J. Eur. Math. Soc. 21, 355–380

DOI 10.4171/JEMS/838

Van Cyr · Bryna Kra

Counting generic measures for a subshift of linear growth

Received October 30, 2015

Abstract. In 1984 Boshernitzan proved an upper bound on the number of ergodic measures for a minimal subshift of linear block growth and asked if it could be lowered without further assumptions on the shift. We answer this question, showing that Boshernitzan's bound is sharp. We further prove that the same bound holds for the, a priori, larger set of nonatomic generic measures, and that this bound remains valid even if one drops the assumption of minimality. Applying these results to interval exchange transformations, we give an upper bound on the number of nonatomic generic measures of a minimal IET, answering a question recently posed by Chaika and Masur.

Keywords. Subshift, automorphism, block complexity, interval exchange transformation

1. Introduction

Let (X, σ) be a subshift, meaning that $X \subset \mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is a finite alphabet, and X is a closed set that is invariant under the left shift $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. A classic problem is to find conditions that imply (X, σ) is uniquely ergodic or, more generally, has a finite number of ergodic measures. In the 1980's, Boshernitzan [1] showed that the complexity of the subshift can be used to obtain such a result. More precisely, if $P_X(n)$ is the number of words of length n which occur in any $x \in X$, he showed that if (X, σ) is minimal and $\limsup_{n\to\infty} P_X(n)/n < 3$, then it is uniquely ergodic (see also related results in [3]). More generally, Boshernitzan showed that if

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < k, \tag{1}$$

then there are at most $k-1$ ergodic measures. Some motivation for studying this problem is generalizing the well-known bound on the number of ergodic measures for an interval exchange transformation (IET), that had been previously proven, independently, by Katok and Veech. Boshernitzan's Theorem applies to a much broader class of dynamical systems than the interval exchange transformations, but the bound he obtains is weaker than that of Katok and Veech in the case of an IET.

V. Cyr: Bucknell University, Lewisburg, PA 17837, USA; e-mail: van.cyr@bucknell.edu

B. Kra: Northwestern University, Evanston, IL 60208, USA; e-mail: kra@math.northwestern.edu

Mathematics Subject Classification (2010): Primary 37B10; Secondary 37A25, 68R15

Boshernitzan asked in [1], and then again in [2], whether his bound could be lowered in this more general setting. One of our main results answers Boshernitzan's question: for the class of minimal subshifts whose complexity function satisfies (1), Boshernitzan's bound is a sharp bound for the number of nonatomic ergodic measures. Our technique also shows that the bound is more general than originally stated: the same bound remains valid (and sharp) even without the assumption of minimality and even if one seeks to bound the (a priori, larger) set of nonatomic generic measures.

The particular case of minimal interval exchange transformations has been well studied (for example Katok [10], Keane [12], and Veech [14]). A minimal k-interval exchange transformation (k-IET) has a natural symbolic cover, its natural coding, and this subshift satisfies the hypothesis of Boshernitzan's Theorem. As an application, this shows that a minimal k-IET (see Section 4 for the definition) has at most $k-1$ ergodic measures. The optimal bound of $\lfloor k/2 \rfloor$ was proven, independently, by Katok [10] and Veech [14]. In a recent paper, Chaika and Masur [4] studied the broader class of generic measures for an IET and asked whether there are bounds on the number of such measures. An interesting facet of this problem is that although several quite different proofs of the bound given by Katok and Veech for the number of ergodic measures exist in the literature, they all use ergodicity in an essential way.

If X is a compact metric space, $\mathcal{B}$ the Borel σ-algebra, μ a Borel probability measure on $\mathcal{B}$, and $T\colon X \to X$ a measurable map preserving the measure μ, then a point $x \in X$ is a *generic point for the measure* μ if for every continuous function $f\colon X \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f \, d\mu.$$

The measure μ is *generic* if it has a generic point. Thus, by the Pointwise Ergodic Theorem, if the measure μ is ergodic then almost every point is generic. However, a generic measure need not be ergodic. Chaika and Masur [4] constructed a 6-interval exchange transformation that has a generic, but not ergodic, measure. They asked if there is a bound on the number of generic measures for a k-IET. We show:

Theorem 1.1. *If (X,σ) is a subshift and there exists $k \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < k,$$

then (X,σ) has at most $k-1$ distinct, nonatomic, generic measures.

In particular, this applies to interval exchange transformations by passing to the natural cover. Theorem 1.1 generalizes Boshernitzan's Theorem [1] in two ways: there is no assumption of minimality and our bound holds for the more general class of generic measures. We also give an analogous bound for lim sup (note the technical assumption is vacuous for minimal subshifts that are not uniquely ergodic).

Theorem 1.2. *Suppose (X,σ) is a subshift and there exists $k \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} < k.$$

If (X, σ) *has a generic measure* μ *and a generic point* x_μ *for which the orbit closure*

$$\overline{\{\sigma^k x_\mu : k \in \mathbb{Z}\}}$$

is not uniquely ergodic, then (X, σ) *has at most* $k - 2$ *distinct, nonatomic, generic measures.*

Recently Damron and Fickenscher [5] proved a related result, showing that any minimal shift (X, σ) whose complexity function satisfies $P_X(n) = kn + c$ for some constant c, $k \geq 4$ and all n sufficiently large has at most $k - 2$ ergodic measures.

Moreover, we show that these theorems are sharp, even if X is assumed to be minimal and the measures are required to be ergodic.

Theorem 1.3. *Suppose* $d > 1$ *is an integer. There exists a minimal subshift* (X, σ) *which has exactly* d *ergodic measures, zero nonergodic generic measures, and which satisfies*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} = d, \qquad \limsup_{n\to\infty} \frac{P_X(n)}{n} = d + 1.$$

We include several other examples in Section 5, showing other senses in which Theorems 1.1 and 1.2 can be said to be sharp.

As an application of Theorem 1.1, we answer Chaika and Masur's question:

Theorem 1.4. *For* $k > 2$*, a minimal* k*-interval exchange transformation has at most* $k - 2$ *generic measures.*

For $k = 2$, a minimal 2-interval exchange is an ergodic rotation, which is uniquely ergodic. For $k = 3$ and 4, Theorem 1.4 is a sharp upper bound, but we do not know if it is sharp for $k \geq 5$. In particular, we do not know if we can improve the symbolic result of Theorem 1.1 for systems that arise as the natural coding of an interval exchange transformation. Similarly, we would like to know if the conclusion of Theorem 1.4 can be strengthened to replace $k - 2$ by $[k/2]$. We also do not know if there can be a second generic measure in the example of Chaika and Masur, nor if a 6-interval exchange with three ergodic measures can also have a generic (and obviously nonergodic) measure.

2. Background and notation

If $\mathcal{A}$ is a finite alphabet, a *word* w in the alphabet is a concatenation of letters in $\mathcal{A}$ and the length $|w|$ of the word is the number (finite or infinite) of letters. A word $w = w_1 \cdots w_\ell$ *occurs* in a word $u = u_1 \cdots u_k$ if there is some $m \in \{1, \ldots, k - \ell\}$ such that $w_1 = u_m, \ldots, w_\ell = u_{m+\ell}$, and we refer to w as a *subword* of u. The analogous definitions hold for a finite word w occurring as a subword of an infinite word u.

A *language* $\mathcal{L}$ is a set of (finite) words such that if $w \in \mathcal{L}$, then any subword is also contained in $\mathcal{L}$. The language *determined by a word* (finite or infinite) is the collection of all finite subwords of the word. We let $\mathcal{L}_n$ denote all the words in the language $\mathcal{L}$ of length n. If $w \in \mathcal{L}$, we write $[w]$ for the *cylinder set* determined by w, meaning that

$$[w] = \{u \in \mathcal{L} : \text{the first } |w| \text{ symbols of } u \text{ agree with } w\}.$$

We assume that the alphabet $\mathcal{A}$ is endowed with the discrete topology and if $x \in \mathcal{A}^{\mathbb{Z}}$, we use $x(n)$ to denote the value of x at $n \in \mathbb{Z}$. The space $\mathcal{A}^{\mathbb{Z}}$ is a compact metric space when endowed with the product topology (and a compatible metric).

A *subshift* (X, σ) is a closed subset $X \subset \mathcal{A}^{\mathbb{Z}}$ that is invariant under the left shift $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $(\sigma x)(n) = x(n+1)$. If $\mathcal{L}$ is the language of the system X, meaning the set of all finite subwords that arise for any $x \in X$, we write $\mathcal{L} = \mathcal{L}(X)$, and we write $\mathcal{L}_n = \mathcal{L}_n(X)$ for the words of length n. We define the *complexity function* $P_X\colon X \to \mathbb{N}$ by

$$P_X(n) = |\mathcal{L}_n(X)|.$$

For a word $w \in \mathcal{L}(X)$, we write $\mathbf{1}_{[w]}$ for the indicator function of the word w. We say that $x = (x(n))_{n\in\mathbb{Z}} \in X$ is *periodic* if there exists $m \neq 0$ such that $x(m+n) = x(n)$ for all $n \in \mathbb{Z}$; otherwise it is *aperiodic*. The point x is *eventually periodic* if there exist $m \neq 0$ and $N \in \mathbb{N}$ such that $x(m+n) = x(n)$ for all $n \geq N$.

For a system (X, σ), the *orbit* of $x \in X$ is defined to be $\{\sigma^n x\colon n \in \mathbb{Z}\}$, and the system is *minimal* if the orbit closure $\overline{\{\sigma^n x\colon n \in \mathbb{Z}\}}$ equals X for any $x \in X$.

For $N, m \in \mathbb{N}$, define $w(N, m) \in \mathcal{L}_N(X)$ by

$$w(N, m) := (x(m), x(m+1), \ldots, x(m+N-1)) \tag{2}$$

to be the word of length N that occurs in x starting at location m. We make use of the following theorem (though stated differently) of Epifanio, Koskas, and Mignosi [7]:

Theorem 2.1 ([7, Theorem 2.2]). *Assume $x \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic and fix $M, N_0 \in \mathbb{N}$. Suppose that for some $N \geq N_0$, there exist $M \leq m_1 < m_2 \leq N$ such that $w_x(N, m_1) = w_x(N, m_2)$. Then there exists $K \geq m_1$ such that*

(i) (Distinct Words Condition) *for all $K \leq k_1 < k_2 \leq K + N - N_0$ we have $w_x(N, k_1) \neq w_x(N, k_2)$;*
(ii) (Prefix First Occurrence Condition) *for all $K \leq k < K + N - N_0$ there exists $M \leq l_k \leq N$ such that $w_x(N_0, k) = w_x(N_0, l_k)$.*

For completeness, we include the proof, but it is merely a translation of the proof in [7] using our hypotheses and emphasizing the stronger conclusion.

Proof of Theorem 2.1. Suppose that $w_x(N, m_1) = w_x(N, m_2)$. Then the word $w_x(N + m_2 - m_1, m_1)$ is periodic of period $m_2 - m_1$. Since x is not eventually periodic, there exists $N' \geq N + m_2 - m_1$ such that $w_x(N', m_1)$ is periodic of period $m_2 - m_1$, while $w_x(N'+1, m_1)$ is not. Let $1 \leq p \leq m_2 - m_1$ be the minimal period of $w_x(N', m_1)$ and define $K := m_1 + N' - N - p \geq m_1$. By minimality of p and the fact that $N \geq p$, if $K \leq i < j \leq K + p - 1$ then $w_x(N, i) \neq w_x(N, j)$ (and all such words are periodic of period p).

