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A new index theorem for monomial ideals by resolutions



Ronald G. Douglas^{a,1}, Mohammad Jabbari^b, Xiang Tang^{b,*},
Guoliang Yu^{a,c}

^a Department of Mathematics, Texas A&M University, College Station, TX, 77843, USA

^b Department of Mathematics, Washington University, St. Louis, MO, 63130, USA

^c Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, 200433, China

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ABSTRACT

We prove an index theorem for the quotient module of a monomial ideal. We obtain this result by resolving the monomial ideal by a sequence of essentially normal Hilbert modules, each of which is a direct sum of (weighted) Bergman spaces on balls.

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1. Introduction

Let $\mathbb{B}^m$ be the unit ball in the complex space $\mathbb{C}^m$, and $L_a^2(\mathbb{B}^m)$ be the Bergman space of square integrable holomorphic functions on $\mathbb{B}^m$, and A be the algebra $\mathbb{C}[z_1, \dots, z_m]$ of polynomials of m variables. The algebra A plays two roles in our study: one is that A

* Corresponding author.

E-mail addresses: rdouglas@math.tamu.edu (R.G. Douglas), jabbari@wustl.edu (M. Jabbari), xtang@math.wustl.edu (X. Tang), guoliangyu@math.tamu.edu (G. Yu).

¹ Recently passed away.

is a dense subspace of the Hilbert space $L_a^2(\mathbb{B}^m)$, the other is that A acts on $L_a^2(\mathbb{B}^m)$ by Toeplitz operators.

In this article we are interested in an ideal I of A generated by monomials. Let $\bar{I}$ be the closure of I in $L_a^2(\mathbb{B}^m)$, and Q_I be the quotient Hilbert space $L_a^2(\mathbb{B}^m)/\bar{I}$. The first author proved in [5, Theorem 2.1] that the Toeplitz operators T_{z_i} , $i = 1, \dots, m$, on $\bar{I}$, and the quotient Q_I are essentially normal,² i.e. the following commutators are compact

$$[T_{z_i}|_{\bar{I}}, (T_{z_j}|_{\bar{I}})^*] \in \mathcal{K}(\bar{I}), \text{ and } [T_{z_i}|_{Q_I}, (T_{z_j}|_{Q_I})^*] \in \mathcal{K}(Q_I), \quad i, j = 1, \dots, m.$$

Let $\mathfrak{T}(Q_I)$ be the unital C^* -algebra generated by the Toeplitz operators $T_{z_i}|_{Q_I}$, $i = 1, \dots, m$. The above essential normality property of the Toeplitz operators gives the following extension sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{T}(Q_I) \longrightarrow C(\sigma_I^e) \longrightarrow 0,$$

where σ_I^e is the essential spectrum of the quotient tuple $(T_{z_1}|_{Q_I}, \dots, T_{z_m}|_{Q_I})$ on Q_I , and $\mathcal{K}$ is the algebra of compact operators. By the Gelfand–Naimark theorem, σ_I^e is the spectrum space of the commutative C^* -algebra $\mathfrak{T}(Q_I)/\mathcal{K}$. Abusing the notion, we will sometimes refer to σ_I^e as the essential spectrum space of the algebra $\mathfrak{T}(Q_I)$. The index problem we want to answer in this article is to provide a good description of the above K -homology class.

The main difficulty in answering the question above is that the ideal I in general fails to be radical. This makes the geometric ideas introduced in [8] and [10] impossible to apply directly. The seed of the main idea in this article is the following observation discussed in [8, Section 5.2]. For $m = 2$, consider the ideal $I = \langle z_1^2 \rangle \subset A = \mathbb{C}[z_1, z_2]$. The quotient Q_I can be written as the sum of two space

$$L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D}),$$

where $\mathbb{D}$ is the unit disk inside the complex plane $\mathbb{C}$, and $L_{a,1}^2(-)$ (and $L_{a,2}^2(-)$) is the weighted Bergman space with respect to the weight defined by the defining function $1 - |z|^2$ (and $(1 - |z|^2)^2$). Define the restriction map $R_I : L_a^2(\mathbb{B}^2) \rightarrow L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D})$ by

$$R_I(f) := (f|_{z_1=0}, \frac{\partial f}{\partial z_1}|_{z_1=0}).$$

It is not hard to introduce a Hilbert $A = \mathbb{C}[z_1, z_2]$ -module structure on $L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D})$ so that the following exact sequence of Hilbert modules holds,

$$0 \longrightarrow \bar{I} \longrightarrow L_a^2(\mathbb{B}^2) \longrightarrow L_{a,1}^2(\mathbb{D}) \oplus L_{a,2}^2(\mathbb{D}) \longrightarrow 0.$$

² Arveson [1, Corollary 2.2] proved the similar result on the Drury–Arveson space.

It is well known that the Toeplitz operators on the (weighted) Bergman space $L^2_{a,1}(\mathbb{D})$ and $L^2_{a,2}(\mathbb{D})$ are essentially normal. In [8, Section 5.2], we observe that one can conclude the essential normality property of the Hilbert module $\bar{I}$ and Q_I from the exact sequence above. Furthermore, it is not hard to prove that the quotient Hilbert module Q_I is isomorphic to the module $L^2_{a,1}(\mathbb{D}) \oplus L^2_{a,2}(\mathbb{D})$, and the extension class $[\mathfrak{T}(Q_I)]$ associated to Q_I can be identified with the one associated to $L^2_{a,1}(\mathbb{D}) \oplus L^2_{a,2}(\mathbb{D})$.

In this article, we extend the example above to an arbitrary ideal I of $L^2_a(\mathbb{B}^m)$ generated by monomials. By considering an ideal generated by two monomials, we realize that it is more natural to work with long exact sequences of Hilbert modules, instead of short ones. The following is our main theorem.

Theorem 1.1. *Let I be an ideal of $\mathbb{C}[z_1, \dots, z_m]$ generated by monomials, and $\bar{I}$ be its closure in the Bergman space $L^2_a(\mathbb{B}^m)$. There are Bergman space like Hilbert A -modules $\mathcal{A}_0 = L^2_a(\mathbb{B}^m), \mathcal{A}_1, \dots, \mathcal{A}_k$ together with bounded A -module morphisms $\Psi_i : \mathcal{A}_i \rightarrow \mathcal{A}_{i+1}$, $i = 0, \dots, k-1$ such that the following exact sequence of Hilbert modules holds*

$$0 \longrightarrow \bar{I} \hookrightarrow L^2_a(\mathbb{B}^m) \xrightarrow{\Psi_0} \mathcal{A}_1 \xrightarrow{\Psi_1} \dots \xrightarrow{\Psi_{k-1}} \mathcal{A}_k \longrightarrow 0.$$

For the example above $I = \langle z_1^2 \rangle$, the number k in Theorem 1.1 is 1, and the Bergman space like Hilbert A -module $\mathcal{A}_1$ is the sum $L^2_{a,1}(\mathbb{D}) \oplus L^2_{a,2}(\mathbb{D})$. We will explain in Section 2.3, Proposition 2.5, that the Hilbert A -module $\mathcal{A}_i$, $i = 1, \dots, k$, has a similar geometric structure as a direct sum of (weighted) Bergman spaces on lower dimensional balls.

As a corollary of Theorem 1.1, we obtain a new proof of the essential normality property of the ideal $\bar{I}$ and its quotient Q_I . Moreover, Theorem 1.1 allows us to identify the extension class associated to $\mathfrak{T}(Q_I)$ geometrically. We compute it in the following theorem.

Theorem 1.2. (Theorem 3.10) *Let $\mathfrak{T}(\mathcal{A}_i)$ be the unital C^* -algebra generated by Toeplitz operators on $\mathcal{A}_i$, and σ_i^e be the associated essential spectrum space. In $K_1(\sigma_1^e \cup \dots \cup \sigma_k^e)$, the following equation holds,*

$$[\mathfrak{T}(Q_I)] = [\mathfrak{T}(\mathcal{A}_1)] - [\mathfrak{T}(\mathcal{A}_2)] + \dots + (-1)^{k-1} [\mathfrak{T}(\mathcal{A}_k)].$$

As it is explained in Section 2.3, every algebra $\mathfrak{T}(\mathcal{A}_i)$, $i = 1, \dots, k$, can be identified as the algebra of Toeplitz operators on square integrable holomorphic sections of a hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}^m$. This geometric interpretation allows us to use the ideas developed in [8] and [10] to study the “geometry” of the algebra $\mathfrak{T}(Q_I)$.

The zero set of a monomial ideal I defines a (singular) projective variety. We think that the K -homology class associated to the extension of $\mathfrak{T}(Q_I)$ is closely related to the fundamental class studied by Baum, Fulton, MacPherson in [2,3]. In particular, in the

near future we plan to study the relation between the Euler characteristic associated to the alternating sum in Theorem 1.2 and the arithmetic genus of the associated projective variety.

We remark that although our Theorem 1.1 is stated for the closure of a monomial ideal inside the Bergman space $L_a^2(\mathbb{B}^m)$, the same proof also works for the closure of any monomial ideal inside more general spaces, e.g. weighted Bergman spaces $L_{a,s}^2(\mathbb{B}^m)$ and the Drury–Arveson space.

Our result in Theorem 1.1 can be viewed as a “resolution” ([12]) of the ideal $\bar{I}$ by essentially normal Hilbert modules $\mathcal{A}_0, \dots, \mathcal{A}_k$. Such an idea of “resolution” goes back to the first author’s work in [5], and the study in this article should not be limited to monomial ideals. In Section 4.3, we explain that a similar construction also works for a more general ideal in $\mathbb{C}[z_1, z_2, z_3]$. In the near future, we hope to report about a systematic study on extending ideas from Theorem 1.1 to more general ideals I with the help of results in [9].

This article is organized as follows. In Section 2, we will introduce the main building block in our construction. In Section 3, we will construct the Hilbert modules $\mathcal{A}_i$ and morphisms Ψ_i , $i = 0, \dots, k$ in Theorem 1.1, and prove the main Theorem. Our proof is inspired by the corresponding algebraic study on monomial ideals [4], [11]. We will end our paper by exhibiting our constructions on concrete examples in Section 4. In particular, a non-monomial ideal example is discussed in Section 4.3.

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2. Generalized Bergman space and the associated Toeplitz operators

In this section, we introduce and study a building block in the construction of the resolution in Theorem 1.1.

2.1. Notations

We start with fixing some notations. For a positive integer q , we use the symbol $S_q(m)$ to denote the set of q -shuffles of the set $[m] = \{1, \dots, m\}$, i.e.

$$S_q(m) := \{j := (j^1, \dots, j^q) | 1 \leq j^1 < j^2 < \dots < j^q \leq m\}.$$

Let $\mathbb{N}$ be the set of all nonnegative integers. For any $\mathbf{i} = (i^1, \dots, i^q) \in \mathbb{N}^q$, we use $|\mathbf{i}|$ to denote the sum $i^1 + \dots + i^q$.

