamma)}^2 = C\|\sigma\|, \quad \varepsilon > 0,$$

where C depends only on A_0, The second fact is the connection between the curvature of a measure and the norm of a Cauchy potential:

$$\|\mathcal{C}_\mu^\varepsilon \mathbf{1}\|_{L^2(\mu)}^2 = \frac{1}{6} c_\varepsilon^2(\mu) + O(\|\mu\|)$$

for any measure $\mu \in \Sigma$ uniformly in ε; see, for example, [To2]. Here, $c_\varepsilon^2(\mu)$ is the truncated version of $c^2(\mu)$ defined in the same way as $c_\varepsilon^2(\mu)$, but the triple integral is taken over the set $\{(x, y, z) \in \mathbb{C}^3 : |x-y|, |y-z|, |x-z| > \varepsilon\}$. This equality with $\mu = c\sigma \in \Sigma$, and the previous relations imply (6.3).

Obviously,

$$I_2 = \left(\iiint_{\Omega_1} + \iiint_{\Omega_2} \right) \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) =: I_{2,1} + I_{2,2},$$

where

$$\Omega_1 := \{D_j \times D_k \times D_l : j = k \neq l \vee j \neq k = l \vee j = l \neq k\},$$
$$\Omega_2 := \{D_j \times D_k \times D_l : j \neq k,\ k \neq l,\ j \neq l\}.$$

To estimate the integral over Ω_1, it suffices to consider the subset

$$\Omega_1' := \{D_j \times D_k \times D_l : j \neq k = l\}.$$

For $x \in D_j = D(x_j, r_j)$, $y, z \in D_k$, $j \neq k$, we have

$$2R(x,y,z) \geq |x-y| \geq c(r_j + r_{j+1} + \cdots + r_k), \quad c = c(\lambda, A_0)$$

(here, we assume that $j < k$; the case $k < j$ is analogous). Then

$$\iiint_{\Omega_1'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) \leq C \left[\sum_{j=1}^{N-1} \|\mu_j\| \sum_{k=j+1}^{N} \frac{\|\mu_k\|^2}{(r_j + r_{j+1} + \cdots + r_k)^2} \right.$$
$$\left. + \sum_{j=1}^{N-1} \|\mu_{N+1-j}\| \sum_{k=j+1}^{N} \frac{\|\mu_{N+1-k}\|^2}{(r_{N+1-j} + r_{j+1} + \cdots + r_{N+1-k})^2} \right] =: C[S_{N,1} + S_{N,2}].$$

Estimates for both terms on the right-hand side are the same. We estimate $S_{N,1}$ using the induction with respect to N.

Base case: $N = 2$. In this case,

$$S_{N,1} = \|\mu_1\| \cdot \frac{\|\mu_2\|^2}{(r_1 + r_2)^2} \leq \|\mu_1\| \leq \|\mu_1\| + \|\mu_2\|.$$

Induction: suppose that the inequality

$$S_{N,1} = \sum_{j=1}^{N-1} \|\mu_j\| \sum_{k=j+1}^{N} \frac{\|\mu_k\|^2}{(r_j + \cdots + r_k)^2} \leq \|\mu_1\| + \cdots + \|\mu_N\| \tag{6.4}$$

holds for some $N \geq 2$. For $N+1$ discs, we have

$$S_{N+1,1} = S_{N,1} + \sum_{j=1}^{N} \|\mu_j\| \frac{\|\mu_{N+1}\|^2}{(r_j + \cdots + r_{N+1})^2}$$
$$\leq S_{N,1} + \|\mu_{N+1}\|^2 \sum_{j=1}^{N} \frac{r_j}{(r_j + \cdots + r_{N+1})^2}.$$

The last sum is dominated by the integral

$$\int_0^\infty \frac{dt}{(r_{N+1}+t)^2} = \frac{1}{r_{N+1}}.$$

Hence,

$$S_{N+1,1} \le S_{N,1} + \|\mu_{N+1}\|^2/r_{N+1} \le \|\mu_1\| + \cdots + \|\mu_{N+1}\|.$$

Thus, (6.4) holds, and we have estimated the triple integral over $\mathbf{\Omega}_1$.