For contradiction, suppose there exist $K \leq i < j \leq K + N - N_0$ such that $w_x(N, i) = w_x(N, j)$. Since i, j cannot both be smaller than $K + p$, it follows that $j \geq K + p$. The word $w_x(N + (j - i), i)$ is periodic of period $j - i$ and its prefix of length $p + j - i$ is periodic of period p. By the Fine–Wilf Theorem [9], this prefix is periodic of period $\gcd(j - i, p)$. Since this prefix has length at least p, it follows that $w_x(N + (j - i), i)$

is periodic of period $\gcd(j-i, p)$ and, in particular, is periodic of period p. Moreover, $K \le i \le K+N-N_0$ and so $w_x(N+(j-i), i)$ begins at least p spaces before $N'+1$ and ends at location

$$i+N+(j-i) = N+j \ge N+(K+p) = m_1+N'+1.$$

Thus the periodicity of $w_x(N+(j-i), i)$ contradicts the fact that $w_x(N', m_1)$ is not periodic of period p, which implies that $w_x(N, i) \ne w_x(N, j)$ for any $M \le i < j \le K+n-N_0$.

Since $w_x(N', m_1)$ is periodic of period $p \le n$ and the length N_0 prefix of $w_i(N, i)$ is a subword of $w_x(N', m_1)$, the second statement follows. □

3. Main results

Theorems 1.1 and 1.2 follow from the following estimate:

Theorem 3.1. *Let (X, σ) be a subshift which has at least $d \ge 1$ distinct, nonatomic, generic measures. Then*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} \ge d.$$

If, in addition, (X, σ) has a generic measure μ and a generic point x_μ whose orbit closure $\overline{\{\sigma^k x_\mu : k \in \mathbb{N}\}}$ is not uniquely ergodic, then

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} \ge d+1.$$

Proof. We show that for arbitrarily small $\delta > 0$, we have

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} > d - 2d\delta \tag{3}$$

and, under the additional hypothesis of a generic measure and associated generic point whose orbit closure is not uniquely ergodic,

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} > d + 1 - 2d\delta. \tag{4}$$

The theorem follows immediately from these estimates.

Fix $\delta > 0$, and for convenience assume that $1/\delta \in \mathbb{N}$. Suppose $\mu_1, \dots, \mu_d$ are distinct, nonatomic, generic measures for (X, σ) and choose $x_1, \dots, x_d \in X$ such that for each $1 \le i \le d$, x_i is generic for μ_i. Observe that if x is eventually periodic and is generic for some measure μ, then μ must be the atomic measure supported on the (eventual) period of x. Therefore, since μ_i is nonatomic, x_i is not eventually periodic for all i. By definition of x_i, for all $w \in \mathcal{L}(X)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_{[w]}(T^k x_i) = \mu_i([w]). \tag{5}$$

For $1 \leq j_1 < j_2 \leq d$, choose words $w_{(j_1,j_2)} \in \mathcal{L}(X)$ such that $\mu_{j_1}([w_{(j_1,j_2)}]) \neq \mu_{j_2}([w_{(j_1,j_2)}])$. Set

$$\varepsilon := \min\{|\mu_{j_1}([w_{(j_1,j_2)}]) - \mu_{j_2}([w_{(j_1,j_2)}])|\colon 1 \leq j_1 < j_2 \leq d\}, \tag{6}$$

$$B := \frac{\delta}{16-4\delta}. \tag{7}$$

By (5), for each $1 \leq i \leq d$ there exists $N_i \in \mathbb{N}$ such that for all $N \geq N_i$ and all $1 \leq j_1 < j_2 \leq d$, we have

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} 1_{[w_{(j_1,j_2)}]}(T^k x_i) - \mu_i([w_{(j_1,j_2)}]) \right| < B \cdot \varepsilon. \tag{8}$$

Set

$$M := \max_{1 \leq i \leq d} N_i. \tag{9}$$

Analogous to (2), for $1 \leq i \leq d$ and $N, m \in \mathbb{N}$, define

$$u_i(N, m) := (x_i(m), x_i(m+1), \ldots, x_i(m+N-1)) \in \mathcal{L}_N(X)$$

to be the word of length N that occurs in x_i starting at location m.

If $u, w \in \mathcal{L}(X)$ and $|u| \geq |w|$, define the frequency with which w occurs as a subword in u to be

$$F(u, w) := \frac{1}{|u| - |w| + 1} \sum_{k=0}^{|u|-|w|} 1_{[w]}(T^k x), \tag{10}$$

where $x \in [u]$. Note that this frequency does not depend on the choice of $x \in [u]$, as it only depends on the first $|u|$ coordinates of x. Suppose

$$N \geq \frac{1}{\delta} \cdot (M + \max\{|w_{(j_1,j_2)}|\colon 1 \leq j_1 < j_2 \leq d\})$$

is fixed and define $L_N := \lfloor (2-\delta)N \rfloor$ and $\ell_N := \lfloor \delta N \rfloor$. By definition, $\ell_N - |w_{(j_1,j_2)}| \geq M$ for all $w_{(j_1,j_2)}$. If $1 \leq i \leq d$, $1 \leq j_1 < j_2 \leq d$, and $M \leq L \leq L_N$, then the frequency with which the word $w_{(j_1,j_2)}$ occurs in the subword of x_i with length ℓ_N and starting from location L is given by (recall that $u_i(\ell_N, L)$ is the word of length ℓ_N that starts at location L in x_i)

$$\begin{aligned} F(u_i(\ell_N, L), w_{(j_1,j_2)}) &= \frac{1}{\ell_N - |w_{(j_1,j_2)}| + 1} \sum_{k=0}^{\ell_N - |w_{(j_1,j_2)}|} 1_{[w_{(j_1,j_2)}]}(T^k(T^L x_i)) \\ &= \frac{1}{\ell_N - |w_{(j_1,j_2)}| + 1} \sum_{k=L}^{L+\ell_N - |w_{(j_1,j_2)}|} 1_{[w_{(j_1,j_2)}]}(T^k x_i) \end{aligned}$$

$$= \frac{1}{\ell_N - |w_{(j_1,j_2)}| + 1}\Big(\sum_{k=0}^{L+\ell_N-|w_{(j_1,j_2)}|} 1_{[w_{(j_1,j_2)}]}(T^k x_i) - \sum_{k=0}^{L-1} 1_{[w_{(j_1,j_2)}]}(T^k x_i)\Big)$$

$$= \frac{L+\ell_N - |w_{(j_1,j_2)}| + 1}{\ell_N - |w_{(j_1,j_2)}| + 1} \cdot \frac{1}{L+\ell_N - |w_{(j_1,j_2)}| + 1} \sum_{k=0}^{L+\ell_N-|w_{(j_1,j_2)}|} 1_{[w_{(j_1,j_2)}]}(T^k x_i)$$

$$- \frac{L}{\ell_N - |w_{(j_1,j_2)}| + 1} \cdot \frac{1}{L} \sum_{k=0}^{L-1} 1_{[w_{(j_1,j_2)}]}(T^k x_i).$$

But by (8),

$$\left| \frac{1}{L+\ell_N - |w_{(j_1,j_2)}| + 1} \sum_{k=0}^{L+\ell_N-|w_{(j_1,j_2)}|} 1_{[w_{(j_1,j_2)}]}(T^k x_i) - \mu_i([w_{(j_1,j_2)}]) \right| < B \cdot \varepsilon,$$

and since $L \geq M$, we have

$$\left| \frac{1}{L} \sum_{k=0}^{L-1} 1_{[w_{(j_1,j_2)}]}(T^k x_i) - \mu_i([w_{(j_1,j_2)}]) \right| < B \cdot \varepsilon.$$

Therefore

$$\begin{aligned}
&|F(u_i(\ell_N, L), w_{(j_1,j_2)}) - \mu_i([w_{(j_1,j_2)}])| \\
&\qquad \leq \frac{L+\ell_N - |w_{(j_1,j_2)}| + 1}{\ell_N - |w_{(j_1,j_2)}| + 1} \cdot B \cdot \varepsilon + \frac{L}{\ell_N - |w_{(j_1,j_2)}| + 1} \cdot B \cdot \varepsilon \\
&\qquad = \frac{2L+\ell_N - |w_{(j_1,j_2)}| + 1}{\ell_N - |w_{(j_1,j_2)}| + 1} \cdot B \cdot \varepsilon \\
&\qquad \leq \frac{2\lfloor (2-\delta)N \rfloor + \lfloor \delta N \rfloor - |w_{(j_1,j_2)}| + 1}{\lfloor \delta N \rfloor - |w_{(j_1,j_2)}| + 1} \cdot B \cdot \varepsilon.
\end{aligned}$$

By definition (7) that $B = \frac{\delta}{16-4\delta}$, for all sufficiently large N this inequality implies

$$|F(u_i(\ell_N, L), w_{(j_1,j_2)}) - \mu_i([w_{(j_1,j_2)}])| < \varepsilon/2. \tag{11}$$

By (6), for all sufficiently large N and all $L_1, L_2 \in \{M, M+1, \ldots, \lfloor (2-\delta)N \rfloor\}$ we see that if $1 \leq i_1 < i_2 \leq d$, then the frequency with which $w_{(i_1,i_2)}$ occurs in $u_{i_1}(\ell_N, L_1)$ is different than its frequency in $u_{i_2}(\ell_N, L_2)$. Therefore $u_{i_1}(\ell_N, L_1) \neq u_{i_2}(\ell_N, L_2)$. For $1 \leq i \leq d$ define

$$\mathcal{W}_i(N) := \{u_i(\ell_N, L) \colon M \leq L \leq \lfloor (2-\delta)N \rfloor\} \subseteq \mathcal{L}_{\ell_N}(X).$$

We have shown that for all sufficiently large N, if $1 \leq i_1 < i_2 \leq d$, then

$$\mathcal{W}_{i_1}(N) \cap \mathcal{W}_{i_2}(N) = \emptyset. \tag{12}$$

Fix i with $1 \leq i \leq d$ and fix N sufficiently large such that (12) holds. If the words

$$u_i(N, M), u_i(N, M+1), \ldots, u_i(N, \lfloor(1-\delta)N\rfloor) \tag{13}$$

are all distinct, then the set

$$\mathcal{S}_i := \{w \in \mathcal{L}_N(X) \colon \text{all subwords of } w \text{ of length } \ell_N \text{ are elements of } \mathcal{W}_i(N)\} \tag{14}$$

contains at least $\lfloor(1-\delta)N\rfloor - M$ elements. If, on the other hand, the words in (13) are not all distinct, then there exist $M \leq L_1 < L_2 \leq \lfloor(1-\delta)N\rfloor$ such that $u_i(N, L_1) = u_i(N, L_2)$. In this case, by Theorem 2.1 there exists $K \in \mathbb{N}$ such that

(i) (Distinct Words Condition) for all $K \leq k_1 < k_2 \leq K+N-\ell_N$ we have $u_i(N, k_1) \neq u_i(N, k_2)$;
(ii) (Prefix First Occurrence Condition) for all $K \leq k \leq K+N-\ell_N$ there exists $\ell_N \leq l_k \leq N$ such that $u_i(\ell_N, k) = u_i(\ell_N, l_k)$.

Thus in this case, the set

$$\mathcal{T}_i := \{w \in \mathcal{L}_N(X) \colon \text{the leftmost subword } w \text{ of length } \ell_N \text{ lies in } \mathcal{W}_i(N)\} \tag{15}$$

contains at least $N - \ell_N$ elements.

By (12), $\mathcal{S}_{i_1} \cap \mathcal{S}_{i_2} = \emptyset$ whenever $i_1 \neq i_2$ (and both sets are defined). A similar statement holds when comparing any $\mathcal{S}_{i_1}$ to $\mathcal{T}_{i_2}$ for any i_2, or when comparing $\mathcal{T}_{i_1}$ to $\mathcal{T}_{i_2}$. Thus for each $1 \leq i \leq d$, we have associated either the set $\mathcal{S}_i$ or the set $\mathcal{T}_i$ and

$$P_X(N) \geq d \cdot \min\{N - \ell_N, \lfloor(1-\delta)N\rfloor - M\} = d \cdot \min\{N - \lfloor\delta N\rfloor, \lfloor(1-\delta)N\rfloor - M\}.$$

Therefore,

$$\frac{P_X(N)}{N} \geq \frac{d \cdot \min\{N - \lfloor\delta N\rfloor, \lfloor(1-\delta)N\rfloor - M\}}{N},$$

which is larger than $d - 2d\delta$ for all sufficiently large N, thus establishing (3).