Fix $\mathbf{j} = (j^1, \dots, j^q) \in S_q(m)$, and $\mathbf{b} = (b^1, \dots, b^q) \in \mathbb{N}^q$. We associate a subset $\mathcal{B}_{\mathbf{j}}^{\mathbf{b}} \subseteq \mathbb{N}^m$ defined by

$$\mathcal{B}_j^b := \{(n^1, \dots, n^m) \in \mathbb{N}^m \mid n^{j^k} \leq b^k, \text{ for } k = 1, \dots, q\}.$$

We call $\mathcal{B}_j^b$ the box associated to j and b .

In the following, we introduce a Hilbert space $\mathcal{H}_j^b$ as a closed subspace of $L_a^2(\mathbb{B}^m)$.

On $\mathbb{B}^m$, consider the weighted Bergman space $L_{a,s}^2(\mathbb{B}^m)$ defined by square integrable holomorphic functions on $\mathbb{B}^m$ with respect to the norm

$$\|f\|_{L_{a,s}^2}^2 = \int_{\mathbb{B}^m} |f(z)|^2 (1 - |z|^2)^s \frac{(m+s)!}{m!s!} dV(z),$$

where dV is the normalized Lebesgue measure on $\mathbb{B}^m$. This Hilbert space has the following standard orthonormal basis

$$\left\{ z^n := \frac{z_1^{n_1} \cdots z_m^{n_m}}{\sqrt{\omega_s(\mathbf{n})}} \mid \mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m \right\} \quad (1)$$

where $\omega_s(\mathbf{n}) := \frac{n_1! \cdots n_m! (m+s)!}{(n_1 + \cdots + n_m + s + m)!}$.

Definition 2.1. The Hilbert space $\mathcal{H}_j^b$ is a closed subspace of $L_a^2(\mathbb{B}^m)$ consisting of functions $f \in L_a^2(\mathbb{B}^m)$ whose expansion $\sum_{\mathbf{n}} f_{\mathbf{n}} z^{\mathbf{n}}$ with respect to the orthonormal basis $z^{\mathbf{n}}$ satisfies

$$f_{\mathbf{n}} = 0, \text{ for } \mathbf{n} \notin \mathcal{B}_j^b.$$

We have the following orthonormal basis for the Hilbert space $\mathcal{H}_j^b$,

$$z^{\mathbf{n}} := \frac{z_1^{n_1} \cdots z_m^{n_m}}{\sqrt{\omega_0(\mathbf{n})}},$$

where $0 \leq n_{j^1} \leq b^1, \dots, 0 \leq n_{j^q} \leq b^q$, and the index n_i for $i \notin j$ belongs to $\mathbb{N}$. In terms of this basis, an element $X \in \mathcal{H}_j^b$ can be written as

$$X = \sum_{0 \leq n^{j^1} \leq b^1, \dots, 0 \leq n^{j^q} \leq b^q} X_{n^1 \dots n^m} z^{\mathbf{n}}. \quad (2)$$

In the following, we define a representation of the polynomial algebra $A = \mathbb{C}[z_1, \dots, z_m]$ on the Hilbert space $\mathcal{H}_j^b$. Let $P_{\mathcal{H}_j^b}$ be the orthogonal projection of $L_a^2(\mathbb{B}^m)$ onto $\mathcal{H}_j^b$. For $p = 1, \dots, m$, the operator $T_{z_p}^{j,b}$ on $\mathcal{H}_j^b$ is defined to be the operator $P_{\mathcal{H}_j^b} T_{z_p}|_{\mathcal{H}_j^b}$. More explicitly, the operator $T_{z_p}^{j,b}$ on $\mathcal{H}_j^b$ is of the following form,

$$T_{z_p}^{j,b}(z^{\mathbf{n}}) = \begin{cases} z_p z^{\mathbf{n}}, & \text{if } (n^1, \dots, n^p + 1, \dots, n^m) \in \mathcal{B}_j^b, \\ 0, & \text{otherwise.} \end{cases}$$

Since the orthogonal complement of the closed subspace $\mathcal{H}_j^b \subset L_a^2(\mathbb{B}^m)$ is invariant under the operator T_{z_p} , $p = 1, \dots, m$, $(T_{z_1}^{j,b}, \dots, T_{z_m}^{j,b})$ is a commuting tuple of bounded linear operators on $\mathcal{H}_j^b$. Hence we have reached the following lemma.

Lemma 2.2. *The above operators $\{T_{z_i}^{j,b}\}_{i=1}^m$ define a bounded representation of the algebra A on the space $\mathcal{H}_j^b$.*

2.2. Essential normality property

In this subsection, we prove the following property of the Hilbert A -module $\mathcal{H}_j^b$.

Proposition 2.3. *The following commutators are compact.*

$$[T_{z_s}^{j,b*}, T_{z_t}^{j,b}] \in \mathcal{K}(\mathcal{H}_j^b), \quad \forall s, t = 1, \dots, m.$$

Therefore, $\mathcal{H}_j^b$ is an essentially normal Hilbert A -module.

Since the Hilbert module $\mathcal{H}_j^b$ can be identified with the quotient module $L_a^2(\mathbb{B}^m) \ominus \langle z_{j^1}^{b^1+1}, \dots, z_{j^q}^{b^q+1} \rangle$, the assertion of Proposition 2.3 follows from [1] and [6]. We outline its proof here for completeness.

Proof. Let P_j^b be the orthogonal projection from $L_a^2(\mathbb{B}^m)$ onto the closed subspace $\mathcal{H}_j^b$. As $T_{z_s}^{j,b}$ can be written as $P_j^b T_{z_s} P_j^b$, it suffices to prove the commutator $[P_j^b, T_{z_s}]$ is compact.

For $\mathbf{n} \in \mathcal{B}_j^b$, $P_j^b T_{z_s}(z^{\mathbf{n}})$ is computed as follows,

$$P_j^b T_{z_s}(z^{\mathbf{n}}) = \begin{cases} \sqrt{\frac{\omega_0(n^1 \dots (n^s+1) \dots n^m)}{\omega_0(n^1 \dots n^m)}} z^{n^1 \dots (n^s+1) \dots n^m}, & \text{if } (n^1 \dots (n^s+1) \dots n^m) \in \mathcal{B}_j^b, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, $T_{z_s} P_j^b(z^{\mathbf{n}})$ is computed as follows,

$$T_{z_s} P_j^b(z^{\mathbf{n}}) = \begin{cases} \sqrt{\frac{\omega_0(n^1 \dots (n^s+1) \dots n^m)}{\omega_0(n^1 \dots n^m)}} z^{n^1 \dots (n^s+1) \dots n^m}, & \text{if } (n^1 \dots n^s \dots n^m) \in \mathcal{B}_j^b, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $[P_j^b, T_{z_s}](z^{\mathbf{n}})$ is computed as follows,

$$[P_j^b, T_{z_s}](z^{\mathbf{n}}) := \begin{cases} -\sqrt{\frac{\omega_0(n^1 \dots (b^k+1) \dots n^m)}{\omega_0(n^1 \dots b^k \dots n^m)}} z^{n^1 \dots (n^s+1) \dots n^m}, & \text{if } (n^1 \dots n^s \dots n^m) \in \mathcal{B}_j^b, \\ & \text{and } s = j^k, \text{ and } n^s = b^k \\ & \text{for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

We observe that the weight $\frac{\omega_0(n^1 \dots (b^k+1) \dots n^m)}{\omega_0(n^1 \dots b^k \dots n^m)}$ converges to zero as $\|(n^1, \dots, b^k, \dots, n^m)\| \rightarrow \infty$. From this, we can conclude that the commutator $[P_j^b, T_{z_s}]$ is compact. $\square$

Remark 2.4. It is not hard to check in the above proof of Proposition 2.3 that the commutators $[P_j^b, T_{z_s}^b]$ and therefore also $[T_{z_s}^{j,b*}, T_{z_t}^{j,b}]$ belong to the Schatten- p ideal for $p > m - q$.

2.3. Geometry of the Hilbert space $\mathcal{H}_j^b$

In this subsection, we discuss briefly the geometry of the Hilbert space $\mathcal{H}_j^b$ introduced in Section 2.1.

Let $\mathbb{B}_j$ be the subset of $\mathbb{B}^m$ cut by the hyperplanes $H_{j^i} := \{z_{j^i} = 0\}$, $i = 1, \dots, q$, i.e.

$$\mathbb{B}_j := \{(z_1, \dots, z_m) \in \mathbb{B}^m \mid z_{j^1} = \dots = z_{j^q} = 0\}.$$

Observe that $\mathbb{B}_j$ is the unit ball inside the subspace

$$H_j := \{(z_1, \dots, z_m) \mid z_{j^1} = \dots = z_{j^q} = 0\},$$

which is isomorphic to the standard complex space $\mathbb{C}^{m-q}$.

For $\mathbf{i} = (i^1, \dots, i^q) \in \mathbb{N}^q$, we consider the following weighted Bergman space $\tilde{\mathcal{H}}_{j,i}^b := L^2_{a,q+|\mathbf{i}|}(\mathbb{B}_j)$. And given any $\mathbf{b} \in \mathbb{N}^q$, we consider the following Hilbert space

$$\tilde{\mathcal{H}}_j^b := \bigoplus_{\mathbf{i} \in \mathbb{N}^q, i^1 \leq b^1, \dots, i^q \leq b^q} \tilde{\mathcal{H}}_{j,i}^b := \bigoplus_{\mathbf{i} \in \mathbb{N}^q, i^1 \leq b^1, \dots, i^q \leq b^q} L^2_{a,q+|\mathbf{i}|}(\mathbb{B}_j).$$

Let $\mathcal{H}_{j,i}^b$ be the closed subspace of $\mathcal{H}_j^b$ spanned by basis vectors z^n for $n_{j^1} = i^1, \dots, n_{j^q} = i^q$. It is not hard to check that $\mathcal{H}_j^b$ is a direct sum of closed subspaces as follows,

$$\mathcal{H}_j^b = \bigoplus_{\mathbf{i} \in \mathbb{N}^q, i^1 \leq b^1, \dots, i^q \leq b^q} \mathcal{H}_{j,i}^b.$$

We define a map $R_j^b : \mathcal{H}_j^b \rightarrow \tilde{\mathcal{H}}_j^b$ by

$$R_j^b(f) := \sum_{i^1 \leq b^1, \dots, i^q \leq b^q} R_{j,i}^b(f),$$

where $R_{j,i}^b : \mathcal{H}_{j,i}^b \rightarrow \tilde{\mathcal{H}}_{j,i}^b$ is defined as follows,

$$R_{j,i}^b(f) := \frac{\partial^{i^1 + \dots + i^q} f}{\partial z_{j^1}^{i^1} \dots \partial z_{j^q}^{i^q}} \Big|_{\mathbb{B}_j}, \quad \forall f \in \mathcal{H}_{j,i}^b.$$

We have the following property of the map R_j^b .