By symmetry,

$$\iiint_{\Omega_2} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) = 6 \iiint_{\Omega_2'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z),$$

where $\mathbf{\Omega}_2' := \{D_j \times D_k \times D_l : j < k < l\}$. Moreover, we may restrict ourselves to integration over

$$\mathbf{\Omega}_{2,1}' := \{D_j \times D_k \times D_l : j < k < l,\ r_j + \cdots + r_k \ge \tfrac{1}{2}(r_j + \cdots + r_l)\}.$$

Indeed, if

$$\iiint_{\Omega_{2,1}'} \frac{1}{R^2(x,y,z)}\, d\mu(x)\, d\mu(y)\, d\mu(z) \le C\|\mu\| \tag{6.5}$$

with $C = C(\lambda, A_0)$, then using the inverse parametrization of Γ, we get the same estimate for the triple integral over

$$\mathbf{\Omega}_{2,2}' := \{D_j \times D_k \times D_l : j < k < l,\ r_k + \cdots + r_l \ge \tfrac{1}{2}(r_j + \cdots + r_l)\}$$

where the discs are enumerated as before. Since $\iiint_{\Omega_2'} \le \iiint_{\Omega_{2,1}'} + \iiint_{\Omega_{2,2}'}$, (6.4) and (6.5) imply (6.2). It remains to prove (6.5).

Fix indices j, k, l. For any triples $(x,y,z), (x',y',z') \in D_j \times D_k \times D_l$, the sine of the angle between the intervals (y,z) and (y',z') does not exceed

$$C\,\frac{r_k + r_l}{r_k + \cdots + r_l}, \quad C = C(\lambda, A_0).$$

For the angle between the intervals (x,z) and (x',z'), we have $C\,\frac{r_j+r_l}{r_j+\cdots+r_l}$. Denote by α, α' the angles at z, z' of the triangles x,y,z and x',y',z', respectively. Since $\sin(\alpha+\beta+\gamma) \le \sin\alpha + \sin\beta + \sin\gamma$ as $\alpha, \beta, \gamma \in [0,\pi]$, we get the estimate

$$\sin\alpha < \sin\alpha' + C\,\frac{r_k + r_l}{r_k + \cdots + r_l} + C\,\frac{r_j + r_l}{r_j + \cdots + r_l}.$$

Hence,

$$\frac{1}{R(x,y,z)} = \frac{2\sin\alpha}{|x-y|} < \frac{C}{|x'-y'|}\left[2\sin\alpha' + \frac{r_k + r_l}{r_k + \cdots + r_l} + \frac{r_j + r_l}{r_j + \cdots + r_l}\right].$$

Moreover, for triples in $\Omega'_{2,1}$ with given j, l, we may consider only those k for which $r_j + \cdots + r_k \geq \frac{1}{2}(r_j + \cdots + r_l)$ (the set of such k can be empty). Suppose that this inequality holds for $p \leq k \leq l-1$ with $p > j$. Below we use the reduced range for k only in the sum $S^{(2)}$. Therefore,

$$\begin{aligned}\iiint_{\Omega'_{2,1}} \frac{1}{R^2(x,y,z)} d\mu(x)d\mu(y)d\mu(z) \leq C\Bigg[\iiint_{\Omega'_{2,1}} \frac{1}{R^2(x',y',z')} d\sigma(x')d\sigma(y')d\sigma(z') \\ + \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=j+1}^{l-1} \frac{r_k^2 \, \|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2} \\ + \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=p}^{l-1} \frac{r_l^2 \, \|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2} \\ + \sum_{j=1}^{N-2} \|\mu_j\| r_j^2 \sum_{l=j+2}^{N} \sum_{k=j+1}^{l-1} \frac{\|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_j + \cdots + r_l)^2}\Bigg] \\ =: C[I + S^{(1)} + S^{(2)} + S^{(3)}].\end{aligned}$$