To prove (4) , suppose that there exists $1 \leq i \leq d$ such that the orbit closure of x_i is not uniquely ergodic. Then for any fixed (and sufficiently large) $N \in \mathbb{N}$, there exist infinitely many $L \in \mathbb{N}$ such that $u_i(\ell_N, L) \notin \mathcal{W}_i(N)$. Fix $N \in \mathbb{N}$.

If the words

$$u_i(N, M), u_i(N, M+1), \ldots, u_i(N, \lfloor(1-\delta)N\rfloor)$$

are all distinct, then we define $\mathcal{S}_i$ as in (14). In this case, choose the smallest $L \geq M$ for which $u_i(\ell_N, L) \notin \mathcal{W}_i(N)$; clearly $L > L_N$. Then each of the words

$$u_i(N, L-N+\ell_N), u_i(N, L-N+\ell_N+1), \ldots, u_i(N, L-\ell_N)$$

has the property that its leftmost subword of length ℓ_N is an element of $\mathcal{W}_i(N)$, and these words are pairwise distinct (in $u_i(N, L-N+\ell_N+j)$ the leftmost occurrence of a subword of length ℓ_N that is not in $\mathcal{W}_i(N)$ begins at location $L-\ell_N-j$). These $N-\ell_N$ words of length N do not lie in $\mathcal{S}_i$, and are not contained in any $\mathcal{S}_j$ or $\mathcal{T}_j$ for any $j \neq i$

(as defined in (15)), since their leftmost subword of length ℓ_N is in $\mathcal{W}_i$. Therefore by the bound on the size of $\mathcal{S}_i$ following (14),

$$\begin{aligned} P_X(N) &\geq d \cdot \min\{N - \ell_N, \lfloor(1-\delta)N\rfloor - M\} \\ &= d \cdot \min\{N - \lfloor\delta N\rfloor, \lfloor(1-\delta)N\rfloor - M\} + (N - \ell_N), \end{aligned}$$

and so in this case,

$$\frac{P_X(N)}{N} \geq \frac{d \cdot \min\{N - \lfloor\delta N\rfloor, \lfloor(1-\delta)N\rfloor - M\}}{N} + \frac{N - \lfloor\delta N\rfloor}{N}.$$

If N is sufficiently large, this is larger than $d + 1 - 2d\delta$.

Thus we are left with showing that there are infinitely many $N \in \mathbb{N}$ for which the words

$$u_i(N, M), u_i(N, M+1), \ldots, u_i(N, \lfloor(1-\delta)N\rfloor) \tag{16}$$

are all distinct. Fix some $N \in \mathbb{N}$ and assume that these words are not all distinct. As before, let $L_1, L_2 \in \{M, M+1, \ldots, \lfloor(1-\delta)N\rfloor\}$ with $L_1 < L_2$ be such that $u_i(N, L_1) = u_i(N, L_2)$. Let p be the minimal period of the word $u_i(N+L_2-L_1, L_1)$ and let K be the largest integer for which $u_i(K, L_1)$ is periodic with period p (note that K is finite since x_i is not eventually periodic). Then the words

$$u_i(K, M), u_i(K, M+1), \ldots, u_i(K, \lfloor(1-\delta)K\rfloor) \tag{17}$$

are all distinct: if $j > L_1 - M$ then the word $u_i(K, M+j)$ begins with a word that is periodic of period p and has length exactly $K - L_1 - j$ (so no two words of this form can coincide), and if $j \leq L_1 - M$ then $u_i(K, M+j)$ either begins with a word of length $K - L_1 + j$ that is periodic of period p, or has a prefix of length at most L_1 followed by a word of length at least $K - L_1 > N$ that is periodic of period p (which occurs in a different location for each such j). Therefore, for each $N \in \mathbb{N}$ there exists $K \geq N$ such that the words in (17) are all distinct, and in particular there are infinitely many N such that the words in (16) are distinct. This establishes (4). □

As immediate corollaries of Theorem 3.1, we have the theorems stated in the introduction:

Corollary (Theorem 1.1). *If (X, σ) is a subshift and there exists $k \in \mathbb{N}$ such that*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < k,$$

then (X, σ) has at most $k-1$ distinct, nonatomic, generic measures.

Corollary (Theorem 1.2). *If (X, σ) is a subshift and there exists $k \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} < k,$$

and if (X, σ) has a generic measure μ and a generic point x_μ whose orbit closure is not uniquely ergodic, then (X, σ) has at most $k-2$ distinct, nonatomic, generic measures.

In Section 5, we show that both of these corollaries are sharp. In particular, the linear growth rate in Theorem 1.1 is optimal, in the sense that a superlinear growth rate does not suffice to show that the set of ergodic measures is finite, and the technical condition of Theorem 1.2 (and in Theorem 3.1) on the existence of a point whose orbit closure is not uniquely ergodic cannot be dropped.

4. The natural coding of an IET

Let $k \geq 1$ be an integer and π be a permutation of $\{1, \ldots, k\}$. Let $I = [0, \lambda)$ be an interval and choose $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \lambda$. The *interval exchange transformation* $T\colon [0, \lambda) \to [0, \lambda)$ is defined to be the map that is an isometry on each subinterval $[\lambda_{i-1}, \lambda_i)$ for $i = 1, \ldots, k$ and rearranges the order of these subintervals according to the permutation π. Without loss of generality, we can assume that k is the smallest number of subintervals needed to define the transformation T (otherwise we can join two consecutive subintervals into a single one). We refer to this interval exchange transformation as a k-*IET* or just an *IET* when k is clear from the context.

Given an interval exchange transformation, there is a natural coding by an associated dynamical system. For $x \in I$, define $\mathbf{x} = (x_n) \in \{1, \ldots, k\}^{\mathbb{N}}$ by setting

$$x_n = i \quad \text{if and only if} \quad T^n x \in [\lambda_{i-1}, \lambda_i).$$

The *language* of $\mathbf{x}$ is the set of all finite words that appear, and the *natural coding* of the interval exchange transformation is the symbolic system, endowed with the shift, that has the same language as $\mathbf{x}$. The *natural symbolic cover* of an interval exchange transformation is the subshift that codes every $x \in I$, meaning it is the symbolic system, endowed with the shift, whose language consists of all finite words that arise in the orbit of any $x \in I$. While the image of the interval $[0, \lambda)$ under this coding is invariant under the shift, it is not necessarily closed, and so we take its orbit closure to produce a semiconjugacy from the coding to the interval exchange. For further details, see [11, Chapter 15, Section 5].

If a point does not lie in the orbit of one of the endpoints of the subintervals defining the IET, then it has a unique preimage under the semiconjugacy, and otherwise it has at most two preimages corresponding to the coding of iterates.

If T is a minimal interval exchange transformation, then any $x \in I$ gives rise to the same language and it suffices to take the orbit of a single point. More generally, the symbolic coding is not topologically conjugate to T, as up to countably many points may have multiple preimages (though a point can only have finitely many preimages). However, since the points with nonunique preimage can only support an atomic measure, it is a measure-theoretic isomorphism for any nonatomic generic measure on X.

Namely, we claim that a generic measure for an interval exchange transformation lifts to a generic measure in the symbolic cover. A basic open set in the symbolic cover is a cylinder set and thus corresponds to an interval or a finite union of intervals in $[0, \lambda)$. Thus it suffices to check the claim for a finite interval $J \subseteq [0, \lambda)$. Let $x \in [0, \lambda)$ be a generic point for the measure μ. Choose continuous functions f and g on $[0,\lambda)$ such that $0 \le f \le \mathbf{1}_J \le g$ and $\int g\,d\mu - \varepsilon/2 \le \mu(J) \le \int f\,d\mu + \varepsilon/2$. Then

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) - \int f\,d\mu\right| < \varepsilon/2$$

for all sufficiently large N, and the same holds for g. Thus

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_J(T^n x) &\le \frac{1}{N}\sum_{n=0}^{N-1} g(T^n x) \le \varepsilon/2 + \int g\,d\mu \le \varepsilon + \int f\,d\mu \\ &\le \varepsilon + \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_J(T^n x).\end{aligned}$$

Hence

$$\left|\mu(J) - \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_J(T^n x)\right| < \varepsilon.$$

Since this holds for all $\varepsilon > 0$, for any open set $J \subset [0,\lambda)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_J(T^n x) = \mu(J).$$

Write $\phi\colon (X,\sigma) \to ([0,\lambda),T)$ for the factor map from the symbolic coding (X,σ) to the interval exchange $([0,\lambda),T)$. Let $\mathcal{L}(X)$ denote the language of the coding and let μ be a generic measure on $([0,\lambda),T)$ with generic point x. Let $x^* \in \phi^{-1}(x)$. Then for any word $w \in \mathcal{L}(X)$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_{[w]}(\sigma^n x^*) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{1}_{\phi([w])}(T^n x) = \mu(\phi([w])),$$

since $\phi([w])$ is a finite union of intervals. Since μ is a nonatomic, generic measure, the pullback $\phi^*(\mu([w])) = \phi^*(\mu(\phi^{-1}(\phi([w]))))$ is also nonatomic, as only countably many points in $([0,\lambda),T)$ have multiple preimages and each of these only has finitely many preimages. (In other words, the pushforward of the pullback of the measure is the measure itself.) Thus a generic measure for the interval exchange transformation corresponds to a generic measure in the symbolic coding.

It is well known that an IET has linear complexity (see for example [8]). We include a proof for completeness:

Proposition 4.1. *The natural coding of a minimal k-IET has complexity*

$$P(n) \le (k-1)n + 1.$$

If the k-IET satisfies the infinite distinct orbits condition (*IDOC*)*, then the complexity is exactly* $P(n) = (k-1)n + 1$.

Proof. We proceed by induction on n. The number $P(1)$ is the size of the alphabet, which has size k, so for $n = 1$ the result is clear. Assume that $P(n) \leq (k-1)n + 1$. If we fix a particular word of length n, the cylinder set defined by this word distinguishes an interval in the exchange, and by considering the cylinder sets associated to each word of length n, we obtain a partition of the exchange. Thus we have associated a partition $\mathcal{I}$ of the exchange to the $(k-1)n+1$ words of length n, and this partition has $(k-1)n+2$ endpoints. Furthermore, these endpoints all arise as iterates of the endpoints of the original $k+1$ endpoints of the interval exchange. Each of the $k+1$ original endpoints lies in some $T(I)$, where T is the exchange map and I is one of the intervals in the partition $\mathcal{I}$. We note that if the exchange satisfies IDOC, then the endpoints arise as distinct iterates, and each of the original endpoints lies in the interior of some $T(I)$; but without this condition there may be overlap in the iterates and this is only an upper bound.

Thus we have $M \leq k-1$ intervals in $(T(I))_{I\in\mathcal{I}}$ which cover all of the original endpoints. These M intervals may each cover more than one of the original endpoints, say m of them, and there are at most $m+1$ distinct ways to continue the orbit of a word of length n. Thus in total, we have $(k-1)n + 1 - M + (k-1) + M$ continuations, which is exactly the bound $P(n+1) \leq (k-1)(n+1) + 1$.

If the exchange satisfies IDOC, then as the endpoints arise as distinct iterates, we see that the complexity is exactly $P(n) = (k-1)n + 1$. □

Combining this with Theorem 3.1, we obtain the statement of Theorem 1.4:

Corollary (Theorem 1.4). *For $k > 2$, a minimal k-IET has at most $k-2$ generic measures.*

5. Sharpness

In this section we show that the bound in Theorem 3.1 is sharp. We recall the statement of Theorem 1.3 for convenience.