Proposition 2.5. *The map R_j^b is an isomorphism of Hilbert spaces.*

Proof. It suffices to prove that the operator $R_{j,i}^b : \mathcal{H}_{j,i}^b \rightarrow \tilde{\mathcal{H}}_{j,i}^b$ is an isomorphism of Hilbert spaces.

For $z^n \in \mathcal{H}_{j,i}^b$,

$$R_{j,i}^b(z^n) = \frac{i^1! \cdots i^q!}{\sqrt{\omega_0(\mathbf{n})}} z_1^{n^1} \cdots \hat{z}_{j^1}^{i^1} \cdots \hat{z}_{j^q}^{i^q} \cdots z_m^{n^m},$$

where by $\hat{z}_{j^s}^{i^s}$, $s = 1, \dots, q$, we mean removing $\hat{z}_{j^s}^{i^s}$ from the expression.

Let $\mathbf{n}'$ be the tuple in $\mathbb{N}^{m-q}$ obtained from $\mathbf{n}$ with $n^{j^1}, \dots, n^{j^q}$ removed. The norm of $R_{j,i}^b(z^n)$ in $L^2_{a,q+|\mathbf{i}|}(\mathbb{B}_j)$ is

$$\frac{i^1! \cdots i^q! \sqrt{\omega_{q+|\mathbf{i}|}(\mathbf{n}')}}{\sqrt{\omega_0(\mathbf{n})}} = \sqrt{\frac{(i^1!)^2 \cdots (i^q!)^2 \omega_{q+|\mathbf{i}|}(\mathbf{n}')}{\omega_0(\mathbf{n})}} = \sqrt{\frac{i^1! \cdots i^q! (m + |\mathbf{i}|)!}{m!}}.$$

Since the above ratio is independent of the index $\mathbf{n}'$, we conclude from the above computation that $R_{j,i}^b$ and therefore R_j^b are isomorphisms of Hilbert spaces. $\square$

Remark 2.6. Proposition 2.5 suggests that our general construction is a proper generalization of the Example of ideal $\langle z_i^2 \rangle$ in the Introduction. The proof of Proposition 2.5 tells that the map R_j^b is an isomorphism but not an isometry of Hilbert spaces. One can properly adjust the definition of the map $R_{j,i}^b$ to make it and therefore R_j^b into isometry, but we have decided to take the above definition of R_j^b as it seems more natural.

We consider the trivial vector bundle $E_j^b := \mathbb{C}^{(b^1+1) \cdots (b^q+1)} \times \mathbb{B}_j$ over $\mathbb{B}_j$. The hermitian structure on E_j^b is defined as follows. We choose the standard basis of $\{e_i\}_{i^1 \leq b^1, \dots, i^q \leq b^q}$ of $\mathbb{C}^{(b^1+1) \cdots (b^q+1)}$. The Hermitian metric on E_j^b at $z \in \mathbb{B}_j$ is

$$\langle e_i, e_{i'} \rangle_z = \delta_{i,i'} (1 - |z|^2)^{q+|\mathbf{i}|}.$$

It is not hard to see that the Hilbert space $\tilde{\mathcal{H}}_j^b$ can be identified with the Bergman space of L^2 -holomorphic sections of the bundle E_j^b . We consider the Toeplitz algebra $\mathfrak{T}(E_j^b)$ generated by matrix valued Toeplitz operators on the Bergman space of L^2 -holomorphic sections. Under the isomorphism R_j^b , one can easily identify the Toeplitz algebra $\mathfrak{T}_j^b$ generated by $T_{z_i}^{j,b}$, $i = 1, \dots, m$ on $\mathcal{H}_j^b$, with the Toeplitz algebra $\mathfrak{T}(E_j^b)$ on $\tilde{\mathcal{H}}_j^b$.

3. Resolutions of monomial ideals

In this section, we present the proof of the main theorem (Theorem 1.1) of this article. In the first three subsections, we generalize the discussion in Section 2 to construct an exact sequence of Hilbert A -modules associated to k boxes in $\mathbb{N}^m$. And in Section 3.4, we apply our construction to prove Theorem 1.1.

3.1. Construction of the Hilbert $\mathbb{C}[z_1, \dots, z_m]$ -modules

Let $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$ be finitely many boxes in $\mathbb{N}^m$ as defined at the beginning of Section 2.1. We start with the following easy property of boxes in $\mathbb{N}^m$.

Lemma 3.1. *Intersections of boxes in $\mathbb{N}^m$ are again boxes.*

Proof. Let us consider two boxes $\mathcal{B}_{j_1}^{b_1}$ and $\mathcal{B}_{j_2}^{b_2}$. Let j_{12} be the union of j_1 and j_2 . Suppose that there are q_{12} elements in j_{12} . Then j_{12} can be viewed as an element of $S_{q_{12}}(m)$. We write j_{12} as $j_{12} := (j_{12}^1, \dots, j_{12}^{q_{12}})$. Define b_{12} to be an element $b_{12} := (b_{12}^1, \dots, b_{12}^{q_{12}})$ in $\mathbb{N}^{q_{12}}$ by

$$b_{12}^k := \begin{cases} \min(b_1^s, b_2^{s'}), & j_{12}^k = j_1^s = j_2^{s'}, \\ b_1^s, & j_{12}^k = j_1^s \notin j_1 \cap j_2, \\ b_2^{s'}, & j_{12}^k = j_2^{s'} \notin j_1 \cap j_2. \end{cases}$$

It is easy to check that the intersection of $\mathcal{B}_{j_1}^{b_1}$ and $\mathcal{B}_{j_2}^{b_2}$ is $\mathcal{B}_{j_{12}}^{b_{12}}$. The general case of the lemma can be proved by induction from the above proof for two boxes. $\square$

For any subset $I \subset \{1, \dots, k\}$, we use $\mathcal{B}_{j_I}^{b_I}$ to denote the following intersection,

$$\mathcal{B}_{j_I}^{b_I} := \bigcap_{i \in I} \mathcal{B}_{j_i}^{b_i}.$$

For each box $\mathcal{B}_{j_I}^{b_I}$, we consider the corresponding Hilbert $A = \mathbb{C}[z_1, \dots, z_m]$ module $\mathcal{H}_{j_I}^{b_I}$ as is introduced in Section 2. It is not hard to see that subsets of size q in $\{1, \dots, k\}$ are in 1-1 correspondence with elements in $S_q(k)$.

Given $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$, for $1 \leq q \leq k$, we define a Hilbert module $\mathcal{A}_q$ as follows.

$$\mathcal{A}_q := \bigoplus_{I \in S_q(k)} \mathcal{H}_{j_I}^{b_I}.$$

For convenience, we use $\mathcal{A}_0$ to denote the Bergman space $L_a^2(\mathbb{B}^m)$. We remark that every Hilbert space $\mathcal{A}_q$ is equipped with a Hilbert A -module structure from the corresponding A -module structure on each component $\mathcal{H}_{j_I}^{b_I}$. It follows from Proposition 2.3 that each $\mathcal{A}_q$ is an essentially normal Hilbert A -module.

3.2. Morphisms

In this subsection, we define the boundary morphism $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$ for $q = 0, \dots, k-1$. In the following, we heavily use the expression introduced in Equation (2). To explain our construction, we start with a few examples with a small number k of boxes $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$ in $\mathbb{N}^m$.

When $k = 1$, there is only one box $\mathcal{B}_j^b$. We have two Hilbert modules $\mathcal{A}_0 = L_a^2(\mathbb{B}^m)$ and $\mathcal{A}_1 = \mathcal{H}_j^b$. We define $\Psi_0 : \mathcal{A}_0 \rightarrow \mathcal{A}_1$ as follows. For $X \in \mathcal{A}_0 = L_a^2(\mathbb{B}^m)$, the map Ψ_0 maps X to $Y := \Psi_0(X) \in \mathcal{H}_j^b$ of the following form,

$$Y_{n^1 \dots n^m} := \begin{cases} X_{n^1 \dots n^m}, & \mathbf{n} \in \mathcal{B}_j^b, \\ 0, & \text{otherwise.} \end{cases}$$

When $k = 2$, there are two boxes $\mathcal{B}_{j_1}^{b_1}$ and $\mathcal{B}_{j_2}^{b_2}$. We denote their intersection by $\mathcal{B}_{j_{12}}^{b_{12}}$. For $X \in \mathcal{A}_0 = L_a^2(\mathbb{B}^m)$, $\Psi_0(X)$ is written as $Y_1 + Y_2$, where $Y_1 \in \mathcal{H}_{j_1}^{b_1}$ and $Y_2 \in \mathcal{H}_{j_2}^{b_2}$ are of the following form,

$$(Y_1)_n := \begin{cases} X_n, & \mathbf{n} \in \mathcal{B}_{j_1}^{b_1}, \\ 0, & \text{otherwise,} \end{cases} \quad (Y_2)_n := \begin{cases} X_n, & \mathbf{n} \in \mathcal{B}_{j_2}^{b_2}, \\ 0, & \text{otherwise.} \end{cases}$$

For $(X_1, X_2) \in \mathcal{H}_{j_1}^{b_1} \oplus \mathcal{H}_{j_2}^{b_2} = \mathcal{A}_1$, define $\Psi_1(X_1, X_2) \in \mathcal{A}_2$ by

$$\Psi_1(X_1, X_2)_n := \begin{cases} (X_1)_n - (X_2)_n, & \mathbf{n} \in \mathcal{B}_{j_{12}}^{b_{12}}, \\ 0, & \text{otherwise.} \end{cases}$$

For a general k , in order to define the morphism $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$, we introduce the following maps $f_{q+1}^i : S_{q+1}(k) \rightarrow S_q(k)$ for $i = 1, \dots, q+1$. An element in $S_{q+1}(k)$ is a subset I_{q+1} of $\{1, \dots, k\}$ of size $q+1$. The map $f_{q+1}^i(I_{q+1})$ is the subset of $\{1, \dots, k\}$ of size q by dropping the i -th smallest element in I_{q+1} . Define $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$ by

$$\Psi_q(X) := \sum_{I_{q+1} \in S_{q+1}(k)} Y^{I_{q+1}}, \quad Y^{I_{q+1}} \in \mathcal{H}_{j_{I_{q+1}}}^{b_{I_{q+1}}},$$

for $X = \sum_{J_k \in S_q(k)} X^{J_k}$ with $X^{J_k} \in \mathcal{H}_{j_{J_k}}^{b_{J_k}}$. The function $Y^{I_{q+1}} \in \mathcal{H}_{j_{I_{q+1}}}^{b_{I_{q+1}}}$ is defined by

$$(Y^{I_{q+1}})_n := \begin{cases} \sum_{i=1}^{q+1} (-1)^{i-1} (X^{f_{q+1}^i(I_{q+1})})_n, & \mathbf{n} \in \mathcal{B}_{j_{I_{q+1}}}^{b_{I_{q+1}}}, \\ 0, & \text{otherwise.} \end{cases}$$

3.3. Properties of the box resolution

In this subsection, we study properties of the box resolutions.