We estimate each of terms in the last line separately. By (6.3),

$$I \leq c^2(\sigma) \leq C\|\sigma\|. \tag{6.6}$$

Write $S^{(1)}$ as

$$S^{(1)} = \sum_{j=1}^{N-2} \|\mu_j\| \sum_{k=j+1}^{N-1} \sum_{l=k+1}^{N} \frac{r_k^2 \, \|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2}.$$

Since the inner sum over l does not exceed

$$\frac{r_k^2 \, \|\mu_k\|}{(r_j + \cdots + r_k)^2} \int_{r_k}^{\infty} \frac{dx}{x^2} = \frac{r_k \, \|\mu_k\|}{(r_j + \cdots + r_k)^2},$$

we get the estimate

$$\begin{aligned}S^{(1)} &\leq \sum_{j=1}^{N-2} \|\mu_j\| \sum_{k=j+1}^{N-1} \frac{r_k \, \|\mu_k\|}{(r_j + \cdots + r_k)^2} \leq \sum_{k=2}^{N-1} \sum_{j=1}^{k-1} \frac{r_j \, r_k \, \|\mu_k\|}{(r_j + \cdots + r_k)^2} \\ &\leq \sum_{k=2}^{N-1} \|\mu_k\| \sum_{j=1}^{k-1} r_k \int_{r_k}^{\infty} \frac{dx}{x^2} = \sum_{k=2}^{N-1} \|\mu_k\| < \|\mu\|.\end{aligned} \tag{6.7}$$

Since $r_j + \cdots + r_k \geq \frac{1}{2}(r_j + \cdots + r_l)$ in $S^{(2)}$, we have

$$
\begin{aligned}
S^{(2)} &= \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=p}^{l-1} \frac{r_l^2 \, \|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_k)^2 (r_k + \cdots + r_l)^2} \\
&\le 4 \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \sum_{k=p}^{l-1} \frac{r_l^2 \, \|\mu_k\| \, \|\mu_l\|}{(r_j + \cdots + r_l)^2 (r_k + \cdots + r_l)^2} \\
&\le 4 \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \frac{r_l \, \|\mu_l\|}{(r_j + \cdots + r_l)^2}
\end{aligned}
$$

(we estimate the sum with respect to k in the same way as above). We may deal with the last double sum as in (6.7), or notice that this sum does not exceed

$$
\sum_{j=1}^{N-2} \|\mu_j\| \left[1 + \sum_{l=j+1}^{N-1} \frac{r_l \, \|\mu_l\|}{(r_j + \cdots + r_l)^2} \right],
$$

and use (6.7) directly to conclude that

$$
S^{(2)} < 8 \|\mu\|. \tag{6.8}
$$

Finally,

$$
S^{(3)} \le \sum_{j=1}^{N-2} \|\mu_j\| \sum_{l=j+2}^{N} \frac{r_j^2 \, \|\mu_l\|}{(r_j + \cdots + r_l)^2 r_j} < \sum_{j=1}^{N-2} \|\mu_j\| < \|\mu\|. \tag{6.9}
$$

The estimates (6.6), (6.7), (6.8), and (6.9) yield (6.5). □

7 Question on almost-additivity

We make more accurate the question posed in Section 1. In Theorems 1.1 and 2.1, discs were λ-separated, where $\lambda > 1$. But what if they are just disjoint? Namely, let D_j be circles with centers on a chord-arc curve with constant A_0 (or even on the real line $\mathbb{R}$), such that $D_j \cap D_k = \varnothing$, $j \ne k$. Let $E_j \subset D_j$ be arbitrary compact sets. Is it true that there exists a constant $c = c(\lambda, A_0) > 0$, such that

$$
\gamma\Big(\bigcup E_j\Big) \ge c \sum_j \gamma(E_j)?
$$

We cannot either prove or construct a counter-example to this claim.

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