Theorem (Theorem 1.3). *Let $d > 1$ be fixed. There exists a minimal subshift (X, σ) such that*

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} = d, \qquad \limsup_{n\to\infty} \frac{P_X(n)}{n} = d+1,$$

and X has exactly d ergodic measures.

Before we delve into the details of the construction, we outline the basic ideas involved. The ideas of this argument were partly inspired by a construction of a minimal and non-uniquely ergodic subshift by Quas on mathoverflow [13] (see also Denker, Grillenberger, and Sigmund [6]).

Fixing $d > 1$ and the alphabet $\mathcal{A} = \{1, \dots, d\}$, we inductively construct d sequences of words $\{w_1^j\}_{j=1}^{\infty}, \dots, \{w_d^j\}_{j=1}^{\infty}$ in $\mathcal{L}(\mathcal{A}^{\mathbb{Z}})$. Roughly speaking, the procedure we use constructs the words in these sequences in the following (somewhat unusual) order: $w_1^1, w_2^1, \dots, w_d^1, w_1^2, w_2^2, \dots, w_d^2, w_1^3, w_2^3, \dots, w_d^3, \dots$. That is, we first construct the first word in each of the sequences, then the second word in each, and so on. The words are constructed such that:

(i) If $i_1, i_2 \in \mathcal{A}$ and $j_1 < j_2$, then $w_{i_1}^{j_1}$ occurs as a subword of $w_{i_2}^{j_2}$ syndetically,[1] with gap size bounded by a constant that depends only on j_1.
(ii) For any $i \in \mathcal{A}$ and $j \in \mathbb{N}$, the frequency with which the letter i occurs in w_i^j (as a percentage of the length of w_i^j) is greater than an absolute constant which is greater than $1/2$.

By taking a limit along a subsequence of $\{w_1^j\}_{j=1}^{\infty}$, we produce a semi-infinite word w_1^{∞}, and taking its orbit closure under the shift σ and passing to the natural two-sided extension, we obtain a closed subshift $X \subset \mathcal{A}^{\mathbb{Z}}$. It follows from the construction that (X, σ) is minimal and that $w_i^j \in \mathcal{L}(X)$ for all $i \in \mathcal{A}$ and $j \in \mathbb{N}$. For fixed $i \in \mathcal{A}$, there are arbitrarily long words in $\mathcal{L}(X)$ for which the frequency of the letter i is at least (a constant greater than) $1/2$ and so the system (X, σ) has an ergodic measure assigning the cylinder set $[i]$ measure greater than $1/2$. Thus (X, σ) has at least $|\mathcal{A}| = d$ ergodic measures. By carefully choosing the lengths of the words, we further show that the system (X, σ) satisfies the desired upper and lower bounds on the complexity. Applying Theorem 3.1, we find that (X, σ) has at most d ergodic measures, and so exactly d ergodic measures.

We now make these ideas precise:

Proof of Theorem 1.3. Set $\mathcal{A} := \{1, \dots, d\}$. Choose a sequence $\kappa_1, \kappa_2, \dots$ of real numbers in $(0, 1)$ such that

$$\prod_{j=1}^{\infty} \kappa_j > 1/2,$$

and choose a strictly decreasing sequence $\delta_1, \delta_2, \dots$ of real numbers in $(0, 1)$ such that $\lim_{j\to\infty} \delta_j = 0$.

Step 1 (Construction of the sequences $\{w_1^j\}_{j=1}^{\infty}, \dots, \{w_d^j\}_{j=1}^{\infty}$): Define the word

$$w_1^1 := \underbrace{1 \cdots 1}_{N_{(1,1)}^{[1]}} 234 \cdots d,$$

where the length $N_{(1,1)}^{[1]} \in \mathbb{N}$ is chosen such that $N_{(1,1)}^{[1]} > \kappa_1 |w_1^1|$. Define

$$s_1^1 := w_1^1 \quad \text{and} \quad p_1^1 := \underbrace{1 \cdots 1}_{N_{(1,1)}^{[1]}}.$$

The word s_1^1 represents the *suffix* of w_1^1 that starts with the beginning of w_1^1, and p_1^1 denotes the *prefix* of w_1^1 consisting of its initial block of 1's (the suffix and prefix terminology is further clarified in the inductive construction, but note that the previous level suffix becomes the beginning of the next level prefix). We refer to the groupings by a length $N_{j,k}^{[n]}$ as a *block* in the word (thus p_1^1 has a single block), and the change from one

[1] A word occurs v occurs *syndetically* in a word w *with gap* g if every subword of w of length g contains a copy of v as a sub-subword.

block to the following block as a *transition* (the word w_2^1 has one transition); and we refer to words that are not entirely within one block as *transition words*.

Next define the word

$$w_2^1 := \underbrace{s_1^1 w_1^1 \cdots w_1^1 p_1^1}_{N_{(2,1)}^{[1]}} \underbrace{2 \cdots 2}_{N_{(2,2)}^{[1]}} \underbrace{3 \cdots 3}_{N_{(2,3)}^{[1]}} \cdots \underbrace{d \cdots d}_{N_{(2,d)}^{[1]}},$$

where the lengths $N_{(2,1)}^{[1]}, \dots, N_{(2,d)}^{[1]} \in \mathbb{N}$ are chosen such that

$$\begin{aligned} |w_1^1| &< (\delta_1)^2 \cdot N_{(2,1)}^{[1]} < (\delta_1)^4 \cdot N_{(2,d)}^{[1]} < (\delta_1)^6 \cdot N_{(2,d-1)}^{[1]} < (\delta_1)^8 \cdot N_{(2,d-2)}^{[1]} \\ &< \cdots < (\delta_1)^{2d-2} \cdot N_{(2,3)}^{[1]} < (\delta_1)^{2d} \cdot N_{(2,2)}^{[1]} \end{aligned} \tag{18}$$

and $N_{(2,2)}^{[1]} > \kappa_1 |w_2^1|$. The ordering of the lengths $N_{(2,k)}^{[1]}$ is important, with the index k passing from 1 to d and then down to $d-1$ and continuing cycling step by step down to 2. The choice of the lengths is used only in estimating the growth of $P_X(n)$; the exact choices of the lengths and the estimates of (18) can be ignored for a first reading of Steps 1 and 2.

Further note that the initial portion of w_2^1 is a concatenation of the word w_1^1 with itself a large number of times, followed by its prefix word p_1^1. We include p_1^1 so that the word $w_1^1 p_1^1 2$ has already occurred as a subword of the concatenation $w_1^1 \cdots w_1^1$, allowing us to make the transition in w_2^1 from the block of w_1^1's to the block of 2's while keeping the possible new words as low as possible. We iterate this technique at each step of the construction.

Next we define the word w_3^1. To do this, we make use of two auxiliary words. Let

$$p_2^1 := \underbrace{s_1^1 w_1^1 \cdots w_1^1 p_1^1}_{N_{(2,1)}^{[1]}} \underbrace{2 \cdots 2}_{N_{(2,2)}^{[1]}}$$

be the prefix of w_2^1 that includes everything through the block of 2's and let

$$s_2^1 := \underbrace{2 \cdots 2}_{N_{(2,2)}^{[1]}} \underbrace{3 \cdots 3}_{N_{(2,3)}^{[1]}} \cdots \underbrace{d \cdots d}_{N_{(2,d)}^{[1]}}$$

be the suffix of w_2^1 that starts with the block of 2's. Finally, we define

$$w_3^1 := \underbrace{s_1^1 w_1^1 \cdots w_1^1 p_1^1}_{N_{(3,1)}^{[1]}} \underbrace{s_2^1 w_2^1 \cdots w_2^1 p_2^1}_{N_{(3,2)}^{[1]}} \underbrace{3 \cdots 3}_{N_{(3,3)}^{[1]}} \cdots \underbrace{d \cdots d}_{N_{(3,d)}^{[1]}},$$

where $N_{(3,1)}^{[1]}, \dots, N_{(3,d)}^{[1]} \in \mathbb{N}$ are chosen such that

$$\begin{aligned} |w_2^1| &< (\delta_1)^2 \cdot N_{(3,2)}^{[1]} < (\delta_1)^4 \cdot N_{(3,1)}^{[1]} < (\delta_1)^6 \cdot N_{(3,d)}^{[1]} < (\delta_1)^8 \cdot N_{(3,d-1)}^{[1]} \\ &< \cdots < (\delta_1)^{2d-2} \cdot N_{(3,4)}^{[1]} < (\delta_1)^{2d} \cdot N_{(3,3)}^{[1]} \end{aligned} \tag{19}$$

and $N^{[1]}_{(3,3)} > \kappa_1|w^1_3|$. Again, the initial block of w^1_3 is periodic (based on the word w^1_1), with the period ending when its pattern dictates the next letter should be 2. The second block is periodic (based on the word w^1_2), begins with a block of 2's and ends when its pattern dictates the next collection of letters should be a block of 3's (w^1_3 then continues with a block of 3's). As in the previous step, this gives two transitions in w^1_3: first from a periodic block of w^1_1's to a periodic block of w^1_2's, and second from a periodic block of w^1_2's to a block of 3's. The choice of the prefixes and suffixes guarantes that these transitions introduce the smallest possible number of new words (the analysis of this number of new words is made precise in Step 3, where we analyze the growth of the complexity $P_X(n)$).

We continue inductively. Let $i < d$ be fixed and suppose we have constructed words $w^1_1, \ldots, w^1_i, p^1_1, \ldots, p^1_i, s^1_1, \ldots, s^1_i$. Define the word (note that many of the words we define are too long to fit in a single line, and the line break is arbitrary)

$$w^1_{i+1} := \underbrace{s^1_1 w^1_1 \cdots w^1_1 p^1_1}_{N^{[1]}_{(i+1,1)}} \underbrace{s^1_2 w^1_2 \cdots w^1_2 p^1_2}_{N^{[1]}_{(i+1,2)}} \cdots \underbrace{s^1_i w^1_i \cdots w^1_i p^1_i}_{N^{[1]}_{(i+1,i)}} \underbrace{(i+1)\cdots(i+1)}_{N^{[1]}_{(i+1,i+1)}} \cdots \underbrace{d \cdots d}_{N^{[1]}_{(i+1,d)}},$$

where $N^{[1]}_{(i+1,1)}, \ldots, N^{[1]}_{(i+1,d)} \in \mathbb{N}$ are chosen such that

$$\begin{aligned} |w^1_i| &< (\delta_1)^2 \cdot N^{[1]}_{(i+1,i)} < (\delta_1)^4 \cdot N^{[1]}_{(i+1,i-1)} < (\delta_1)^6 \cdot N^{[1]}_{(i+1,i-2)} \\ &< (\delta_1)^8 \cdot N^{[1]}_{(i+1,i-3)} < \cdots < (\delta_1)^{2i} \cdot N^{[1]}_{(i+1,1)} < (\delta_1)^{2i+2} \cdot N^{[1]}_{(i+1,d)} \\ &< (\delta_1)^{2i+4} \cdot N^{[1]}_{(i+1,d-1)} < \cdots < (\delta_1)^{2d-2} \cdot N^{[1]}_{(i+1,i+2)} < (\delta_1)^{2d} \cdot N^{[1]}_{(i+1,i+1)} \end{aligned} \tag{20}$$

and $N^{[1]}_{(i+1,i+1)} > \kappa_1|w^1_{i+1}|$. Again, the lengths are chosen to control the growth of the complexity, and the index k in $N^{[1]}_{(i+1,k)}$ is taken in a cyclical order. Also define a prefix and a suffix of w^1_{i+1} by

$$p^1_{i+1} := \underbrace{s^1_1 1 w^1_1 \cdots w^1_1 p^1_1}_{N^{[1]}_{(i+1,1)}} \underbrace{s^1_2 w^1_2 \cdots w^1_2 p^1_2}_{N^{[1]}_{(i+1,2)}} \cdots \underbrace{s^1_i w^1_i \cdots w^1_i p^1_i}_{N^{[1]}_{(i+1,i)}} \underbrace{(i+1)\cdots(i+1)}_{N^{[1]}_{(i+1,i+1)}},$$

$$s^1_{i+1} := \underbrace{(i+1)\cdots(i+1)}_{N^{[1]}_{(i+1,i+1)}} \underbrace{(i+2)\cdots(i+2)}_{N^{[1]}_{(i+1,i+2)}} \cdots \underbrace{d \cdots d}_{N^{[1]}_{(i+1,d)}},$$

so that p^1_{i+1} is the prefix of w^1_{i+1} that includes everything through the block of $(i+1)$'s, and s^1_{i+1} is the suffix of w^1_{i+1} that begins with the block of $(i+1)$'s. By induction, this defines words $w^1_1, \ldots, w^1_d, p^1_1, \ldots, p^1_d, s^1_1, \ldots, s^1_d$.