Proposition 3.2. $\forall q \geq 0$, the morphism $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$ is bounded.

Proof. We write $X \in \mathcal{A}_q$ as a sum

$$X = \sum_{I_q \in S_q(k)} X^{I_q}, \quad X^{I_q} \in \mathcal{H}_{j_{I_q}}^{b_{I_q}}.$$

By using the definition above we have $\Psi_q(X) = \sum_{I'_{q+1}} Y^{I'_{q+1}}$, where $Y^{I'_{q+1}} \in \mathcal{H}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}}$ is equal to

$$(Y^{I'_{q+1}})_n := \begin{cases} \sum_{i=1}^{q+1} (-1)^{i-1} (X^{f_{q+1}^i(I'_{q+1})})_n, & n \in \mathcal{B}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}}, \\ 0, & \text{otherwise.} \end{cases}$$

The norm of $\Psi_q(X)$ is computed as

$$\begin{aligned} \|\Psi_q(X)\|^2 &= \sum_{I'_{q+1}} \|Y^{I'_{q+1}}\|^2 = \sum_{I'_{q+1}} \sum_{n \in \mathcal{B}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}}} |Y_n^{I'_{q+1}}|^2 \\ &= \sum_{I'_{q+1}} \sum_{n \in \mathcal{B}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}}} \left| \sum_{i=1}^{q+1} (-1)^{i-1} (X^{f_{q+1}^i(I'_{q+1})})_n \right|^2 \end{aligned}$$

by the Cauchy–Schwartz inequality

$$\leq \sum_{I'_{q+1}} \sum_{n \in \mathcal{B}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}}} (q+1) |(X^{f_{q+1}^i(I'_{q+1})})_n|^2$$

$$\text{as } \mathcal{B}_{j_{I'_{q+1}}}^{b_{I'_{q+1}}} \subseteq \mathcal{B}_{j_{I_q}}^{b_{I_q}}$$

$$\leq \sum_{I'_{q+1}} \sum_{n \in \mathcal{B}_{j_{I_q}}^{b_{I_q}}} (q+1) |(X^{I_q})_n|^2$$

as every I_q is contained in at most $(k-q)$ number of I'_{q+1}

$$\leq (k-q)(q+1) \sum_{I \in S_q(k)} \sum_{n \in \mathcal{B}_{j_{I_q}}^{b_{I_q}}} |(X^{I_q})_n|^2$$

$$= (k-q)(q+1) \|X\|^2. \quad \square$$

Proposition 3.3. *The map $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$ is an $A = \mathbb{C}[z_1, \dots, z_m]$ -module morphism.*

Proof. For every $I \in S_q(k)$, for $X^I \in \mathcal{H}_{j_I}^{b_I}$, $\Psi_q(X^I)$ is a sum

$$\sum_{1 \leq s \leq k, s \notin I} (-1)^{\text{sign}(I, s)} Y^{I \cup \{s\}},$$

where $Y^{I \cup \{s\}} \in \mathcal{H}_{j_{I \cup \{s\}}}^{b_{I \cup \{s\}}}$, and s is the α -th smallest number in $I \cup \{s\}$, and $\text{sign}(I, s) = \alpha - 1$, and the function $Y^{I \cup \{s\}}$ has the following form,

$$Y_{\mathbf{n}}^{I \cup \{s\}} = \begin{cases} X_{\mathbf{n}}^I, & \mathbf{n} \in \mathcal{B}_{\mathbf{j}_{I \cup \{s\}}}^{\mathbf{b}_{I \cup \{s\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

If $p \in \{1, \dots, m\}$, the z_p action on $\mathcal{H}_{\mathbf{j}_I}^{\mathbf{b}_I}$ is as follows,

$$T_{z_p}^{\mathbf{j}_I, \mathbf{b}_I}(X^I)_{n_1 \dots (n_p+1) \dots n_m} = \begin{cases} \sqrt{\frac{\omega_0(n_1 \dots (n_p+1) \dots n_m)}{\omega_0(n_1 \dots n_m)}} X_{n_1 \dots n_p \dots n_m}^I, & p \notin \mathbf{j}_I, \\ \sqrt{\frac{\omega_0(n_1 \dots (n_p+1) \dots n_m)}{\omega_0(n_1 \dots n_m)}} X_{n_1 \dots n_p \dots n_m}^I, & p = j^s \in \mathbf{j}_I, \\ & \text{and } n_p + 1 \leq b^s, \\ 0, & \text{otherwise.} \end{cases}$$

From this, we observe that the operator $T_{z_p}^{\mathbf{j}_I, \mathbf{b}_I}$ preserves the component $\mathcal{H}_{\mathbf{j}_I}^{\mathbf{b}_I}$. Similarly, the z_p action on $\mathcal{H}_{\mathbf{j}_{I \cup \{s\}}}^{\mathbf{b}_{I \cup \{s\}}}$ is as follows,

$$\begin{aligned} & T_{z_p}^{\mathbf{j}_{I \cup \{s\}}, \mathbf{b}_{I \cup \{s\}}}(Y^{I \cup \{s\}})_{n_1 \dots (n_p+1) \dots n_m} \\ &= \begin{cases} \sqrt{\frac{\omega_0(n_1 \dots (n_p+1) \dots n_m)}{\omega_0(n_1 \dots n_m)}} Y_{n_1 \dots n_p \dots n_m}^{I \cup \{s\}}, & p \notin \mathbf{j}_I, \quad p \neq s, \\ \sqrt{\frac{\omega_0(n_1 \dots (n_p+1) \dots n_m)}{\omega_0(n_1 \dots n_m)}} Y_{n_1 \dots n_p \dots n_m}^{I \cup \{s\}}, & p = j^t \in \mathbf{j}_{I \cup \{s\}}, \quad n_p + 1 \leq b^t, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Using the above definition of $\Psi_q(X^I)$, we can directly check that on each component $\mathcal{H}_{\mathbf{j}_{I \cup \{s\}}}^{\mathbf{b}_{I \cup \{s\}}}$,

$$\left(\Psi_q(T_{z_p}^{\mathbf{j}_I, \mathbf{b}_I}(X^I)) \right)^{I \cup \{s\}} = T_{z_p}^{\mathbf{j}_{I \cup \{s\}}, \mathbf{b}_{I \cup \{s\}}} \left(\Psi_q(X^I)^{I \cup \{s\}} \right),$$

which shows that Ψ_q is compatible with the A -module structure. $\square$

Proposition 3.4. $\text{Im}(\Psi_{q-1}) \subseteq \ker(\Psi_q)$.

Proof. For every $I \in S_{q-1}(k)$ and any $X^I \in \mathcal{H}_{\mathbf{j}_I}^{\mathbf{b}_I}$, the image of X^I under Ψ_{q-1} is of the form

$$\sum_{1 \leq s \leq k, s \notin I} (-1)^{\text{sign}(I, s)} Y^{I \cup \{s\}},$$

where $Y^{I \cup \{s\}} \in \mathcal{H}_{\mathbf{j}_{I \cup \{s\}}}^{\mathbf{b}_{I \cup \{s\}}}$, and s is the α -th smallest number in $I \cup \{s\}$, and $\text{sign}(I, s) = \alpha - 1$, and the function $Y^{I \cup \{s\}}$ has the following form,

$$Y_{\mathbf{n}}^{I \cup \{s\}} = \begin{cases} X_{\mathbf{n}}^I, & \mathbf{n} \in \mathcal{B}_{\mathbf{j}_{I \cup \{s\}}}^{\mathbf{b}_{I \cup \{s\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the image of $Y^{I \cup \{s\}}$ under the map Ψ_q is of the form

$$\sum_{1 \leq t \leq k, t \notin I \cup \{s\}} (-1)^{\text{sign}(I \cup \{s\}, t)} Z^{I \cup \{s, t\}},$$

where $Z^{I \cup \{s, t\}} \in \mathcal{H}_{I \cup \{s, t\}}^{\mathbf{b}_{I \cup \{s, t\}}}$, and t is the β -th smallest number in $I \cup \{s, t\}$, and $\text{sign}(I \cup \{s\}, t) = \beta - 1$, and the function $Z^{I \cup \{s, t\}}$ has the following form,

$$Z_{\mathbf{n}}^{I \cup \{s, t\}} = \begin{cases} Y_{\mathbf{n}}^{I \cup \{s\}}, & \mathbf{n} \in \mathcal{B}_{I \cup \{s, t\}}^{\mathbf{b}_{I \cup \{s, t\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Combining the above computation, we have the following expression for $\Psi_q(\Psi_{q-1}(X^I))$,

$$\begin{aligned} \Psi_q(\Psi_{q-1}(X^I)) &= \sum_{1 \leq s \leq k, s \notin I} (-1)^{\text{sign}(I, s)} \Psi_q(Y^{I \cup \{s\}}) \\ &= \sum_{1 \leq s \leq k, s \notin I} (-1)^{\text{sign}(I, s)} \sum_{1 \leq t \leq k, t \notin I \cup \{s\}} (-1)^{\text{sign}(I \cup \{s\}, t)} Z^{I \cup \{s, t\}} \\ &= \sum_{1 \leq s \neq t \leq k, s, t \notin I} (-1)^{\text{sign}(I, s) + \text{sign}(I \cup \{s\}, t)} Z^{I \cup \{s, t\}} \\ &= \sum_{1 \leq s < t \leq k, s, t \notin I} \left((-1)^{\text{sign}(I, s) + \text{sign}(I \cup \{s\}, t)} + (-1)^{\text{sign}(I, t) + \text{sign}(I \cup \{t\}, s)} \right) \\ &\quad \times Z^{I \cup \{s, t\}}. \end{aligned}$$

When $s < t$, it is not hard to check the following equations

$$\text{sign}(I, s) = \text{sign}(I \cup \{t\}, s), \quad \text{sign}(I \cup \{s\}, t) = \text{sign}(I, t) + 1.$$

We conclude that $\Psi_q(\Psi_{q-1}(X^I)) = 0$, and complete the proof of this proposition. $\square$

Lemma 3.5.

$$\text{Im}(\Psi_0) \supseteq \ker(\Psi_1).$$

Proof. Consider $X = (X^1, \dots, X^p) \in \mathcal{A}_1$ such that $X \in \ker \Psi_1$. Define a function $\xi \in \mathcal{A}_0 = L_a^2(\mathbb{B}^m)$ as follows.

$$\xi_{\mathbf{n}} := \begin{cases} X_{\mathbf{n}}^s, & \text{there is } s \text{ such that } \mathbf{n} \in \mathcal{B}_{\mathbf{j}_s}^{\mathbf{b}_s}, \\ 0, & \text{otherwise.} \end{cases}$$

We observe that if there are s, t such that $\mathbf{n}$ belongs to both $\mathcal{B}_{\mathbf{j}_s}^{\mathbf{b}_s}$ and $\mathcal{B}_{\mathbf{j}_t}^{\mathbf{b}_t}$, then the $\mathcal{H}_{\mathbf{j}_{st}}^{\mathbf{b}_{st}}$ component of $\Psi_1(X)$ is

$$X_n^s - X_n^t = 0,$$

as $X \in \ker(\Psi_1)$. Hence, the value ξ_n is independent of the choices of s . Therefore ξ is well defined. Furthermore, $\|\xi\|^2$ is no more than the sum

$$\|X^1\|^2 + \cdots + \|X^p\|^2. \quad (3)$$

Hence, $\xi \in \mathcal{A}_0$ and $\Psi_0(\xi) = X \in \ker(\Psi_1)$. $\square$

Proposition 3.6.

$$\text{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q), q = 1, \dots, k.$$

Proof. We prove the proposition by induction on the number k .

For $k = 1$, we consider the map $\Psi_0 : \mathcal{A}_0 \rightarrow \mathcal{A}_1$. With the orthonormal basis, it is not hard to observe that $\mathcal{A}_1$ can be identified with a closed subspace of $\mathcal{A}_0 = L_a^2(\mathbb{B}^m)$, and the map Ψ_0 is the corresponding orthogonal projection map. Therefore, Ψ_0 is a surjective map.

Suppose that the following is true

$$\text{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q), q = 1, \dots, k,$$

for all $1 \leq k < p$. We prove the statement for $k = p$.

The case for $q = 1$ is proved in Lemma 3.5. We are left to show the cases for $2 \leq q \leq p$. We consider the following two collections of $p - 1$ boxes,

(1) the first $p - 1$ boxes

$$\{\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_{p-1}}^{b_{p-1}}\}.$$

We follow the construction in Section 3.1–3.2 and consider the associated A -modules $\mathcal{A}_s^1$ together with the A -module morphisms $\Psi_s^1 : \mathcal{A}_s^1 \rightarrow \mathcal{A}_{s+1}^1$, $s = 1, \dots, p - 2$. Set $\mathcal{A}_p^1 := \{0\}$, and $\Psi_{p-1}^1 = 0$;

(2) the intersections of the first $p - 1$ boxes with the last one $\mathcal{B}_{i_p}^{b_p}$,

$$\{\mathcal{B}_{j_{1p}}^{b_{1p}}, \dots, \mathcal{B}_{j_{p-1p}}^{b_{p-1p}}\}.$$

We follow the construction in Section 3.1–3.2 and consider the associated A -modules $\mathcal{A}_s^2$ together with the A -module morphisms $\Psi_s^2 : \mathcal{A}_s^2 \rightarrow \mathcal{A}_{s+1}^2$, $s = 1, \dots, p - 2$. Set $\mathcal{A}_p^2 := \{0\}$, and $\Psi_{p-1}^2 = 0$.

By the induction assumption, we know that

$$\text{Im}(\Psi_{q-1}^1) \supseteq \ker(\Psi_q^1), \quad \text{Im}(\Psi_{q-1}^2) \supseteq \ker(\Psi_q^2), \quad q = 1, \dots, p - 1.$$

We define a map $\Phi_t : \mathcal{A}_t^1 \rightarrow \mathcal{A}_t^2$ by

$$\Phi_t(X^I) = Y^{I \cup \{p\}}, \quad I \in S_t(p-1)$$

where by $Y^{I \cup \{p\}}$ we refer to the component corresponding to the intersection of the boxes $\mathcal{B}_{j_{i_1 p}}^{b_{i_1 p}}, \dots, \mathcal{B}_{j_{i_t p}}^{b_{i_t p}}$ with

$$Y_n^{I \cup \{p\}} := \begin{cases} (-1)^t X_n^I, & n \in \mathcal{B}_{j_{I \cup \{p\}}}^{b_{I \cup \{p\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Similar to Proposition 3.3, Φ_t is an A -module morphism. We leave the details to the reader.

With the above construction, we can easily check the following identities.

- (1) $\mathcal{A}_q = \mathcal{A}_q^1 \oplus \mathcal{A}_{q-1}^2$, for $q = 2, \dots, p$, where $\mathcal{A}_p^1 = \{0\}$.
- (2) $\Psi_q = \begin{pmatrix} \Psi_q^1 & 0 \\ \Phi_q & \Psi_{q-1}^2 \end{pmatrix}$, for $q = 2, \dots, p-1$, where $\Psi_{p-1}^1 = 0$.

We use the above identifications to prove $\text{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q)$. The proof consists of the following three cases.

$$i) \ q = 2, \quad ii) \ 3 \leq q \leq p-1, \quad iii) \ q = p.$$

i) $q = 2$.

Suppose $(X_1, X_2) \in \mathcal{A}_2^1 \oplus \mathcal{A}_1^2 = \mathcal{A}_2$ is in the kernel of the morphism Ψ_2 . By the above identification of Ψ_q , we have

$$\Psi_2^1(X_1) = 0, \quad \Phi_2(X_1) + \Psi_1^2(X_2) = 0.$$

By the induction assumption, $\ker(\Psi_2^1) \subseteq \text{Im}(\Psi_1^1)$. So there exists $Y_1 \in \mathcal{A}_1^1$ such that $\Psi_1^1(Y_1) = X_1$. By Proposition 3.4 for the morphism $\Psi_\bullet$, we have

$$\begin{aligned} (0, 0) &= \Psi_2(\Psi_1(Y_1, 0)) = \Psi_2(\Psi_1^1(Y_1), \Phi_1(Y_1)) = \left(\Psi_2^1(\Psi_1^1(Y_1)), \Phi_2(\Psi_1^1(Y_1)) + \Psi_1^2(\Phi_1(Y_1)) \right) \\ &\text{as } \Psi_1^1(Y_1) = X_1, \Psi_2^1(\Psi_1^1(Y_1)) = 0 \\ &= \left(0, \Phi_2(X_1) + \Psi_1^2(\Phi_1(Y_1)) \right). \end{aligned}$$

Hence, $\Phi_2(X_1) + \Psi_1^2(\Phi_1(Y_1)) = 0$. Consider $X'_2 = X_2 - \Phi_1(Y_1)$. We compute

$$\Psi_1^2(X'_2) = \Psi_1^2(X_2 - \Phi_1(Y_1)) = \Psi_1^2(X_2) - \Psi_1^2(\Phi_1(Y_1)) = \Psi_1^2(X_2) + \Phi_2(X_1) = 0,$$

as $\Psi_2(X_1, X_2) = (\Psi_2^1(X_1), \Phi_2(X_1) + \Psi_1^2(X_2)) = 0$.

Using the property that $\Psi_1^2(X'_2) = 0$, we construct an element $Y_2 \in \mathcal{H}_{j_p}^{b_p}$ by setting

$$(Y_2)_n := \begin{cases} (X'_2)^{i_p}_n, & n \in \mathcal{B}_{j_{ip}}^{b_{ip}}, \text{ for some } i = 1, \dots, p-1, \\ 0, & \text{otherwise.} \end{cases}$$

As $\Psi_1^2(X'_2) = 0$, the above definition of Y_2 is independent of the choices of i . It is not hard to check the norm of Y_2 is bounded following the similar estimate as Equation (3). Therefore, Y_2 is in $\mathcal{H}_{j_p}^{b_p} \subset L_a^2(\mathbb{B}^m)$, and $\Psi_0^2(Y_2) = X'_2$.

In summary, we have constructed an element $(Y_1, Y_2) \in \mathcal{A}_1 = \mathcal{A}_1^1 \oplus \mathcal{H}_{j_p}^{b_p}$. And it is not hard to check that

$$\Psi_1(Y_1, Y_2) = (\Psi_1^1(Y_1), \Phi_1(Y_1) + \Psi_0^2(Y_2)) = (X_1, \Phi_1(Y_1) + X'_2) = (X_1, X_2),$$

which shows that $(X_1, X_2) \in \text{Im}(\Psi_1)$.

ii) $3 \leq q \leq p-1$.

Suppose $(X_1, X_2) \in \mathcal{A}_q^1 \oplus \mathcal{A}_{q-1}^2 = \mathcal{A}_q$ is in the kernel of the morphism Ψ_q . By the above identification of Ψ_q , we have

$$\Psi_q^1(X_1) = 0, \quad \Phi_q(X_1) + \Psi_{q-1}^2(X_2) = 0.$$

As $\text{Im}(\Psi_{q-1}^1) \supseteq \ker(\Psi_q^1)$, there is $Y_1 \in \mathcal{A}_{q-1}^1$ such that $X_1 = \Psi_{q-1}^1(Y_1)$. As $\Psi_q(\Psi_{q-1}(Y_1, 0)) = 0$, $\Phi_q(X_1) + \Psi_{q-1}^2(\Phi_{q-1}(Y_1)) = 0$. Hence, we have

$$\Psi_{q-1}^2(X_2 - \Phi_{q-1}(Y_1)) = 0.$$

As $\text{Im}(\Psi_{q-2}^2) \supseteq \ker(\Psi_{q-1}^2)$, there exists $Y_2 \in \mathcal{A}_{q-2}^2$ such that

$$\Psi_{q-2}^2(Y_2) = X_2 - \Phi_{q-1}(Y_1).$$

Therefore, we have found $(Y_1, Y_2) \in \mathcal{A}_q$ such that

$$\Psi_{q-1}(Y_1, Y_2) = (\Psi_{q-1}^1(Y_1), \Phi_{q-1}(Y_1) + \Psi_{q-2}^2(Y_2)) = (X_1, X_2).$$

iii) $q = p$.

We notice that $\mathcal{A}_p$ is the same as $\mathcal{A}_{p-1}^2$. As the map $\Psi_{p-2}^2 : \mathcal{A}_{p-2}^2 \rightarrow \mathcal{A}_{p-1}^2$ is surjective, it follows that the map $\Psi_{p-1} : \mathcal{A}_{p-1} = \mathcal{A}_{p-1}^1 \oplus \mathcal{A}_{p-2}^2 \rightarrow \mathcal{A}_p = \mathcal{A}_{p-1}^2$ is surjective. $\square$

3.4. Proof of Theorem 1.1

In this subsection, we prove the main theorem of the paper. We assume that I is an ideal of the polynomial algebra $\mathbb{C}[z_1, \dots, z_m]$ generated by monomials $z^{\alpha_i} := z_1^{\alpha_i^1} \cdots z_m^{\alpha_i^m}$, for $\alpha_i \in \mathbb{N}^m$, $i = 1, \dots, l$.