For each $i \in \mathcal{A}$, it follows immediately from the construction that:

(a) Every letter in $\mathcal{A}$ appears in w^1_i.
(b) The frequency with which the letter i occurs in w^1_i is at least κ_1, by choice of the integer $N^{[1]}_{(i+1,i+1)}$ and the relations in (20).

We continue to define the sequences of words inductively. Assuming that we have already defined words $w_1^j, \ldots, w_d^j, p_1^j, \ldots, p_d^j, s_1^j, \ldots, s_d^j$, we define

$$w_1^{j+1} := \underbrace{s_1^j w_1^j \cdots w_1^j p_1^j}_{N_{(1,1)}^{[j+1]}} \underbrace{s_2^j w_2^j \cdots w_2^j p_2^j}_{N_{(1,2)}^{[j+1]}} \cdots \underbrace{s_d^j w_d^j \cdots w_d^j p_d^j}_{N_{(1,d)}^{[j+1]}},$$

where

$$\begin{aligned} |w_d^j| &< (\delta_{j+1})^2 \cdot N_{(1,d)}^{[j+1]} < (\delta_{j+1})^4 \cdot N_{(1,d-1)}^{[j+1]} < (\delta_{j+1})^6 \cdot N_{(1,d-2)}^{[j+1]} \\ &< (\delta_{j+1})^8 \cdot N_{(1,d-3)}^{[j+1]} < \cdots < (\delta_1)^{2d-2} \cdot N_{(1,2)}^{[j+1]} < (\delta_1)^{2d} \cdot N_{(1,1)}^{[j+1]} \end{aligned} \tag{21}$$

and $N_{(1,1)}^{[j+1]} > \kappa_{j+1}|w_1^{j+1}|$. Define prefixes and suffixes by

$$p_1^{j+1} := \underbrace{s_1^j w_1^j \cdots w_1^j p_1^j}_{N_{(1,1)}^{[j]}}, \qquad s_1^{j+1} := w_1^{j+1}.$$

We have analogs of properties (a) and (b) for the base case of the construction: each of the words $w_1^j, \ldots, w_d^j$ occurs as a subword of w_1^{j+1} and the frequency with which the letter 1 occurs in w_1^{j+1} is at least $\prod_{k=1}^{j+1} \kappa_k$, provided that the frequency with which it occurs in w_1^j was at least $\prod_{k=1}^{j} \kappa_k$.

Continuing inductively, for $i < d$, we define the word (note the change in superscript half way through)

$$\begin{aligned} w_{i+1}^{j+1} := &\underbrace{s_1^{j+1} w_1^{j+1} \cdots w_1^{j+1} p_1^{j+1}}_{N_{(i+1,1)}^{[j+1]}} \underbrace{s_2^{j+1} w_2^{j+1} \cdots w_2^{j+1} p_2^{j+1}}_{N_{(i+1,2)}^{[j+1]}} \cdots \\ &\underbrace{s_i^{j+1} w_i^{j+1} \cdots w_i^{j+1} p_i^{j+1}}_{N_{(i+1,i)}^{[j+1]}} \underbrace{s_{i+1}^j w_{i+1}^j \cdots w_{i+1}^j p_{i+1}^j}_{N_{(i+1,i+1)}^{[j+1]}} \cdots \underbrace{s_d^j w_d^j \cdots w_d^j p_d^j}_{N_{(i+1,d)}^{[j+1]}}, \end{aligned}$$

where

$$\begin{aligned} |w_i^{j+1}| &< (\delta_{j+1})^2 \cdot N_{(i+1,i)}^{[j+1]} < (\delta_{j+1})^4 \cdot N_{(i+1,i-1)}^{[j+1]} < (\delta_{j+1})^6 \cdot N_{(i+1,i-2)}^{[j+1]} \\ &< (\delta_{j+1})^8 \cdot N_{(i+1,i-3)}^{[j+1]} < \cdots < (\delta_{j+1})^{2i} \cdot N_{(i+1,1)}^{[j+1]} \\ &< (\delta_{j+1})^{2i+2} \cdot N_{(i+1,d)}^{[j+1]} < (\delta_{j+1})^{2i+4} \cdot N_{(i+1,d-1)}^{[j+1]} < \cdots \\ &< (\delta_{j+1})^{2d-2} \cdot N_{(i+1,i+2)}^{[j+1]} < (\delta_{j+1})^{2d} \cdot N_{(i+1,i+1)}^{[j+1]} \end{aligned} \tag{22}$$

and $N_{(i+1,i+1)}^{[j+1]} > \kappa_{j+1}|w_{i+1}^{j+1}|$, and define prefixes and suffixes by

$$p_{i+1}^{j+1} := \underbrace{s_1^{j+1} w_1^{j+1} \cdots w_1^{j+1} p_1^{j+1}}_{N_{(i+1,1)}^{[j+1]}} \underbrace{s_2^{j+1} w_2^{j+1} \cdots w_2^{j+1} p_2^{j+1}}_{N_{(i+1,2)}^{[j+1]}} \cdots \underbrace{s_i^{j+1} w_i^{j+1} \cdots w_i^{j+1} p_i^{j+1}}_{N_{(i+1,i)}^{[j+1]}} \underbrace{s_{i+1}^{j} w_{i+1}^{j} \cdots w_{i+1}^{j} p_{i+1}^{j}}_{N_{(i+1,i+1)}^{[j+1]}},$$

$$s_{i+1}^{j+1} := \underbrace{s_{i+1}^{j} w_{i+1}^{j} \cdots w_{i+1}^{j} p_{i+1}^{j}}_{N_{(i+1,i+1)}^{[j+1]}} \underbrace{s_{i+2}^{j} w_{i+2}^{j} \cdots w_{i+2}^{j} p_{i+2}^{j}}_{N_{(i+1,i+2)}^{[j+1]}} \cdots \underbrace{s_{d}^{j} w_{d}^{j} \cdots w_{d}^{j} p_{d}^{j}}_{N_{(i+1,d)}^{[j+1]}}.$$

Again, we point out that the words $w_1^j, \dots, w_d^j$ occur as subwords of w_{i+1}^{j+1}, and the frequency with which the letter $i+1$ occurs in w_{i+1}^{j+1} is at least $\prod_{k=1}^{j+1} \kappa_k$, provided that the frequency with which it occurs in w_{i+1}^j was at least $\prod_{k=1}^{j} \kappa_k$.

By induction, we obtain sequences $\{w_1^j\}_{j=1}^\infty, \dots, \{w_d^j\}_{j=1}^\infty$ satisfying:

(a) For any $j > 2$, any $1 \le k < j-2$, and any $i_1, i_2 \in \mathcal{A}$, the word $w_{i_1}^k$ occurs in each of the words $w_1^{k+1}, \dots, w_d^{k+1}$ by construction. Each of the words

$$w_1^{k+2}, \dots, w_d^{k+2}, p_1^{k+2}, \dots, p_d^{k+2}, s_1^{k+2}, \dots, s_d^{k+2} \tag{23}$$

contains at least one of the words $w_1^{k+1}, \dots, w_d^{k+1}$ and so $w_{i_1}^k$ occurs in each of these words. Since $w_{i_2}^j$ is a concatenation of the words appearing in (23), it follows that $w_{i_1}^k$ occurs in $w_{i_2}^j$ syndetically, and the maximal gap length is at most

$$g_k := \max\{|w_l^{k+2}| : l \in \mathcal{A}\} \cup \{|p_l^{k+2}| : l \in \mathcal{A}\} \cup \{|s_l^{k+2}| : l \in \mathcal{A}\} = |w_d^{k+2}|.$$

(b) For any $i \in \mathcal{A}$ and any $j \in \mathbb{N}$, the frequency with which the letter i occurs as a subword of w_i^j is at least $\prod_{k=1}^{j} \kappa_k \ge \prod_{k=1}^{\infty} \kappa_k > 1/2$.

We further note that given the freedom with which the lengths are chosen, we can assume that $N_{(i,k)}^{[j]}$ divides $N_{(i,k)}^{[j+1]}$ for all $i, k \in \mathcal{A}$ and all $j \in \mathbb{N}$. We make this assumption for the remainder of the proof.

Step 2 (Construction and ergodic properties of the subshift (X, σ)): Observe that w_1^j is the leftmost subword of w_1^{j+1} for all $j \in \mathbb{N}$, and so we can define a (one-sided) infinite word w_1^∞ by declaring that for all j, the leftmost subword of w_1^∞ of length $|w_1^j|$ is w_1^j. Then for any $i \in \mathcal{A}$ and any $j \in \mathbb{N}$, the word w_i^j occurs as a subword of w_1^∞ syndetically. Moreover, every subword of w_1^∞ occurs as a sub-subword of w_1^j for some j. Therefore all subwords of w_1^∞ occur syndetically.

Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be the set of all bi-infinite sequences whose language consists only of subwords of w_1^∞, meaning it is the natural extension of the closure of w_1^∞ under σ. Since all words in $\mathcal{L}(X)$ occur syndetically in every element of X, (X, σ) is minimal.

Moreover, $w_i^j \in \mathcal{L}(X)$ for all $i \in \mathcal{A}$ and $j \in \mathbb{N}$. Therefore, for fixed $i \in \mathcal{A}$, there are arbitrarily long words in $\mathcal{L}(X)$ for which the frequency with which the letter i occurs is at least $\prod_{k=1}^{\infty} \kappa_k > 1/2$. For each such word w_i^n, if $y_n \in [w_i^n]$ and

$$\mu_n := \frac{1}{n} \sum_{k=0}^{|w_i^n|-1} \delta_{\sigma^k(y_n)},$$

then any weak-* limit point ν of the sequence (μ_n) is a σ-invariant probability measure supported on X such that $\nu([i]) > 1/2$ (such a construction of ν is, for example, a standard way to prove the Krylov–Bogolyubov Theorem). Consequently, as the ergodic measures are the extreme points in the (convex) set of invariant probability measures on X, there exists an ergodic measure μ_i supported on X for which $\mu_i([i]) > 1/2$. It follows that $\mu_i([j]) < 1/2$ for all $j \neq i$, and so $\mu_j \neq \mu_i$ for any $j \neq i$. Thus (X, σ) has at least d ergodic measures. If we can show that

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} < d + 1,$$

then there are at most d ergodic measures by Theorem 3.1; hence exactly d.