Following [11, Theorem 1.1.2], monomials inside the ideal I form a linear basis of the ideal I over $\mathbb{C}$. We consider the lattice $\mathbb{N}^m$ and the subset $C(I)$ consisting of exponents of monomials that do not belong to I . According to [11, Proposition 1.1.5], a monomial v belongs to I if and only if there is a monomial w such that $v = wz^{\alpha_i}$ for some $i = 1, \dots, l$. Therefore, the monomial $z_1^{n^1} \cdots z_m^{n^m}$ does not belong to I if and only if for any $i = 1, \dots, l$, z^{α_i} is not a factor of $z_1^{n^1} \cdots z_m^{n^m}$. Equivalently, $z_1^{n^1} \cdots z_m^{n^m}$ does not belong to I if and only if for every i , there is s_i such that $n^{s_i} < \alpha_i^{s_i}$.

Consider the finite collection $S(\alpha_1, \dots, \alpha_l)$ of l -tuple of natural numbers $\mathfrak{s} = (s_1, \dots, s_l)$ such that $1 \leq s_i \leq m$. For each $\mathfrak{s}$, let $j_{\mathfrak{s}} \subseteq \{1, \dots, m\}$ be the subset consisting of those numbers appearing in the array $(s_1, \dots, s_l)$. For every $k \in j_{\mathfrak{s}}$, let b_k be the minimum of all $\alpha_i^{s_i}$ such that $s_i = k$ for $i = 1, \dots, l$. From the above conditions on monomials not in I , we conclude that $C(I)$ is the union of all boxes $\mathcal{B}_{\mathfrak{s}}$. We refer the reader to [4, Section 9.2, Theorem 3] for a related discussion. We conclude from the discussion above that a polynomial f belongs to I if and only if f has no nonzero component in any of the boxes $\mathcal{B}_{j_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ for any $\mathfrak{s} \in S(\alpha_1, \dots, \alpha_l)$. Let $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$ be the collection of nonempty boxes in $\{\mathcal{B}_{j_{\mathfrak{s}}}^{b_{\mathfrak{s}}} : \mathfrak{s} \in S(\alpha_1, \dots, \alpha_l)\}$ associated to the ideal I .

Associated to the above collection of boxes $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$, we apply the constructions in Section 3.1–3.2 to construct a sequence of Hilbert A -modules $\mathcal{A}_0, \dots, \mathcal{A}_k$ together with module morphisms $\Psi_q : \mathcal{A}_q \rightarrow \mathcal{A}_{q+1}$, i.e.

$$\mathcal{A}_0 = L_a^2(\mathbb{B}^m) \xrightarrow{\Psi_0} \mathcal{A}_1 \xrightarrow{\Psi_1} \cdots \xrightarrow{\Psi_{k-1}} \mathcal{A}_k \longrightarrow 0.$$

Proposition 2.3 shows that each $\mathcal{A}_q$ ($q = 0, 1, \dots, k$) is a Hilbert $A = \mathbb{C}[z_1, \dots, z_m]$ -module. Proposition 3.2–3.3 show that Ψ_q ($q = 0, \dots, k-1$) is a bounded module morphism. Proposition 3.4–3.6 show that the above sequence is exact at $\mathcal{A}_q$ for $q = 1, \dots, k$.

We are left to prove that the kernel of the morphism Ψ_0 is the completion of I in $L_a^2(\mathbb{B}^m)$, i.e.

$$\overline{I} = \ker(\Psi_0).$$

By the above discussion, if $f \in I$, then f has no nonzero component in any of the boxes $\mathcal{B}_{j_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ for any $\mathfrak{s} \in S(\alpha_1, \dots, \alpha_l)$. This shows that $f \in \ker(\Psi_0)$. Therefore, I and its closure $\overline{I}$ are contained inside $\ker(\Psi_0)$.

Suppose that f is in the kernel $\ker(\Psi_0)$. Write f in terms of the orthonormal basis,

$$f = \sum_{\mathbf{n} \in \mathbb{N}^m} f_{\mathbf{n}} z^{\mathbf{n}}.$$

As $\Psi_0(f) = 0$, by the definition of Ψ_0 , for any $s = 1, \dots, k$, and any $\mathbf{n} \in \mathcal{B}_{j_s}^{b_s}$, $f_{\mathbf{n}} = 0$. For any positive integer M , let f_M be the truncation of the above expansion of f by requiring $n^1, \dots, n^m < M$, i.e.

$$f_M := \sum_{\mathbf{n} \in \mathbb{N}^m, n^1 < M, \dots, n^m < M} f_{\mathbf{n}} z^{\mathbf{n}}.$$

It is not hard to see that f_M is a polynomial and has no component in the boxes $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$. By the construction of the boxes $\mathcal{B}_{j_1}^{b_1}, \dots, \mathcal{B}_{j_k}^{b_k}$, f_M belongs to the ideal I . As $M \rightarrow \infty$, f_M converges to f in $L_a^2(\mathbb{B}^m)$. Hence, we have shown that f belongs to the closure $\bar{I}$, and $\ker(\Psi_0)$ is a subset of $\bar{I}$. So we conclude that $\bar{I} = \ker(\Psi_0)$.

In summary, we have completed the proof of Theorem 1.1 for general monomial ideals.

3.5. K -homology class

As a corollary of Theorem 1.1, by applying [6, Theorem 1], we can conclude that the closure $\bar{I}$ of the ideal I and the quotient $Q_I := L_a^2(\mathbb{B}^m)/\bar{I}$ are both essentially normal Hilbert modules.

Let $\mathfrak{T}(\bar{I})$ (and $\mathfrak{T}(Q_I)$) be the unital C^* -algebra generated by Toeplitz operators on the module $\bar{I}$ (and the quotient module Q_I). We now discuss properties of the K -homology class associated to the following Toeplitz extension,

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{T}(Q_I) \longrightarrow C(\sigma_I^e) \longrightarrow 0,$$

where σ_I^e is the essential spectrum space of the algebra $\mathfrak{T}(Q_I)$ and $\mathcal{K}$ is the algebra of compact operators.

By Theorem 1.1, for $i = 1, \dots, k$, we introduce the following closed subspace of $\mathcal{A}_i$,

$$\mathcal{A}_i^- := \text{Im}(\Psi_{i-1}) = \ker(\Psi_i).$$

As Ψ_{k-1} is surjective, $\mathcal{A}_k^- = \mathcal{A}_k$.

As $\Psi_i : \mathcal{A}_i \rightarrow \mathcal{A}_{i+1}$ is a morphism of $A = \mathbb{C}[z_1, \dots, z_m]$ -modules, the kernel $\mathcal{A}_i^- = \ker(\Psi_i)$ is naturally an A -module. Furthermore, we have the following exact sequence of Hilbert A -modules,

$$0 \longrightarrow \mathcal{A}_i^- \longrightarrow \mathcal{A}_i \longrightarrow \mathcal{A}_{i+1}^- \longrightarrow 0, \quad i = 1, \dots, k-1,$$

where the first map is the inclusion, and the second map is Ψ_i .

Lemma 3.7. *$\forall i = 1, \dots, k$, the A -module $\mathcal{A}_i^-$ is essentially normal, and therefore the quotient module $Q_i := \mathcal{A}_i/\mathcal{A}_i^-$ is also essentially normal.*

Proof. When $i = k-1$, as Ψ_{k-1} is surjective, we have the following short exact sequence,

$$0 \longrightarrow \mathcal{A}_{k-1}^- \longrightarrow \mathcal{A}_{k-1} \longrightarrow \mathcal{A}_k \longrightarrow 0.$$

As it is explained in Section 3.1, both $\mathcal{A}_{k-1}$ and $\mathcal{A}_k$ are essentially normal A -modules. It follows from [6, Theorem 1] that $\mathcal{A}_{k-1}^-$ is an essentially normal A -module.

Repeating the above arguments to the exact sequence

$$0 \longrightarrow \mathcal{A}_{k-2}^- \longrightarrow \mathcal{A}_{k-1} \longrightarrow \mathcal{A}_{k-1}^- \longrightarrow 0,$$

we conclude that $\mathcal{A}_{k-2}^-$ is an essentially normal A -module. Similarly, the iterated arguments show that every $\mathcal{A}_i^-$ is an essentially normal A -module. $\square$

Modeled by the above short exact sequence of essentially normal Hilbert A -modules $\mathcal{A}_i^-$ and $\mathcal{A}_i$, we prove the following property.

Proposition 3.8. *Let M_1, M_2, M_3 be essentially normal Hilbert A -modules, and $W_1 : M_1 \rightarrow M_2, W_2 : M_2 \rightarrow M_3$ be morphisms of Hilbert A -modules satisfying the following short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{W_1} M_2 \xrightarrow{W_2} M_3 \longrightarrow 0.$$

Suppose that the essential spectrum of the tuples $(T_{z_1}^i, \dots, T_{z_m}^i), i = 1, 2, 3$, is contained in the closed unit ball $\overline{\mathbb{B}}^m \subset \mathbb{C}^m$, and let $\alpha_i : C(\overline{\mathbb{B}}^m) \rightarrow \mathcal{C}(M_i) = B(M_i)/\mathcal{K}(M_i)$ be the induced representations of $C(\overline{\mathbb{B}}^m)$ on the Calkin algebra $\mathcal{C}(M_i)$. There are co-isometry operators $U : M_2 \rightarrow M_1$ and $V : M_2 \rightarrow M_3$ such that

$$UV^* = 0 = VU^*, U^*U + V^*V = I,$$

and commute with A -module structures up to compact operators, i.e.

$$[U]\alpha_2[U]^* = \alpha_1, \quad [V]\alpha_2[V]^* = \alpha_3,$$

where $\alpha_i(p) = [T_p] \in \mathcal{C}(M_i)$ denotes the equivalence class of the multiplication operator $T_p \in L(M_i)$.

Proof. As W_2 is surjective, $W_2W_2^*$ is positive definite. Let $W_2 = A_3V$ be the polar decomposition of the morphism W_2 such that A_3 is a positive definite operator on M_3 , s.t. $A_3 = (W_2W_2^*)^{\frac{1}{2}}$, and V is a co-isometry, i.e. $VV^* = I$.

As W_2 is an A -module morphism, for any $f \in A = \mathbb{C}[z_1, \dots, z_m]$, we have

$$A_3VT_f^2 = W_2T_f^2 = T_f^3W_2 = T_f^3A_3V,$$

where T_f^2 and T_f^3 are the Toeplitz operators on M_2 and M_3 associated to f . As M_2 and M_3 are essentially normal, T_f^2 and T_f^3 are both normal in the respective Calkin algebras. It follows from the Fuglede–Putnam theorem that the following equation holds modulo compact operators

$$A_3V(T_f^2)^* = W_2(T_f^2)^* = (T_f^3)^*W_2 = (T_f^3)^*A_3V.$$

Taking the adjoint of both sides of the above equation, we reach

$$T_f^2V^*A_3 = V^*A_3T_f^3.$$

Multiplying A_3V to the left of each term in the above equation, with the property $VV^* = I$, we have

$$A_3VT_f^2V^*A_3 = A_3VV^*A_3T_f^3 = A_3^2T_f^3.$$

As

$$A_3VT_f^2 = T_f^3A_3V,$$

we conclude from the above equation that modulo compact operators

$$A_3VT_f^2V^*A_3 = T_f^3A_3VV^*A_3 = T_f^3A_3^2 = A_3^2T_f^3.$$

As A_3 is positive definite, it is safe to conclude that modulo compact operators

$$T_f^3A_3 = A_3T_f^3.$$

The above commutativity plus the equation $A_3VT_f^2 = T_f^3A_3V$ gives the following identity, modulo compact operators,

$$VT_f^2 = T_f^3V.$$

The property of $VV^* = I$ confirms that modulo compact operators

$$VT_f^2V^* = T_f^3,$$

which is exactly

$$V\alpha_2V^* = \alpha_3.$$

As $W_1 : M_1 \rightarrow M_2$ is injective with closed range, $W_1^*W_1$ is positive definite. Let $W_1 = WA_1$ be the polar decomposition of the operator W_1 , where $A_1 = (W_1^*W_1)^{\frac{1}{2}}$ and $W : M_1 \rightarrow M_2$ is a partial isometry, i.e. $W^*W = I$. An argument similar to the one above can be made for W_2 to show that modulo compact operators, for any $f \in A = \mathbb{C}[z_1, \dots, z_m]$,

$$A_1T_f^1 = T_f^1A_1,$$

and

$$W^*T_f^2W = T_f^1.$$

If we set $U = W^*$, then $UT_f^2U^* = T_f^1$, and $UU^* = I$, which shows

$$U\alpha_2U^* = \alpha_1.$$

As $W_2W_1 = A_3VU^*A_1 = 0$, $VU^* = 0$ follows from the invertibility of A_1 and A_3 . Therefore, U^*U and V^*V are commuting two orthogonal projections on M_2 . To prove that their sum is the identity operator, it is enough to show that the kernel of their sum is trivial. If $\xi \in M_2$ satisfies $U^*U\xi + V^*V\xi = 0$, it follows that $U^*U\xi = V^*V\xi = 0$, and $U\xi = V\xi = 0$. Then $W_2\xi = A_3V\xi = 0$, and $W_1^*\xi = A_1U\xi = 0$. As $W_2\xi = 0$, ξ belongs to the kernel of W_2 , and the exactness of the morphisms shows that there is $\eta \in W_1$ such that $W_1\eta = \xi$. As $W_1^*\xi = 0$, $W_1^*W_1\eta = 0$, and $\xi = W_1\eta = 0$. Hence the kernel of $U^*U + V^*V$ is trivial, and $U^*U + V^*V = I$. $\square$

Let $\mathfrak{T}(M_i)$ be the unital C^* -algebra generated by Toeplitz operators on M_i , and σ_i^e be the associated essential spectrum space of the tuple $(T_{z_1}, \dots, T_{z_m})$ on M_i . The morphisms α_1 and α_2 factor to injective algebra homomorphisms from $C(\sigma_1^e)$ and $C(\sigma_2^e)$ to $\mathcal{C}(M_1)$ and $\mathcal{C}(M_2)$. By Proposition 3.8, $\alpha_1 = [U]\alpha_2[U]^*$. The composition of $[U]\alpha_2[U]^*$ with α_1^{-1} is an algebra homomorphism from $C(\sigma_2^e)$ to $C(\sigma_1^e)$. Accordingly, this induces a natural map from σ_1^e to σ_2^e . Similar arguments also give a natural map from σ_3^e to σ_2^e . Accordingly, α_1 and α_3 induce classes in $K_1(\sigma_2^e)$. The following property is a quick corollary of Proposition 3.8.

Corollary 3.9. *Under the same assumption as Proposition 3.8, the following equation holds in $K_1(\sigma_2^e)$,*

$$[\alpha_2] = [\alpha_1] + [\alpha_3],$$

where $[\alpha_1]$ and $[\alpha_3]$ are identified as classes in $K_1(\sigma_2^e)$ by the co-isometry operators U and V introduced in Proposition 3.8.

Proof. As we have explained, α_1 and α_3 induces classes in $K_1(\sigma_2^e)$. Furthermore, as K -homology classes of σ_2^e , $\alpha_1 = [U]\alpha_2[U]^*$, $\alpha_3 = [V]\alpha_2[V]^* : C(\sigma_2^e) \rightarrow \mathcal{C}(M_2)$ with $UU^* = I = VV^*$, $UV^* = 0 = VU^*$, and $U^*U + V^*V = I$. Such a pair U, V proves the following equation,

$$[\alpha_2] = [\alpha_1] + [\alpha_3]. \quad \square$$

We are now ready to apply Proposition 3.8 and Corollary 3.9 to the exact sequence constructed in Theorem 1.1. Let $\mathfrak{T}(\mathcal{A}_i)$ (and $\mathfrak{T}(\mathcal{A}_i^-)$) be the unital C^* -algebra generated by Toeplitz operators on $\mathcal{A}_i$ (and $\mathcal{A}_i^-$), and $\sigma_i^e \subset \mathbb{B}^m$ (and $\sigma_{i-}^e \subset \mathbb{B}^m$) be the associated essential spectrum space, and α_i (and α_i^-) be the associated representation of $C(\sigma_i^e)$ (and $C(\sigma_{i-}^e)$) into the Calkin algebra $\mathcal{C}(\mathcal{A}_i)$ (and $\mathcal{C}(\mathcal{A}_i^-)$).

Theorem 3.10. (Theorem 1.2) *The K -homology class associated to $[\mathfrak{T}(Q_I)]$ is the same as $[\alpha_1^-]$, and in $K_1(\sigma_1^e \cup \dots \cup \sigma_k^e)$,*

$$[\mathfrak{T}(Q_I)] = [\alpha_1] - [\alpha_2] + \dots + (-1)^{k-1}[\alpha_k].$$

Proof. We apply Corollary 3.9 to the following short exact sequence of essentially normal Hilbert A -modules

$$0 \longrightarrow \mathcal{A}_i^- \longrightarrow \mathcal{A}_i \longrightarrow \mathcal{A}_{i+1}^- \longrightarrow 0, \quad i = 1, \dots, k-1.$$

We have the following equation in $K_1(\sigma_i)$,

$$[\alpha_i] = [\alpha_i^-] + [\alpha_{i+1}^-].$$

When $i = k-1$, $\mathcal{A}_k^- = \mathcal{A}_k$, and in $K_1(\sigma_{k-1}^e)$,

$$[\alpha_{k-1}] = [\alpha_{k-1}^-] + [\alpha_k].$$

Similarly, for $i = k-2$, in $K_1(\sigma_{k-2}^e)$,

$$[\alpha_{k-2}] = [\alpha_{k-2}^-] + [\alpha_{k-1}^-].$$

Combining the above two equations on K -homology groups, we conclude that in $K_1(\sigma_{k-1}^e \cup \sigma_{k-2}^e)$,

$$[\alpha_{k-1}] + [\alpha_{k-2}^-] = [\alpha_k] + [\alpha_{k-2}],$$

by pushing forward the respective equations in $K_1(\sigma_{k-1}^e)$ and $K_1(\sigma_{k-2}^e)$ into the ones in $K_1(\sigma_{k-1}^e \cup \sigma_{k-2}^e)$ via the natural inclusion maps $\sigma_{k-1}^e, \sigma_{k-2}^e \hookrightarrow \sigma_{k-1}^e \cup \sigma_{k-2}^e$.

Repeating the above arguments inductively, we conclude that in $K_1(\sigma_1^e \cup \dots \cup \sigma_k^e)$,

$$[\alpha_1^-] = [\alpha_1] - [\alpha_2] + \dots + (-1)^{k-1}[\alpha_k].$$

By the short exact sequence of essentially normal Hilbert A -modules,

$$0 \longrightarrow \bar{I} \longrightarrow L_a^2(\mathbb{B}^m) \longrightarrow \mathcal{A}_1^- \longrightarrow 0,$$

we conclude that there is a natural A -module isomorphism between the quotient Hilbert modules $Q_I = L_a^2(\mathbb{B}^m)/\bar{I}$ and $\mathcal{A}_1^-$. We conclude from [8, Proposition 4.4] that they must define the same K -homology class. Therefore, we conclude that in $K_1(\sigma_1^e \cup \dots \cup \sigma_k^e)$,

$$[\mathfrak{T}(Q_I)] = [\alpha_1] - [\alpha_2] + \dots + (-1)^{k-1}[\alpha_k]. \quad \square$$

4. Examples

In this section, we explain our construction of the boxes in examples.

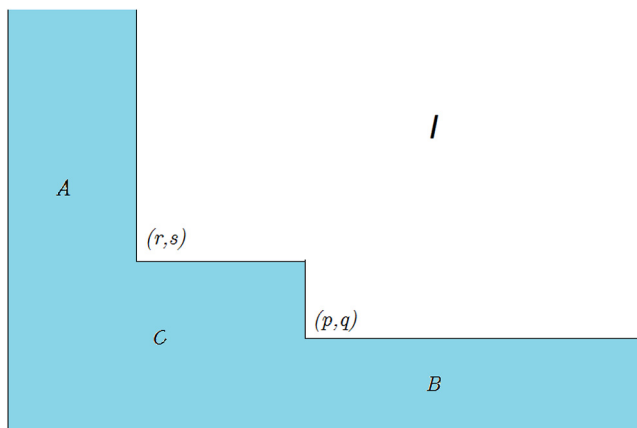


Fig. 1. Staircase diagram corresponding to $I = \langle z_1^p z_2^q, z_1^r z_2^s \rangle$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

4.1. Ideal $I = \langle z_1^2 z_2^2 \rangle \subset \mathbb{C}[z_1, z_2]$

The exponents of monomials in the ideal $I = \langle z_1^2 z_2^2 \rangle$ comprises the region $\{(n^1, n^2) | n^1, n^2 \geq 2\}$. In this example, there is only one $\alpha = (2, 2)$. There are two boxes associated to the ideal, $\mathcal{B}_{j_1}^{b_1} := \{(n^1, n^2) | n^1 \leq 1\}$, $\mathcal{B}_{j_2}^{b_2} := \{(n^1, n^2) | n^2 \leq 1\}$. The intersection $\mathcal{B}_{j_{12}}^{b_{12}}$ of $\mathcal{B}_{j_1}^{b_1}$ and $\mathcal{B}_{j_2}^{b_2}$ is of the form $\{(n^1, n^2) | n^1, n^2 \leq 1\}$.