Thus we are left with showing that

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} = d, \qquad \limsup_{n\to\infty} \frac{P_X(n)}{n} = d + 1.$$

Step 3 (Analysis of the growth rate of $P_X(n)$): Let $n > |w_1^2|$ be a fixed integer. We estimate the number of words in $\mathcal{L}_n(X)$, which is, by definition, $P_X(n)$. By construction,

$$|w_1^1| < |w_2^1| < \cdots < |w_d^1| < |w_1^2| < |w_2^2| < \cdots < |w_d^2| < |w_1^3| < \cdots .$$

We make the convention that $w_{d+1}^j := w_1^{j+1}$, $w_{d+2}^j := w_2^{j+1}$, and so on (with the analogous convention for $N_{(i_1,i_2)}^{[j]}$ when $i_2 > d$). Therefore, there exist $i_1 \in \mathcal{A}$ and $j_1 \in \mathbb{N}$ such that

$$|w_{i_1}^{j_1}| \leq n < |w_{i_1+1}^{j_1}|.$$

With this convention, observe that

$$\begin{aligned} w_{i_1+1}^{j_1+1} := {} & \underbrace{s_1^{j_1+1} w_1^{j_1+1} \cdots w_1^{j_1+1} p_1^{j_1+1}}_{N_{(i_1+1,1)}^{[j_1+1]}} \underbrace{s_2^{j_1+1} w_2^{j_1+1} \cdots w_2^{j_1+1} p_2^{j_1+1}}_{N_{(i_1+1,2)}^{[j_1+1]}} \cdots \\ & \underbrace{s_{i_1}^{j_1+1} w_{i_1}^{j_1+1} \cdots w_{i_1}^{j_1+1} p_{i_1}^{j_1+1}}_{N_{(i_1+1,i_1)}^{[j_1+1]}} \underbrace{s_{i_1+1}^{j_1} w_{i_1+1}^{j_1} \cdots w_{i_1+1}^{j_1} p_{i_1+1}^{j_1}}_{N_{(i_1+1,i_1+1)}^{[j_1]}} \cdots \underbrace{s_d^{j_1} w_d^{j_1} \cdots w_d^{j_1} p_d^{j_1}}_{N_{(i_1+1,d)}^{[j_1]}}, \end{aligned}$$

where

$$n < |w_{i_1+1}^{j_1}| < |w_{i_1+2}^{j_1}| < \cdots < |w_d^{j_1}| < |w_1^{j_1+1}| < \cdots < |w_{i_1}^{j_1+1}|. \tag{24}$$

It follows from the construction that if $i_2 \in \mathcal{A}$ and $j_2 \in \mathbb{N}$ are such that $|w_{i_2}^{j_2}| \geq |w_{i_1+1}^{j_1+1}|$, then $w_{i_2}^{j_2}$ can also be written as a concatenation of words from the sets

$$\{w_1^{j_1+1}, w_2^{j_1+1}, \dots, w_{i_1}^{j_1+1}, w_{i_1+1}^{j_1}, w_{i_1+2}^{j_1}, \dots, w_d^{j_1}\} = \{w_{i_1}^{j_1+1}, w_{i_1+1}^{j_1}, \dots, w_{i_1+d-1}^{j_1}\}$$

and

$$\{p_i^{j_1+1} : 1 \leq i \leq i_1\} \cup \{p_i^{j_1} : i_1 < i \leq d\} \cup \{s_i^{j_1+1} : 1 \leq i \leq i_1\} \cup \{s_i^{j_1} : i_1 < i \leq d\}.$$

Moreover, there are restrictions on the order in which these words may be concatenated in $w_{i_2}^{j_2}$:

(i) If $i_1 + 1 \leq i < i_1 + d$, then the only words that may be concatenated with $w_i^{j_1}$ are $w_i^{j_1}$ itself and $p_i^{j_1} s_{i+1}^{j_1} w_{i+1}^{j_1}$.

(ii) The only words that may be concatenated to the right end of $w_{i_1+d}^{j_1}$ $(= w_{i_1}^{j_1+1})$ are $w_{i_1}^{j_1+1}$ itself and $p_{i_1}^{j_1+1} s_{i_1+1}^{j_1} w_{i_1+1}^{j_1}$.

Therefore, by (24), the only words of length n that appear as subwords of $w_{i_2}^{j_2}$ are those which appear as subwords of words from the set

$$\{w_i^{j_1} w_i^{j_1} : i_1 < i \leq i_1 + d\} \cup \{w_i^{j_1} p_i^{j_1} s_{i+1}^{j_1} w_{i+1}^{j_1} : i_1 < i < i_1 + d\} \cup \{w_{i_1+d}^{j_1} p_{i_1+d}^{j_1} s_{i_1+1}^{j_1} w_{i_1+1}^{j_1}\}, \tag{25}$$

with superscripts following the convention that if the subscript is larger than d, we increment the superscript by 1. Since all words in $\mathcal{L}_n(X)$ occur as subwords of $w_1^{j_2}$ for all sufficiently large j_2, it follows that all words in $\mathcal{L}_n(X)$ appear as subwords of the $2d$ words in the set (25).

We now analyze the words that appear in (25) by decomposing them into words of length comparable to n. By construction, if $1 \leq m \leq d$ then $w_{i_1+m}^{j_1}$ can be written as a concatenation of words from the set (recall the divisibility of the lengths assumed at the end of Step 1)

$$\begin{aligned}\{w_{i_1+1}^{j_1}\} \cup \bigcup_{k=2}^{d} \Big\{ &\underbrace{s_{i_1+k-1}^{j_1} w_{i_1+k-1}^{j_1} \cdots w_{i_1+k-1}^{j_1} p_{i_1+k-1}^{j_1}}_{N_{(i_1+k,i_1+k-1)}^{[j_1]}}, \underbrace{s_{i_1+k}^{j_1-1} w_{i_1+k}^{j_1-1} \cdots w_{i_1+k}^{j_1-1} p_{i_1+k}^{j_1-1}}_{N_{(i_1+k,i_1+k)}^{[j_1]}}, \\ &\underbrace{s_{i_1+k+1}^{j_1-1} w_{i_1+k+1}^{j_1-1} \cdots w_{i_1+k+1}^{j_1-1} p_{i_1+k+1}^{j_1-1}}_{N_{(i_1+k,i_1+k+1)}^{[j_1]}}, \dots, \underbrace{s_{d-1}^{j_1-1} w_{d-1}^{j_1-1} \cdots w_{d-1}^{j_1-1} p_{d-1}^{j_1-1}}_{N_{(i_1+k,d-1)}^{[j_1]}}, \\ \underbrace{s_d^{j_1-1} w_d^{j_1-1} \cdots w_d^{j_1-1} p_d^{j_1-1}}_{N_{(i_1+k,d)}^{[j_1]}}, &\underbrace{s_1^{j_1} w_1^{j_1} \cdots w_1^{j_1} p_1^{j_1}}_{N_{(i_1+k,1)}^{[j_1]}}, \underbrace{s_2^{j_1} w_2^{j_1} \cdots w_2^{j_1} p_2^{j_1}}_{N_{(i_1+k,2)}^{[j_1]}}, \dots, \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N_{(i_1+k,i_1)}^{[j_1]}} \Big\}\end{aligned} \tag{26}$$

obeying the analogous rules for concatenation ($w_{i_1+1}^{j_1}$ may be concatenated with itself or any block that begins with $s_{i_1+2}^{j_1}$, and any word that ends with $p_{i_1+k}^{j_1}$ may be concatenated with a word that begins with $s_{i_1+k+1}^{j_1}$, again understood cyclically). In turn, the words of length n that occur in words produced from this set are, themselves, words that occur when words from the set

$$\Big\{w_{i_1+1}^{j_1}, \underbrace{s_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} \cdots w_{i_1+2}^{j_1-1} p_{i_1+2}^{j_1-1}}_{N^{[j_1]}_{(i_1+2,i_1+2)}}, \dots, \underbrace{s_d^{j_1-1} w_d^{j_1-1} \cdots w_d^{j_1-1} p_d^{j_1-1}}_{N^{[j_1]}_{(i_1+2,d)}}, \underbrace{s_1^{j_1} w_1^{j_1} \cdots w_1^{j_1} p_1^{j_1}}_{N^{[j_1]}_{(i_1+2,1)}}, \dots, \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N^{[j_1]}_{(i_1+2,i_1)}}\Big\} \tag{27}$$

are concatenated (with the analogous rules for concatenation). Moreover,

$$|w_{i_1+2}^{j_1-1}| < |w_{i_1+3}^{j_1-1}| < \cdots < |w_d^{j_1-1}| < |w_1^{j_1}| < \cdots < |w_{i_1}^{j_1}| \le n < |w_{i_1+1}^{j_1}| \tag{28}$$

and

$$n < |w_{i_1+1}^{j_1}| < N^{[j_1]}_{(i_1+2,m)} \tag{29}$$

for all $1 \le m \le d$ (again, if $i_1+2 > d$ then we increment the superscript of $N^{[j_1]}_{(i_1+2,m)}$ by 1 and reduce the subscript by d). In particular, every word in the set (25) can be obtained by concatenating words from the set (26), and any word of length n that occurs in a word in (25) can also be found in a word obtained by concatenating words from (27).

For $i_1 + 2 \le i < i_1 + d + 1$, define

$$q_i := \cdots w_i^{j_1-1} w_i^{j_1-1} w_i^{j_1-1} p_i^{j_1-1} s_{i+1}^{j_1-1} w_{i+1}^{j_1-1} w_{i+1}^{j_1-1} w_{i+1}^{j_1-1} \cdots$$

to be the bi-infinite word whose restriction to the set of nonnegative indices is the infinite concatenation $s_{i+1}^{j_1-1} w_{i+1}^{j_1-1} w_{i+1}^{j_1-1} w_{i+1}^{j_1-1} \cdots$, and whose restriction to the set of negative indices is the infinite concatenation $\cdots w_i^{j_1-1} w_i^{j_1-1} w_i^{j_1-1} p_{i_1+1}^{j_1}$ (note that the dependence of q_i on j_1 is suppressed in our notation). Similarly define

$$q_{i_1+d+1} := \cdots w_{i_1+1}^{j_1} w_{i_1+1}^{j_1} w_{i_1+1}^{j_1} p_{i_1+1}^{j_1} s_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} \cdots .$$

The set of words length n that arise by concatenating words from the set (27) is precisely the set of words of length n that appear in $q_{i_1+1}, q_{i_1+2}, \dots, q_{i_1+d}$, by (29). By the estimates in (19)–(22), we have

$$|w_i^{j_1-1}| < \delta_{j_1} \cdot |w_{i_1}^{j_1}| \le \delta_{j_1} \cdot n$$

for all $i_1 + 1 < i < d + i_1$. It follows that:

(i) If $i_1 + 1 < i < i_1 + d - 1$, then the number of words of q_i of length n is at least $n+1$ (since q_i is aperiodic) and at most $n + 2\delta_{j_1} n$, as there are at most $\delta_{j_1} n$ words in each block (periodic part) of q_i and at most n transition words obtained from words that overlap the origin.

(ii) The number of words of q_{i_1+d-1} of length n is at least $n+1$ and at most $n+\delta_{j_1}n + |w_{i_1}^{j_1}|$.

(iii) The only new words of q_{i_1+d} are the $n+1$ transition words which appear in $\underbrace{w_{i_1}^{j_1}\cdots w_{i_1}^{j_1}p_{i_1}^{j_1}}_{N^{[j_1]}_{(i_1+2,i_1)}} s_{i_1+1}^{j_1}w_{i_1+1}^{j_1}$ as well as words that appear in $w_{i_1+1}^{j_1}w_{i_1+1}^{j_1}$.

(iv) The only new words of q_{i_1+d+1} are the $n+1$ transition words which appear in $w_{i_1+1}^{j_1}p_{i_1+1}^{j_1}\underbrace{s_{i_1+2}^{j_1-1}w_{i_1+2}^{j_1-1}\cdots w_{i_1+2}^{j_1-1}}_{N^{[j_1]}_{(i_1+2,i_1+2)}}$.

Thus we are left with counting subwords of $w_{i_1+1}^{j_1}w_{i_1+1}^{j_1}$ that have not already appeared.