We have following subsets $\mathbb{B}_{j_1} := \{(0, z_2) | |z_2| < 1\}$, $\mathbb{B}_{j_2} := \{(z_1, 0) | |z_1| < 1\}$.

The Hilbert module $\mathcal{A}_1$ is the direct sum of two submodules $\mathcal{A}_1^1$ and $\mathcal{A}_1^2$, where $\mathcal{A}_1^1$ is the closed subspace of $L_a^2(\mathbb{B}^2)$ spanned by $\{z_2^n, z_1 z_2^n\}_{n \in \mathbb{N}}$, and $\mathcal{A}_1^2$ is the closed subspace of $L_a^2(\mathbb{B}^2)$ spanned by $\{z_1^n, z_1^n z_2\}_{n \in \mathbb{N}}$.

The Hilbert A -module on $\mathcal{A}_2$ is the subspace of $L_a^2(\mathbb{B}^2)$ spanned by $\{1, z_1, z_2, z_1 z_2\}$. It is easy to see that $\mathcal{A}_2 = \mathcal{A}_1^1 \cap \mathcal{A}_1^2$.

4.2. Ideal $I = \langle z_1^p z_2^q, z_1^r z_2^s \rangle \subset \mathbb{C}[z_1, z_2]$

We consider the ideal $I = \langle z_1^p z_2^q, z_1^r z_2^s \rangle \subset \mathbb{C}[z_1, z_2]$, $p, q, r, s \geq 0$. We explain our construction of boxes and the associated Hilbert modules in Theorem 1.1 in this example.

The exponents of monomials in I belong to the white region of the Fig. 1 marked by I . The subset $C(I) \subset \mathbb{N}^2$ consisting of exponents of monomials not in the ideal I is the blue region in Fig. 1. In this example, $\alpha_1 = (r, s)$, $\alpha_2 = (p, q)$, $S(\alpha_1, \alpha_2)$ consists of 4 arrays $(1, 1), (1, 2), (2, 1), (2, 2)$. The boxes associated to these arrays are described below.

- (1) For $\mathfrak{s} = (1, 1)$, the box $\mathcal{B}_{j_{11}}^{b_{11}}$ is $\{(n^1, n^2) | n^1 < r\}$;
- (2) For $\mathfrak{s} = (1, 2)$, the box $\mathcal{B}_{j_{12}}^{b_{12}} = \{(n^1, n^2) | n^1 < p, n^2 < s\}$;
- (3) For $\mathfrak{s} = (2, 1)$, the box $\mathcal{B}_{j_{21}}^{b_{21}} = \{(n^1, n^2) | n^1 < r, n^2 < q\}$;
- (4) For $\mathfrak{s} = (2, 2)$, the box $\mathcal{B}_{j_{22}}^{b_{22}} = \{(n^1, n^2) | n^2 < q\}$.

In the Fig. 1, the box $\mathcal{B}_{j_{11}}^{b_{11}}$ is marked as region A , and the box $\mathcal{B}_{j_{12}}^{b_{12}}$ is marked as region C , and the box $\mathcal{B}_{j_{22}}^{b_{22}}$ is marked as region B . As the box $\mathcal{B}_{j_{21}}^{b_{21}}$ is contained inside $\mathcal{B}_{j_{12}}^{b_{12}}$, we do not need to include the box $\mathcal{B}_{j_{21}}^{b_{21}}$ in our construction. However, our theorem still works even if we include it.

The Hilbert space $\mathcal{A}_1$ is a direct sum of three spaces $\mathcal{A}_1^{11}, \mathcal{A}_1^{12}, \mathcal{A}_1^{22}$. The Hilbert space $\mathcal{A}_1^{11}$ is the subspace of $L_a^2(\mathbb{B}^2)$ spanned by $\{z_2^n, z_1 z_2^n, \dots, z_1^{r-1} z_2^n\}_{n \in \mathbb{N}}$, and the Hilbert space $\mathcal{A}_1^{12}$ is the finite dimension subspace of $L_a^2(\mathbb{B}^2)$ spanned by

$$\begin{array}{cccc} 1 & , z_1 & , \dots & , z_1^{p-1}, \\ z_2 & , z_1 z_2 & , \dots & , z_1^{p-1} z_2, \\ & \dots & \dots & \\ z_2^{s-1} & , z_1 z_2^{s-1} & , \dots & , z_1^{p-1} z_2^{s-1}. \end{array}$$

The Hilbert space $\mathcal{A}_2$ is the subspace of $L_a^2(\mathbb{B}^2)$ spanned by $\{z_1^n, z_1^n z_2, \dots, z_1^n z_2^{q-1}\}_{n \in \mathbb{N}}$.

The Hilbert space $\mathcal{A}_2$ is a direct sum of three spaces, $\mathcal{A}_1^{11} \cap \mathcal{A}_1^{12}$, $\mathcal{A}_1^{11} \cap \mathcal{A}_1^{22}$, and $\mathcal{A}_1^{12} \cap \mathcal{A}_1^{22}$.

The Hilbert space $\mathcal{A}_3$ is the subspace of $L_a^2(\mathbb{B}^2)$ spanned by

$$\begin{array}{cccc} 1 & , \dots & , z_1^{r-1}, \\ z_2 & , \dots & , z_1^{r-1} z_2, \\ & \dots & \\ z_2^{q-1} & , \dots & , z_1^{r-1} z_2^{q-1}. \end{array}$$

4.3. Ideal $\langle z_1^2, z_3 - z_2^2 \rangle \subset \mathbb{C}[z_1, z_2, z_3]$

We consider the biholomorphic transformation $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ by $(z_1, z_2, z_3) \mapsto (u_1, u_2, u_3) = (z_1, z_2, z_3 - z_2^2)$. Under the biholomorphic transformation T , the ideal $I = \langle z_1^2, z_3 - z_2^2 \rangle$ is changed to $I' = \langle z_1^2, z_3 \rangle$. T maps the unit ball $\mathbb{B}^3$ to the domain Ω defined as follows,

$$\Omega = \{(u_1, u_2, u_3) \mid |u_1|^2 + |u_2|^2 + |u_3 + u_2^2|^2 < 1\} \subset \mathbb{C}^3.$$

Let dV_Ω be the normalized Lebesgue measure on Ω , and $L_a^2(\Omega)$ be the Bergman space on Ω with respect to the measure dV_Ω . Under the biholomorphic transformation T , the Bergman space $L_a^2(\mathbb{B}^3)$ is mapped isometrically to $L_a^2(\Omega)$. And the closure $\bar{I}$ of the ideal I in $L_a^2(\mathbb{B}^3)$ is mapped to the closure $\bar{I}'$ of the ideal I' in $L_a^2(\Omega)$. We notice that the transformation T maps the polynomial algebra to itself. As A -modules, $L_a^2(\mathbb{B}^3)$ (and $\bar{I}$) and $L_a^2(\Omega)$ (and $\bar{I}'$) are isomorphic.

On $L_a^2(\Omega)$, for the ideal $I' = \langle z_1^2, z_3 \rangle$, we apply the similar construction as in Section 3. There is only one box $\mathcal{B}_j^b$ associated to the ideal I' , where $j = (1, 3)$ and $b = (1, 0)$, and $\mathcal{B} = \{(n_1, n_2, 0) \mid n_1 \leq 1\}$. We consider the subset $\Omega_j = \{(z_1, z_2, z_3) \in \Omega \mid z_1 = z_3 = 0\}$. Ω_j can be identified with an open subset of $\mathbb{C}$ of the form $\{z_2 \mid |z_2|^2 + |z_2|^4 < 1\}$. Let $L_{a,s}^2(\Omega_j)$

be the Hilbert space of square integrable holomorphic functions on Ω_j with respect to the norm

$$(1 - |z_2|^2 - |z_2|^4)^s dV_{\Omega_j},$$

where dV_{Ω_j} is the normalized Lebesgue measure on Ω_j .

Define the Hilbert space $\mathcal{A}$ to be the following direct sum of two weighted Bergman spaces, i.e.

$$\mathcal{A} = L_{a,2}^2(\Omega_j) \oplus L_{a,3}^2(\Omega_j).$$

We define the $A = \mathbb{C}[z_1, z_2, z_3]$ -module structure on $\mathcal{A}_{I'}$ by the following formulas. For $(X, Y) \in L_{a,2}^2(\Omega_j) \oplus L_{a,3}^2(\Omega_j)$,

$$T_{z_1}(X, Y) = (0, X), \quad T_{z_2}(X, Y) = (z_2 X, z_2 Y), \quad T_{z_3}(X, Y) = (0, 0).$$

We notice that monomials $\{z_2^i\}_{i \in \mathbb{N}}$ form an orthogonal basis of both $L_{a,2}^2(\Omega_j)$ and $L_{a,3}^2(\Omega_j)$. A straightforward computation shows that $\mathcal{A}_{I'}$ is an essentially normal Hilbert $\mathbb{C}[z_1, z_2, z_3]$ -module (see also [7]).

We define a map $\Psi_{I'} : L_a^2(\Omega) \rightarrow \mathcal{A}_{I'} = L_{a,2}^2(\Omega_j) \oplus L_{a,3}^2(\Omega_j)$ by

$$\Psi(f) := (f|_{z_1=z_3=0}, \frac{\partial f}{\partial z_1}|_{z_1=z_3=0}).$$

Similar to the argument in the proof of Theorem 1.1 in Section 3.4, we can prove that $\Psi_{I'}$ is a bounded surjective A -module morphism and its kernel is exactly $\bar{I}'$. Hence we have the following short exact sequence of essentially normal A -modules,

$$0 \longrightarrow \bar{I}' \longrightarrow L_a^2(\Omega) \xrightarrow{\Psi_{I'}} \mathcal{A}_{I'} \longrightarrow 0.$$

The biholomorphic transformation T maps $W := \{(0, z_2, z_2^2) \in \mathbb{B}^3\}$ isomorphically to $\Omega_j = \{(0, z_2, 0) \in \Omega\}$. We can replace Ω_j by W in the corresponding definitions of $\mathcal{A}_I$ the map Ψ_I . Using the transformation T , we have a short exact sequence of essentially normal A -modules,

$$0 \longrightarrow \bar{I} \longrightarrow L_a^2(\mathbb{B}^3) \xrightarrow{\Psi_I} \mathcal{A}_I \longrightarrow 0.$$

As a corollary, we can conclude that the ideal $\bar{I}$ and associated quotient module Q_I are both essentially normal. We also have the following geometric description for the corresponding Toeplitz extension,

$$[\mathfrak{T}(Q_I)] = [\mathfrak{T}(L_{a,2}^2(\Omega_j) \oplus L_{a,3}^2(\Omega_j))].$$

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