Write

$$w_{i_1+1}^{j_1} := \underbrace{s_1^{j_1}w_1^{j_1}\cdots w_1^{j_1}p_1^{j_1}}_{N^{[j_1]}_{(i_1+1,1)}}\underbrace{s_2^{j_1}w_2^{j_1}\cdots w_2^{j_1}p_2^{j_1}}_{N^{[j_1]}_{(i_1+1,2)}}\cdots$$
$$\underbrace{s_{i_1}^{j_1}w_{i_1}^{j_1}\cdots w_{i_1}^{j_1}p_{i_1}^{j_1}}_{N^{[j_1]}_{(i_1+1,i_1)}}\underbrace{s_{i_1+1}^{j_1-1}w_{i_1+1}^{j_1-1}\cdots w_{i_1+1}^{j_1-1}p_{i_1+1}^{j_1-1}}_{N^{[j_1]}_{(i_1+1,i_1+1)}}\cdots\underbrace{s_d^{j_1-1}w_d^{j_1-1}\cdots w_d^{j_1-1}p_d^{j_1-1}}_{N^{[j_1]}_{(i_1+1,d)}}.$$

Then by (22) and the observation that $|w_{i_1+1}^{j_1}| < N^{[j_1]}_{(i_1+1,i_1+1)}/\delta_{j_1}$ for all sufficiently large j_1, we have

$$\begin{aligned}
|w_{i_1}^{j_1}| &< \delta_{j_2}\cdot N^{[j_1]}_{(i_1+1,i_1)} < N^{[j_1]}_{(i_1+1,i_1)}/\delta_{j_1} < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,i_1-1)} < N^{[j_1]}_{(i_1+1,i_1-1)}/\delta_{j_1}\\
&< \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,i_1-2)} < N^{[j_1]}_{(i_1+1,i_1-2)}/\delta_{j_1} < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,i_1-3)} < N^{[j_1]}_{(i_1+1,i_1-3)}/\delta_{j_1}\\
&< \cdots < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,1)} < N^{[j_1]}_{(i_1+1,1)}/\delta_{j_1} < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,d)} < N^{[j_1]}_{(i_1+1,d)}/\delta_{j_1}\\
&< \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,d-1)} < N^{[j_1]}_{(i_1+1,d-1)}/\delta_{j_1} < \cdots < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,i_1+2)}\\
&< N^{[j_1]}_{(i_1+1,i_1+2)}/\delta_{j_1} < \delta_{j_1}\cdot N^{[j_1]}_{(i_1+1,i_1+1)} < |w_{i_1+1}^{j_1}| < N^{[j_1]}_{(i_1+1,i_1+1)}/\delta_{j_1}.
\end{aligned}$$

Thus there are four possibilities:

(i) $|w_{i_1}^{j_1}| \le n < N^{[j_1]}_{(i_1+1,i_1)}$;

(ii) $N^{[j_1]}_{(i_1+1,i_1)} \le n < N^{[j_1]}_{(i_1+1,i_1-1)}$ (indices taken modulo d);

(iii) $N^{[j_1]}_{(i_1+1,i_1-1)} \le n < N^{[j_1]}_{(i_1+1,i_1+1)}$ and there exists $i_2 \in \mathcal{A}\setminus\{i_1, i_1+1\}$ such that

$$N^{[j_1]}_{(i_1+1,i_2)} \le n < N^{[j_1]}_{(i_1+1,i_2-1)} \qquad \text{(indices taken modulo } d\text{);}$$

(iv) $N^{[j_1]}_{(i_1+1,i_1+1)} \le n < |w_{i_1+1}^{j_1}|$ (by condition (29)).

In case (i), there are no words of length n in $w_{i_1+1}^{j_1} w_{i_1+1}^{j_1}$ that were not previously counted (all blocks in its decomposition are of length larger than n). In this case

$$P_X(n) \leq (d-1)\delta_{j_1} n + |w_{i_1}^{j_1}| + dn. \tag{30}$$

In particular, since $|w_{i_1}^{j_1}| < \delta_{j_1} N_{(i_1+1,i_1)}^{[j_1]}$, we are in case (i) when $n = \lfloor |w_{i_1}^{j_1}|/\delta_{j_1} \rfloor$ and so (30) holds. This implies that

$$P_X\left(\left\lfloor \frac{|w_{i_1}^{j_1}|}{\delta_{j_1}} \right\rfloor\right) \leq (d + d\delta_{j_1}) \cdot \left\lfloor \frac{|w_{i_1}^{j_1}|}{\delta_{j_1}} \right\rfloor.$$

This situation arises infinitely often (once for each δ_j), and since $\delta_j \xrightarrow{j\to\infty} 0$,

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} \leq d.$$

Combining this with the fact that (X, σ) has at least d distinct nonatomic ergodic measures and applying Theorem 3.1, we find that

$$\liminf_{n\to\infty} \frac{P_X(n)}{n} = d.$$

In particular, this implies that there are exactly d ergodic measures.

In case (ii), we have

$$N_{(i_1+1,i_1)}^{[j_1]} \leq n < N_{(i_1+1,i_1-1)}^{[j_1]}$$

and $n < N_{(i_1+1,k)}^{[j_1]}$ for all $k \in \mathcal{A} \setminus \{i_1\}$. In this case, the only new words of length n that arise in $w_{i_1+1}^{j_1} w_{i_1+1}^{j_1}$ are the $n - N_{(i_1+1,i_1)}^{[j_1]}$ transition words that arise in

$$\underbrace{s_{i_1-1}^{j_1} w_{i_1-1}^{j_1} \cdots w_{i_1-1}^{j_1} p_{i_1-1}^{j_1}}_{N_{(i_1+1,i_1-1)}^{[j_1]}} \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N_{(i_1+1,i_1)}^{[j_1]}} \underbrace{s_{i_1+1}^{j_1-1} w_{i_1+1}^{j_1-1} \cdots w_{i_1+1}^{j_1-1} p_{i_1+1}^{j_1-1}}_{N_{(i_1+1,i_1+1)}^{[j_1]}},$$

where a word is a transition word if it completely contains the middle block (all other blocks have length larger than n and so contribute no new words). Thus, in case (ii),

$$P_X(n) \leq dn + 2d\delta_{j_1} n + (n - N_{(i_1+1,i_1)}^{[j_1]}) \leq (d+1)n + 2d\delta_{j_1}. \tag{31}$$

In case (iii),

$$n \geq N_{(i_1+1,i_2)}^{[j_1]} > \delta_{j_1} \cdot N_{(i_1+1,i_2+1)}^{[j_1]} > \delta_{j_1} \cdot N_{(i_1+1,i_2+2)}^{[j_1]} > \cdots > \delta_{j_1} \cdot N_{(i_1+1,i_1)}^{[j_1]} \tag{32}$$

and

$$n < N_{(i_1+1,i_2-1)}^{[j_1]} < N_{(i_1+1,i_2-2)}^{[j_1]} < \cdots < N_{(i_1+1,i_1+1)}^{[j_1]}$$

by (19)–(22). Therefore the only new words of length n that arise in $w_{i_1+1}^{j_1} w_{i_1+1}^{j_1}$ are the transition words that arise in

$$\underbrace{s_{i_2-1}^{j_1} w_{i_2-1}^{j_1} \cdots w_{i_2-1}^{j_1} p_{i_2-1}^{j_1}}_{N_{(i_1+1,i_2-1)}^{[j_1]}} \underbrace{s_{i_2}^{j_1} w_{i_2}^{j_1} \cdots w_{i_2}^{j_1} p_{i_2}^{j_1}}_{N_{(i_1+1,i_2)}^{[j_1]}} \cdots \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N_{(i_1+1,i_1)}^{[j_1]}} \underbrace{s_{i_1+1}^{j_1-1} w_{i_1+1}^{j_1-1} \cdots w_{i_1+1}^{j_1-1} p_{i_1+1}^{j_1-1}}_{N_{(i_1+1,i_1+1)}^{[j_1]}}.$$

That is, this word decomposes into blocks such that the first and last have length larger than n; the transition words are those that fully contain one of the blocks of length smaller than n. There are at most

$$(n - N_{(i_1+1,i_2)}^{[j_1]}) + N_{(i_1+1,i_2+1)}^{[j_1]} + N_{(i_1+1,i_2+2)}^{[j_1]} + \cdots + N_{(i_1+1,i_1-1)}^{[j_1]} + N_{(i_1+1,i_1)}^{[j_1]}$$

such blocks. By (32), this is at most $n - N_{(i_1+1,i_2)}^{[j_1]} + d\delta_{j_1} n$. So in case (iii),

$$P_X(n) \le dn + 2d\delta_{j_1} n + n - N_{(i_1+1,i_2)}^{[j_1]} + d\delta_{j_1} n \le (d+1)n + 3d\delta_{j_1} n. \tag{33}$$

Finally, in case (iv), the only new words are the transition words that occur in

$$\underbrace{s_{i_1+1}^{j_1-1} w_{i_1+1}^{j_1-1} \cdots w_{i_1+1}^{j_1-1} p_{i_1+1}^{j_1-1}}_{N_{(i_1+1,i_1+1)}^{[j_1]}} \underbrace{s_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} \cdots w_{i_1+2}^{j_1-1} p_{i_1+2}^{j_1-1}}_{N_{(i_1+1,i_1+2)}^{[j_1]}} \cdots \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N_{(i_1+1,i_1)}^{[j_1]}} \underbrace{s_{i_1+1}^{j_1-1} w_{i_1+1}^{j_1-1} \cdots w_{i_1+1}^{j_1-1} p_{i_1+1}^{j_1-1}}_{N_{(i_1+1,i_1+1)}^{[j_1]}} \underbrace{s_{i_1+2}^{j_1-1} w_{i_1+2}^{j_1-1} \cdots w_{i_1+2}^{j_1-1} p_{i_1+2}^{j_1-1}}_{N_{(i_1+1,i_1+2)}^{[j_1]}} \cdots \underbrace{s_{i_1}^{j_1} w_{i_1}^{j_1} \cdots w_{i_1}^{j_1} p_{i_1}^{j_1}}_{N_{(i_1+1,i_1)}^{[j_1]}},$$

where a word is a transition word if it completely contains any of the blocks of length smaller than $N_{(i_1+1,i_1+1)}^{[j_1]}$. However, by (19)–(22), we have

$$\delta_{j_1} n \ge \delta_{j_1} N_{(i_1+1,i_1+1)}^{[j_1]} > N_{(i_1+1,k)}^{[j_1]}$$

for all $k \in \mathcal{A} \setminus \{i_1 + 1\}$, and so there are at most $n + d\delta_{j_1} n$ such words. Thus in case (iv),

$$P_X(n) \le dn + 2d\delta_{j_1} n + n + d\delta_{j_1} n = (d+1)n + 3d\delta_{j_1} n. \tag{34}$$

It follows from (30), (31), (33), and (34) that

$$\limsup_{n\to\infty} \frac{P_X(n)}{n} \le d + 1,$$

and therefore the lim sup is equal to $d + 1$ by Theorem 3.1. □

We end with several constructions showing various senses in which our results cannot be improved. We first review some standard facts about Sturmian shifts. A *Sturmian shift* (Y, σ) is a minimal subshift of $\{0, 1\}^{\mathbb{Z}}$ whose complexity function satisfies $P_Y(n) = n+1$ for all $n \in \mathbb{N}$. Any Sturmian shift is uniquely ergodic, and for any $\alpha \in (0, 1) \setminus \mathbb{Q}$ there exists a Sturmian shift (Y_α, σ) whose unique invariant probability measure μ satisfies $\mu([0]) = \alpha$. In particular, there are uncountably many distinct Sturmian shifts.

We first show that the technical condition (that there exists a generic measure μ and a generic point x_μ such that the orbit closure of x_μ is not uniquely ergodic) cannot be dropped from the second statement in Theorem 3.1:

Proposition 5.1. *For $d \geq 1$, there exists a subshift (X, σ) which has precisely d ergodic measures and zero nonergodic generic measures, and whose complexity function satisfies $P_X(n) = dn + d$ for all $n \in \mathbb{N}$. This subshift has the property that every $x \in X$ is generic for some ergodic measure and the orbit closure of any point is uniquely ergodic.*

Proof. Fix $d \in \mathbb{N}$ and fix a Sturmian shift (Y, σ) on the alphabet $\{0, 1\}$. Let $\mathcal{A} := \{0_1, 1_1, 0_2, 1_2, \ldots, 0_d, 1_d\}$ and for $1 \leq i \leq d$ let $Y_i \subset \mathcal{A}^{\mathbb{Z}}$ be the image of (Y, σ) under the 1-block code that sends $0 \mapsto 0_i$ and $1 \mapsto 1_i$. Let

$$X := \bigcup_{i=1}^{d} Y_i \subset \mathcal{A}^{\mathbb{Z}}$$

and observe that X is closed and σ-invariant. Moreover, we have $P_X(n) = dn + d$ for all $n \in \mathbb{N}$. Each subshift $Y_i \subset X$ supports a unique ergodic measure, and so there are at least d ergodic measures for (X, σ). Conversely, for each $x \in X$ there exists $1 \leq i \leq d$ such that $x \in Y_i$. Since Y_i is uniquely ergodic, x is generic for the (unique) ergodic measure supported on Y_i. Thus there can be no other measures that have a generic point. □

Finally, we show that the assumption of linear growth in Theorem 1.1 is optimal, in the sense that there is no analog of Theorem 1.1 with an assumption of a superlinear growth rate and conclusion that the set of ergodic measures is finite for all subshifts whose complexity function grows at most at that rate.

Proposition 5.2. *Let $(p_n)_{n=1}^{\infty}$ be a sequence of real numbers such that*

$$\liminf_{n\to\infty} \frac{p_n}{n} = \infty.$$

Then there exists a subshift (X, σ) which has infinitely many nonatomic ergodic measures and satisfies $P_X(n) \leq p_n$ for all but finitely many n.

Proof. For each $n \in \mathbb{N}$, there exists a set $\mathcal{F}_n \subset \{0, 1\}^n$ such that $|\mathcal{F}_n| = n + 1$ and $\mathcal{L}_n(Y_\alpha) = \mathcal{F}_n$ for uncountably many $\alpha \in (0, 1)$. For $N \leq n$, let $\mathcal{X}_N(\mathcal{F}_n)$ be the set of words of length N that arise as a subword of a word in $\mathcal{F}_n$. Clearly if $\mathcal{L}_n(Y_\alpha) = \mathcal{F}_n$ then $\mathcal{L}_N(Y_\alpha) = \mathcal{X}_N(\mathcal{F}_n)$. Let $\mathcal{G}_1 \subset \{0, 1\}$ be such that $\mathcal{G}_1 = \mathcal{X}_1(\mathcal{F}_n)$ for infinitely many $n \in \mathbb{N}$. Inductively, we assume that we have defined $\mathcal{G}_i \subset \{0, 1\}^i$ for all $1 \leq i < j$ such that:

(i) For all $1 \le j_1 < j_2 < j$ we have $\mathcal{G}_{j_1} = \mathcal{X}_{j_1}(\mathcal{G}_{j_2})$.
(ii) There are infinitely many n for which $\mathcal{G}_{j-1} = \mathcal{X}_{j-1}(\mathcal{F}_n)$.

We then choose $\mathcal{G}_j \subset \{0,1\}^j$ such that among those n for which $\mathcal{G}_{j-1} = \mathcal{X}_{j-1}(\mathcal{F}_n)$, there are infinitely many n for which $\mathcal{G}_j = \mathcal{X}_j(\mathcal{F}_n)$. In this way, we obtain an infinite sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ such that if $1 \le j_1 < j_2$, then $\mathcal{G}_{j_1} = \mathcal{X}_{j_1}(\mathcal{G}_{j_2})$ and there are uncountably many $\alpha \in (0,1)$ for which $\mathcal{L}_{j_2}(Y_\alpha) = \mathcal{G}_{j_2}$.

For each $n \in \mathbb{N}$, set

$$A_n := \{\alpha \in (0,1) : \mathcal{L}_n(Y_\alpha) = \mathcal{G}_n\}.$$

Then by construction, A_n is uncountable for all $n \in \mathbb{N}$,

$$A_1 \supseteq A_2 \supseteq \cdots,$$

and $A_n \neq A_{n+1}$ for infinitely many $n \in \mathbb{N}$. (If not, there exist distinct $\alpha_1, \alpha_2 \in \bigcap A_n$, and so $\mathcal{L}_n(Y_{\alpha_1}) = \mathcal{L}_n(Y_{\alpha_2})$ for all n, contradicting the fact that the frequency with which the letter 0 occurs as a subword of any word in $\mathcal{L}_n(Y_{\alpha_i})$ tends to α_i for $i = 1, 2$.)

We now construct the subshift. Find $N_1 \in \mathbb{N}$ such that $p_n > 2n + 2$ for all $n \ge N_1$. Choose the smallest $M_1 \ge N_1$ for which $A_{M_1+1} \neq A_{M_1}$ and let $\alpha_1 \in A_{M_1} \setminus A_{M_1+1}$. Set $X_1 := Y_{\alpha_1}$ and observe that $P_{X_1}(n) = n + 1$ for all n. Now find $N_2 \in \mathbb{N}$ such that $p_n > 3n + 3$ for all $n \ge N_2$. Find the smallest $M_2 \ge N_2$ for which $A_{M_2+1} \neq A_{M_2}$ and let $\alpha_2 \in A_{M_2} \setminus A_{M_2+1}$. Set $X_2 := Y_{\alpha_1} \cup Y_{\alpha_2}$. Then, by construction, $\alpha_2 \in A_{M_1}$ and so $\mathcal{L}_{M_1}(Y_{\alpha_2}) = \mathcal{L}_{M_1}(Y_{\alpha_1})$, but since $\alpha_2 \in A_{M_2} \subset A_{M_1+1}$ we know that $\mathcal{L}_{M_1+1}(Y_{\alpha_2}) \neq \mathcal{L}_{M_1+1}(Y_{\alpha_1})$. Consequently, $P_{X_2}(n) = n+1$ for all $n \le M_1$, and $n+1 < P_{X_2}(n) \le 2n+2$ for all $n > M_1$.

Now recursively suppose we have chosen integers $M_1 < \cdots < M_i$ such that for each $1 \le k \le i$ we have $p_n > (k+1)n + (k+1)$ for all $n \ge M_k$ and moreover $A_{M_k+1} \neq A_{M_k}$. Suppose further that we have chosen $\alpha_1, \dots, \alpha_i$ such that $\alpha_k \in A_{M_k} \setminus A_{M_k+1}$ for each k. Finally, define $X_i := Y_{\alpha_1} \cup \cdots \cup Y_{\alpha_i}$. Then, by construction, we have $P_{X_i}(n) \le kn + k$ for all $n \le M_k$. Find $N_{i+1} \in \mathbb{N}$ such that $p_n > (i+2)n + (i+2)$ for all $n \ge N_{i+1}$, and let M_{i+1} be the smallest integer larger than N_{i+1} for which $A_{M_{i+1}} \neq A_{M_{i+1}+1}$. Let $\alpha_{i+1} \in A_{M_{i+1}} \setminus A_{M_{i+1}+1}$. Define $X_{i+1} := Y_{\alpha_1} \cup \cdots \cup Y_{\alpha_i} \cup Y_{\alpha_{i+1}}$. Since $\alpha_{i+1} \in A_{M_{i+1}} \subset A_{M_i}$, we know that $\mathcal{L}_k(X_i) = \mathcal{L}_k(X_{i+1})$ for all $k \le M_i$, and since $\alpha_{i+1} \in A_{M_i+1}$ we know that $\mathcal{L}_{M_i+1}(X_i) \neq \mathcal{L}_{M_i+1}(X_{i+1})$. Consequently, $P_{X_{i+1}}(n) = P_{X_i}(n)$ for all $n \le M_i$, and $P_{X_{i+1}}(n) \le (i+1)n + (i+1)$ for all $n > M_i$. By construction, $p_n > (i+1)n + (i+1)$ for all $n \ge M_i$. Therefore our recursive construction continues for all $i \in \mathbb{N}$.

Thus we obtain a sequence of subshifts $X_1 \subset X_2 \subset \cdots$ such that $P_{X_i}(n) < p_n$ for all $i \in \mathbb{N}$ and all $n \ge N_1$. Setting

$$X := \overline{\bigcup_{i=1}^{\infty} X_i},$$

we see that $\mathcal{L}_n(X) = \bigcup_{i=1}^{\infty} \mathcal{L}_n(X_i)$ for all $n \in \mathbb{N}$. Therefore, $P_X(n) < p_n$ for all $n \ge N_1$. On the other hand, $Y_{\alpha_i} \subset X$ for all $i \in \mathbb{N}$ and there is an ergodic probability supported on Y_{α_i}. Since $Y_{\alpha_i} \neq Y_{\alpha_j}$ for all $i \neq j$ by construction, X has infinitely many ergodic measures. □

Acknowledgments. The second author was partially supported by NSF grant 1500670.

References

[1] Boshernitzan, M.: A unique ergodicity of minimal symbolic flows with linear block growth. J. Anal. Math. **44**, 77–96 (1984) Zbl 0602.28008 MR 0801288

[2] Boshernitzan, M.: A condition for unique ergodicity of minimal symbolic flows. Ergodic Theory Dynam. Systems **12**, 425–428 (1992) Zbl 0756.58030 MR 1182655

[3] Boshernitzan, M.: A condition for minimal interval exchange maps to be uniquely ergodic. Duke Math. J. **52**, 723–752 (1985) Zbl 0602.28009 MR 0808101

[4] Chaika, J., Masur, H.: There exists an interval exchange with a non-ergodic generic measure. J. Modern Dynam. **9**, 289–304 (2015), Zbl 1356.37059 MR 0808101

[5] Damron, M., Fickenscher, J.: On the number of ergodic measures for minimal shifts with eventually constant complexity growth. Ergodic Theory Dynam. Systems **37**, 2099–2130 (2017) Zbl 1381.37021 MR 3693122

[6] Denker, M., Grillenberger, C., Sigmund, K.: Ergodic Theory on Compact Spaces. Lecture Notes in Math. 527, Springer, Berlin (1976) Zbl 0328.28008 MR 0457675

[7] Epifanio, C., Koskas, M., Mignosi, F.: On a conjecture on bidimensional words. Theoret. Computer Sci. **299**, 123–150 (2003) Zbl 1040.68076 MR 1973149

[8] Ferenczi, S., Zamboni, L.: Languages of k-interval exchange transformations. Bull. London Math. Soc. **40**, 705–714 (2008) Zbl 1147.37008 MR 2441143

[9] Fine, N., Wilf, H.: Uniqueness theorem for periodic functions. Proc. Amer. Math. Soc. **16**, 109–114 (1965) Zbl 0131.30203 MR 0174934

[10] Katok, A.: Invariant measures of flows on orientable surfaces. Dokl. Akad. Nauk SSSR **211**, 775–778 (1973) (in Russian) Zbl 0298.28013 MR 0331438

[11] Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems. Encyclopedia Math. Appl. 54, Cambridge Univ. Press, Cambridge (1995) Zbl 0878.58019 MR 1326374

[12] Keane, M.: Interval exchange transformations. Math. Z. **141**, 25–31 (1975) Zbl 0278.28010 MR 0357739

[13] Quas, A.: Math Overflow. http://mathoverflow.net/questions/152470/getting-unique-ergodicity-from-minimality/152524#152524

[14] Veech, W.: Interval exchange transformations. J. Anal. Math. **33**, 222–272 (1978) Zbl 0455.28006 MR 0516048